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ANALYTIC PERTURBATIONS OF THE $\bar{\partial}$-OPERATOR AND INTEGRAL REPRESENTATION FORMULAS IN HILBERT SPACES

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1. INTRODUCTION

In this paper we present the construction of some operator-valued kernels which occur naturally in the study of certain integral representation formulas, in particular in the analytic functional calculus for several commuting operators in Hilbert spaces. These integral kernels are obtained in connection with the analytic perturbations of a specific type of the $\bar{\partial}$-operator, when $\bar{\partial}$ is regarded as a closed operator on Hilbert spaces of square integrable vector-valued exterior forms.

Let H be a complex Hilbert space and $\mathscr{C}(H)$ ($\mathscr{L}(H)$) the set of all densely defined closed (bounded) operators, acting in H. For any $T \in \mathscr{C}(H)$ we denote by $\mathscr{D}(T)$, $\mathscr{R}(T)$, $\mathscr{K}(T)$ the domain of definition, the range and the kernel of T, respectively.

In what follows we shall deal mainly with operators $T \in \mathscr{C}(H)$ having the property $\mathscr{R}(T) \subset \mathscr{K}(T)$, i.e., roughly speaking, with operators T satisfying $T^2 = 0$. Such an operator T will be called *exact* when one actually has $\mathscr{R}(T) = \mathscr{K}(T)$. The exactness of an operator $T \in \mathscr{C}(H)$ with $\mathscr{R}(T) \subset \mathscr{K}(T)$ is equivalent to the invertibility in $\mathscr{L}(H)$ of the operator $T + T^*$, where T^* denotes the adjoint of T; this is a simple and useful criterion from which some of the main results of this paper will be derived.

Let us consider a finite system of indeterminates $\sigma = (\sigma_1, \ldots, \sigma_n)$. The exterior algebra over the complex field $\mathbf{C}$ generated by $\sigma_1, \ldots, \sigma_n$ will be denoted by $\Lambda[\sigma]$. For any integer p, $0 \leqslant p \leqslant n$, we denote by $\Lambda^p[\sigma]$ the space of all homogeneous exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$. The space $\Lambda[\sigma]$ has a natural structure of Hilbert space in which the elements

$$\sigma_{j_1} \wedge \ldots \wedge \sigma_{j_p} \quad (1 \leqslant j_1 < \ldots < j_p \leqslant n;\ p = 1, \ldots, n)$$

as well as $1 \in \mathbf{C} = \Lambda^0[\sigma]$ form an orthogonal basis (the symbol "$\wedge$" stands for the exterior product).

Let us define the operators

$$S_j \xi = \sigma_j \wedge \xi \quad (\xi \in \Lambda[\sigma];\ j = 1, \ldots, n). \tag{1.1}$$

Then the adjoints of the operators (1.1) are given by the formula

$$S_j^*(\xi_j' + \sigma_j \wedge \xi_j'') = \xi_j'', \quad (j = 1, \ldots, n), \tag{1.2}$$

where $\xi_j' + \sigma_j \wedge \xi_j''$ is the canonical decomposition of an arbitrary element $\xi \in \Lambda[\sigma]$, with ξ_j' and ξ_j'' not containing σ_j. Note the anticommutations relations

$$\begin{aligned} S_j S_k + S_k S_j &= 0 \\ S_j S_k^* + S_k^* S_j &= \varepsilon_{jk} \end{aligned} \qquad (j, k = 1, \ldots, n), \tag{1.3}$$

where ε_{jk} is the Kronecker symbol, which can be readily obtained from (1.1) and (1.2).

For an arbitrary complex linear space L we denote by $\Lambda[\sigma, L]$ the tensor product $L \otimes \Lambda[\sigma]$. If λ is any endomorphism of L then the action of λ is extended on $\Lambda[\sigma, L]$ by the endomorphism $\lambda \otimes 1$. We identify these endomorphisms and keep the notation λ for both of them. Analogously, if θ is any endomorphism of $\Lambda[\sigma]$ then the endomorphism $1 \otimes \theta$, acting on $\Lambda[\sigma, L]$, will be also denoted by θ.

Any commuting system of endomorphisms $\alpha = (\alpha_1, \ldots, \alpha_n)$ acting on L will be associated with the endomorphism δ_α on $\Lambda[\sigma, L]$, defined by the relation

$$\delta_\alpha \xi = (\alpha_1 S_1 + \ldots + \alpha_n S_n)\xi \quad (\xi \in \Lambda[\sigma, L]). \tag{1.4}$$

From (1.3) we have that $(\delta_\alpha)^2 = 0$.

Assume now that L is a Hilbert space H. Then $\Lambda[\sigma, H]$ is also a Hilbert space. The action of any $T \in \mathscr{C}(H)$ will be extended by $T \otimes 1$, denoted simply by T, defined on $\mathscr{D}(T) \otimes \Lambda[\sigma] = \Lambda[\sigma, \mathscr{D}(T)]$. Clearly, for any endomorphism θ of $\Lambda[\sigma]$ we have $\theta T \subset T\theta$ (for T_1, T_2 in $\mathscr{C}(H)$ the notation $T_1 \subset T_2$ means that T_2 is an extension of T_1).

Let Ω be an open set in $\mathbf{C}^m$ and $C^\infty(\Omega, H)$ $(A(\Omega, H))$ the set of all H-valued indefinitely differentiable (analytic) functions on Ω. Consider a commuting system $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $A(\Omega, \mathscr{L}(H))$, i.e. a system of operator-valued analytic functions such $\alpha_j(z)\alpha_k(w) = \alpha_k(w)\alpha_j(z)$ for any $j, k = 1, \ldots, n$ and z, w in Ω. The corresponding endomorphism (1.4) for $L = C^\infty(\Omega, H)$ will be then given by

$$\delta_\alpha \xi(z) = (\alpha_1(z) S_1 + \ldots + \alpha_n(z) S_n)\xi(z), \quad (z \in \Omega), \tag{1.5}$$

where $\xi \in \Lambda[\sigma, C^\infty(\Omega, H)]$. We can consider also the usual $\bar{\partial}$-operator

$$\bar{\partial} = \frac{\partial}{\partial \bar{z}_1} \mathrm{d}\bar{z}_1 + \ldots + \frac{\partial}{\partial \bar{z}_m} \mathrm{d}\bar{z}_m, \tag{1.6}$$

acting in the space $\Lambda[\mathrm{d}\bar{z}, C^\infty(\Omega, H)]$, where $z = (z_1, \ldots, z_m) \in \Omega$ are the complex coordinates and $\mathrm{d}\bar{z} = (\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_m)$ is the corresponding system of differentials. Then the endomorphism $\delta_\alpha + \bar{\partial}$ acts in the space $\Lambda[(\sigma, \mathrm{d}\bar{z}), C^\infty(\Omega, H)]$, with $(\sigma, \mathrm{d}\bar{z}) = (\sigma_1, \ldots, \sigma_n, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_m)$, and has the property $(\delta_\alpha + \bar{\partial})^2 = 0$, since $\delta_\alpha \bar{\partial} = -\bar{\partial}\delta_\alpha$.

The aim of this paper is to study the exactness of the operators of the type $\delta_\alpha + \bar{\partial}$, as well as some of its consequences, in certain Hilbert spaces of square integrable exterior forms. Unlike in some works dealing with harmonic forms on strongly pseudoconvex manifolds [3], [1], or in the Hodge theory [8], we shall try to emphasize the role played by $T + T^*$ rather than of $TT^* + T^*T$, where $T \in \mathscr{C}(H)$ is an operator with the property $\mathscr{R}(T) \subset \mathscr{K}(T)$. Indeed, it is such an operator which leads us to a class of natural kernels yielding integral representations formulas in Hilbert spaces (see also [6]). Among some applications, we show that the usual multiplicativity of the analytic functional calculus for commuting systems of operators follows from a more general characteristic trait, namely from a property of module homomorphism over the algebra of complex-valued analytic functions. Let us mention that the results of this paper have been partially announced in [7].

2. THE $\bar{\partial}$-OPERATOR IN HILBERT SPACES

From now on H will be a fixed complex Hilbert space. Let Ω be an open relatively compact subset of $\mathbf{C}^m$ and $L^2(\Omega)$ the usual Hilbert space of all (classes of) complex-valued square integrable functions on Ω, with respect to the Lebesgue measure. Let us denote by H_Ω the completion $L^2(\Omega) \bar{\otimes} H$ of the tensor product $L^2(\Omega) \otimes H$ with respect to the canonical hilbertian norm. In other words, H_Ω is the space of all (classes of) H-valued functions, strongly measurable on Ω and whose norm is a square integrable function [5]. We shall use also the notation $\mathbf{C}_\Omega$ for $L^2(\Omega)$.

Let us fix a system of indeterminates $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m)$ and define the operator $\bar{\partial}$ in $\Lambda[\bar{\zeta}, H_\Omega]$. As in the scalar case [2], we shall use the way of the theory of distributions. Every element $\xi \in \Lambda[\bar{\zeta}, H_\Omega]$ can be associated with a $\Lambda[\bar{\zeta}, H]$-valued distribution v_ξ by the formula

$$v_\xi(\varphi) = \int \varphi(z)\xi(z)\, \mathrm{d}\lambda(z), \quad (\varphi \in C_0^\infty(\Omega)), \tag{2.1}$$

where $\mathrm{d}\lambda$ is the Lebesgue measure and $C_0^\infty(\Omega)$ is the usual subspace of $C^\infty(\Omega)$ ($= C^\infty(\Omega, \mathbf{C})$) of all functions having compact support. We may therefore consider the areolar derivatives $\partial v_\xi/\partial\bar{z}_j$ as well as the operations $\bar{\zeta}_j \wedge v_\xi := v_{\bar{\zeta}_j \wedge \xi}$ ($j = 1, \ldots, m$). In this way the formula

$$\bar{\partial} v_\xi = \left(\frac{\partial}{\partial\bar{z}_1}\bar{\zeta}_1 + \ldots + \frac{\partial}{\partial\bar{z}_m}\bar{\zeta}_m\right) \wedge v_\xi$$

makes sense and defines the operator $\bar{\partial}$ within the theory of distributions.

We denote by $\mathscr{D}(\bar{\partial}) \subset \Lambda[\bar{\zeta}, H_\Omega]$ the set of those $\xi \in \Lambda[\bar{\zeta}, H_\Omega]$ such that there exists an $\eta \in \Lambda[\bar{\zeta}, H_\Omega]$ satisfying $\bar{\partial} v_\xi = v_\eta$; we set $\bar{\partial}\xi = \eta$. In other words we have

$$\int \varphi(z)\eta(z)\, \mathrm{d}\lambda(z) = -\int \left(\frac{\partial\varphi}{\partial\bar{z}_1}(z)\bar{\zeta}_1 + \ldots + \frac{\partial\varphi}{\partial\bar{z}_m}(z)\bar{\zeta}_m\right) \wedge \xi(z)\, \mathrm{d}\lambda(z), \tag{2.2}$$

for any $\varphi \in C_0^\infty(\Omega)$. The formula (2.2) shows that the operator $\bar\partial$ is a weak extension of the operator (1.6). (We prefer to use the system $\bar\zeta = (\bar\zeta_1, \ldots, \bar\zeta_m)$ instead of $d\bar z = (d\bar z_1, \ldots, d\bar z_m)$ in order to stress the independence of the former on the points in Ω.) As in the scalar case, the operator $\bar\partial$ is closed and densely defined. In fact, if ξ is in $\Lambda[\bar\zeta, C^\infty(\Omega, H)]$ and both ξ and $\bar\partial\xi$ belong to $\Lambda[\bar\zeta, H_\Omega]$, where $\bar\partial\xi$ is defined as in (1.6) with an obvious identification, then $\xi \in \mathscr{D}(\bar\partial)$ and $\bar\partial\xi$ satisfies also (2.2). A nother useful remark is that if $\gamma \in \Lambda[\bar\zeta, C^\infty(\Omega, \mathscr{L}(H))]$ and γ with its derivatives are bounded on Ω then for any $\xi \in \mathscr{D}(\bar\partial)$ we have also $\gamma \wedge \xi \in \mathscr{D}(\bar\partial)$ and $\bar\partial(\gamma \wedge \xi)$ can be calculated according to the rules of the exterior derivative. Indeed, the formula (2.2) is still valid for $\varphi \in C_0^\infty(\Omega, \mathscr{L}(H))$; this last assertion follows from the density of $C_0^\infty(\Omega) \otimes \mathscr{L}(H)$ in $C_0^\infty(\Omega, \mathscr{L}(H))$ [5].

The most important feature of the operator $\bar\partial$ is that $\mathscr{R}(\bar\partial) \subset \mathscr{K}(\bar\partial)$, as one can see from the formula (2.2). Therefore $\bar\partial^*$ has a similar property. Let us denote by $\bar\partial_s$ the "scalar" operator $\bar\partial$, i.e. the operator $\bar\partial$ obtained for $H = \mathbf{C}$. We shall see that $\bar\partial$ is the closure of $\bar\partial_s \otimes 1$, defined on $\mathscr{D}(\bar\partial_s) \otimes H$.

2.1. LEMMA. *For any $\xi \in \mathscr{D}(\bar\partial)$ there is a sequence $\xi_j \in \mathscr{D}(\bar\partial_s) \otimes H$ such that $\xi_j \to \xi$ and $(\bar\partial_s \otimes 1)\xi_j \to \bar\partial\xi$ as $j \to \infty$, in $\Lambda[\bar\zeta, H_\Omega]$.*

Analogously, if $\xi \in \mathscr{D}(\bar\partial^*)$ then there is a sequence $\xi_j \in \mathscr{D}(\bar\partial_s^*) \otimes H$ such that $\xi_j \to \xi$ and $(\bar\partial_s^* \otimes 1)\,\xi_j \to \bar\partial^*\xi$ as $j \to \infty$, in $\Lambda[\bar\zeta, H_\Omega]$.

Proof. Let us fix $\xi \in \mathscr{D}(\bar\partial)$. Since the coefficients of ξ and $\bar\partial\xi$ are strongly measurable functions and we are interested to approximate their values with elements of H, with no loss of generality we may suppose that H is separable. Assume that $\{e_k\}_{k=1}^\infty$ is an orthonormal basis of H. Let us represent $\xi = \sum_I \xi_I \bar\zeta_I$, where $I = (i_1, \ldots, i_p)$ is an arbitrary multi-index with $1 \leqslant i_1 < \ldots < i_p \leqslant m$ and $\bar\zeta_I = \bar\zeta_{i_1} \wedge \ldots \wedge \bar\zeta_{i_p}$. (The symbol "$\otimes$" will be generally omitted when representing exterior forms; it will be used only to stress the aspect of certain forms.) Analogously, $\eta = \bar\partial\xi$ will be written as $\eta = \sum_I \eta_I \bar\zeta_I$. Let us define the operators

$$u_k\gamma(z) = \sum_I \langle \gamma_I(z), e_k\rangle e_k \bar\zeta_I, \quad (z \in \Omega;\, k = 1, 2, 3, \ldots), \tag{2.3}$$

for any $\gamma = \sum_I \gamma_I \bar\zeta_I \in \Lambda[\bar\zeta, H_\Omega]$, where the scalar product is in H. According to the definitions (2.1) and (2.2) we can write

$$v_{u_k\eta}(\varphi) = \int \varphi(z) \sum_I \langle \eta_I(z), e_k\rangle\, e_k \bar\zeta_I \,d\lambda(z) =$$
$$= u_k\left(\sum_I \left(\int \varphi(z)\eta_I(z)\,d\lambda(z)\right)\bar\zeta_I\right) = -u_k\left(\sum_I \left(\int \bar\partial\varphi(z)\;\xi_I(z)\,d\lambda(z)\right) \wedge \bar\zeta_I\right) =$$
$$= -\sum_{j=1}^m \bar\zeta_j \wedge \int \frac{\partial\varphi}{\partial\bar z_j}(z) u_k\xi(z)\,d\lambda(z) = \bar\partial v_{u_k\xi}(\varphi),$$

for any $\varphi \in C_0^\infty(\Omega)$. In this way $u_k\xi \in \mathscr{D}(\bar\partial)$ and $\bar\partial u_k\xi = u_k\bar\partial\xi$.

Let us define now $\xi_j = \sum_{k=1}^{j} u_k \xi$ and $\eta_j = \sum_{k=1}^{j} u_k \eta$. We have $\lim_{j\to\infty} \xi_j(z) = \xi(z)$, $\lim_{j\to\infty} \eta_j(z) = \eta(z)$ almost everywhere and $\|\xi_j(z)\| \leq \|\xi(z)\|$, $\|\eta_j(z)\| \leq \|\eta(z)\|$, by the Bessel inequalities. Therefore the Lebesgue theorem of dominated convergence implies that $\xi_j \to \xi$ and $\eta_j = \bar{\partial}\xi_j \to \eta = \bar{\partial}\xi$ as $j \to \infty$, in $\Lambda[\bar{\zeta}, H_\Omega]$.

Next we show that $\sum_I \langle \xi_I(*), e_k\rangle \bar{\zeta}_I \in \mathscr{D}(\bar{\partial}_s)$ for any natural k. Indeed, if we define the operator

$$w_k(\sum_I x_I \bar{\zeta}_I) = \sum_I \langle x_I, e_k\rangle \bar{\zeta}_I \in \Lambda[\bar{\zeta}, \mathbf{C}]$$

for any $\sum_I x_I \bar{\zeta}_I \in \Lambda[\bar{\zeta}, H]$, then we have for each $\varphi \in C_0^\infty(\Omega)$

$$\int \varphi(z) \sum_I \langle \eta_I(z), e_k\rangle \bar{\zeta}_I \, d\lambda(z) =$$

$$= w_k\left(\sum_I \left(\int \varphi(z)\eta_I(z)\, d\lambda(z)\right)\bar{\zeta}_I\right) =$$

$$= -w_k\left(\sum_I \int \sum_{j=1}^{m} \frac{\partial \varphi}{\partial \bar{z}_j}(z)\xi_I(z)\, d\lambda(z)\, \bar{\zeta}_j \wedge \bar{\zeta}_I\right) =$$

$$= -\int \bar{\partial}\varphi(z) \wedge \sum_I \langle \xi_I(z), e_k\rangle\, \bar{\zeta}_I d\lambda(z),$$

which proves that $\bar{\partial}_s \sum_I \langle \xi_I(z), e_k\rangle \bar{\zeta}_I = \sum_I \langle \eta_I(z), e_k\rangle \bar{\zeta}_I$. Consequently the elements ξ_j constructed above belong to $\mathscr{D}(\bar{\partial}_s) \otimes H$, which finishes the proof of the first part of Lemma 2.1.

Consider now $\xi \in \mathscr{D}(\bar{\partial}^*)$ and notice that we may still suppose H separable and that $\{e_k\}_{k=1}^\infty$ is an orthonormal basis in H. It is clear that the operator u_k given by (2.3) is self-adjoint (in fact, u_k is a self-adjoint projection), therefore $\xi_j = \sum_{k=1}^{j} u_k \xi$ is an element of $\mathscr{D}(\bar{\partial}^*)$ since

$$\langle \xi_j, \bar{\partial}\gamma\rangle = \left\langle \sum_{k=1}^{j} u_k \bar{\partial}^* \xi, \gamma \right\rangle,$$

for any $\gamma \in \mathscr{D}(\bar{\partial})$, on account of the first part of the proof. The same argument using the Lebesgue theorem of dominated convergence shows that $\xi_j \to \xi$ and $\bar{\partial}^*\xi_j \to \bar{\partial}^*\xi$ as $j \to \infty$, in $\Lambda[\bar{\zeta}, H_\Omega]$.

Take $\theta = \sum_J \theta_J \bar{\zeta}_J \in \mathscr{D}(\bar{\partial}_s)$ arbitrary. Then from the formula (2.2) we infer easily that $\theta \otimes x \in \mathscr{D}(\bar{\partial})$ for any $x \in H$, therefore assuming $\xi = \sum_I \xi_I \bar{\zeta}_I$ and $\bar{\partial}^*\xi =$

$= \sum \eta_I \bar{\zeta}_I$ we shall have

$$\langle \sum_I \langle \xi_I(*), e_k \rangle \bar{\zeta}_I, \bar{\partial}_s \sum_J \theta_J \bar{\zeta}_J \rangle =$$
$$= \langle \sum_I \langle \xi_I(*), e_k \rangle e_k \bar{\zeta}_I, \bar{\partial} \sum_J \theta_J \otimes e_k \bar{\zeta}_J \rangle =$$
$$= \langle \sum_I \langle \eta_I(*), e_k \rangle e_k \bar{\zeta}_I, \sum_J \theta_J \otimes e_k \bar{\zeta}_J \rangle =$$
$$= \langle \sum_I \langle \eta_I(*), e_k \rangle \bar{\zeta}_I, \sum_J \theta_J \bar{\zeta}_I \rangle,$$

hence $\sum_I \langle \xi_I(*), e_k \rangle \bar{\zeta}_I \in \mathscr{D}(\bar{\partial}_s^*)$. In this way $\xi_j \in \mathscr{D}(\bar{\partial}_s^*) \otimes H$ for any j, and the proof is complete.

2.2. THEOREM. *The operator $\bar{\partial}$ is the closure of the operator $\bar{\partial}_s \otimes 1$.*

Analogously, the operator $\bar{\partial}^*$ is the closure of the operator $\bar{\partial}_s^* \otimes 1$.

Proof. As we have already noticed in the previous proof, $\mathscr{D}(\bar{\partial}_s) \otimes H \subset \mathscr{D}(\bar{\partial})$; by Lemma 2.1 we obtain that $\bar{\partial}$ is precisely the closure of $\bar{\partial}_s \otimes 1$.

Concerning the second assertion we have only to prove that $\mathscr{D}(\bar{\partial}_s^*) \otimes H \subset \mathscr{D}(\bar{\partial}^*)$. Indeed, if $(\sum_I \theta_I \bar{\zeta}_I) \otimes x$ is in $\mathscr{D}(\bar{\partial}_s^*) \otimes H$ then for any $(\sum_J \xi_J \bar{\zeta}_J) \otimes y$ in $\mathscr{D}(\bar{\partial}_s) \otimes H$ we have

$$\langle (\sum \theta_I \bar{\zeta}_I) \otimes x, \ \bar{\partial} (\sum_J \xi_J \bar{\zeta}_J) \otimes y \rangle =$$
$$\langle (\bar{\partial}_s^* \sum_I \theta_I \bar{\zeta}_I) \otimes x, (\sum_J \xi_J \bar{\zeta}_J) \otimes y \rangle$$

and approximating any $\xi \in \mathscr{D}(\bar{\partial})$ with elements from $\mathscr{D}(\bar{\partial}_s) \otimes H$ in the sense of Lemma 2.1 we obtain the desired conclusion.

Theorem 2.2 suggests that many significant properties of the operator $\bar{\partial}_s$ can be formulated and proved for the operator $\bar{\partial}$ too. As a sample, we shall show that if Ω is a strongly pseudoconvex domain in $\mathbf{C}^m$ (in the sense of [1]) then the range of $\bar{\partial}$ in $\Lambda[\bar{\zeta}, H_\Omega]$ is closed. For the operator $\bar{\partial}_s$ such a result is a consequence of the deep theory concerning the $\bar{\partial}$-Neumann problem, developed by J.J. Kohn [3], [1]. We need some auxiliary results, which can be formulated in a more general context.

Let us fix an operator $T \in \mathscr{C}(H)$ such that $\mathscr{R}(T) \subset \mathscr{K}(T)$.

2.3. LEMMA. *The operator $L = TT^* + T^*T$ is self-adjoint.*

Proof. The result is given in [3, Prop. 2.3], so that we only sketch its proof. It is enough to show the relation

$$(L + 1)^{-1} = (1 + T^*T)^{-1} + (1 + TT^*)^{-1} + 1,$$

herefore $(L + 1)^{-1}$ is self-adjoint, whence L is self-adjoint.

2.4. LEMMA. *The operator $B = T + T^*$ is self-adjoint.*

Proof. Obviously, $B^* \supset B$. Let us show that B is closed. For, take $x_k \in \mathscr{D}(B) = \mathscr{D}(T) \cap \mathscr{D}(T^*)$ such that $x_k \to x$ and $Bx_k \to y$ as $k \to \infty$. We can write $x_k = x_k' + x_k''$, with $x_k' \in \mathscr{K}(T)$ and $x_k'' \in \overline{\mathscr{R}(T^*)}$. Then $x_k' \to x' \in \mathscr{K}(T)$, $x_k'' \to x'' \in \overline{\mathscr{R}(T^*)}$, $Tx_k'' \to y' \in \mathscr{K}(T)$, $T^*x_k' \to y'' \in \overline{\mathscr{R}(T^*)}$ and $y = y' + y'' = Tx'' + T^*x' = Bx$, hence B is closed.

Assume now that $x_0 \in \mathscr{D}(B^*)$ is such that the pair $\{x_0, B^*x_0\}$ is orthogonal in $H \oplus H$ on the graph of B. Then we have $\langle x_0, x\rangle + \langle B^*x_0, Bx\rangle = 0$ for any $x \in \mathscr{D}(B)$, whence $B^*x_0 \in \mathscr{D}(B^*)$ and $B^{*2}x_0 = -x_0$. By Lemma 2.3 we obtain $(1 + L)x_0 = 0$, thus $x_0 = 0$. Since B is closed we must have $B = B^*$.

2.5. COROLLARY. *We have the orthogonal decomposition*

$$H = \overline{\mathscr{R}(T)} \oplus \overline{\mathscr{R}(T^*)} \oplus \mathscr{K}(T + T^*). \tag{2.4}$$

Proof. The equality (2.4) follows from the relation

$$\overline{\mathscr{R}(T + T^*)} = \overline{\mathscr{R}(T)} \oplus \overline{\mathscr{R}(T^*)}, \tag{2.5}$$

whose proof is straightforward.

2.6. LEMMA. *The space $\mathscr{R}(T)$ is closed if an only if the space $\mathscr{R}(T + T^*)$ is closed.*

Proof. The assertion is a consequence of the equality (2.5).

Let us return to the operator $\bar{\partial}$ when acting in strongly pseudoconvex domains.

2.7. THEOREM. *Assume that $\Omega \subset \mathbf{C}^m$ is strongly pseudoconvex. Then $\mathscr{R}(\bar{\partial})$ is closed in $\Lambda[\zeta, H_\Omega]$.*

Proof. Let us consider the self-adjoint operator $L = \bar{\partial}_s\bar{\partial}_s^* + \bar{\partial}_s^*\bar{\partial}_s$. It is known that $\mathscr{R}(L)$ is closed in $\Lambda[\zeta, \mathbf{C}_\Omega]$ [3], [1]. Therefore we can write $L = L_0 \oplus 0$ with respect to the decomposition $\Lambda[\zeta, \mathbf{C}_\Omega] = \mathscr{R}(L) \oplus \mathscr{K}(L)$, and L_0 is self-adjoint and has a bounded inverse on $\mathscr{R}(L)$. Note the identification

$$\Lambda[\zeta, H_\Omega] = (\mathscr{R}(L) \mathbin{\overline{\otimes}} H) \oplus (\mathscr{K}(L) \mathbin{\overline{\otimes}} H). \tag{2.6}$$

The operator $L_0^{-1} \otimes 1$ has a bounded self-adjoint extension $L_0^{-1} \mathbin{\overline{\otimes}} 1$ on $\mathscr{R}(L) \mathbin{\overline{\otimes}} H$, which must be injective since the range of $L_0^{-1} \otimes 1$ is dense in $\mathscr{R}(L) \mathbin{\overline{\otimes}} H$. Then the operator $L_0 \otimes 1$ has a closed extension $L_0 \mathbin{\overline{\otimes}} 1$, whose inverse is $L_0^{-1} \mathbin{\overline{\otimes}} 1$. In this way the operator $L \otimes 1$ has a closed extension $L \mathbin{\overline{\otimes}} 1 = (L_0 \mathbin{\overline{\otimes}} 1) \oplus 0$ on $\Lambda[\zeta, H_\Omega]$ by (2.6), and the range of $L \mathbin{\overline{\otimes}} 1$ is closed. Obviously, $L \mathbin{\overline{\otimes}} 1$ is also self-adjoint.

Let us prove now that $L \mathbin{\overline{\otimes}} 1$ is exactly the operator $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Indeed, if $\xi \in \mathscr{D}(L)$ then by Theorem 2.2 we have $\xi \otimes x \in \mathscr{D}(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$ for any $x \in H$, therefore $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \supset L \otimes 1$. Since both $L \mathbin{\overline{\otimes}} 1$ and $\bar{\partial}\bar{\partial}^* \otimes \bar{\partial}^*\bar{\partial}$ are self-adjoint (the latter by Lemma 2.3) we have also that $L \mathbin{\overline{\otimes}} 1 \supset \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, hence they must coincide. In particular, the range of $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ is closed in $\Lambda[\bar{\rho}, H_\Omega]$. Since $\bar{\partial} + \bar{\partial}^*$ is self-adjoint (Lemma 2.4) and the range of a self-adjoint operator is closed if and only if zero is an isolated point of its spectrum, we infer that the range of $\bar{\partial} + \bar{\partial}^*$ is closed in $\Lambda[\bar{\zeta}, H_\Omega]$, therefore the range of $\bar{\partial}$ is closed, by Lemma 2.6.

Since for $\Omega \neq \emptyset$ and $H \neq \{0\}$ the operator $\bar{\partial}$ cannot be exact, Theorem 2.7 is the best information about $\bar{\partial}$ on this line.

3. ANALYTIC PERTURBATIONS OF $\bar{\partial}$

Let U be an arbitrary open set in $\mathbf{C}^m$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ a commuting system in $A(U, \mathscr{L}(H))$. We denote by $\mathscr{S}_U(\alpha, H)$ the set of all points $z \in U$ such that the system $\alpha(z) = (\alpha_1(z), \ldots, \alpha_n(z))$ is *singular* as a commuting system of linear operators [4]. The set $\mathscr{S}_U(\alpha, H)$ is closed in U (it may be either empty or equal to U in certain cases), therefore the set $U \setminus \mathscr{S}_U(\alpha, H)$ is open [4], [6]. We associate the system $\alpha = (\alpha_1, \ldots, \alpha_n)$ with the system of indeterminates $\sigma = (\sigma_1, \ldots, \sigma_n)$. The system $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m)$ will be associated, in the sense of the previous section, with the operator $\bar{\partial}$. It is known that for $z \in U \setminus \mathscr{S}_U(\alpha, H)$ the operator $\delta_{\alpha(z)} + \delta^*_{\alpha(z)}$, where $\delta_{\alpha(z)}$ is given by (1.4), has a bounded inverse on $\Lambda[\sigma, H]$, therefore $(\delta_{\alpha(z)} + \delta^*_{\alpha(z)})^{-1}$ is an element of $C^\infty(\Omega, \mathscr{L}(\Lambda[\sigma, H]))$, for any open $\Omega \subset U \setminus \mathscr{S}_U(\alpha, H)$ ([6]; see also Lemma 3.1 below). When Ω is an open relatively compact subset of $U \setminus \mathscr{S}_U(\alpha, H)$ (i.e., the closure of Ω is also contained in $U \setminus \mathscr{S}_U(\alpha, H)$) then we may consider the operator $\delta_\alpha + \bar{\partial}$, acting in $\Lambda[(\sigma, \bar{\zeta}), H_\Omega]$, where δ_α is given by (1.5). When defining the operator $\delta_\alpha + \bar{\partial}$ we take into account the following canonical identifications:

$$\Lambda[(\sigma, \bar{\zeta}), H_\Omega] = \Lambda[\sigma, \Lambda[\bar{\zeta}, H_\Omega]] = \Lambda(\bar{\zeta}, \Lambda([\sigma, H_\Omega]] =$$
$$= \Lambda[\sigma, \Lambda[\bar{\zeta}, H]_\Omega] = \Lambda[\bar{\zeta}, \Lambda[\sigma, H]_\Omega].$$

We start with the unbounded variant of a result in [6], stated in the general case.

3.1. LEMMA. *Assume that $T \in \mathscr{C}(H)$ has the property $\mathscr{R}(T) \subset \mathscr{K}(T)$. Then T is exact if and only if $T + T^*$ has a bounded inverse on H.*

Proof. If $\mathscr{R}(T) = \mathscr{K}(T)$ then, by Lemma 2.6, $\mathscr{R}(T + T^*)$ is closed. If $x \in \mathscr{K}(T + T^*)$, as Tx and T^*x are orthogonal, we have $Tx = 0 = T^*x$. But $x = Ty$, therefore $T^*Ty = 0$, whence $x = 0$. In this way $(T + T^*)^{-1}$ exists and is everywhere defined, hence $(T + T^*)^{-1} \in \mathscr{L}(H)$.

Conversely, if $(T + T^*)^{-1} \in \mathscr{L}(H)$ then, by Lemma 2.6, $\mathscr{R}(T)$ is closed and $H = \mathscr{R}(T) \oplus \mathscr{R}(T^*)$, from (2.4). Consequently, $\mathscr{K}(T) = \mathscr{R}(T)$.

3.2. COROLLARY. *If T is exact then we have the relations*

$$(T + T^*)^{-1}Tx = T^*(T + T^*)^{-1}x, \quad (x \in \mathscr{D}(T)),$$
$$(T + T^*)^{-1}T^*y = T(T + T^*)^{-1}y, \quad (y \in \mathscr{D}(T^*)).$$

Proof. If $v \in \mathscr{K}(T)$ then $v = Tv_1$ with $v_1 \in \mathscr{K}(T^*)$, hence $(T + T^*)^{-1} v = v_1 \in \mathscr{K}(T^*)$. This means that $(T + T^*)^{-1}\mathscr{K}(T) \subset \mathscr{K}(T^*)$. Analogously, we have $(T + T^*)^{-1} \mathscr{K}(T^*) \subset \mathscr{K}(T)$.

Take now $x \in \mathscr{D}(T)$. Then $x = x_0 + x_1$, $x_0 \in \mathscr{K}(T)$ and $x_1 \in \mathscr{K}(T^*)$. We have then

$$(T + T^*)^{-1}Tx = (T + T^*)^{-1}Tx_1 = (T + T^*)^{-1}(T + T^*)x_1 =$$
$$= (T + T^*)(T + T^*)^{-1}x_1 = T^*(T + T^*)^{-1}x_1 = T^*(T + T^*)^{-1}x.$$

The second relation can be obtained in a similar way.

Let us return to the case specified at the beginning of this section.

3.3. LEMMA. *Consider an open set* $U \subset \mathbf{C}^m$ *and a commuting system* $\alpha = (\alpha_1, \ldots, \alpha_n) \subset A(U, \mathscr{L}(H))$. *If* Ω *is any open relatively compact subset in* $U \setminus \mathscr{S}_U(\alpha, H)$ *then the operator* $D_\alpha = \delta_\alpha + \bar{\partial}$ *is exact in* $\Lambda[(\sigma, \bar{\zeta}), H_\Omega]$.

Proof. We use an argument similar to that of Theorem 3.1 from [6], with some modifications due to the unboundedness of $\bar{\partial}$.

Consider $\eta \in \mathscr{D}(D_\alpha)$ such that $D_\alpha \eta = 0$. With no loss of generality we may suppose that η is homogeneous of degree $p \leqslant n + m$ in $\sigma_1, \ldots, \sigma_n, \bar{\zeta}_1, \ldots, \bar{\zeta}_m$. Then we represent $\eta = \eta_0 + \eta_1 + \ldots + \eta_p$, where η_j is of degree j in $\bar{\zeta}_1, \ldots, \bar{\zeta}_m$ and of degree $p - j$ in $\sigma_1, \ldots, \sigma_n$; moreover, by (2.2), each η_j is in $\mathscr{D}(\bar{\partial})$. We shall be looking for a solution ξ of the equation $D_\alpha \xi = \eta$, where $\xi = \xi_0 + \xi_1 + \ldots + \xi_{p-1}$, ξ_j being of degree j in $\bar{\zeta}_1, \ldots, \bar{\zeta}_m$ and of degree $p - j - 1$ in $\sigma_1, \ldots, \sigma_n$. By identifying the forms of the same type we obtain the system of equations

$$\delta_\alpha \xi_0 = \eta_0$$
$$\delta_\alpha \xi_j + \bar{\partial}\xi_{j-1} = \eta_j \quad (j = 1, \ldots, p-1),$$
$$\bar{\partial}\xi_{p-1} = \eta_p$$

with the conditions

$$\delta_\alpha \eta_0 = 0$$
$$\delta_\alpha \eta_j + \bar{\partial}\eta_{j-1} = 0 \qquad (j = 1, \ldots, p)$$
$$\bar{\partial}\eta_p = 0.$$

Let us define $Q(\alpha(z)) = (\delta_{\alpha(z)} + \delta^*_{\alpha(z)})^{-1} (z \in \Omega)$ and note that $Q(\alpha(z))$ and its derivatives are bounded on Ω, according to the choice of Ω. Define also $\xi_0(z) = Q(\alpha(z))\eta_0(z)$. By Corollary 3.2, applied to $\delta_{\alpha(z)}$, we have $\xi_0(z) \in \mathscr{K}(\delta^*_{\alpha(z)})$, for every $z \in \Omega$. Moreover, as $\eta_0 \in \mathscr{D}(\bar{\partial})$ we have also $\xi_0 \in \mathscr{D}(\bar{\partial})$ and $\delta_0 \bar{\partial}\xi_0 = -\bar{\partial}\delta_\alpha \xi_0 = -\bar{\partial}\eta_0 = \delta_\alpha \eta_1$, whence $\delta_\alpha(\eta_1 - \bar{\partial}\xi_0) = 0$. Define then $\xi_1(z) = Q(\alpha(z))(\eta_1(z) - \bar{\partial}\xi_0(z)) \in \mathscr{K}(\delta^*_{\alpha(z)})$ $(z \in \Omega)$, hence $\delta_\alpha \xi_1 = \eta_1 - \bar{\partial}\xi_0$. We have also $\delta_\alpha \xi_1 \in \mathscr{D}(\bar{\partial})$, therefore $\delta_\alpha(\eta_2 - \bar{\partial}\xi_1) = \delta_\alpha \eta_2 + \bar{\partial}\delta_\alpha \xi_1 = 0$, which allows the continuation of the procedure. One has, in general, $\eta_j(z) - \bar{\partial}\xi_{j-1}(z) \in \mathscr{K}(\delta_{\alpha(z)})$, hence $\xi_j(z) = Q(\alpha(z))(\eta_j(z) - \bar{\partial}\xi_{j-1}(z)) \in \mathscr{K}(\delta^*_{\alpha(z)})$ and $\delta_\alpha \xi_j + \bar{\partial}\xi_{j-1} = \eta_j$, for any $j = 1, \ldots, p-1$. Note also that $\xi_{j-1} \in \mathscr{D}(\bar{\partial})$ implies that $\xi_j \in \mathscr{D}(\bar{\partial})$ too. From the structure of the operator δ_α and from the Corollary 2.2. we obtain that the degree of ξ_j in $\sigma_1, \ldots, \sigma_n$ must be $p - j - 1$. In particular, the degree of $\eta_p(z) - \bar{\partial}\xi_{p-1}(z)$ in $\sigma_1, \ldots, \sigma_n$ is zero. Since $\delta_{\alpha(z)}$ is exact, the kernel of $\delta_{\alpha(z)}$ on the space $\Lambda^0[\sigma, \Lambda[\bar{\zeta}, H]]$ must be zero, therefore $\bar{\partial}\xi_{p-1} = \eta_p$, and the proof is complete.

3.4. COROLLARY. *With the conditions of Lema 3.3, if $\eta \in \Lambda^p[(\sigma, \bar\zeta), H_\Omega]$ has the property $D_\alpha\eta = 0$ in Ω then*

$$B_\alpha\eta(z) = \sum_{j=0}^{p-1}\sum_{k=0}^{j}(-1)^k Q(\alpha(z))(\bar\partial Q(\alpha(z)))^k\eta_{j-k}(z), \qquad (z \in \Omega)$$

is a solution of the equation $D_\alpha\xi = \eta$ in Ω, where $Q(\alpha(z)) = (\delta_{\alpha(z)} + \delta^*_{\alpha(z)})^{-1}$ and η_j is the part of η of degree j in $\bar\zeta_1, \cdots, \bar\zeta_m$. Moreover, $B_\alpha\eta \in \Lambda^{p-1}[(\sigma, \bar\zeta), H_\Omega]$.

Proof. The solution $B_\alpha\eta$ of the equation $D_\alpha\xi = \eta$ is the explicit form of the solution constructed in the previous lemma.

Let us consider the differential operator

$$\vartheta = -\left(\frac{\partial}{\partial z_1}\bar Z^1_* + \cdots + \frac{\partial}{\partial z_m}\bar Z^*_m\right), \tag{3.1}$$

acting in $\Lambda[\bar\zeta, C^\infty(\Omega, H)]$, where $\Omega \subset \mathbf{C}^m$ and $\bar Z^*_j$ corresponds to $\bar\zeta_j$ by the relation (1.2). It is easily seen that (3.1) is the formal adjoint of the operator $\bar\partial$. It is also clear that the operator (3.1) has an extension, in the theory of distributions sense, in the space $\Lambda[\bar\zeta, H_\Omega]$. Furthermore, we have the following

3.5. LEMMA. *Assume that $\Omega \subset \mathbf{C}^m$ is open and relatively compact. If $\eta \in \Lambda[\bar\zeta, H_\Omega]$ has compact support and $\eta \in \mathscr{D}(\bar\partial^*)$ then $\bar\partial^*\eta = \vartheta\eta$.*

Proof. Variants of this result for the scalar case can be found in [1] and [2]. As we need parts of the argument in the sequel, we shall give a complete proof.

Let us take $\chi \in C_0^\infty(\mathbf{C}^m)$ such that $\operatorname{supp}\chi = \{z; \|z\| \leqslant 1\}$ (where "supp" stands for the support), $\chi(z) = \chi(-z)$, $\chi \geqslant 0$ and $\int\chi(z)\,\mathrm{d}\lambda(z) = 1$. Define then $\chi_\varepsilon(z) = \varepsilon^{-2m}\chi(z/\varepsilon)$, for any $\varepsilon > 0$. Let us denote by η_ε the convolution product $\chi_\varepsilon * \eta$; analogously, $\gamma_\varepsilon = \chi_\varepsilon * \gamma$, where $\gamma = \bar\partial^*\eta$. Note also that γ is still with compact support. Indeed, if $\psi \in C_0^\infty(\Omega)$ is arbitrary then a direct calculation from the definition of $\bar\partial^*$ shows that

$$\bar\partial^*(\psi\eta) = -\left(\frac{\partial\psi}{\partial z_1}\bar Z^*_1 + \ldots + \frac{\partial\psi}{\partial z_m}\bar Z^*_m\right)\eta + \psi\bar\partial^*\eta. \tag{3.2}$$

In particular, if $\psi = 1$ in a neighbourhood of $\operatorname{supp}\eta$, we obtain that $\operatorname{supp}\gamma$ has to be compact.

As in the scalar case [2], we have $\eta_\varepsilon \to \eta$ and $\gamma_\varepsilon \to \gamma$ in $\Lambda[\bar\zeta, H_\Omega]$, as $\varepsilon \to 0$. We shall prove that $\bar\partial^*\eta_\varepsilon = \gamma_\varepsilon$. Indeed, according to the properties of χ_ε, we can write

$$\langle\chi_\varepsilon * \gamma, \theta\rangle = \langle\gamma, \chi_\varepsilon * \theta\rangle = \langle\eta, \bar\partial(\chi_\varepsilon * \theta)\rangle =$$
$$= \langle\eta, \chi_\varepsilon * \bar\partial\theta\rangle = \langle\eta_\varepsilon, \bar\partial\theta\rangle = \langle\bar\partial^*\eta_\varepsilon, \theta\rangle,$$

for any $\theta \in \mathscr{D}(\bar\partial)$; the equality $\bar\partial(\chi_\varepsilon * \theta) = \chi_\varepsilon * \bar\partial\theta$ is true in a neighbourhood of supp η, on account of the relation (2.2). If the coefficients of θ are in $C^\infty(\Omega, H)$ then

$$\langle \bar\partial^* \eta_\varepsilon, \theta \rangle = - \left\langle \sum_{j=1}^m \frac{\partial}{\partial z_j} \bar Z_j^* \eta_\varepsilon, \theta \right\rangle,$$

therefore $\bar\partial^* \eta_\varepsilon = - \sum_{j=1}^m (\partial/\partial z_j)\bar Z_j^* \eta_\varepsilon$ because of the density of such θ's in $\Lambda[\bar\zeta, H_\Omega]$. Finally, if $\varphi \in C_0^\infty(\Omega)$ is arbitrary then, with the notation (2.1), one has that

$$v_\gamma(\varphi) = \int \varphi(z)\gamma(z) d\lambda(z) = \lim_{\varepsilon \to 0} \int \varphi(z)\, \bar\partial^* \eta_\varepsilon(z)\, d\lambda(z) =$$

$$= \lim_{\varepsilon \to 0} \int \left(\sum_{j=1}^m \frac{\partial \varphi}{\partial z_j} \bar Z_j^* \right) \eta_\varepsilon(z)\, d\lambda(z) = \left(- \sum_{j=1}^m \frac{\partial}{\partial z_j} \bar Z_j^*\, v_\eta \right)(\varphi),$$

i.e. $\bar\partial^* \eta = \vartheta \eta$ in the theory of distributions sense.

3.6. LEMMA. *With the conditions of the previous lemma, if η is also in $\mathscr{D}(\bar\partial)$ and $\eta = \sum_I \eta_I \bar\zeta_I$ then $\partial \eta_I / \partial \bar z_j \in H$ for any I, j and*

$$\sum_I \sum_{j=1}^m \int \left\| \frac{\partial \eta_I}{\partial \bar z_j} \right\|^2 d\lambda(z) = \|\bar\partial \eta\|^2 + \|\vartheta \eta\|^2.$$

Proof. Assume first that the coefficients of η are functions from $C^\infty(\Omega, H)$, with compact support. Then, by the relations (1.3), we can write in $\Lambda[\bar\zeta, H_\Omega]$

$$\|\bar\partial \eta\|^2 + \|\vartheta \eta\|^2 = \langle (\vartheta \bar\partial + \bar\partial \vartheta)\eta, \eta \rangle =$$

$$= - \left\langle \sum_{j=1}^m \frac{\partial^2}{\partial z_j \partial \bar z_j} \sum_I \eta_I \bar\zeta_I, \sum_I \eta_I \bar\zeta_I \right\rangle =$$

$$= \sum_I \sum_{j=1}^m \int \left\| \frac{\partial \eta_I}{\partial \bar z_j} \right\|^2 d\lambda(z).$$

Take now χ_ε as in the previous lemma. Then for $\eta_\varepsilon = \chi_\varepsilon * \eta$ we have $\eta_\varepsilon \to \eta$, $\bar\partial \eta_\varepsilon = \chi_\varepsilon * \bar\partial \eta \to \bar\partial \eta$ and $\vartheta \eta_\varepsilon = \chi_\varepsilon * \vartheta \eta \to \vartheta \eta$ as $\varepsilon \to 0$, in $\Lambda[\bar\zeta, H_\Omega]$. By the preceding calculation we obtain that $\partial \eta_I / \partial \bar z_j$ are elements of H_Ω and the equality still holds.

We can prove now the main result of this section. By a *smooth form* we mean any element $\xi \in \Lambda[(\sigma, \bar\zeta), C^\infty(\Omega, H)]$ with $\xi, \bar\partial \xi$ in $\Lambda[(\sigma, \bar\zeta), H_\Omega]$.

3.7. THEOREM. *Consider an open set $U \subset \mathbf{C}^m$ and a commuting system $= (\alpha_1, \cdots, \alpha_n) \subset A(U, \mathscr{L}(H))$. If Ω is any open relatively compact subset of $U \setminus \mathscr{S}_U(\alpha, H)$ then the operator $D_\alpha + D_\alpha^*$ has a bounded inverse R_α on $\Lambda[(\sigma, \bar\zeta), H_\Omega]$. Moreover, if ξ is a smooth form then $R_\alpha \xi$ is also smooth.*

Proof. The invertibility of $D_\alpha + D_\alpha^*$ follows from Lemmas 3.1 and 3.3. The second assertion is a reflection of the regularity of the solutions of the elliptic differential operators. Let us reduce our problem to a manageable case.

Assume first that $\xi \in \Lambda[(\sigma, \bar{\zeta}), H_\Omega]$ is a smooth form such that $D_\alpha\xi = 0$. By Corollary 3.2 we have that $D_\alpha R_\alpha \xi = \xi$ and $D_\alpha^* R_\alpha \xi = 0$. It will be enough to prove that for any $\psi \in C_0^\infty(\Omega)$ the form $\eta = \psi R_\alpha \xi$ is smooth. Since $R_\alpha\xi \in \mathscr{D}(\bar{\partial}^*)$, we have also $\eta \in \mathscr{D}(\bar{\partial}^*)$, by the formula (3.2). From the equalities satisfied by $R_\alpha\xi$ we obtain that

$$\bar{\partial}\eta = \bar{\partial}\psi \wedge R_\alpha\xi + \psi(\xi - \delta_\alpha R_\alpha \xi),$$

$$\bar{\partial}^*\eta = (U_\psi - \psi\delta_\alpha^*)R_\alpha\xi,$$

where $U_\psi = -((\partial\psi/\partial z_1)\bar{Z}_1^* + \ldots + (\partial\psi/\partial z_m)\bar{Z}_m^*)$. By Lemmas 3.5 and 3.6 we obtain that if $\eta = \sum_I \eta_I \bar{\zeta}_I$, with $\eta_I \in \Lambda[\sigma, H]_\Omega$, then $\partial\eta_I/\partial\bar{z}_j$ are still in $\Lambda[\sigma, H]_\Omega$, for any I and j. Analogously, $\partial\eta_I/\partial z_j$ are in $\Lambda[\sigma, H]_\Omega$ (note that Lemma 3.6 can be stated with $\partial\eta_I/\partial z_j$ instead of $\partial\eta_I/\partial\bar{z}_j$, with a similar proof). We can apply now an induction argument (see [2], Th. 4.2.5(b)) in order to show that the coefficients η_I belong to any Sobolev space $W^q(\Omega, \Lambda[\sigma, H])$ $(q \geqslant 0)$ of those $\Lambda[\sigma, H]$-valued functions on Ω, whose derivatives up to the order q are square integrable, therefore η_I are smooth functions, by the Sobolev lemma.

Let us obtain the assertion in its full generality. Consider a smooth form $\xi \in \Lambda[(\sigma, \bar{\zeta}), H_\Omega]$. By Corollary 3.2 and the previous case, $\xi_1 = D_\alpha^* R_\alpha \xi = R_\alpha D_\alpha \xi$ is smooth, hence $\xi_2 = D_\alpha R_\alpha \xi = \xi - \xi_1$ is also smooth. We have therefore the relations $\bar{\partial} R_\alpha \xi = \xi_2 - \delta_\alpha R_\alpha \xi$ and $\bar{\partial}^* R_\alpha \xi = \xi_1 - \delta_\alpha^* R_\alpha \xi$. It is clear that the preceding procedure can be again applied to $\psi R_\alpha \xi$ for every $\psi \in C_0^\infty(\Omega)$, which finishes the proof.

3.8. Corollary. *With the conditions of the previous theorem we have the equalities*

$$R_\alpha(\mathscr{K}(D_\alpha) \cap \Lambda^p[(\sigma, \bar{\zeta}), H_\Omega]) = \mathscr{K}(D_\alpha^*) \cap \Lambda^{p-1}[(\sigma, \bar{\zeta}), H_\Omega]$$

and

$$R_\alpha(\mathscr{K}(D_\alpha^*) \cap \Lambda^p[(\sigma, \bar{\zeta}), H_\Omega]) = \mathscr{K}(D_\alpha) \cap \Lambda^{p+1}[(\sigma, \bar{\zeta}), H_\Omega],$$

for any integer p, $0 \leqslant p \leqslant n + m$, *where* $\Lambda^p[(\sigma, \bar{\zeta}), H_\Omega]$ *is zero* for $p = -1$ *and* $p = m + n + 1$.

Proof. The equalities follow from Corollary 3.2 and from the structure of the operator D_α, mapping $\Lambda^p[(\sigma, \bar{\zeta}), H_\Omega]$ into $\Lambda^{p+1}[(\sigma, \bar{\zeta}), H_\Omega]$.

4. SOME INTEGRAL FORMULAS

Let U be any open subset of $\mathbf{C}^m$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_m)$ systems of indeterminates. Let us denote by H_U^{loc} the space of all (classes of) strongly measurable H-valued function in U, whose norms are locally square integrable. If

$\xi \in A[(\sigma, \bar\zeta), H_U^{\mathrm{loc}}]$ then we define its integral on a bounded Borel set $M \subset U$ in the following way: Denote by ξ_m the part of ξ of degree m in $\bar\zeta_1, \ldots, \bar\zeta_m$ and by $\tilde\xi_m$ the form obtained from ξ_m by substituting $\bar\zeta_1, \ldots, \bar\zeta_m$ with $d\bar z_1, \ldots, d\bar z_m$, respectively. Then we put, by definition

$$\int_M \xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \int_M \tilde\xi_m(z) \wedge dz_1 \wedge \ldots \wedge dz_m. \tag{4.1}$$

Clearly, the right side of (4.1) make sense and it is an element of $A[\sigma, H]$.

The integral (4.1) does not suffice for our purpose. We need a more complicated concept, valid for some smooth surfaces of real dimension $2m - 1$. First of all notice that the operator $\bar\partial$ makes sense as a closed operator in $A[\bar\zeta, H_U^{\mathrm{loc}}]$ (hence also in $A[(\sigma, \bar\zeta), H_U^{\mathrm{loc}}]$), when $A[\bar\zeta, H_U^{\mathrm{loc}}]$ is endowed with its natural topology of Fréchet-Hilbert space, giving $\bar\partial$ a similar meaning with that from the second section. Then for any open relatively compact $\Delta \subset U$, whose boundary Σ is a smooth surface, and for any $\xi \in \mathscr{D}(\bar\partial)$ in $A[(\sigma, \bar\zeta), H_U^{\mathrm{loc}}]$ we define

$$\int_\Sigma \xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \int_\Delta \bar\partial\xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m, \tag{4.2}$$

where the right side is given by (4.1). Plainly, the formula (4.2) is suggested by the Stokes formula. In particular, if $\xi \in \mathscr{D}(\bar\partial)$ has compact support and we denote by ξ_ε the convolution product $\chi_\varepsilon * \xi$, where χ_ε has been defined in the proof of Lemma 3.5, then $\xi_\varepsilon \to \xi$ and $\bar\partial\xi_\varepsilon \to \bar\partial\xi$ as $\varepsilon \to 0$, therefore

$$\int_\Sigma \xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \lim_{\varepsilon\to 0}\int_\Sigma \xi_\varepsilon(z) \wedge dz_1 \wedge \ldots \wedge dz_m.$$

Consequently, if $\eta = \bar\partial\xi$ and the support of ξ is contained in Δ then

$$\int_\Delta \eta(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \int_U \eta(z) \wedge dz_1 \wedge \ldots \wedge dz_m = 0. \tag{4.3}$$

The definition (4.2) makes sense also for forms defined only in neighbourhoods of Σ in U. Indeed, if $\Omega \supset \Sigma$ is an open relatively compact subset of U and $\varphi \in C_0^\infty(\Omega)$ is equal to 1 in a neighbourhood of Σ then for any $\xi \in \mathscr{D}(\bar\partial)$ in $A[(\sigma, \bar\zeta), H_\Omega]$ we have $\varphi\xi \in \mathscr{D}(\bar\partial)$ in $A[(\sigma, \bar\zeta), H_U^{\mathrm{loc}}]$ by natural extension, hence we may define

$$\int_\Sigma \xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \int_\Delta \bar\partial\varphi\xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m. \tag{4.2*}$$

By the above remarks, the formula (4.2)* does not depend on the particular choice of the function φ.

Suppose now that $\alpha = (\alpha_1, \ldots, \alpha_n) \subset A(U, \mathscr{L}(H))$ is a commuting system such that $\mathscr{S}_U(\alpha, H)$ is compact in U. Then the operator B_α, given by Corollary 3.4, maps the kernel $Z[(\sigma, \bar{\zeta}), H_V^{\mathrm{loc}}]$ of $\delta_\alpha \dotplus \bar{\partial}$, when acting in the space $\Lambda[(\sigma, \bar{\zeta}), H_V^{\mathrm{loc}}]$, into the space $\Lambda[(\sigma, \bar{\zeta}), H_V^{\mathrm{loc}}]$, where $V = U \setminus \mathscr{S}_U(\alpha, H)$. From now on we shall denote by $P_{\bar{\zeta}}$ the projection of $\Lambda[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ onto $\Lambda[\bar{\zeta}, H_U^{\mathrm{loc}}]$; note that $P_{\bar{\zeta}}\bar{\partial} \subset \bar{\partial}P_{\bar{\zeta}}$. Let us also assume that $\mathscr{S}_U(\alpha, H) \subset \Delta$, with Δ as above. Then for any $\xi \in Z[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ we define the H-valued linear map

$$\mu_\alpha(\xi) = \int_\Sigma P_{\bar{\zeta}} B_\alpha \xi(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m - \int_\Delta P_{\bar{\zeta}}\xi(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m. \tag{4.4}$$

Let us remark that the map (4.4) may be not null only on the space $Z^m[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ of those forms of $Z[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ which are homogeneous of degree m in $\sigma_1, \ldots, \sigma_n, \bar{\zeta}_1, \ldots, \bar{\zeta}_m$. Indeed, if ξ is of degree $\leqslant m-1$ then, by Corollary 3.4 $B_\alpha\xi$ is of degree $\leqslant m-2$ and the integral (4.4) is plainly null. When ξ is of degree $\geqslant m+1$ then $P_{\bar{\zeta}}\xi = 0$ and the part η_{m-1} of $B_\alpha\xi$ of degree $m-1$ in $\bar{\zeta}_1, \ldots, \bar{\zeta}_m$, which is the only one participating at the integration, is of degree $\geqslant 1$ in $\sigma_1, \ldots, \sigma_n$, hence $P_{\bar{\zeta}}\eta_{m-1} = 0$.

4.1. LEMMA. *The map μ_α does not depend on the particular choice of the set $\Delta \supset \mathscr{S}_U(\alpha, H)$ and is continuous.*

Proof. Indeed, if $\eta \in Z^m[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ and its support is compact and disjoint of $\mathscr{S}_U(\alpha, H)$ then, by Corollary 3.4, $B_\alpha\eta$ satisfies $(\delta_\alpha + \bar{\partial})B_\alpha\eta = \eta$ in U and the support of $B_\alpha\eta$ is contained in the support of η. Then we can write $P_{\bar{\zeta}}\eta = \bar{\partial}\eta_1$ in U, where the support of η_1 is compact, hence by (4.3)

$$\int_U P_{\bar{\zeta}}\eta(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m = 0.$$

In particular, if we take

$$\eta = ((\delta_\alpha + \bar{\partial})\psi_1 B_\alpha\xi - \xi) - ((\delta_\alpha + \bar{\partial})\psi_2 B_\alpha\xi - \xi),$$

where $\psi_j \in C^\infty(U)$ is zero in an open neighbourhood of $\mathscr{S}_U(\alpha, H)$ and is one outside another (relatively compact) open neighbourhood of $\mathscr{S}_U(\alpha, H)$ in $U (j = 1, 2)$, then we obtain that the integral (4.4) does not depend on Δ.

A similar argument shows that we have also, for any $\xi \in Z^m[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$,

$$\mu_\alpha(\xi) = \int_\Sigma P_{\bar{\zeta}} R_{\alpha,\Omega}\xi(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m - \int_\Delta P_{\bar{\zeta}}\xi(z) \wedge \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m,$$

where $\Omega \supset \Delta$ is a relatively compact open set in U, whose closure is disjoint of $\mathscr{S}_U(\alpha, H)$, and $R_{\alpha,\Omega}$ is given by Theorem 3.7. If $\psi \in C_0^\infty(\Omega)$ has the property that $\psi = 1$ in a neighbourhood of Σ then by (4.2)

$$\int_\Sigma P_{\bar{\zeta}} R_{\alpha,\Omega}\xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m = \int_\Delta P_{\bar{\zeta}}\bar{\partial}\, \psi R_{\alpha,\Omega}\xi(z) \wedge dz_1 \wedge \ldots \wedge dz_m.$$

Since $R_{\alpha,\Omega}$ is continuous and $\bar{\partial} R_{\alpha,\Omega} = \xi - \delta_\alpha R_{\alpha,\Omega}\xi$ in Ω, we infer the continuity of μ_α on $Z^m[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$.

Let us mention that the continuity of μ_α can be also proved by showing that B_α is closed, hence continuous, on $Z^m[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$. However, the formula (4.5) makes a connection between μ_α and Theorem 3.7.

Note that $Z^m[\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ is an $A(U)$-module. In some important cases the space H can be also given a structure of $A(U)$-module by means of the maps

$$\nu_\alpha(f)x = \mu_\alpha(fx\sigma_1 \wedge \ldots \wedge \sigma_n) \quad (f \in A(U); \quad x \in H). \tag{4.6}$$

The map μ_α itself becomes an $A(U)$-module homomorphism. More precisely, let us denote by $Z_0^n[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ the closure in $Z^n[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ of those exterior forms $\sum_j \eta_j x_j$, where the coefficients of η_j are smooth functions having values in the commutant of the set $\{\alpha_1(z), \ldots, \alpha_n(z); z \in U\}$ in $\mathscr{L}(H)$, $x_j \in H$ and $(\delta_\alpha + \bar{\partial})\eta_j = 0$ for any j. Then we have the following

4.2. THEOREM. *Assume that* $a = (a_1, \ldots, a_n)$ *is a commuting system of operators in* $\mathscr{L}(H)$ *and define* $\alpha_j(z) = z_j - a_j$, $j = 1, \ldots, n$, $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ *and* $\alpha = (\alpha_1, \ldots, \alpha_n)$. *If* $U \subset \mathbf{C}^n$ *is any open set containing* $\mathscr{S}_{\mathbf{C}^n}(\alpha, H)$ *then for any* $\xi \in Z_0^n[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ *and* $f \in A(U)$ *we have* $\mu_\alpha(f\xi) = \nu_\alpha(f)\mu_n(\xi)$.

It is known that (4.6) provides, in this case, an analytic functional calculus, i.e. ν_α is a continuous homomorphism of the algebra $A(U)$ into the algebra $\mathscr{L}(H)$ such that $\nu_\alpha(p) = p(a)$, for any complex polynomial p [6], since $\mathscr{S}_{\mathbf{C}^n}(\alpha, H)$ is exactly the joint spectrum of the system $a = (a_1, \ldots, a_n)$ when acting on H [4], [6]. Theorem 4.2 asserts that the usual multiplicativity of the analytic functional calculus is, in fact, a property of $A(U)$-module homomorphism. This feature will follow from a Fubini type property of the integral map (4.4), which will be described in the sequel (see also [4], [6]).

Suppose that $U \subset \mathbf{C}^{n+m}$ is open and write a point in U as a pair $(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_m)$. The corresponding system of indeterminates when defining $\bar{\partial}$ will be $(\bar{\zeta}, \bar{\omega}) = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n, \bar{\omega}_1, \ldots, \bar{\omega}_m)$. We shall consider two systems $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$ in $A(U, \mathscr{L}(H))$ such that $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is a commuting system. The operator δ_α will be defined with the system of indeterminates $\sigma = (\sigma_1, \ldots, \sigma_n)$ while the operator δ_β will be defined with the system of indeterminates $\tau = (\tau_1, \ldots, \tau_m)$.

A closed set $F \subset U$ is said to be $\mathbf{C}^n$*-compact* in U if for every compact $K \subset V$ the set $F \cap (\mathbf{C}^n \times K)$ is compact, where V is the projection of U on the last m coordinates [4].

Let us extend the definition (4.4) for commuting systems $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\mathscr{S}_U(\alpha, H)$ $\mathbf{C}^n$-compact in U. Namely, we shall fix an arbitrary function $\varphi \in C^\infty(U)$ such that $\varphi = 0$ in a neighbourhood of $\mathscr{S}_U(\alpha, H)$, $\varphi = 1$ outside another neighbourhood of $\mathscr{S}_U(\alpha, H)$ and $\operatorname{supp}(1-\varphi)$ is $\mathbf{C}^n$-compact in U. Then the support of the form $(\delta_\alpha + \bar\partial)\varphi B_\alpha \xi - \xi$ is $\mathbf{C}^n$-compact in U and if P_σ is the map of $Z^n[(\sigma, \bar\zeta, \bar\omega), H_U^{\mathrm{loc}}]$ which annihilates the monomials containing $\sigma_1, \ldots, \sigma_n$ and letting the other invariant (which agrees with $P_{\bar\zeta}$ from (4.4)), then we define

$$\mu_\alpha(\xi)(w) = \int_{\mathbf{C}^n} P_\sigma(\bar\partial \varphi B_\alpha \xi(z, w) - \xi(z, w)) \wedge dz_1 \wedge \ldots \wedge dz_n, \tag{4.7}$$

for any $\xi \in Z^n[(\sigma, \bar\zeta, \bar\omega), H_U^{\mathrm{loc}}]$. We shall see that the definition (4.7) is independent of φ and $w \to \mu_\alpha(\xi)(w)$ is analytic.

4.3. LEMMA. *Assume that $\eta \in \Lambda[(\bar\zeta, \bar\omega), H_U^{\mathrm{loc}}]$ is in the domain of $\bar\partial$ and its support is $\mathbf{C}^n$-compact. Then the form $\int \eta(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n$ is in the domain of $\bar\partial$ in $\Lambda[\bar\omega, H_V^{\mathrm{loc}}]$ and*

$$\bar\partial \int \eta(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n = \int \bar\partial\eta(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n,$$

where V is the projection of U on the last m coordinates.

In particular, if the degree of η in $\bar\zeta_1, \ldots, \bar\zeta_n$ is $\leqslant n-1$ then $\int \bar\partial\eta(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n = 0$.

Proof. Take $\chi_\varepsilon \in C_0^\infty(\mathbf{C}^{n+m})$ with the properties of Lemma 3.5. Then for every relatively compact open set $Q \subset V$ we have $\eta_\varepsilon = \chi_\varepsilon * \eta \to \eta$ and $\bar\partial\eta_\varepsilon \to \bar\partial\eta$ in $\Lambda[(\bar\zeta, \bar\omega), H_W]$ as $\varepsilon \to 0$, where $W = U \cap (\mathbf{C}^n \times Q)$. If $\eta = \eta_{n-1} + \eta_n$, where η_{n-1} is of degree $\leqslant n-1$ in $\bar\zeta_1, \ldots, \bar\zeta_n$ then we have for $w \in Q$

$$\int \bar\partial\eta(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n = \lim_{\varepsilon \to 0} \int \bar\partial\eta_\varepsilon(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n =$$

$$\lim_{\varepsilon \to 0} \int \bar\partial\eta_{n,\varepsilon}(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n = \lim_{\varepsilon \to 0} \bar\partial \int \eta_{n,\varepsilon}(z, w)\, dz_1 \wedge \ldots \wedge dz_n,$$

where $\eta_{n-1,\varepsilon} = \chi_\varepsilon * \eta_{n-1}$ and $\eta_{n,\varepsilon} = \chi_\varepsilon * \eta_n$, since

$$\int \bar\partial\eta_{n-1,\varepsilon}(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n = 0$$

for a sufficiently small ε, by the Stokes formula. Consequently, $\int \eta(z, w) \wedge dz_1 \wedge \wedge \ldots \wedge dz_n = \int \eta_n(z, w) \wedge dz_1 \wedge \ldots \wedge dz_n$ is in the domain of $\bar{\partial}$ in $\Lambda[\bar{\omega}, H_V^{\mathrm{loc}}]$ and the first assertion holds. The second assertion follows easily from the first.

4.4. LEMMA. *The map μ_α given by* (4.7) *does not depend on the particular choice of the function φ. Moreover, $\mu_\alpha(\xi) \in A(V, H)$, for every $\xi \in Z^n[(\sigma, \bar{\zeta}, \bar{\omega}), H_U^{\mathrm{loc}}]$.*

Proof. The independence of μ_α on the choice of φ follows by refining the first part of the proof of Lemma 4.1, via Corollary 3.4 and Lemma 4.3. Note also that the integrand of (4.7) satisfies

$$\bar{\partial} P_\sigma(\bar{\partial}\varphi B_\alpha \xi(z, w) - \xi(z, w)) = - P_\sigma \bar{\partial} \xi(z, w) = P_\sigma \delta_\alpha \xi(z, w) = 0,$$

hence, by Lemma 4.3, $\bar{\partial}\mu_\alpha(\xi) = 0$.

The next result is an extension, in Hilbert spaces, of Theorem 3.6 from [4].

4.4. THEOREM. *Let U be an open set in $\mathbf{C}^{n+m}$ and V the projection of U on the last m coordinates. Consider also $\alpha = (\alpha_1, \ldots, \alpha_n) \subset A(U, \mathscr{L}(H))$ and $\beta = (\beta_1, \ldots, \beta_m) \subset A(V, \mathscr{L}(H))$ such that (α, β) is a commuting system. If $\mathscr{S}_U(\alpha, H)$ is $\mathbf{C}^n$-compact in U and $\mathscr{S}_V(\beta, H)$ is compact then for every $\xi \in Z^n[(\sigma, \bar{\zeta}, \bar{\omega}), H_U^{\mathrm{loc}}]$ and $\eta \in \Lambda^m[(\tau, \bar{\omega}), C^\infty(V, (\alpha, \beta)^c)]$, with $(\delta_\alpha + \bar{\partial})\eta = 0$ in V, where $(\alpha, \beta)^c$ is the commutant of (α, β) in $\mathscr{L}(H)$, we have*

$$\mu_{(\alpha, \beta)}(\eta \wedge \xi) = \mu_\beta(\eta \mu_\alpha(\xi)).$$

Proof. The formal part of the proof is not essentially different from that of Theorem 3.8 from [6], which in turn is an explicite variant of the proof of Theorem 3.6 from [4], so that we only sketch it.

Consider $\varphi \in C^\infty(U)$ with the properties from the definition (4.7) for $\mathscr{S}_U(\alpha, H)$. Analogously, take $\psi \in C^\infty(V)$ for $\mathscr{S}_V(\beta, H)$ and $\theta \in C^\infty(U)$ for $\mathscr{S}_U((\alpha, \beta), H)$, with similar properties. Then we can obtain the relation

$$\bar{\partial}\theta P_{(\sigma, \tau)} B_{(\alpha, \beta)}(\eta \wedge \xi) - P_\tau \eta \wedge P_\sigma \xi =$$

$$\bar{\partial}\psi P_\tau B_\beta (\eta \wedge (\bar{\partial}\varphi P_\sigma B_\alpha \xi - P_\sigma \xi) - P_\tau \eta \wedge (\bar{\partial}\varphi P_\sigma B_\alpha \xi - P_\sigma \xi) + \bar{\partial}\eta_0,$$

where the support of η_0 is compact. By integrating this relation and using the equality

$$\bar{\partial}\psi \int B_\beta \eta \wedge (\bar{\partial}\varphi P_\sigma B_\alpha \xi - P_\sigma \xi) \wedge dz_1 \wedge \ldots \wedge dz_n =$$

$$= \bar{\partial}\psi B_\beta \eta \int (\bar{\partial}\varphi P_\sigma B_\alpha \xi - P_\sigma \xi) \wedge dz_1 \wedge \ldots \wedge dz_n + \bar{\partial}\eta_1,$$

where the support of η_1 is compact in V, we obtain the desired conclusion.

4.5. PROPOSITION. *Let $U \subset \mathbf{C}^{n+m}$ be an open set and $\alpha = (\alpha_1, \ldots, \alpha_n) \subset \subset A(U, \mathscr{L}(H))$ is a matrix of commuting elements which commute also with $\alpha_1, \ldots, \alpha_n$. Denote by $\beta_j = \sum_{k=1}^{n} u_{jk}\alpha_k$, $\beta = (\beta_1, \ldots, \beta_n)$. If $\mathscr{S}_U(\alpha, H)$ and $\mathscr{S}_U(\beta, H)$ are both $\mathbf{C}^n$-compact then $\mu_\alpha(\xi) = \mu_\beta(\tilde{u}\xi)$, for any $\xi \in Z^n[(\sigma, \bar{\zeta}, \bar{\omega}), H_U^{\mathrm{loc}}]$, where $\tilde{u}$ is the map induced by the formula*

$$\tilde{u}(z, w)(x\sigma_{j_1} \wedge \ldots \wedge \sigma_{j_p}) = \sum_{k_1 < \cdots < k_p} (\det u_{k_h j_q}(z, w))_{h, q} x\sigma_{k_1} \wedge \ldots \wedge \sigma_{k_p},$$

for $x \in H$, $\tilde{u}(z, w)$ being the identity on the other terms (here "det" stands for the determinant).

The proof of Proposition 4.5 is similar to that of Proposition 3.12 from [4] (see also [6]), so that we omit it.

Proof of Theorem 4.2. By Lemma 4.1, it will be sufficient to verify the property $\mu_\alpha(f\xi) = \nu_\alpha(f)\mu_\alpha(\xi)$ for $\xi = \eta x$, where the coefficients of η are smooth functions with values in the commutant of α in $\mathscr{L}(H)$ and $x \in H$. By Theorem 4.4 and Lemma 4.1 we can write

$$\nu_\alpha(f)\mu_\alpha(\eta x) = \mu_\alpha(\nu_\alpha(f)\eta x) = \mu_\alpha(\eta \nu_\alpha(f) x) = \mu_{(\alpha, \alpha)}(\eta \wedge f x \sigma_1 \wedge \ldots \wedge \sigma_n).$$

If we transform the system $(z_1 - a_1, \ldots, z_n - a_n, w_1 - a_1, \ldots, w_n - a_n)$ into the system $(w_1 - z_1, \ldots, w_n - z_n, w_1 - a_1, \ldots, w_n - a_n)$ with a suitable matrix (see Theorem 4.3 from [4] or Theorem 4.1 from [6]) then the form $\eta \wedge f x \sigma_1 \wedge \ldots \wedge \sigma_n$ remains unchanged and we obtain by Proposition 4.5,

$$\mu_{(\alpha, \alpha)}(\eta \wedge f x \sigma_1 \wedge \ldots \wedge \sigma_n) = \mu_\alpha(\eta \nu_\beta(f) x) = \mu_\alpha(f \eta x),$$

where $\beta(z, w) = w - z$ and $\mu_\beta(f)(w) = f(w)$ [6].

Theorem 4.2 is, of course, related to the existence of the analytic functional calculus for commuting systems in $\mathscr{L}(H)$. Moreover, the formula (4.4) is given by a canonical kernel of Martinelli type (see also [6]), while (4.5) is connected to a family of canonical kernels, seemingly depending upon the parameter Ω. However, Theorem 4.2 gives more than the analytic functional calculus, actually in the one-dimensional case. Let us illustrate this assertion.

Consider $b \in \mathscr{L}(H)$ and take an open set $U \subset \mathbf{C}$ containing the spectrum of b. In this case the space $Z^1[(\sigma, \bar{\zeta}), H_U^{\mathrm{loc}}]$ can be identified with the space of those pairs (f_1, f_2) from $H_U^{\mathrm{loc}} \times H_U^{\mathrm{loc}}$ such that $(\partial f_1/\partial \bar{z})(z) = (z - b) f_2(z)$ almost everywhere in U. If we denote by ξ the form $f_1\sigma + f_2\bar{\zeta}$ then for $\alpha(z) = z - b$ the formula (4.4) can be written as

$$\mu_\alpha(\xi) = \int_\Gamma (z - b)^{-1} f_1(z)\, \mathrm{d}z + \int_\Delta f_2(z)\, \mathrm{d}z \wedge \mathrm{d}\bar{z},$$

where $\Delta \subset U$ contains the spectrum of b and Γ is its boundary. By Theorem 4.2 we obtain $\mu_\alpha(f\xi) = f(b)\mu_\alpha(\xi)$, for any $f \in A(U)$ and $\xi \in Z_0^1[(\sigma, \bar{\zeta}), H_U^{loc}]$, where $f(b)$ is given by the Riesz-Dunford functional calculus. In fact, this property is valid in this case for a larger class of forms, namely for those obtained by asking only the integrability of f_2.

In the scalar case, the formula (4.4), more precisely an extension of it, is connected to an integral representation formula of Martinelli type with an additional term. It is therefore plausible that these techniques can be applied in order to obtain integral representation formulas of a more general type, in particular for exterior forms whose coefficients are vector-valued functions, with respect to commuting systems of operator-valued analytic functions.

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