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COMMUTING WEIGHTED SHIFTS AND ANALYTIC FUNCTION THEORY IN SEVERAL VARIABLES

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1. INTRODUCTION

Given a separable complex Hilbert space H with orthonormal basis $\{e_n\}$ and a bounded sequence of complex numbers $\{w_n\}$, a weighted shift operator T is a (bounded linear) operator which satisfies $Te_n = w_n e_{n+1}$ for all n. T is called unilateral or bilateral according as the index n ranges over the non-negative integers or over all the integers. An excellent introduction to the theory of such operators and an extensive bibliography can be found in the recent comprehensive survey article by A. L. Shields [12]. It is shown there that each weighted shift is unitarily equivalent to multiplication by the function z on a weighted H^2 or L^2 space. This identification has been the cornerstone of an extensive interplay between operator theory and analytic function theory and weighted shift operators have been a rich source of examples and counter-examples in both areas.

In this paper we begin to extend the theory of single (i.e., one-variable) weighted shifts to systems of (N-variable) weighted shifts (which we define below) and we show an analogous identification between such systems and multiplications on certain H^2 or L^2 spaces in several variables. We concentrate our attention on the unilateral case where we will develop the basic analytic function theory. We will follow the outline of [12] as a model for our theory, and we omit the details of proofs which are obvious extensions of the single operator case. We assume that the reader is familiar with the basic theory of operators in Hilbert space. We present several applications of our theory to the theory of general commuting contractions, commuting subnormal operators, Toeplitz operators in several complex variables, and several variable analytic function theory. It is these applications which, to a great extent, motivate this presentation of the general theory. We also note that [12] contains a number of open problems most of which have natural extensions to the multivariable case.

2. DEFINITIONS AND ELEMENTARY PROPERTIES

Let N be a fixed positive integer throughout. We will use multi-index notation, i.e., let I be a multi-index $(i_1, \ldots, i_N)$ of integers. We write $I \geqslant 0$ whenever $i_j \geqslant 0$, $i = 1, \ldots, N$. We also use the notation

$$|I| = |i_1 + \ldots + i_n|, \quad I! = i_1! \ldots i_N!.$$

For $I \geqslant 0$ we write

$$z^I \equiv z_1^{i_1} \ldots z_N^{i_N}$$

where $z = (z_1, \ldots, z_N) \in \mathbf{C}^N$ and

$$T^I = T_1^{i_1} \ldots T_N^{i_N}$$

whenever $T = \{T_1, \ldots, T_N\}$ is a family of N commuting operators. We let $\varepsilon_k = (0, \ldots, 1, \ldots, 0)$ be the multi-index having $i_j = 1$ or 0 according as $j = k$ or otherwise and 0 be the multi-index $(0, 0, \ldots, 0)$ whose every entry is zero. For I the multi-index $(i_1, \ldots, i_N)$, $I \pm \varepsilon_k$ denotes the multi-index $(i_1, \ldots, i_k \pm 1, \ldots, i_N)$.

Let $\{e_I\}$ be an orthonormal basis of a complex Hilbert space H and let $\{w_{I,j}: j = 1, \ldots, N\}$ be a bounded net of complex numbers such that

$$(*) \qquad w_{I,k} w_{I+\varepsilon_k, l} = w_{I,l} w_{I+\varepsilon_l, k} \quad \text{for all } I,\ 1 \leqslant k, l \leqslant N.$$

$\mathscr{B}(H)$ denotes the algebra of all bounded linear operators on H.

DEFINITION. *A system of N-variable weighted shifts* is a family of N operators, $T = \{T_1, \ldots, T_N\}$ on H such that

$$T_j e_I = w_{I,j} e_{I+\varepsilon_j} \qquad (j = 1, \ldots, N).$$

Clearly the condition (*) on the set $\{w_{I,j}\}$ implies that T is a commuting family of operators. The family, T, is called a unilateral shift or bilateral shift according as I ranges over $\{I: I \geqslant 0\}$ or all the multi-indices of integers.

In the following we will restrict our attention primarily to systems of N-variable unilateral weighted shifts (which we will just call a unilateral shift) since these yield our main applications of the theory. Some of the results which are stated only for unilateral shifts have analogous statements for the bilateral case and, for the most part, we leave it to the reader to investigate when this is possible. So from now on, unless stated otherwise, we assume that T is a unilateral shift. (Similarly, we could generalize further and omit condition (*) to define non-commuting shifts, but our applications all deal with the commutative case.)

PROPOSITION 1. *If $\{\lambda_I\}$ are complex numbers of modulus 1, then T is unitarity equivalent to the weighted shift $S = \{S_1, \ldots, S_N\}$ with weights*

$$\mu_{I,j} = \bar{\lambda}_{I+\varepsilon_j} \lambda_I w_{I,j}.$$

[We note that $T = \{T_1, \ldots, T_N\}$, where each T_j acts on a Hilbert space H, is said to be unitarily equivalent to $S = \{S_1, \ldots, S_N\}$ where each S_j acts on a Hilbert space K, if there exists a unitary operator $U\colon H \to K$ such that

$$U^* S_j U = T_j, \quad j = 1, \ldots, N.]$$

Proof. Let U be the unitary operator defined by $Ue_I = \lambda_I e_I$.

COROLLARY 2. *Suppose all the weights $w_{I,j}$ of T are non-zero. Then T is unitarily equivalent to the unilateral shift S with weights given by*

$$\mu_{I,j} = |w_{I,j}|.$$

Proof. In Proposition 1 let $\lambda_0 = 1$ and $\lambda_{I+\varepsilon_j} = \lambda_I w_{I,j}/|w_{I,j}|$. The corollary follows once we show that $\{\lambda_I\colon I \geqslant 0\}$ is well-defined. Let

$$J = I_1 + \varepsilon_j = I_2 + \varepsilon_k = I + \varepsilon_j + \varepsilon_k.$$

Then

$$\lambda_J = \lambda_{I_1+\varepsilon_j} = \lambda_{I_1}\frac{w_{I_1,j}}{|w_{I_1,j}|} = \frac{\lambda_{I+\varepsilon_k} w_{I+\varepsilon_k,j}}{|w_{I+\varepsilon_k,j}|} =$$

$$= \frac{\lambda_I w_{I,k} w_{I+\varepsilon_k,j}}{|w_{I+\varepsilon_k,j}|\,|w_{I,k}|} = \frac{\lambda_I w_{I,j} w_{I+\varepsilon_j,k}}{|w_{I,j} w_{I+\varepsilon_j,k}|} =$$

$$= \frac{\lambda_{I+\varepsilon_j} w_{I+\varepsilon_j,k}}{|w_{I+\varepsilon_j,k}|} = \lambda_{I_2+\varepsilon_k}.$$

Hence, by induction (over $|I|$), $\{\lambda_I\colon I \geqslant 0\}$ is well-defined.

Note. Provided all the weights are non-zero, Corollary 2 is valid for bilateral shifts also. In this case we define $\{\lambda_J\}$ inductively as follows: (note that Proposition 1 is also true for bilateral shifts),

$$\lambda_0 = 1, \quad \lambda_{I+\varepsilon_j} = \lambda_I w_{I,j}/|w_{I,j}|, \quad \text{and} \quad \lambda_{I-\varepsilon_j} = \lambda_I \bar{w}_{I-\varepsilon_j,j}/|w_{I-\varepsilon_j,j}|.$$

As above λ_J is well-defined for

$$J = I_1 + \varepsilon_j = I_2 + \varepsilon_k \quad \text{and} \quad J = I_1 - \varepsilon_j = I_2 - \varepsilon_k;$$

suppose

$$J = I_1 + \varepsilon_j = I_2 - \varepsilon_k = I + \varepsilon_j - e_k.$$

Then

$$\begin{aligned}\lambda_J = \lambda_{I_1+\varepsilon_j} &= \frac{\lambda_{I_1} w_{I_1,j}}{|w_{I_1,j}|} = \frac{\lambda_{I-\varepsilon_k} w_{I-\varepsilon_k,j}}{|w_{I-\varepsilon_k,j}|} = \\ &= \frac{\lambda_I \bar{w}_{I-\varepsilon_k,k}}{|w_{I-\varepsilon_k,k}|} \frac{w_{I-\varepsilon_k,j}}{|w_{I-\varepsilon_k,j}|} = \\ &= \frac{\lambda_I \bar{w}_{I-\varepsilon_k,k}}{|w_{I-\varepsilon_k,k}|} \frac{w_{I-\varepsilon_k,k} w_{I,j}/w_{I-\varepsilon_k+\varepsilon_j,k}}{|w_{I-\varepsilon_k,k} w_{I,j}|/|w_{I-\varepsilon_k+\varepsilon_j,k}|} = \\ &= \lambda_I \frac{|w_{I-\varepsilon_k,k}|^2 w_{I,j} |w_{I-\varepsilon_k+\varepsilon_j,k}|}{|w_{I-\varepsilon_k,k}|^2 |w_{I,j}| w_{I-\varepsilon_k+\varepsilon_j,k}} = \\ &= \lambda_I \frac{w_{I,j}}{|w_{I,j}|} \frac{\bar{w}_{I-\varepsilon_k+\varepsilon_j,k}}{|w_{I-\varepsilon_k+\varepsilon_j,k}|} = \lambda_{I+\varepsilon_j} \frac{\bar{w}_{I-\varepsilon_k+\varepsilon_j,k}}{|w_{I-\varepsilon_k+\varepsilon_j,k}|} = \\ &= \lambda_{J_2-\varepsilon_k} = \lambda_J.\end{aligned}$$

This calculation shows that $\{\lambda_J\}$ is well-defined for all J.

COROLLARY 3. *Suppose* $\lambda = (\lambda_1, \ldots, \lambda_N)$ *where* $|\lambda_1| = \ldots = |\lambda_N| = 1$. *Then*

$$T = \{T_1, \ldots, T_N\} \quad and \quad \lambda T = \{\lambda_1 T_1, \ldots, \lambda_N T_N\}$$

are unitarily equivalent.

Proof. Let $\lambda_I = \bar{\lambda}_1^{i_1} \ldots \bar{\lambda}_N^{i_N}$ in Proposition 1.

From Corollary 3 we see that the spectrum of each $T_j, j = 1, \ldots, N$ as well as the various parts of the spectrum have circular symmetry about the origin, and the joint spectrum is invariant under "torus" rotations, i.e., $(\mu_1, \ldots, \mu_N) \in$ (joint spectrum of T) implies that $(\mu_1 e^{i\theta_1}, \ldots, \mu_N e^{i\theta_N}) \in$ (joint spectrum of T) for all $\theta_j \in$ $\in [0, 2\pi]$, $j = 1, \ldots, N$. Also, Corollary 2 shows that, for shifts with non-zero weights, we may always assume that $\{w_{I,j}\}$ is a set of positive real numbers.

PROPOSITION 4.

$$\|T^I\| = \sup_{J \geqslant 0} |w_{J,N} w_{J+\varepsilon_N,N} \cdots w_{J+(i_N-1)\varepsilon_N,N} w_{J+i_N\varepsilon_N,N-1} \cdots$$
$$w_{J+i_N\varepsilon_N+(i_{N-1}-1)\varepsilon_{N-1},N-1} \cdots w_{J+i_N\varepsilon_N+i_{N-1}\varepsilon_{N-1}+\ldots+i_2\varepsilon_2+(i_1-1)\varepsilon_1,1}|.$$

PROPOSITION 5.

$$T_j^* e_I = \begin{cases} \bar{w}_{I-\varepsilon_j,j} e_{I-\varepsilon_j} & \text{if } \ i_j \geqslant 1, \ \ j = 1, \ldots, N \\ 0 & \text{if } \ i_j = 0 \end{cases}$$

(If T is a bilateral shift, then

$$T_j^* e_I = \bar{w}_{I-\varepsilon_j,j} e_{I-\varepsilon_j}, \ j = 1, \ldots, N.)$$

PROPOSITION 6. *T_j is compact if and only if $|w_{I,j}| \to 0$ as $|I| \to \infty$. $T_j \in \mathscr{C}^p (0 < p < \infty)$ if and only if $\sum_I |w_{I,j}|^p < \infty$.*

PROPOSITION 7. *Let A be an operator on H having matrix (a_{IJ}) with respect to the basis $\{e_I\}$*, i.e., $a_{IJ} = (Ae_J, e_I)$, *and let S be a weighted shift with weight sequence $\{v_{I,j}\}$. Then, for any j, $1 \leqslant j \leqslant N$,*

$$AS_j = T_j A$$

if and only if

$$\begin{cases} v_{J,j} a_{I,J+\varepsilon_j} = 0 & \text{if } i_j = 0 \quad (I, J \geqslant 0) \\ v_{J,j} a_{I+\varepsilon_j, J+\varepsilon_j} = w_{I,j} a_{IJ} & \text{otherwise.} \end{cases}$$

(If S and T are bilateral, then $AS_j = T_j A$ if and only if

$$v_{J,j} a_{I+\varepsilon_j, J+\varepsilon_j} = w_{I,j} a_{I,j}$$

for all I, J.)

Proof. Compare the action of AS_j, $T_j A$ on e_J.

This proposition can be used to derive necessary and sufficient conditions for two shifts T and S to be similar or unitarily equivalent, e.g., two unilateral shifts, both with positive weights sets, $w_{I,j}$ and $v_{I,j}$, are unitarily equivalent if and only if $w_{I,j} = v_{I,j}$ for all $I \geqslant 0$ and $1 \leqslant j \leqslant N$.

EXAMPLES. (1) Let $L^2(T^N)$ denote the standard Lebesgue space of square summable functions from the N-torus, T^N, into **C**. Let $H^2(T^N)$ denote the standard Hardy space of $L^2(T^N)$ functions with analytic extension to the N-polydisc. Then $\{e_I = z^I\}$ is an orthonormal basis for $H^2(T^N)$ or $L^2(T^N)$ according as $I \geqslant 0$ or I is all multi-indices. The system $M = \{M_{z_1}, \ldots, M_{z_N}\}$ where M_{z_j} acts on $H^2(T^N)$ or $L^2(T^N)$ by multiplication by z_j $(1 \leqslant j \leqslant N)$ gives a system of N-variable weighted shifts, unilateral or bilateral, respectively, with weights $w_{I,j} = 1$ for all I and j.

(2) Let S^N denote the unit sphere in $\mathbf{C}^N$. Let $H^2(S^N)$ denote the standard Hardy space given by the closure in $L^2(S^N)$ of the polynomials in the coordinate functions $z_1, \ldots, z_N$. For each j, $1 \leqslant j \leqslant N$, let M_{z_j} act on $H^2(S^N)$ by multiplication by z_j. We can parametrize the sphere in such a way that the system $M = \{M_{z_1}, \ldots, M_{z_N}\}$ is identified as a system of N-variable weighted shifts with weights given by

$$w_{I,j} = (i_j + 1)^{\frac{1}{2}} / (|I| + N)^{\frac{1}{2}}$$

(see [7]). Note that M_{z_j} is the Toeplitz operator acting as $H^2(S^N)$ with symbol the jth coordinate function.

(3) Let $\{e_I : I \geqslant 0\}$ be an orthonormal basis for a Hilbert space H and let $T = \{T_1, \ldots, T_N\}$ be a system of N-variable weighted unilateral shifts with weights

$$w_{I,j} = (i_j + 1)^{\frac{1}{2}} / (|I| + 1)^{\frac{1}{2}}.$$

Then T can be used as a universal model for a large class of commuting contractions in the sense that if $S = \{S_1, \ldots, S_N\}$ is a system of commuting contractions such that

$$\sum_{j=1}^{N} \|S_j\|^2 < 1,$$

then S is unitarily equivalent to a compression of T to the orthogonal complement of some joint invariant subspace [8]. This yields analogues of some well-known theorems modelling (single) contractions on the adjoint of the standard unilateral shift. For some related work on compressions of systems of unilateral shifts and their dilations see [2, 3].

Furthermore, it can be shown that each T_j is subnormal, i.e, (for each j, $1 \leqslant \leqslant j \leqslant N$) there exists a Hilbert space $K_j \supseteq H$ and normal operators $N_j \in \mathscr{B}(K_j)$ with $N_j | H = T_j$. The lifting problem asks whether T has a commuting subnormal extension, i.e., whether there exists a Hilbert space $K \supseteq H$ and commuting normal operators $M_1, \ldots, M_N \in \mathscr{B}(K)$ such that

$$M_j | H = T_j, \quad 1 \leqslant j \leqslant N.$$

It was formerly unknown whether the lifting problem always had a solution, but T answers this negatively [8]. Two additional examples of commuting subnormals without commuting normal extension follow; we note that, at present, all known examples of this phenomenon use weighted shifts. In this context, Carl Cowen has recently described an analytic Toeplitz operator (which is thus subnormal) whose commutant does not dilate. In fact its commutant contains a compact operator. See [5].

(4) Let $N = 2$ and $\{e_I : I \geqslant 0\}$ be an orthonormal basis for H. Let $T = \{T_1, T_2\}$ be a two-variable system of weighted unilateral shifts with weights

$$w_{I,1} = \begin{cases} 2 \text{ if } i_2 = 0, \\ 1 \text{ if } i_2 \neq 0, \end{cases} \qquad w_{I,2} = \begin{cases} 2^{-n} \text{ if } i_2 = 0,\ i_1 = n \\ 1 \text{ if } i_2 \neq 0. \end{cases}$$

Then T_1 and T_2 are both subnormal, but do not have a commuting normal extension. In fact, T_2 does not have any bounded extension commuting with the minimal normal extension of T_1. This example is due to M. B. Abrahamse [1], although it was not given in the context of weighted shifts.

(5) Let $N = 2$ and $\{e_I : I \geqslant 0\}$ be an orthonormal basis for H. Let $T = \{T_1, T_2\}$ be a two-variable system of weighted shifts with weights

$$\begin{aligned} w_{I,1} &= 1 \quad \text{if} \quad i_2 = 0 \\ w_{I,2} &= 1 \quad \text{if} \quad i_1 = 0 \\ w_{I,j} &= 0 \quad \text{otherwise.} \end{aligned}$$

Then T_1 and T_2 are both subnormal, and, in fact, are both quasinormal; also, each element of the two parameter semigroup $\{T^I\}$ is subnormal. However, T does not have a commuting normal extension [10].

Although Examples 4 and 5 are of interest as counter-examples to some natural conjectures, shifts having some of their weights zero represent, in some sense, a degenerate case. Hence, unless specified otherwise, we assume all weights $w_{I,j}$ are non-zero. Then, as already noted, Corollary 2 implies that we may assume that all the weights are positive real numbers.

3. WEIGHTED SEQUENCE SPACES

As in the single operator case, we now find that we may view a system of N-variable weighted unilateral shifts as multiplication operators on certain weighted sequence spaces.

DEFINITION. Let $\{\beta_I : I \geqslant 0\}$ be a set of strictly positive numbers with $\beta_0 = 1$. Then, let

$$H^2(\beta) = \{f(z) = \sum_{I \geqslant 0} f_I z^I : \|f\|_\beta^2 = \sum_{I \geqslant 0} |f_I|^2 \beta_I^2 < \infty\}.$$

Clearly $H^2(\beta)$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{I \geqslant 0} f_I \bar{g}_I \beta_I^2.$$

We note that the elements of $H^2(\beta)$ are considered as formal power series without regard to convergence at any point $z \in \mathbf{C}^N$. $\{z^I : I \geqslant 0\}$ forms an orthogonal basis for $H^2(\beta)$ which is, in general, not orthonormal.

Let $M = \{M_{z_1}, \ldots, M_{z_N}\}$ denote the multiplication operators given by

$$M_{z_j} f(z) = z_j f(z) \qquad (j = 1, \ldots, N)$$

defined on the "polynomials" in the coordinate functions, z_j, of $H^2(\beta)$. Then M_{z_j}, which may not be bounded on $H^2(\beta)$, shifts the weighted basis $\{z^I\}$ of $H^2(\beta)$ and, as the following proposition shows, this is equivalent to a weighted shift acting on an orthonormal basis.

PROPOSITION 8. *The linear transformations M_{z_j} $(j = 1, \ldots, N)$ acting on $H^2(\beta)$ form a system which is unitarily equivalent to a system of injective weighted unilateral shift linear transformations with weights $w_{I,j}$ defined in terms of β as below. Conversely, every system of injective weighted unilateral shifts with weights $w_{I,j}$ is unitarily equivalent to $\{M_{z_1}, \ldots, M_{z_N}\}$ acting on some $H^2(\beta)$.*

Proof. For the first half define $w_{I,j} = \beta_{I+\varepsilon_j}/\beta_I$. On the other hand, given $\{T_1, \ldots, T_N\}$ define β_I by

$$T^I e_0 = \beta_I\, e_I,\ I \geqslant 0.$$

$\beta_I \neq 0$ for all I since each T_j is injective and, in fact, $\beta_I > 0$ since we may assume that the shifts have positive weights. In either case define $U: H \to H^2(\beta)$ by $Ue_I = \beta_I^{-1} z^I$. Then U is unitary and

$$U^* M_j U = T_j, \; j = 1, \ldots, N.$$

Note that the above proposition holds only for commuting weighted shifts. For the bilateral case, define again

$$w_{I,j} = \beta_{I+\varepsilon_j}/\beta_I.$$

In the other direction, given $\{w_{I,j}\}$, define

$$\beta_0 = 1, \beta_{I+\varepsilon_j} = \beta_I w_{I,j},$$

(which reduces to $T^I e_0 = \beta_I e_I$ in the unilateral case) and

$$\beta_{I-\varepsilon_j} = \beta_I / w_{I-\varepsilon_j, j}.$$

We must show that $\{\beta_J\}$ is well-defined. This follows, since, if

$$J = I_1 + \varepsilon_i = I_2 - \varepsilon_j = I + \varepsilon_i - \varepsilon_j,$$

then

$$\beta_J = \beta_{I_1+\varepsilon_i} = \beta_{I_1} w_{I_1, i} = \beta_{I-\varepsilon_j} w_{I-\varepsilon_j, i} = \beta_I w_{I-\varepsilon_j, i} / w_{I-\varepsilon_j, j} =$$
$$= \beta_J w_{I, i} / w_{I+\varepsilon_i-\varepsilon_j, j} = \beta_{I+\varepsilon_i} / w_{I+\varepsilon_i-\varepsilon_j, j} = \beta_{I_2} - \varepsilon_j = \beta_J.$$

COROLLARY 9. *M_{z_j} is bounded ($j = 1, \ldots, N$) if and only if*

$$\{\beta(I+\varepsilon_j)/\beta(I) : I \geqslant 0\}$$

is bounded for each j, $1 \leqslant j \leqslant N$.

From now on we will assume that each M_{z_j} is bounded and we note that we can now interchange freely between either viewing the operators as weighted shifts or as multiplication operators. By examining the conditions for similarity of shifts we note, amongst other facts, that

$$H^2(T^N) \neq H^2(S^N) \quad (N > 1).$$

EXAMPLES. (1) In Example 1 of Section 2, $\beta_I = 1$ for all I and

$$H^2(\beta) = H^2(T^N).$$

(2) In Example 2 of Section 2,

$$\beta_I = [I!/(|I| + N - 1)!]^{1/2} \text{ and } H^2(\beta) = H^2(S^N).$$

(3) In Example 3 of Section 2,

$$\beta_I = [I!/|I|!]^{1/2}.$$

(4) In Example 4 of Section 2,

$$\beta_I = 1 \text{ if } i_2 \neq 0, \; \beta_I = 2^{i_1} \text{ if } i_2 = 0.$$

Note that $\{\beta_I\}$ is unbounded, but M_{z_1}, M_{z_2}, are each bounded operators. This is possible since

$$\|M_{z_1}\| = 2 > 1.$$

4. THE COMMUTANT

Given a formal power series, $\varphi(z)$, in N variables, φ induces a map on $H^2(\beta)$ by formal power series multiplication $f \to \varphi f$. We denote by $H^\infty(\beta)$ the set $\{\varphi\colon \varphi f \in H^2(\beta)$ for all $f \in H^2(\beta)\}$ and, for $\varphi \in H^\infty(\beta)$ we denote by M_φ the map taking f to φf. Since $z^0 \in H^2(\beta)$ we have $\varphi z^0 = \varphi \in H^2(\beta)$ for any $\varphi \in H^\infty(\beta)$, i.e., $H^\infty(\beta) \subseteq H^2(\beta)$. So φ has the representation

$$\varphi(z) = \sum_{I \geqslant 0} \varphi_I z^I.$$

A linear operator A on $H^2(\beta)$ can be represented by the matrix (A_{IJ}) with respect to the orthogonal basis z^I, where

$$A_{IJ} = \frac{\langle Az^J, z^I\rangle}{\|z^I\|^2}.$$

If A and B are operators with corresponding matrices (A_{IJ}) and (B_{IJ}), then AB is represented by the matrix whose (I, J)th entry is $\sum_{K \geqslant 0} A_{I,K} B_{K,J}$.

PROPOSITION 10. (1) *M_φ is a bounded map on $H^2(\beta)$;*

(2) $M_{\varphi\psi} = M_\varphi M_\psi$ $(\varphi, \psi \in H^\infty(\beta))$.

Proof. (1) Note that

$$\varphi(z) z^J = \sum_{I \geqslant 0} \varphi_I z^{I+J} = \sum_{I \geqslant J} \varphi_{I-J} z^I.$$

Thus

$$(\varphi z^J, z^K) = \varphi_{K-J} \beta_K^2 \quad (K \geqslant J).$$

Hence the matrix of M_φ is given by $A_{IJ} = \varphi_{I-J}$ $(I \geqslant J)$, 0 elsewhere. This implies that M_φ is bounded since its matrix is everywhere defined.

(2) For $f \in H^2(\beta)$,

$$M_{\varphi\psi} f = (\varphi\psi) f$$

(note that $\varphi\psi$ is a well-defined element of $H^\infty(\beta)$).
Thus

$$M_{\varphi\psi} f = \varphi(\psi f) = M_\varphi M_{\psi f}.$$

This last proposition shows that $H^\infty(\beta)$ is a commutative algebra of bounded operators on $H^2(\beta)$ containing M_{z_j}, $j = 1, \ldots, N$. Hence the commutant, $\{M_{z_j}\colon j = 1, \ldots, N\}'$, contains $H^\infty(\beta)$. We will show that equality holds.

THEOREM 11. *If A is a bounded operator on $H^2(\beta)$ which commutes with M_{z_j}, $(j = 1, \ldots, N)$, then $A = M_\varphi$ for some $\varphi \in H^\infty(\beta)$.*

Proof. Let $\varphi = Az^0$. Then $\varphi \in H^2(\beta)$ and $Az^K = AM_{z^K} z^0$ where $M_z = \{M_{z_1}, \ldots, M_{z_N}\}$. Thus $Az^K = M_{z^K} Az^0 = z^K \varphi$ $(K \geqslant 0)$. Thus $Af = \varphi f$ for all polynomials f. For arbitrary $f \in H^2(\beta)$ we approximate f by polynomials f_n, and, by the algebraic properties of power series multiplication, we see that $Af = \varphi f$. Thus $\varphi \in H^\infty(\beta)$ and $A = M_\varphi$.

COROLLARY 12. *Suppose T is a system of injective unilateral shifts. Then $\{T_1, \ldots, T_N\}'$ is a maximal abelian subalgebra of $\mathscr{B}(H)$.*

COROLLARY 13. *$\{T_1, \cdots, T_N\}$ have no common reducing subspace.*

EXAMPLES. (1) Consider Example 2 of Section 2. The operators $T_1, \ldots, T_N$ are the Toeplitz operators on $H^2(S^N)$ with symbols given by the coordinate functions. Theorem 11 shows that $T \in \mathscr{B}(H^2(S^N))$ commutes with $T_1, \ldots, T_N$ if and only if $T = M_\varphi$ for some $\varphi \in H^\infty(\beta)$. Since $H^2(\beta) = H^2(S^N)$, it is easy to see that $H^\infty(\beta) = H^\infty(S^N)$, i.e., functions which are boundary values of bounded analytic functions as the open unit ball in $\mathbf{C}^N$. This was first proved for general N in [7]. Corollary 12 shows that the C^*-algebra generated by $\{T_j : j = 1, \ldots, N\}$ is irreducible; this was first proved by Coburn [4] using properties of the Szegö reproducing kernel for $H^2(S^2)$. An alternative proof in the spirit of this paper is given in [7].

(2) In the case $N = 1$ it is easy to see that M_{z_1} does not have a square root. For $N > 1$, since it is only $\{M_{z_1}, \ldots, M_{z_N}\}'$ and not $\{M_{z_j}\}'$ that is well-behaved, it is not surprising that roots exist. For $f \in H^2(\beta)$ we write

$$f(z) = \sum z_N^k f_k(z_1, \ldots, z_{N-1}).$$

Define A by

$$A(z_N^{2j} g(z_1, \ldots, z_{N-1})) = z_N^{2j+1} g(z_1, \ldots, z_{N-1})$$

$$A(z_N^{2j+1} g) = z_1 z_N^{2j} g.$$

We choose β so that we can extend A to $H^2(\beta)$ by linearity and continuity (this is possible for many choices of β). Then

$$A^2 f = M_{z_1} f.$$

We use the notation $\|\varphi\|_\infty = \|M_\varphi\|$ for $\varphi \in H^\infty(\beta)$. Note that $\|\varphi\|_\beta \leqslant \|\varphi\|_\infty$ (since $\|\varphi\|_\beta = \|M_\varphi z^0\|_\beta \leqslant \|M_\varphi\| \|z^0\|_\beta = \|M_\varphi\|$).

COROLLARY 14. *$H^\infty(\beta)$ is a commutative Banach algebra.*

5. THE SPECTRUM

For any operator A let $\sigma(A)$ denote its spectrum and $r(A)$ its spectral radius. For a system of N commuting operators $T = (T_1, \ldots, T_N)$, let $j\sigma(T)$ denote the joint spectrum of $(T_1, \ldots, T_N)$.

THEOREM 15. *Let T be a system of N-variable weighted unilateral shifts (not necessarily injective). Then*

$$\sigma(T_j) = \{\lambda \in \mathbf{C}: |\lambda| \leqslant r(T_j)\} \quad (j = 1, \ldots, N).$$

Proof. Consider T_1 and assume the system is injective. Suppose $\lambda \notin \sigma(T_1)$. Then $(T_1 - \lambda)^{-1}$ exists and commutes with $T_1, \ldots, T_N$. Hence $(T_1 - \lambda)^{-1} = M_\varphi$ for some $\varphi \in H^\infty(\beta)$ by Theorem 11. Thus $(z_1 - \lambda)\varphi = 1$ which implies

$$\varphi_I = \begin{cases} -\lambda^{(-i_1+1)} \text{ if } i_2 = \ldots = i_N = 0 \\ 0 \text{ otherwise.} \end{cases}$$

Therefore,

$$|\varphi_I \beta(I + J)^2| = |\langle M_\varphi z^J,\ z^{I+J}\rangle| \leqslant \|M_\varphi\| \beta(J)\beta(I + J).$$

Thus

$$\beta(I + J)/\beta(J) \leqslant \|M_\varphi\|\ |\lambda|^{i_1+1}$$

for all I with $i_2 = \ldots = i_N = 0$ and for all J. Using Proposition 4 we have

$$\|M_{z_1}^k\| = \sup_{J \geqslant 0} |w_{J,1}\, w_{J+\varepsilon_1,1} \ldots w_{J+(k-1)\varepsilon_1,1}| = \sup_{J \geqslant 0} \left\{\frac{\beta(J + k\varepsilon_1)}{\beta(J)}\right\}.$$

So $\|M_{z_1}^k\| \leqslant \|M_\varphi\| |\lambda|^{k+1}$. Taking kth roots and letting $k \to \infty$, we have $r(T_1) \leqslant |\lambda|$. Thus $\sigma(T_1) = \{\lambda : |\lambda| \leqslant r(T_1)\}$. If T_1 has some zero weights, then T_1 is a norm limit of operators of the form S_1 where $\{S_j: 1 \leqslant j \leqslant N\}$ is a system of injective unilateral shifts and hence the result follows.

At this point it seems valuable to point out that information concerning $\{T_1, \ldots, T_N\}$ can be gleaned by regarding each T_j as a countable direct sum of one-variable weighted shift operators. Let us return to our original orthonormal basis $\{e_I: I \geqslant 0\}$ and for simplicity consider the case $N = 2$; the case for general N holds analogously.

Write X_m for the closed linear span of $\{e_{nm} : n \geqslant 0\}$ for each $m \geqslant 0$ and let Y_n be the closed linear span of $\{e_{nm} : m \geqslant 0\}$ for each $n \geqslant 0$. Clearly for each $n, m \geqslant 0$, X_m reduces T_1 and Y_n reduces T_2. Thus, we can write $T_1 = \bigoplus_{m=0}^{\infty} T_1|X_m$ and $T_2 = \bigotimes_{n=0}^{\infty} T_2|Y_n$ and each of the summands is a one-variable weighted shift with respect to the corresponding basis.

THEOREM 16. *Let $T : \{T_1, T_2\}$ be a system of injective two-variable weighted unilateral shifts. Then T_1 and T_2 have empty point spectrum.*

Proof. Suppose x is an eigenvector for T_1 corresponding to eigenvalue λ. We write

$$x = \sum_{n,m=0}^{\infty} x_{nm}\, e_{nm}.$$

Then

$$x_0 = \sum_{n=0}^{\infty} x_{n0} e_{n0}$$

is an eigenvector for the injective shift, $T_1|x_0$, with eigenvalue λ. However, injective (one-variable) unilateral shifts have no eigenvalues [9] and so $x_0 = 0$. Similarly,

$$0 = x_m = \sum_{n=0}^{\infty} x_{nm} e_{nm}$$

for each m and, therefore, $x = 0$. Thus T_1, and similarly, T_2, have empty point spectrum.

We can use the direct sum decomposition to improve Corollary 13 by dropping the commutativity assumption.

PROPOSITION 17. *Let $\{T_1, T_2\}$ be a system of not necessarily commuting weighted unilateral shifts having no zero weights. Then T_1, T_2 have no common reducing subspace.*

Proof. Consider $H = \bigoplus_{m=0}^{\infty} X_m$. With respect to this decomposition, we have

$$T_1 = \begin{bmatrix} T_1|X_0 & & & 0 \\ & T_1|X_1 & & \\ & & * & \\ 0 & & & * \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 & & & 0 \\ S_0 & 0 & & \\ & S_1 & * & \\ 0 & & * & * \end{bmatrix}$$

where S_j maps X_j onto X_{j+1} since all of the weights are non-zero. Let P be a projection onto a common reducing subspace, so that $PT_1 = T_1P$ and $PT_2 = T_2P$. Writing the matrix $P = (P_{ij})$ with $P_{ij}^* = P_{ji}$ and comparing entries of PT_2 and T_2P, we see, since each S_j is onto, that $P_{ij} = 0$ if $i \neq j$. Thus, each P_{ii} is a projection on X_{i-1} and $P_{ii}T_1|X_{i-1} = (T_1|X_{i-1}) P_{ii}$ $(i \geqslant 1)$. An injective weighted (one-variable) unilateral shift is irreducible and so $P_{ii} = 0$ or $I|X_{i-1}$ for each i. $PT_2 = T_2P$ implies that $P_{ii}S_{i-2} = S_{i-2}P_{i-1,\, i-1}$ $(i \geqslant 2)$ and hence $P = 0$ or I according as $P_{11} = 0$ or $I|X_0$, i.e., there are no nontrivial common reducing subspaces.

6. ANALYTIC STRUCTURE

For any $w = (w_1, \ldots, w_N) \in \mathbf{C}^N$, let λ_w denote the linear functional of evaluation at w defined on the polynomials by $\lambda_w(p) = p(w)$.

DEFINITION. w is said to be a *bounded point evaluation* (bpe) on $H^2(\beta)$ if λ_w extends to a bounded linear functional on $H^2(\beta)$, i.e., there exists some $c > 0$ such that

$$|\lambda_w(p)| = |p(w)| \leqslant c\|p\|_\beta$$

for all polynomials p.

If w is a bpe, then, by the Riesz theorem, there exists some $k_w \in H^2(\beta)$ such that $\lambda_w(f) = (f, k_w)$ for all $f \in H^2(\beta)$. We call k_w the *reproducing kernel* for $H^2(\beta)$ at w. Since $(z^J, k_w) = w^J$ we see that we must have

$$k_w(z) = \sum_{J \geqslant 0} \bar{w}^J z^J / \beta(J)^2.$$

PROPOSITION 18. *w is a* bpe *if and only if $k_w(z) \in H^2(\beta)$, i.e., if and only if*

$$\sum_{J \geqslant 0} |w_1|^{2j_1} \ldots |w_N|^{2j_N} / \beta(J)^2 < \infty.$$

PROPOSITION 19. *w is a* bpe *if and only if $\bar{w}_j$ is in the point spectrum of T_j^* for each j, $1 \leqslant j \leqslant N$, and each $\bar{w}_j$ corresponding to a common eigenvector in $H^2(\beta)$. If $|w_j| = \|T_j\|$ for each j, $1 \leqslant j \leqslant N$, then w is not a* bpe.

Proof. If w is a bpe, then, for $f \in H^2(\beta)$, we have

$$(f, T_j^* k_w) = (z_j f, k_w) = w_j (f, k_w) = (f, \bar{w}_j k_w).$$

Thus $T_j^* k_w = \bar{w}_j k_w$ and so $\bar{w}_j$ is in the point spectrum of T_j^* for each j. By Corollary 3, $\bar{w}_j$ is in the point spectrum of T_j^* for each j with a common eigenvector.
Conversely, suppose x is a non-zero vector in $H^2(\beta)$ and $T_j^* x = \bar{w}_j x$ for $j = 1, \ldots, N$. Let $\lambda(f) = (f, cx)$, for $f \in H^2(\beta)$ where c is a non-zero constant to be determined. λ is a bounded linear functional on $H^2(\beta)$. Also

$$\lambda(z_j f) = (z_j f, cx) = (f, c\, T_j^* x) = w_j \lambda(f).$$

So $\lambda(z^J) = w^J \lambda(z^0)$, $\lambda \not\equiv 0$ and so $\lambda(z^0) \neq 0$. Put $c = 1/\overline{(z_0, x)}$ so that $\lambda(z^0) = 1$. Then $\lambda(p) = p(w)$ for all polynomials p and so w is a bpe. If $|w_j| = \|T_j\|$ for $j = 1, \ldots, N$, then

$$\sum_{J \geqslant 0} \frac{|w_1|^{2j_1} \ldots |w_N|^{2j_N}}{\beta(J)^2} \geqslant \sum_{\substack{J = n\varepsilon_j \\ j = 1, \ldots, N \\ n = 0, 1, 2, \ldots}} \frac{|w_1|^{2j_1} \ldots |w_N|^{2j_N}}{\beta(J)^2} =$$

$$= \sum_{\substack{n=0 \\ j=1, \ldots, N}}^{\infty} \|T_j\|^{2n} / \beta(n\varepsilon_j)^2 = \infty$$

since $\beta(n\varepsilon_j) \leqslant \|T_j^n\|^2 \leqslant \|T_j\|^{2n}$.

PROPOSITION 20. *If w is a* bpe *and $f = \sum f_J z^J \in H^2(\beta)$, then the series $\sum f_J w^J$ converges absolutely to $\lambda_w(f)$. Thus we can unambiguously denote $\lambda_w(f) = f(w)$. Further this property characterizes a* bpe.

Proof. Suppose w is a bpe and $f = \sum f_J z^J$. Let $S_n = \sum_{|J| \leqslant n} f_J z^J$ be the nth partial sum of f. Then S_n converges to f in $H^2(\beta)$ and so $S_n(w) = \lambda_w(S_n) \to \lambda_w(f)$ and hence the power series $\sum f_J w^J$ converges. By symmetry, it also converges at $(|w_1|, \ldots, |w_n|)$ and since $f \in H^2(\beta)$ implies that $\sum |f_J| z^J \in H^2(\beta)$, the convergence must be absolute

Conversely, suppose the power series $\sum f_J w^J$ converges for all $f \in H^2(\beta)$. Then we can define a linear functional on $H^2(\beta)$ by $\lambda(f) = \sum f_J w^J$ and let

$$k_w^{(n)} = \sum_{|J| \leq n} \bar{w}^J z^J / \beta(J)^2.$$

Then

$$(f, k_w^{(n)}) = \sum_{|J| \leq n} f_J w^J \to \lambda(f) \text{ as } n \to \infty.$$

By the uniform boundedness principle

$$k_w = \sum_{J \geqslant 0} \bar{w}^J z^J / \beta_J^2 \in H^2(\beta)$$

and thus w is a bpe.

PROPOSITION 21. *If the power series $\varphi(z)$ converges at w for all $\varphi \in H^\infty(\beta)$, then*

$$|\varphi(w)| \leqslant \|M_\varphi\|.$$

Further, if w is a bpe *and $f \in H^2(\beta)$, then*

$$\lambda_w(\varphi f) = \lambda_w(\varphi)\lambda_w(f).$$

Also k_w is an eigenvector for all operators in $\{T_1^*, \ldots, T_N^*\}'$.

Proof. Since $H^\infty(\beta)$ is a commutative Banach algebra with identity and evaluation at w is a multiplicative linear functional, we have $|\varphi(w)| \leqslant \|M_\varphi\|$ from general Banach algebra theory. Note that this holds for all bpe's w, but in general will hold for a larger class of w's. For

$$\varphi = \sum \varphi_J z^J \in H^\infty(\beta),$$

we have

$$|\varphi_J| \leqslant \|M_\varphi\| / \|T^J\|,$$

and this can be used to compute which w's give convergence of the power series $\varphi(w)$ in special cases. The second statement follows from a formal power series argument and the third by reasoning similar to the proof of Proposition 19.

THEOREM 22. *If φ represents a bounded analytic function on the polydisc $\{z \colon |z_i| < \sqrt{n}\}$ then $\varphi \in H^\infty(\beta)$ and*

$$\|M_\varphi\| \leqslant \sup \{|\varphi(z)| : |z_i| < \sqrt{n}. \|T_i\|\}.$$

Proof. The key to the proof is an analogue of von Neumann's inequality for commuting contractions, namely

$$\|p(T_1, \ldots, T_n)\| \leqslant \sup \{|p(z)| : |z_i| < \sqrt{n}. \|T_i\|\} \text{ [9]}.$$

We note that this result is not in general best possible and we do not know if, in fact, $\|p(T_1, \ldots, T_N)\| \leqslant \sup \{|p(z)| : |z_i| < 1\}$ (which is in general false for commuting contractions) does hold for contractive weighted shifts.

This establishes the theorem for φ a polynomial. For a general bounded analytic function we approximate φ by its rectangular Cesaro sums which converge strongly as in the one-variable case [13, p. 310].

Question. Under what conditions is $H^\infty(\beta)=H^\infty(D)$ for some $D\subseteq \mathbf{C}^N$, and in this case how do we describe D?

7. SUBNORMAL SHIFTS

The results for one-variable weighted shifts carry through to N-variable case almost without change and so we only quote the most important of these which generalizes Berger's characterization of one-variable subnormal weighted shifts.

PROPOSITION 23. *$\{T_1, \ldots, T_N\}$ has a commuting normal extension if and only if there exists a probability measure μ defined on the N-dimensional rectangle $R = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_N]$, where $a_i = \|T_i\|$, such that $\int_R t_1^{2j_1}\ldots t_n^{2j_n}\mathrm{d}\mu(t)=$ $= \int_R t^{2J}\mathrm{d}\mu(t) = \beta_J^2$ for all $J \geqslant 0$.*

The proof is identical to the one-variable case given in [12].

8. ALGEBRAS GENERATED BY SHIFTS

Let $\mathscr{A}_T$ ~~be~~ the closure in $\mathscr{B}(H)$, in the weak operator topology, of the polynomials in $T_1, \ldots, T_N$. It is clear that $\mathscr{A}_T$ is contained in the commutant of $\{T_1, \ldots, T_N\}$ which is equal to $\{M_\varphi : \varphi\in H^\infty(\beta)\}$ by Theorem 11. We wish to show that equality holds. This follows from Theorem 26 below since a subspace of $\mathscr{B}(H)$ is closed in the weak operator topology if and only if it is closed in the strong operator topology.

For

$$w \in T^N = \{(w_1, \ldots, w_N) : |w_j| = 1, 1 \leqslant j \leqslant N\},$$

define a map on $H^2(\beta)$, $f \to f_w$, by

$$f_w(z_1, \ldots, z_N) = f(w_1 z_1, \ldots, w_N z_N),$$

i.e.,

$$(f_w)_J = w^J f_J.$$

PROPOSITION 24. (1) *If $\varphi \in H^\infty(\beta)$ and $w \in T^N$, then $\varphi_w \in H^\infty(\beta)$ and*

$$\|\varphi_w\|_\infty = \|\varphi\|_\infty;$$

(2) *$w \to \varphi_w$ is continuous from T^N into $H^\infty(\beta)$ in the strong operator topology for each $\varphi \in H^\infty(\beta)$.*

Proof. (1) Let $f \in H^2(\beta)$. Then $(\varphi f)_w = \varphi_w f_w$ and $(f_{w_1})_{w_2} = f_{w_1 w_2}$. Therefore $(f_{\bar{w}})_w = f$ and so $\varphi_w f = \varphi_w (f_{\bar{w}})_w = (\varphi(f_{\bar{w}}))_w \in H^2(\beta)$ and so $\varphi_w \in H(\beta)$. Also $\|f_w\|_\beta = \|f\|_\beta$ and so

$$\|\varphi_w f\| = \|\varphi f_{\bar{w}}\|_\beta \leq \|\varphi\|_\infty \|f\|_\beta, \text{ i.e., } \|\varphi_w\|_\infty \leq \|\varphi\|_\infty = \|(\varphi_w)_{\bar{w}}\|_\infty \leq \|\varphi_w\|_\infty.$$

$$(2)\ \|\varphi e_J - \varphi_w e_J\|_\beta = \| \sum_K \varphi_K e_{J+K} - (\varphi_w)_K e_{J+K}\|_\beta =$$

$$= \| \sum_{I \geqslant J} (1 - w^{I-J}) \varphi_{I-J} e_I\|_\beta \to 0 \text{ as } w \to (1, 1, \ldots, 1)$$

for fixed J. Thus, using (1), the required continuity holds at $(1, 1, \ldots, 1)$. By translation the result follows.

This proposition allows us to define the vector-valued Riemann integral $\int \varphi_w p(w) f \mathrm{d}s$ (when $f \in H^2(\beta)$, $\varphi \in H^\infty(\beta)$, p is continuous on T^N and $\mathrm{d}s$ is normalized Lebesgue measure on T^N) as in the one-variable case.

PROPOSITION 25. *If $\varphi \in H^\infty(\beta)$ and p is of the form:*

$$p(w) = \sum_K p_K w^K (w \in T^N)$$

where only finitely many coefficients are different from zero, then

$$\int \varphi_w p(\bar{w})\, \mathrm{d}s = M_{\varphi * p} \text{ when } (\varphi * p)(z) = \sum \varphi_J p_J z^J \in H^\infty(\beta).$$

Proof. Identical to the one-variable case.

Let $K_J(t) = K_{j_1}(t_1) \ldots K_{j_N}(t_N)$ be the multiple (rectangular) Fejer kernel [13, p. 303] where K_n is the usual one-variable Fejer kernel. Then, for $\varphi \in H^\infty(\beta)$,

$$\varphi * K_J = [1/(j_1 + 1) \ldots (j_{N+1})] \sum_{K \leqslant J} S_K(\varphi) = \sigma_J(\varphi)$$

where

$$S_K(\varphi) = \sum_{I \leqslant K} \varphi_I z^I.$$

THEOREM 26. *If $\varphi \in H^\infty(\beta)$, then*

(1) $\sigma_J(\varphi) \in H^\infty(\beta)$;

(2) $\|\sigma_J(\varphi)\|_\infty \leqslant \|\varphi\|_\infty$;

(3) $\sigma_J(\varphi) \to \varphi$ *in the strong operator topology.*

Proof. Identical to the one-variable case.

9. REMARKS

In the one-variable case there are several results concerning invariant subspaces of a weighted shift. In the present situation it is the common invariant subspaces of $\{T_1, \ldots, T_N\}$ that are of interest, but it seems too much to hope for much general information concerning these subspaces. However, in particular cases, (such as Example 2 of Section 2) a description of the common invariant subspaces of

$\{T_1, \ldots, T_N\}$ of finite codimension can be obtained [7] and this result can be extended to the general situation with a little work. Similarly, results concerning reflexivity of $\mathscr{A}_T$ can be proved [7] and again these can be extended to the general situation with suitable hypotheses. However, little extra is gained by looking at the proofs in the general situation and so we omit them here.

In [11] O'Donovan gave a beautiful description of the C^*-algebras generated by a single weighted shift in terms of certain covariance algebras when the shift is either essentially normal or has closed range. It would seem of interest to extend these ideas to the N-variable case since a general result would extend results already known for particular examples (Example 1 — see [6], Example 2 — see [7]). This appears to be a non-trivial problem.

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