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PERTURBATIONS OF NEST ALGEBRAS

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INTRODUCTION

Let $\{P_n: n \geqslant 1\}$ be a sequence of finite dimensional projections on a Hilbert space $\mathscr{H}$ such that $P_n \uparrow 1$. An operator T on $\mathscr{H}$ is said to be *triangular* (resp. *quasi-triangular*) relative to $\{P_n\}$ if $P_n^{\perp} T P_n = 0$ for each $n \geqslant 1$ (resp. $\|P_n^{\perp} T P_n\| \to 0$ as $n \to \infty$). Let $\mathscr{T}$ and $\mathscr{Q}\mathscr{T}$ denote the corresponding two algebras of operators on $\mathscr{H}$. The algebra $\mathscr{Q}\mathscr{T}$ was introduced in [1], where it was shown that every operator in $\mathscr{Q}\mathscr{T}$ is a compact perturbation of an operator in $\mathscr{T}$. One deduces easily from this that $\mathscr{Q}\mathscr{T} = \mathscr{T} + \mathscr{K}$, which can be regarded as a characterization of the compact perturbations of operators in $\mathscr{T}$ (here $\mathscr{K}$ denotes the algebra of all compact operators on $\mathscr{H}$).

It is natural to ask the extent to which a characterization like this is valid for other nest algebras in place of $\mathscr{T}$. Specifically, consider the triangular algebra $\mathscr{T}[0, 1]$ of the unit interval, defined as the set of all operators on $L^2[0, 1]$ which leave invariant each subspace $L^2[0, t]$, $0 \leqslant t \leqslant 1$. What we seek is a characterization of operators in $\mathscr{T}_{[0,1]} + \mathscr{K}$ in terms of the projections $\{P_t: 0 \leqslant t \leqslant 1\}$, where P_t denotes the projection onto $L^2[0, t]$. This is accomplished in section 2 (Corollary of 2.3), following a rather general discussion in section 1 which implies that $\mathscr{T}_{[0,1]} + \mathscr{K}$ is norm-closed. In fact, we present a characterization of compact perturbations of arbitrary nest algebras which covers both cases $\mathscr{T}_{[0,1]}$ and $\mathscr{T}$ at once.

1. CLOSURE PROPERTIES OF PERTURBED ALGEBRAS

It is easy to see that for any C^*-algebra $\mathscr{A}$ of operators on a Hilbert space, the algebra $\mathscr{A} + \mathscr{K}$ of all compact perturbations of operators in $\mathscr{A}$ is norm-closed. Indeed,

$$\mathscr{A} + \mathscr{K} = \pi^{-1}(\pi(\mathscr{A}))$$

where π is the Calkin map and $\pi(\mathscr{A})$ is closed since π is a C^*-homomorphism ([5], 1.8.3). This is false for general norm closed non-self-adjoint operator algebras [3].

In this section we obtain a result which contains the required information about nest algebras, as well as a much broader class of operator algebras. In view of the negative result of [3], this appears to be the best state of affairs one could hope for.

Let $\mathscr{S}$ be a complex-linear space of operators on a Hilbert space $\mathscr{H}$ and let $\mathscr{S}^*$ denote tee space of all ultraweakly continuous linear functionals on $\mathscr{S}$. $\mathscr{S}$ is called *local* if there are enough compact operators in $\mathscr{S}$ to separate points in $\mathscr{S}_*$; equivalently, $\mathscr{S}$ is local if, and only if, the ultraweak closure of $\mathscr{S} \cap \mathscr{K}$ contains $\mathscr{S}$. Thus, the most general local linear space of operators is obtained by starting with a linear space $\mathscr{S}_0$ of compact operators and taking for $\mathscr{S}$ any linear space lying between $\mathscr{S}_0$ and the ultraweak closure of $\mathscr{S}_0$.

THEOREM 1.1. *Let $\mathscr{S}$ be a norm-closed local linear space of operators. Then $\mathscr{S} + \mathscr{K}$ is norm-closed, and the canonical isomorphism between the Banach space quotients $\mathscr{S} + \mathscr{K}/\mathscr{S}$ and $\mathscr{K}/\mathscr{S} \cap \mathscr{K}$ is an isometry.*

Proof. Fix a compact operator K. We claim:

$$\inf_{S \in \mathscr{S}} \|K + S\| = \inf_{S \in \mathscr{S} \cap \mathscr{K}} \|K + S\|.$$

We prove only the nontrivial inequality $\geqslant$. By the Hahn-Banach theorem, there is a linear functional f in $\mathscr{K}'$ satisfying $\|f\| = 1$, $f = 0$ on $\mathscr{S} \cap \mathscr{K}$, and

$$|f(K)| = \inf_{S \in \mathscr{S} \cap \mathscr{K}} \|K + S\|.$$

If we express f in the form $f(X) = \text{trace}\,(XT)$, $X \in \mathscr{K}$, where T is a trace class operator satisfying trace $|T| = 1$, then we may consider f as extended to a linear functional $\tilde{f}$ on all bounded operators X by the same formula. $\tilde{f}$ is ultraweakly continuous of norm 1, it vanishes on $\mathscr{S} \cap \mathscr{K}$, and so it vanishes on the ultraweak closure of $\mathscr{S} \cap \mathscr{K}$. Since the latter contains $\mathscr{S}$ by hypothesis, we conclude that $\tilde{f}(\mathscr{S}) = 0$. So for each S in $\mathscr{S}$,

$$|f(K)| = |\tilde{f}(K + S)| \leqslant \|K + S\|.$$

The desired inequality follows by taking the infimum over all S in $\mathscr{S}$.

Now form the Banach quotient space $\mathscr{K}/\mathscr{S} \cap \mathscr{K}$ and the normed linear space $\mathscr{S} + \mathscr{K}/\mathscr{S}$ (the latter should be regarded as a subspace of the quotient $\mathscr{L}(\mathscr{H})/\mathscr{S}$). Define a linear map

$$\alpha : \mathscr{K}/\mathscr{S} \cap \mathscr{K} \to \mathscr{S} + \mathscr{K}/\mathscr{S}$$

by $\alpha(K + \mathscr{S} \cap \mathscr{K}) = K + \mathscr{S}$. The preceding paragraph shows that α is an isometry. Since $\mathscr{K}/\mathscr{S} \cap \mathscr{K}$ is a Banach space, $\mathscr{S} + \mathscr{K}/\mathscr{S}$ is complete and therefore closed as a linear subset of $\mathscr{L}(\mathscr{H})/\mathscr{S}$. Letting $\pi:\mathscr{L}(\mathscr{H})\to\mathscr{L}(\mathscr{H})/\mathscr{S}$ be the canonical projection, we conclude that

$$\mathscr{S}+\mathscr{K}=\pi^{-1}(\mathscr{S}+\mathscr{K}/\mathscr{S})$$

is norm-closed as a linear subspace of $\mathscr{L}(\mathscr{H})$. ▨

COROLLARY 1. *Let $\mathscr{A}$ be norm-closed local algebra of operators. Then $\mathscr{A}+\mathscr{K}$ is a norm-closed algebra and the natural homomorphism of $\mathscr{A}/\mathscr{A}\cap\mathscr{K}$ on $\mathscr{A}+\mathscr{K}/\mathscr{K}$ is an isomorphism of Banach algebras.*

Proof. $\mathscr{A}+\mathscr{K}$ is an algebra because $\mathscr{K}$ is an ideal in $\mathscr{L}(\mathscr{H})$, and the preceding theorem implies that it is norm-closed. Thus we may form the two indicated Banach algebra quotients, and the natural map $\alpha\colon \mathscr{A}/\mathscr{A}\cap\mathscr{K}\to\mathscr{A}+\mathscr{K}/\mathscr{K}$, defined by

$$\alpha\colon A+\mathscr{A}\cap\mathscr{K}\mapsto A+\mathscr{K},$$

is a surjective homomorphism satisfying $\|\alpha\|\leqslant 1$. Since α has trivial kernel, the closed graph theorem implies that α^{-1} is bounded, and the assertion follows. ▨

Notice that, unlike the situation of Theorem 1.1, one cannot assert that the isomorphism of $\mathscr{A}/\mathscr{A}\cap\mathscr{K}$ on $\mathscr{A}+\mathscr{K}/\mathscr{K}$ is an isometry, but merely a contradiction with bounded inverse.

By a *nest* we mean a linearly ordered family of projections on a Hilbert space which contains 0 and 1 and is closed in the strong operator topology. Every nest $\mathscr{P}$ has an associated *nest algebra*, defined as the algebra of all operators which leave each element of $\mathscr{P}$ invariant. It is shown in the appendix (see Corollary 2 of Proposition A) that a nest algebra is the ultraweakly closed linear span of its rank one operators. Thus, every nest algebra is local, and we have

COROLLARY 2. *For every nest algebra $\mathscr{A}$, $\mathscr{A}+\mathscr{K}$ is norm-closed.*

2. COMPACT PERTURBATIONS OF NEST ALGEBRAS

Consider the algebra $\mathscr{I}_{[0,1]}+\mathscr{K}$ of all compact perturbations of operators in the nest algebra $\mathscr{I}_{[0,1]}$ of the unit interval. There is an obvious condition that every operator A in this algebra must satisfy, namely $P_t^{\perp}AP_t$ should be compact for every t in the unit interval [0, 1], where P_t is the projection on $L^2[0,t]$. Indeed, if $A=T+K$ with K compact and T in $\mathscr{I}_{[0,1]}$, then

$$P_t^{\perp}AP_t=P_t^{\perp}KP_t\in\mathscr{K}.$$

This necessary condition for membership in $\mathscr{I}_{[0,1]}+\mathscr{K}$ is not sufficient, as the following example (adapted from pp. 18—19 of [7]) shows. Let $P_1, P_2, \ldots$ be any strictly increasing sequence of nonzero projections in $\{P_t : 0\leqslant t\leqslant 1\}$ such that $P_n \uparrow 1$, and for each $n \geq 1$ put $E_n = P_n - P_{n-1}$, where P_0 is taken as zero. Choose a unit vector e_n in the range of E_n, and define an operator T on $L^2[0, 1]$ by

$$T\zeta = \sum_{n=1}^{\infty} (\zeta, e_n) e_{n+1}.$$

T is clearly a partial isometry with infinite rank which satisfies $TE_n = E_{n+1}T$ and $\|TE_n\| = 1$ for each $n \geq 1$. Since $P_n = E_1 + \dots + E_n$ and since $P_1 T = 0$, the preceding identity implies that $TP_n = (P_{n+1} - P_1)T = P_{n+1} T$, for each $n \geq 0$.

Note first that $P^\perp TP$ has finite rank for each P in $\{P_t : 0 \leq t \leq 1\}$. Indeed, if $P \neq 1$ then we may choose $n \geq 1$ so that $P_{n-1} \leq P < P_n$, hence

$$TP = T(P - P_{n-1}) + TP_{n-1} = T(P - P_{n-1}) + P_n T,$$

$$PTP = PT(P - P_{n-1}) + PT,$$

and so

$$TP - PTP = P^\perp T(P - P_{n-1}) + (P_n - P)T.$$

But each term on the right is at most of rank one, since $P^\perp T(P - P_{n-1}) = P^\perp TE_n(P - P_{n-1})$, $(P_n - P)T = (P_n - P)E_n T$, and both operators TE_n and $E_n T$ have rank one. Thus $TP - PTP$ has rank at most two.

Second, we claim that there is no compact operator K such that $T - K$ belongs to $\mathscr{I}_{[0,1]}$. Indeed, if K is any operator for which $T - K$ leaves each P_n invariant, then notice that $TE_n = E_{n+1}KE_n$. For

$$TE_n = E_{n+1}TE_n = E_{n+1}(T - K)E_n + E_{n+1}KE_n = E_{n+1}KE_n,$$

because

$$E_{n+1}(T - K)E_n = E_{n+1} P_n^\perp (T - K)P_n E_n = 0.$$

It follows that

$$\|KE_n\| \geq \|E_{n+1} KE_n\| = \|TE_n\| = 1$$

for each n. This inequality implies that K cannot be compact because the E_n's are mutually orthogonal.

What we need for a second condition is an appropriate generalization of the notion of quasitriangularity described in the introduction which applies to arbitrary nests.

DEFINITION 2.1. *Let $\mathscr{P}$ be a nest of projections on $\mathscr{H}$. An operator T on $\mathscr{H}$ is said to be quasitriangular relative to $\mathscr{P}$ if*

(i) $P^\perp TP$ is compact for each P in $\mathscr{P}$, and

(ii) the function $P \in \mathscr{P} \mapsto P^\perp TP \in \mathscr{K}$ is continuous.

Here, $\mathscr{K}$ is topologized by its *norm* topology and $\mathscr{P}$ has the strong operator topology. To see that this does generalize the preceding definition of quasitriangularity, let $P_n \uparrow 1$ be as in the introduction and let $\mathscr{P}$ be the nest $\{P_n: n \geqslant 1\} \cup \{0, 1\}$. Notice that the projection 1 is the only accumulation point of $\mathscr{P}$; since $P^\perp TP = 0$ for $P = 1$, we see that 2.1 (*ii*) reduces to the assertion

$$\lim_{n\to\infty} \|P_n^\perp TP_n\| = 0.$$

It is of some interest that definition 2.1 can be given in an alternate form which involves only the *range* of the mapping $P \to P^\perp TP$, and we digress momentarily to present this. Throughout this section, $\mathscr{H}$ will denote a complex Hilbert space.

PROPOSITION 2.2. *Let $\mathscr{P}$ be a nest of projections on $\mathscr{H}$ and let T be an operator such that $P^\perp TP$ is compact for every P in $\mathscr{P}$.*

Then T is quasitriangular relative to $\mathscr{P}$ if, and only if $\{P^\perp TP: P \in \mathscr{P}\}$ is a norm-compact set of operators.

Proof. The "only if" assertion follows from the elementary fact that the continuous image of a compact space is compact.

For the converse, assume that the set

$$\mathscr{S} = \{P^\perp TP: P \in \mathscr{P}\}$$

is norm-compact. Note that the relative norm and strong operator topologies on $\mathscr{S}$ must coincide; for the identity map of ($\mathscr{S}$, norm) onto ($\mathscr{S}$, strong) is a continuous injective map of a compact space onto a Hausdorff space, and is therefore a homeomorphism.

Since operator multiplication is (jointly) strongly continuous on the unit ball of $\mathscr{L}(\mathscr{H})$, we see that the function $P \mapsto P^\perp TP$ is continuous, considered as a map of $\mathscr{P}$ into ($\mathscr{S}$, strong). By the preceding paragraph, we deduce the required property 2.1 (*ii*). ▨

Let $\mathscr{P}$ be a nest of projections on $\mathscr{H}$, let $\mathscr{I}_{\mathscr{P}}$ be its associated nest algebra, and let $\mathscr{Q}\mathscr{I}_{\mathscr{P}}$ be the algebra of operators which are quasitriangular relative to $\mathscr{P}$. In the proof of Theorem 2.3, we will make use of the distance formula from [1]:

$$\inf_{T\in\mathscr{I}_{\mathscr{P}}} \|A - T\| = \sup_{P\in\mathscr{P}} \|P^\perp AP\|,$$

for every operator A on $\mathscr{H}$.

THEOREM 2.3. *For any nest $\mathscr{P}$,*

$$\mathscr{Q}\mathscr{I}_{\mathscr{P}} = \mathscr{I}_{\mathscr{P}} + \mathscr{K}.$$

Proof. We first prove the inclusion $\supseteq$. If T belongs to $\mathscr{I}_{\mathscr{P}}$ and K is any compact operator, then as we have already observed $P^\perp(T + K)P = P^\perp KP \in \mathscr{K}$, for each P in $\mathscr{P}$. So it suffices to show that, for fixed K, $P \mapsto P^{\perp}KP$ is a continuous function from $\mathscr{P}$ to $\mathscr{K}$.

Assume first that K has rank one, that is $K\xi = (\xi, e)f$, where e and f are vectors in the underlying Hilbert space. For any two projections P and Q, we can write

$$P^{\perp}KP\xi - Q^{\perp}KQ\xi = (P\xi, e)\, P^{\perp}f - (Q\xi, e)\, Q^{\perp}f$$

$$= (\xi, Pe - Qe)\, P^{\perp}f + (\xi, Qe)\,(P^{\perp} f - Q^{\perp}f).$$

It follows that

$$\|P^{\perp}KP - Q^{\perp}KQ\| \leqslant \|Pe - Qe\| \cdot \|f\| + \|Pf - Qf\| \cdot \|e\|,$$

and hence the left side goes to zero as P converges strongly to Q.

By taking finite linear combinations, it follows that $P \mapsto P^{\perp}KP$ is continuous for any finite rank K. Finally, if K is an arbitrary compact operator, choose finite rank operators $K_1, K_2, \ldots$ so that $\|K - K_n\| \to 0$ as $n \to \infty$. Then since

$$\sup_{P \in \mathscr{P}} \|P^{\perp}K_nP - P^{\perp}KP\| \leqslant \|K_n - K\|$$

tends to zero as $n \to \infty$, the function $P \mapsto P^{\perp}KP$ appears as a uniform limit of continuous functions from $\mathscr{P}$ to $\mathscr{K}$. Hence, $P \mapsto P^{\perp}KP$ is continuous.

For the opposite inclusion $\subseteq$, let A belong to $\mathscr{Q}\mathscr{I}_{\mathscr{P}}$ and choose $\varepsilon > 0$. We will find a compact operator $K = K_{\varepsilon}$ such that the distance from $A - K$ to $\mathscr{I}_{\mathscr{P}}$ is at most ε. Because ε is arbitrary, this implies that A belongs to the norm closure of $\mathscr{I}_{\mathscr{P}} + \mathscr{K}$, and the theorem will then follow from the results of the previous section.

To obtain K, we make use of the fact that the set $\{P^{\perp}AP : P \in \mathscr{P}\}$, being compact in the norm topology, contains on ε — net $\{P_j^{\perp}AP_j : 0 \leqslant j \leqslant n\}$. Thus, for every P in there is a j, $0 \leqslant j \leqslant n$, so that

$$\|P^{\perp}AP - P_j^{\perp}AP_j\| \leqslant \varepsilon.$$

There is clearly no loss if we assume that

$$0 = P_0 < P_1 < \ldots < P_n = 1.$$

Putting $E_j = P_j - P_{j-1}$, define

$$K = \sum_{j=1}^{n} P_j^{\perp}AE_j.$$

By property 2.1(i), each operator $P_j^{\perp}AE_j = P_j^{\perp}AP_jE_j$ is compact, and hence K is compact.

By the remark preceding this theorem, to prove that the distance from $A - K$ to $\mathcal{I}_{\mathcal{P}}$ is at most ε it suffices to show that $\|Q^{\perp}(A - K)Q\| \leqslant \varepsilon$ for each Q in $\mathcal{P}$.

For that, fix Q. Then there is a j, $0 \leqslant j \leqslant n$, so that $P_{j-1} \leqslant Q \leqslant P_j$, and we claim first that

$$Q^{\perp}(A - K)\, Q = E_j Q^{\perp} AQE_j.$$

Indeed,

$$A - K = \sum_{k=1}^{n} AE_k - \sum_{k=1}^{n} P_k^{\perp} AE_k = \sum_{k=1}^{n} P_k AE_k .$$

But since $E_k Q = 0$ if $k > j$ and $Q^{\perp}P_k = 0$ if $k < j$, we have $Q^{\perp}P_k AE_k Q = 0$ for all $k \neq j$, and hence

$$Q^{\perp}(A - K)Q = Q^{\perp}P_j AE_j Q = E_j Q^{\perp} AQE_j.$$

Now choose k, $0 \leqslant k \leqslant n$, so that

$$\|Q^{\perp}AQ - P_k^{\perp}AP_k\| \leqslant \varepsilon.$$

Then $E_j Q^{\perp} AQE_j$ is within ε of $E_j P_k^{\perp} AP_k E_j$, and so

$$\|Q^{\perp}(A - K)Q\| = \|E_j Q^{\perp} AQE_j\| \leqslant \|E_j P_k^{\perp} AP_k E_j\| + \varepsilon.$$

But since $P_k E_j = 0$ for $k < j$ and $E_j P_k^{\perp} = 0$ for $k \geqslant j$, it follows that $E_j P_k^{\perp} AP_k E_j$ vanishes for all k, and we have the desired conclusion that $\|Q^{\perp}(A - K)Q\| \leqslant \varepsilon$. ▨

Let $\{P_n\}$ be a sequence of finite dimensional projections such that $P_n \uparrow 1$, and let $\mathcal{I}$ be the nest algebra alg $\{P_n\}$. It is clear that theorem 2.3 implies the characterization given in [1] of compact perturbations of operators in $\mathcal{I}$, namely that A belongs to $\mathcal{I} + \mathcal{K}$ if and only if $\|P_n^{\perp} AP_n\|$ tends to zero as $n \to \infty$. On the other hand, it is a simple matter to deduce the following characterization of compact perturbations of operators in the triangular algebra $\mathcal{I}_{[0,\,1]}$ of the unit interval. Letting P_t denote the projection of $L^2[0, 1]$ onto the subspace $L^2[0, t]$, we have

COROLLARY. *Let A be a bounded operator on $L^2[0, 1]$. In order that A should have a decomposition $A = T + K$ with $T \in \mathcal{I}_{[0,\,1]}$ and K compact, it is necessary and sufficient that $P_t^{\perp} AP_t$ be compact for all $t \in [0, 1]$ and the function $t \mapsto P_t^{\perp} AP_t$ be (norm) continuous.*

Proof. This follows from 2.3 after noting that the map $t \mapsto P_t$ is a homeomorphism of the closed unit interval onto the nest $\{P_t : 0 \leqslant t \leqslant 1\}$. ▨

APPENDIX: THE PREDUAL OF A NEST ALGEBRA

Let $\mathcal{P}$ be a nest of projections on a Hilbert space $\mathcal{H}$, and let $\mathcal{A} = \text{alg}\, \mathcal{P}$ be its associated nest algebra. The purpose of this section is to describe the annihilator of $\mathscr{A}$ in the space of all ultraweakly continuous linear functionals on $\mathscr{L}(\mathscr{H})$, and thereby identify the predual of $\mathscr{A}$.

By an *atom* (associated with $\mathscr{P}$) we mean a nonzero projection of the form $Q-P$, where P and Q are projections in $\mathscr{P}$ such that P is the immediate predecessor of Q; that is, $P < Q$, and if $R \in \mathscr{P}$ satisfies $P \leqslant R \leqslant Q$, then $R = P$ or $R = Q$. It is easily seen that the atoms are simply the minimal projections in the von Neumann algebra generated by $\mathscr{P}$. It follows that distinct atoms are orthogonal, and for each operator A in $\mathscr{L}(\mathscr{H})$ we can form the operator

$$\delta(A) = \Sigma EAE,$$

the sum extended over all atoms E associated with $\mathscr{P}$ (if $\mathscr{P}$ has no atoms then δ is defined to be the zero mapping). δ is a normal positive linear map of $\mathscr{L}(\mathscr{H})$ into the diagonal $\mathscr{A} \cap \mathscr{A}^*$ of $\mathscr{A}$. Moreover, since each projection E appearing in the sum is a nested difference of $\mathscr{A}$-invariant projections, we have $EABE = EAEBE$ for all A, B in $\mathscr{A}$. It follows that *the restriction of δ to $\mathscr{A}$ is a homomorphism.*

The Banach space $\mathscr{L}(\mathscr{H})_*$ of all ultraweakly continuous linear functionals on $\mathscr{L}(\mathscr{H})$ is identified with the space $\mathscr{L}^1(\mathscr{H})$ of all trace class operators X having the natural norm

$$\|X\|_1 = \mathrm{tr}((X^*X)^{1/2}),$$

and this identification associates the linear functional ρ with the operator X via the formula $\rho(T)=\mathrm{tr}(XT)$, $T \in \mathscr{L}(\mathscr{H})$. The annihilator of $\mathscr{A}$ in $\mathscr{L}(\mathscr{H})_*$ is described as follows.

PROPOSITION A. *Let $\rho \in \mathscr{L}(\mathscr{H})_*$. Then ρ annihilates $\mathscr{A}$ iff ρ has the form*

$$\rho(T) = \mathrm{tr}(XT),$$

where X is a trace class operator in $\mathscr{A}$ satisfying $\delta(X) = 0$.

Proof. $\Rightarrow$: Let ρ be an ultraweakly continuous linear function on $\mathscr{L}(\mathscr{H})$ which annihilates $\mathscr{A}$ and let X be a trace class operator such that $\rho(\cdot) = \mathrm{tr}(X\cdot)$. We will show that $X \in \mathscr{A}$ and $\delta(X) = 0$.

Fix $P \in \mathscr{P}$ and $T \in \mathscr{L}(\mathscr{H})$. Note that $PTP^{\perp}$ belongs to $\mathscr{A}$. Indeed, if Q is any projection in $\mathscr{P}$, then either $Q \leqslant P$ or $Q \geqslant P$ and hence $P^{\perp}Q = 0$ or $Q^{\perp}P = 0$; in either case we have $Q^{\perp}(PTP^{\perp})Q = 0$, which shows that $PTP^{\perp} \in \mathrm{alg}\,\mathscr{P} = \mathscr{A}$. Since $\mathrm{tr}(X\cdot)$ vanishes on $\mathscr{A}$ we have

$$\mathrm{tr}(P^{\perp}XPT) = \mathrm{tr}(XPTP^{\perp}) = 0,$$

and therefore $P^{\perp}XP=0$ since T is arbitrary. The conclusion $X \in \mathscr{A}$ now follows.

To prove that $\delta(X) = 0$ it suffices to show that $EXE = 0$ for every atom E. But since $ETE \in \mathscr{A} \cap \mathscr{A}^* \subseteq \mathscr{A}$ for each T in $\mathscr{L}(\mathscr{H})$, we have

$$\operatorname{tr}(EXET) = \operatorname{tr}(XETE) = 0,$$

and hence $EXE = 0$ since T was arbitrary.

In order to prove the converse, we require some preliminaries. We remark that the referee has kindly pointed out that the essential ingredients of Lemmas 2 and 3 below are contained in pp. 23 — 28 of [9]. We have retained this discussion for completeness.

LEMMA 1. *Let e be a vector in $\mathscr{H}$ such that $Ee = 0$ for every atomic projection E associated with $\mathscr{P}$. Then $\{\|Pe\| : P \in \mathscr{P}\}$ is the entire interval* $[0, \|e\|]$.

Proof. Clearly $\{\|Pe\| : P \in \mathscr{P}\}$ is a subset of $[0, \|e\|]$ which contains both end-points. So if it is not the entire interval then there is a t, $0 < t < \|e\|$, for which $\|Pe\| \neq t$ for every $P \in \mathscr{P}$. Define

$$\mathscr{S}_1 = \{P \in \mathscr{P} \colon \|Pe\| < t\}$$

$$\mathscr{S}_2 = \{P \in \mathscr{P} \colon \|Pe\| > t\}.$$

$\mathscr{S}_1$ and $\mathscr{S}_2$ are complementary sets in $\mathscr{P}$, and $P_1 < P_2$ for any pair of projections $P_i \in \mathscr{S}_1$. Put.

$$P_1 = \bigvee \mathscr{S}_1 \text{ and } P_2 = \bigwedge \mathscr{S}_2.$$

Then each P_i belongs to $\mathscr{P}$ and $P_1 \leqslant P_2$. Moreover, by a familiar lemma ([4], Appendice II), P_i must belong to the strong closure of $\mathscr{S}_1$, $i = 1, 2$. Since $\|Pe\| < t$ for each. $P \in \mathscr{S}_1$, we have $\|P_1e\| \leqslant t$; and since $\|P_1e\|$ cannot equal t, it follows that $\|P_1e\| < t$. Similarly, $\|P_2e\| > t$.

In particular, $P_1 < P_2$. We claim now that $P_2 - P_1$ is atomic. For if $Q \in \mathscr{P}$ satisfies $P_1 < Q$, then Q cannot belong to $\mathscr{S}_1$, so that $Q \in \mathscr{S}_2$, and hence $Q \geqslant P_2$ as asserted. By hypothesis we conclude that $\|(P_2 - P_1)e\| = 0$, and hence $\|P_1e\| = \|P_2e\|$. This is absurd, however, since $\|P_1e\| < t < \|P_2e\|$. ▨

For every finite subset $\mathscr{F}$ of $\mathscr{P}$ having the form

$$\mathscr{F} = \{0 = P_0 < P_1 < \ldots < P_n = 1\},$$

define a mapping $\delta_{\mathscr{F}} \colon \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ by

$$\delta_{\mathscr{F}}(T) = \sum_{j=1}^{n} (P_j - P_{j-1})\, T(P_j - P_{j-1}).$$

Each $\delta_{\mathscr{F}}$ is a unital, completely positive expectation of $\mathscr{L}(\mathscr{H})$ onto the commutant $\mathscr{F}'$ of $\mathscr{F}$; the restriction of $\delta_{\mathscr{F}}$ to $\mathscr{A}$ is an algebra homomorphism; and for $\mathscr{F} \subseteq \mathscr{G}$ we have

$$\delta_{\mathscr{G}} \circ \delta_{\mathscr{F}} = \delta_{\mathscr{F}} \circ \delta_{\mathscr{G}} = \delta_{\mathscr{G}}.$$

Considering the finite sets $\mathscr{F}$ as an increasing directed family relative to set inclusion, we have

LEMMA 2. *For every compact operator* X,

$$\lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(X) - \delta(X)\| = 0.$$

Proof. Let A be the sum of all atomic projections associated with $\mathscr{P}$. We claim first that

$$\lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(A^{\perp}X)\| = \lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(XA^{\perp})\| = 0,$$

for every compact operator X. Since $XA^{\perp} = (A^{\perp}X^*)^*$ and each $\delta_{\mathscr{F}}$ maps adjoints to adjoints, it suffices to show that

$$\lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(XA^{\perp})\| = 0.$$

Suppose first that X has rank one, that is $X\xi = (\xi, e)f$ where e and f are vectors in $\mathscr{H}$. Then

$$XA^{\perp}\xi = (A^{\perp}\xi, e)f = (\xi, A^{\perp}e)f.$$

Choose numbers $0 = t_0 < t_1 < \ldots < t_n = \|A^{\perp}e\|$ so that $|t_i^2 - t_{i-1}^2| \leqslant \varepsilon^2$ for each i. Applying Lemma 1, we may find projections $0 = P_0 < P_1 < \ldots < P_n = 1$ in $\mathscr{P}$ so that $\|P_j A e\| = t_j$, $0 \leqslant j \leqslant n$. This clearly implies

$$\|(P_j - P_{j-1})A^{\perp}e\|^2 = t_j^2 - t_{j-1}^2 \leqslant \varepsilon^2.$$

Letting

$$\mathscr{F} = \{P_0, P_1, \ldots, P_n\},$$

we claim that

$$\|\delta_{\mathscr{F}}(XA^{\perp})\| \leqslant \varepsilon \|f\|.$$

Indeed, if $\xi \in \mathscr{H}$, then

$$\begin{aligned}\|\delta_{\mathscr{F}}(XA^{\perp})\,\xi\|^2 &= \Big\|\sum_{j=1}^{n} (\xi, (P_j - P_{j-1})\,A^{\perp}e)\,(P_j - P_{j-1})f\,\Big\|^2 = \\ &= \sum_{j=1}^{n} |(\xi, (P_j - P_{j-1})\,A^{\perp}e)|^2 \|\,(P_j - P_{j-1})f\,\|^2 \leqslant \\ &\leqslant \|\xi\|^2\varepsilon^2 \sum_{j=1}^{n} \|(P_j - P_{j-1})f\,\|^2 = \\ &= \|\xi\|^2\varepsilon^2\|f\,\|^2,\end{aligned}$$

as asserted.

Now if $\mathscr{G} \supseteq \mathscr{F}$, then since $\delta_{\mathscr{G}} = \delta_{\mathscr{G}} \circ \delta_{\mathscr{F}}$, we have

$$\begin{aligned}\|\delta_{\mathscr{G}}(XA^{\perp})\| &= \|\delta_{\mathscr{G}}(\delta_{\mathscr{F}}(XA^{\perp}))\| \\ &\leqslant \|\delta_{\mathscr{F}}(XA^{\perp})\| \leqslant \varepsilon \, \|f\|.\end{aligned}$$

It follows that $\lim \|\delta_{\mathscr{F}}(XA)\| = 0$ for every rank one operator X.

By taking finite linear combinations, the same conclusion holds for all finite rank operators X; since the maps $\delta_{\mathscr{F}}$ are all uniformly bounded in norm by 1 and since the finite rank operators are norm dense in the space of compact operators, we conclude that

$$\lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(XA^{\perp})\| = 0$$

for every compact operator X.

To finish the proof, let X be an arbitrary compact operator. Then we can write

$$X = XA^{\perp} + A^{\perp}XA + AXA.$$

Since both terms $\delta_{\mathscr{F}}(XA^{\perp})$ and $\delta_{\mathscr{F}}(A^{\perp}XA)$ tend to zero in norm as $\mathscr{F}$ increases, it suffices to show that

$$\lim_{\mathscr{F}} \|\delta_{\mathscr{F}}(AXA) - \delta(X)\| = 0.$$

Fix $\varepsilon > 0$. Since AXA is a compact operator which lives in $A\mathscr{H}$, we can find a finite set of atoms $A_1, \ldots, A_n$ so that $\|PAXAP\| \leqslant \varepsilon$ for every projection P satisfying $P \perp A_1 + \ldots + A_n$. We may assume that each A_j has the form $A_j = Q_j - P_j$, where

$$P_1 < Q_1 \leqslant P_2 < Q_2 \leqslant \ldots \leqslant P_n < Q_n.$$

Let

$$\mathscr{F} = \{0, P_1, Q_1, \ldots, P_n, Q_n, 1\}.$$

Then every finite set which contains $\mathscr{F}$ has the property that Q_j is the immediate successor to P_j in $\mathscr{G}$, for each $j = 1, 2, \ldots, n$. Thus $\delta_{\mathscr{G}}(AXA)$ has the form

$$\delta_{\mathscr{G}}(AXA) = \sum_{j=1}^{n} A_j X A_j + \sum_{E} EAXAE,$$

where the projections E in the second sum are all orthogonal to $A_1 + \ldots + A_n$. Thus

$$\|\Sigma EAXAE\| = \sup_E \|EAXAE\| \leqslant \varepsilon.$$

Similarly,

$$\delta(X) = \sum_{j=1}^{n} A_j X A_j + \sum_{F} FXF,$$

where each F in the second sum satisfies $F \leqslant A$ and $F \perp A_1 + \ldots + A_n$. Hence $\|\Sigma FXF\| \leqslant \varepsilon$. It follows that

$$\|\delta_{\mathscr{G}}(AXA) - \delta(X)\| \leqslant 2\varepsilon,$$

whenever $\mathscr{G} \supseteq \mathscr{F}$. ▨

LEMMA 3. *Let X be a compact operator in $\mathscr{A}$ such that $\delta(X) = 0$. Then X belongs to the radical of $\mathscr{A}$.*

Proof. Assume first that $\mathscr{P}$ is finite, say

$$\mathscr{P} = \{0 = P_0 < P_1 < \ldots < P_n = 1\}.$$

In this case δ has the form

$$\delta(T) = \sum_{j=1}^{n} E_j T E_j,$$

where $E_j = P_j - P_{j-1}$. We claim first that each $X \in \mathscr{A}$, for which $\delta(X) = 0$, must satisfy $X^n = 0$. Indeed, since $P_j^\perp X P_j = 0$ for all j we have $E_p X E_q = 0$ for all $p > q$, and since $\delta(X) = 0$ we in fact have $E_p X E_q = 0$ for all $p \geqslant q$. Because

$$E_1 + \ldots + E_n = 1,$$

we can write X in the form

$$X = \sum_{p,\, q=1}^{n} E_p X E_q = \sum_{p<q} E_p X E_q.$$

It follows that X^n is a finite sum of operators of the form

$$E_{p_1} X E_{q_1} E_{p_2} X E_{q_2} \ldots E_{pn} X E_{q_n},$$

where $p_1 < q$, $p_2 < q_2, \ldots, p_n < q_n$. Note that each product of this form must be zero. For if it is not, then by the orthogonality of the E's we must have

$$p_1 < q_1 = p_2 < q_2 = \ldots = p_n < q_n,$$

and in particular $1 \leqslant p_1 < p_2 < \ldots < p_n \leqslant n$, an absurdity.

Notice that, in fact, $(AX)^n = (AX)^n = 0$ for all A in $\mathscr{A}$. For

$$\delta(AX) = \delta(A)\,\delta(X) = 0 \text{ and } \delta(XA) = \delta(X)\,\delta(A) = 0,$$

and the preceding argument implies that both AX and XA are nilpotent of index n. In particular, this shows that X belongs to the radical of $\mathscr{A}$.

In the case of a general nest $\mathscr{P}$, we can argue as follows. Fix a compact $X \in \mathscr{A}$ with $\delta(X) = 0$, and choose $\varepsilon > 0$. By lemma 2, there is a finite subset

$$\mathscr{F} = \{0 = P_0 < P_1 < \ldots < P_n = 1\}$$

of $\mathscr{P}$ such that

$$\|\delta_{\mathscr{F}}(X)\| \leqslant \varepsilon.$$

Consider $X_0 = X - \delta_{\mathscr{F}}(X)$. This is a compact operator in alg $\mathscr{F}$ for which $\delta_{\mathscr{F}}(X_0)=0$. By the argument just given, we see that

$$(AX_0)^n = (X_0A)^n = 0$$

for every A in alg $\mathscr{F}$, and in particular this holds for all A in $\mathscr{A} \subseteq \text{alg}\,\mathscr{F}$. This, implies that X_0 belongs to the radical of $\mathscr{A}$. Since $\|X - X_0\| \leqslant \varepsilon$, ε is arbitrary and since the radical of $\mathscr{A}$ is norm-closed, we conclude that X belongs to the radical of $\mathscr{A}$. ▨

Returning now to the proof of Proposition A, let X be a trace class operator, in $\mathscr{A}$ satisfying $\delta(X) = 0$. We claim that tr $(XA) = 0$ for each A in $\mathscr{A}$. Indeed, for each A in $\mathscr{A}$ XA is a trace-class operator satisfying

$$\delta(XA) = \delta(X)\,\delta(A) = 0.$$

By lemma 3, XA must belong to the radical of $\mathscr{A}$ and in particular is quasinilpotent. Since the trace of a quasinilpotent trace class operator is zero ([6] XI.9.18), we conclude that

$$\text{tr}\,(XA) = 0. ▨$$

It is well-known that the Hardy space H^∞ of the unit disc is the dual of the Banach space quotient

$$L^1/H_0^1,$$

where H_0^1 denotes all functions in H^1 which vanish at the origin. In view of Proposition A, we may infer an analogous result about any nest algebra $\mathscr{A} = \text{alg}\,\mathscr{P}$. Letting $\mathscr{H}_0^1(\mathscr{A})$ denote the space of all trace class operators X in $\mathscr{A}$ satisfying $\delta(X) = 0$, we have

COROLLARY 1. *$\mathscr{A}$ is the dual of the quotient space $\mathscr{L}^1(\mathscr{H})/\mathscr{H}_0^1(\mathscr{A})$.*

Proof. Since $\mathscr{L}(\mathscr{H})$ is the dual of $\mathscr{L}^1(\mathscr{H})$ and $\mathscr{H}_0^1(\mathscr{A})$ is the annihilator of $\mathscr{A}$ in $\mathscr{L}^1(\mathscr{H})$, the formula is evident from first principles. ▨

COROLLARY 2. *Every nest algebra $\mathscr{A}$ is the ultraweakly closed linear span of its rank one operators.*

Proof. Let ρ be an ultraweakly continuous linear functional which annihilates each rank one operator in $\mathscr{A}$, and write $\rho(\cdot) = \text{tr}\,(X\cdot)$, where $X \in \mathscr{L}^1(\mathscr{H})$. By proposition A it suffices to show that $X \in \mathscr{A}$ and $\delta(X) = 0$.

To see that $X \in \mathscr{A}$, fix P in the lattice lat $\mathscr{A} = \mathscr{P}$. Choose vectors $e \in P^\perp\mathscr{H}$ and $f \in \mathscr{H}$, and let $e \otimes f$ be the rank one operator $\xi \mapsto (\xi, e)f$. Since

$$e \otimes f = P(e \otimes f)P^\perp \in P\mathscr{L}(\mathscr{H})\,P^\perp,$$

it follows as in the proof of proposition A that $e \otimes f \in \mathscr{A}$. Hence

$$\operatorname{tr}(X(e \otimes f)) = 0.$$

But

$$X(e \otimes f) = e \otimes Xf,$$

and so

$$0 = \operatorname{tr}(X(e \otimes f)) = \operatorname{tr}(e \otimes Xf) = (Xf, e).$$

Since e (resp. f) is arbitrary in $P^{\perp}\mathscr{H}$ (resp. $P\mathscr{H}$), if follows that $P^{\perp}XP = 0$. Hence, $X \in \operatorname{alg} \mathscr{P} = \mathscr{A}$.

We claim now that $\delta(X) = 0$; equivalently, $EXE = 0$ for every atomic projection E associated with $\mathscr{P}$. Indeed, since E is atomic we must have either $E \leqslant P$ or $E \perp P$ for every $P \in \mathscr{P}$. So if e, f are any vectors in $E\mathscr{H}$, then $P^{\perp}(e \otimes f)P = Pe \otimes P^{\perp} f = 0$, and hence $e \otimes f \in \mathscr{A}$. It follows that

$$(Xf, e) = \operatorname{tr}(e \otimes Xf) = \operatorname{tr}(X(e \otimes f)) = 0.$$

Arguing as above, we conclude that $EXE = 0$. ▨

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