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A PROOF OF A THEOREM ON TRACE REPRESENTATION OF STRONGLY POSITIVE LINEAR FUNCTIONALS ON $OP*$-ALGEBRAS

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1. INTRODUCTION

In [4], the following theorem was shown.

THEOREM I. *Let $\mathscr{D}$ be a dense linear subspace of a Hilbert space. Suppose that $\mathscr{D}[t_+]$ is a Fréchet space. The following are equivalent:*

(1) *$\mathscr{D}[t_+]$ is a Montel space.*

(2) *For any Op$*$-algebra $\mathscr{A}$ on $\mathscr{D}$ with $t_{\mathscr{A}} = t_+$, each strongly positive linear functional f on $\mathscr{A}$ is a trace functional, i.e. f is of the form $f(a) = \mathrm{Tr}\, ta$, $a \in \mathscr{A}$, where $t \in \mathfrak{S}_1(\mathscr{D})_+$.*

The proof given in [4] for the main part (1) $\Rightarrow$ (2) of the theorem relied on a method developed by Sherman [6]. Because Sherman's proof is very long, it is desirable to make it simpler. The purpose of the present note is to give another proof of the above-noted result (1) $\Rightarrow$ (2) which will be stated separately as

THEOREM II. *Let $\mathscr{A}$ be an Op$*$-algebra on $\mathscr{D}$ and let f be a strongly positive linear functional on $\mathscr{A}$. Suppose that $\mathscr{D}[t_{\mathscr{A}}]$ is a Fréchet-Montel space.*

Then there exists an operator $t \in \mathfrak{S}_1(\mathscr{D})_+$ such that $f(a) = \mathrm{Tr}\, ta$ for all $a \in \mathscr{A}$.

Our proof of Theorem II is based on topological arguments. It makes use of Theorem 4.4 (more precisely, of Proposition 4.1) in [4]. Let us recall some part of Theorem 4.4 from [4] in a convenient formulation for later use.

PROPOSITION. *Let $\mathscr{A}$ be an Op$*$-algebra on $\mathscr{D}$ such that $\mathscr{D}[t_{\mathscr{A}}]$ is a Fréchet-Montel space. Let f be a $\tau_{\mathscr{D}}$-continuous linear functional on $\mathscr{A}$.*

Then there is an operator $t \in \mathfrak{S}_1(\mathscr{D})$ so that $f(a) = \mathrm{Tr}\, ta$ for all $a \in \mathscr{A}$.

The present approach to Theorem II is shorter than Sherman's original proof and it gives a more general result.

Some arguments used here independently appear in [1]. In the case of Fréchet-Montel domains which have an unconditional basis another proof of Theorem II was also given in [5].

2. DEFINITIONS AND NOTATIONS

We collect the definitions we use in what follows (for more details about $Op*$-algebras we refer to [2]).

Let $\mathscr{D}$ be a dense linear subspace of a Hilbert space $\mathscr{H}$ and let $L^+(\mathscr{D}) := \{a \in \operatorname{End} \mathscr{D} : a\mathscr{D} \subseteq \mathscr{D}, a^*\mathscr{D} \subseteq \mathscr{D}\}$. Endowed with the involution $a \to a^+ := a^*|\mathscr{D}$, $L^+(\mathscr{D})$ is a $*$-algebra. An $Op*$-algebra $\mathscr{A}$ on $\mathscr{D}$ is a $*$-subalgebra of $L^+(\mathscr{D})$ containing the identity $I = I_{\mathscr{D}}$. $\mathscr{A}$ is said to be self-adjoint on $\mathscr{D}$ if

$$\mathscr{D} = \bigcap_{a \in \mathscr{A}} \mathscr{D}(a^*).$$

By the graph topology $t_{\mathscr{A}}$ on $\mathscr{D}$ we mean the locally convex topology defined by the seminorms $\|\varphi\|_a := \|a\varphi\|$, $a \in \mathscr{A}$. In the case $\mathscr{A} = L^+(\mathscr{D})$ we write t_+. The uniform topology $\tau_{\mathscr{D}}$ on $\mathscr{A}$ ([2]) is generated by the family of seminorms

$$p_{\mathfrak{m}}(a) := \sup_{\varphi, \psi \in \mathfrak{m}} |\langle a\varphi, \psi \rangle|$$

taken for all bounded subsets $\mathfrak{m}$ of $\mathscr{D}[t_{\mathscr{A}}]$.

Let $\mathscr{A}_h = \{a \in \mathscr{A} : a^+ = a\}$, $\mathscr{A}_+ = \{a \in \mathscr{A} : \langle a\varphi, \varphi \rangle \geqslant 0 \ \ \forall \varphi \in \mathscr{D}\}$ and $a \geqslant b$ iff $a - b \in \mathscr{A}_+$ for $a, b \in \mathscr{A}_h$. A linear functional f on $\mathscr{A}$ is called strongly positive if $f(a) \geqslant 0 \quad \forall\, a \in \mathscr{A}_+$.

Further, let $\mathfrak{S}_1(\mathscr{A}) = \{t \in \mathscr{B}(\mathscr{H}) : \overline{ta}$ and $\overline{t^*a}$ are of trace class for all $a \in \mathscr{A}; t\mathscr{H} \subseteq \mathscr{D}$, $t^*\mathscr{H} \subseteq \mathscr{D}\}$, $\mathfrak{S}_1(\mathscr{A})_+ = \{t \in \mathfrak{S}_1(\mathscr{A}) : t \geqslant 0\}$, $\mathfrak{S}_1(\mathscr{D}) = \mathfrak{S}_1(L^+(\mathscr{D}))$ and $\mathfrak{S}_1(\mathscr{D})_+ = \mathfrak{S}_1(L^+(\mathscr{D}))_+$.

A Montel space is a barreled locally convex space in which each bounded set is relatively compact.

3. PROOF OF THEOREM II

For convenience, the proof will be divided into several steps stated as lemmas.

LEMMA 1. *Let a be a symmetric operator on a unitary space $\mathscr{D}$. Let $\psi, \eta \in \mathscr{D}$ and $\varepsilon \in \mathbf{R}$, $0 < \varepsilon < 1$. If*

$$(\lambda\bar{\lambda} + \|\eta\|^2)^{-1} \langle a(\lambda\psi + \eta), \lambda\psi + \eta \rangle \geqslant (1 - \varepsilon) \langle a\psi, \psi \rangle, \quad \forall\, \lambda \in \mathbf{C}, \lambda \neq 0,$$

then

$$\langle a\eta, \eta \rangle \leqslant (1 + \varepsilon) \langle a(\lambda\psi + \eta), \lambda\psi + \eta \rangle, \quad \forall\, \lambda \in \mathbf{C}.$$

Proof. From our assumption it follows that

$$\lambda\bar{\lambda} \langle a\psi, \psi \rangle + \lambda \langle a\psi, \eta \rangle + \bar{\lambda} \langle a\eta, \psi \rangle - (1 - \varepsilon) \langle a\psi, \psi \rangle (\lambda\bar{\lambda} + \|\eta\|^2)$$

is non-negative for all $\lambda \in \mathbf{C}$. Hence the discriminant must be non-negative, that is,

$$\varepsilon \langle a\,\psi, \psi \rangle [\langle a\eta, \eta \rangle - (1 - \varepsilon) \langle a\psi, \psi \rangle \|\eta\|^2] - |\langle a\psi, \eta \rangle|^2 \geqslant 0.$$

Because $0 < \varepsilon < 1$, this gives

$$(1+\varepsilon)\langle a\psi,\psi\rangle\varepsilon\langle a\eta,\eta\rangle-(1+\varepsilon)\,|\langle a\psi,\eta\rangle|^2\geqslant 0.$$

Therefore

$$\lambda\bar{\lambda}(1+\varepsilon)\langle a\psi,\psi\rangle+(1+\varepsilon)\lambda\langle a\psi,\eta\rangle+(1+\varepsilon)\bar{\lambda}\langle a\eta,\psi\rangle+\varepsilon\langle a\eta,\eta\rangle=$$

$$=(1+\varepsilon)\langle a(\lambda\psi+\eta),\lambda\psi+\eta\rangle-\langle a\eta,\eta\rangle$$

takes only non-negative values for all $\lambda\in\mathbf{C}$. This proves the assertion. ▨

LEMMA 2. *Let f be a strongly positive linear functional on an Op*-algebra $\mathscr{A}$ on $\mathscr{D}$ and let $a, x\in\mathscr{A}$.*

If $|\langle x\varphi,\varphi\rangle)\rangle| \leqslant \langle a\varphi,\varphi\rangle$ for all $\varphi\in\mathscr{D}$, then $|f(x)|\leqslant\sqrt{2}f(a)$.

Proof. Let $x=x_1+\mathrm{i}x_2$, $x_1=x_1^+\in\mathscr{A}$, $x_2=x_2^+\in\mathscr{A}$. $|\langle x\,\varphi,\varphi\rangle|\leqslant\langle a\varphi,\varphi\rangle\ \ \forall\ \varphi\in\mathscr{D}$ implies that $\pm\, x_1\leqslant a$ and $\pm x_2\leqslant a$. Hence $\pm\, f(x_1)\leqslant f(a)$ and $\pm\, f(x_2)\leqslant f(a)$ which gives

$$|f(x)|^2=f(x_1)^2+f(x_2)^2\leqslant 2f(a)^2. ▨$$

LEMMA 3. *Let $\{a_i, i\in\mathbf{N}\}$ be a sequence of operators $a_i\in L^+(\mathscr{D})$ and $\{\alpha_i, i\in\mathbf{N}\}$ a sequence of positive real numbers.*

There is a real sequence $\{\beta_i, i\in\mathbf{N}\}$ with $\beta_1=\alpha_1^2$, $0<\beta_i<\alpha_i^2$, $\forall\ i\in\mathbf{N}$ and an orthonormal system $\{\varphi_i, i\in\mathbf{N}\}$ of vectors $\varphi_i\in\mathscr{D}$ so that

$$\sum_{i=1}^{n}\beta_i\|a_i(I-E_n)\varphi\|^2\leqslant 4\sum_{i=1}^{n}\alpha_i^2\|a_i\varphi\|^2 \tag{1}$$

for all $\varphi\in\mathscr{D}$, $n\in\mathbf{N}$, whereby $E_n=P_1+\ldots+P_n$ and P_i denotes the projection on the one-dimensional subspace generated by φ_i.

Proof. Let $\{\varepsilon_i, i\in\mathbf{N}\}$ be a real sequence with $0<\varepsilon_i<1/2$ and $\prod_{i=1}^{n}(1+\varepsilon_i)\leqslant 4$.

By induction on n we prove that

$$\sum_{i=1}^{n}\beta_i\|a_i(I-E_n)\varphi\|^2\leqslant\left[\sum_{i=1}^{n}(1+\varepsilon_i)\right]\sum_{i=1}^{n}\alpha_i^2\|a_i\varphi\|^2,\ \ \forall\ \varphi\in\mathscr{D} \tag{2}$$

which implies (1).

Suppose that $\beta_1,\ldots,\beta_n$ and $\varphi_1,\ldots,\varphi_n$ are already chosen so that (2) is fulfilled. Let

$$\beta_{n+1}=(1+\|a_{n+1}E_n\|)^{-2}\alpha_{n+1}^2. \tag{3}$$

Further, let

$$I_{n+1}=\inf\left\{\sum_{i=1}^{n+1}\beta_i\|a_i\varphi\|^2;\ \|\varphi\|=1,\ \varphi\in(I-E_n)\mathscr{D}\right\}.$$

We choose a unit vector $\varphi_{n+1} \in (I - E_n)\mathscr{D}$ so that

$$(4) \qquad (1 - \varepsilon_{n+1}) \sum_{i=1}^{n+1} \beta_i \|a_i \varphi_{n+1}\|^2 \leqslant I_{n+1}.$$

We prove that (2) is true for $n+1$. Let $\varphi \in \mathscr{D}$. We apply Lemma 1 with

$$a = \sum_{i=1}^{n+1} \beta_i a_i^+ a_i, \psi = \varphi_{n+1}, \ \eta = (1 - E_{n+1})\varphi$$

and $\varepsilon = \varepsilon_{n+1}$. By the definition of I_{n+1} we have

$$(\lambda\bar{\lambda} + \|\eta\|^2)^{-1} \langle a(\lambda\psi + \eta), \lambda\psi + \eta \rangle \equiv$$

$$\equiv (\lambda\bar{\lambda} + \|(I - E_{n+1})\varphi\|^2)^{-1} \sum_{i=1}^{n+1} \beta_i \|a_i(\lambda\varphi_{n+1} + (I - E_{n+1})\varphi\|^2 \geqslant I_{n+1} \underset{(4)}{\geqslant}$$

$$\underset{(4)}{\geqslant} (1 - \varepsilon_{n+1}) \sum_{i=1}^{n+1} \beta_i \|a_i \varphi_{n+1}\|^2 \equiv (1 - \varepsilon)\langle a\psi, \psi \rangle \text{ for all } \lambda \in \mathbf{C}, \lambda \neq 0.$$

Putting $\lambda = \langle \varphi, \varphi_{n+1} \rangle$ it follows from Lemma 1 that

$$\sum_{i=1}^{n+1} \beta_i \|a_i(I - E_{n+1})\varphi\|^2 \equiv \langle a\eta, \eta \rangle \leqslant (1 + \varepsilon)\langle a(\lambda\psi + \eta), \lambda\psi + \eta \rangle \equiv$$

$$(5) \qquad \equiv (1 + \varepsilon_{n+1}) \sum_{i=1}^{n+1} \beta_i \|a_i(P_{n+1}\varphi + (I - E_{n+1})\varphi\|^2.$$

Applying the induction hypothesis we obtain

$$\sum_{i=1}^{n+1} \beta_i \|a_i(I - E_{n+1})\varphi\|^2 \underset{(5)}{\leqslant} (1 + \varepsilon_{n+1}) \sum_{i=1}^{n+1} \beta_i \|a_i(I - E_n)\varphi\|^2 \leqslant$$

$$\leqslant (1 + \varepsilon_{n+1}) \left[\left[\prod_{i=1}^{n} (1 + \varepsilon_i) \right] \sum_{i=1}^{n} \alpha_i^2 \|a_i\varphi\|^2 + \beta_{n+1}(\|a_{n+1}\varphi\| + \|a_{n+1}E_n\| \|a_{n+1}\varphi\|)^2 \right] \underset{(3)}{\leqslant}$$

$$\underset{(3)}{\leqslant} \left[\prod_{i=1}^{n+1} (1 + \varepsilon_i) \right] \sum_{i=1}^{n+1} \alpha_i^2 \|a_i\varphi\|^2$$

which gives (2) for $n + 1$.

We have to say some words about the first step of induction. We take $\beta_1 = \alpha_1^2$ and choose $\varphi_1 \in \mathscr{D}, \|\varphi_1\| = 1$, such that (4) is fulfilled for $n = 0, E_0 = 0$. Then, by the same arguments, (1) is true in the case $n = 1$. ▨

Now let $\mathscr{A}$ be an $Op*$-algebra on $\mathscr{D}$. Suppose that $\mathscr{D}[t_\mathscr{A}]$ is a Fréchet-Montel space. Then there is a sequence $\{a_i, i \in \mathbf{N}\}$ of operators $a_i \in \mathscr{A}$ so that $\|\varphi\| \leqslant \leqslant \|a_i\varphi\| \leqslant \|a_{i+1}\varphi\| \ \ \forall \varphi \in \mathscr{D}, i \in \mathbf{N}$, and the seminorms $\|\varphi\|_{a_i}, i \in \mathbf{N}$, generate $t_\mathscr{A}$. Using Lemma 3 for these operators a_i, the next Lemma is mainly a consequence of the Montel property of $\mathscr{D}[t_\mathscr{A}]$.

LEMMA 4. *For each $n \in \mathbf{N}$ there is a number $k_n \in \mathbf{N}$ such that*

$$n^2\|a_n(I - E_{k_n})\varphi\|^2 \leqslant \sum_{i=1}^{k_n} \beta_i\|a_i(I - E_{k_n})\varphi\|^2 \quad \forall \varphi \in \mathscr{D}. \tag{6}$$

Proof. Assume that this is not the case for a number $n \in \mathbf{N}$. Then for every $k \in \mathbf{N}$ there is a vector $\xi_k \in \mathscr{D}$ so that

$$n^2\|a_n(I - E_k)\xi_k\|^2 > \sum_{i=1}^{k} \beta_i\|a_i(I - E_k)\xi_k\|^2.$$

Let

$$\psi_k = (I - E_k)\xi_k.$$

After norming the vectors we obtain $\|a_n\psi_k\| = 1$. Then

$$n^2 > \sum_{i=1}^{k} \beta_i\|a_i\psi_k\|^2 \geqslant \beta_i\|a_i\psi_k\|^2 \tag{7}$$

for $i \leqslant k$. Consequently,

$$\sup_{k\in\mathbf{N}} \|a_i\psi_k\|^2 \leqslant \sup(\|a_i\psi_1\|^2, \ldots, \|a_i\psi_{i-1}\|^2, n^2/\beta_i) < +\infty.$$

Thus, $\{\psi_k, k \in \mathbf{N}\}$ is a $t_{\mathscr{A}}$-bounded sequence. Since $\mathscr{D}[t_{\mathscr{A}}]$ is a Montel space, there is a $t_{\mathscr{A}}$-convergent subsequence $\psi_{r_k} \to \psi_0 \in \mathscr{D}$. By (7), we have

$$n^2 > \sum_{i=1}^{s} \beta_i\|a_i\psi_{r_k}\|^2 \quad \text{for } s \leqslant r_k, s \in \mathbf{N},$$

which gives

$$n^2 \geqslant \sum_{i=1}^{s} \beta_i\|a_i\psi_0\|^2 \quad \text{for all } s \in \mathbf{N}.$$

Since

$$1 = \lim_{k\to\infty} \|a_n\psi_{r_k}\| = \|a_n\psi_0\|,$$

it follows that $\psi_0 \neq 0$.
Clearly, $\psi_0 \in (I - E_s)\mathscr{D}$ $\forall s \in \mathbf{N}$ because $\psi_{r_k} \in (I - E_{r_k})\mathscr{D}$. Hence by the definition of I_s we get

$$\|\psi_0\|^{-2} n^2 \geqslant \|\psi_0\|^{-2} \sum_{i=1}^{s} \beta_i\|a_i\psi_0\|^2 \geqslant I_s \underset{(4)}{\geqslant} (1 - \varepsilon_s) \sum_{i=1}^{s} \beta_i\|a_i\varphi_s\|^2 \geqslant$$
$$\geqslant 1/2\, \beta_i\|a_i\varphi_s\|^2 \quad \text{for } i \leqslant s.$$

Obviously,

$$\|a_i\varphi_s\|^2 \leqslant 2n^2\|\psi_0\|^{-2}/\beta_i$$

for $i \leqslant s$ implies the $t_{\mathscr{A}}$-boundedness of the sequence $\{\varphi_s, s \in \mathbf{N}\}$. Using again the Montel property of $\mathscr{D}[t_{\mathscr{A}}]$, it follows that this sequence would have a cluster point. This is a contradiction because $\{\varphi_s, s \in \mathbf{N}\}$ is an orthonormal system. ▨

LEMMA 5.

$$n\|a_n(I - E_{k_n})\varphi\|\|\varphi\| \leqslant \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|^2 \quad \textit{for all } \varphi \in \mathscr{D},\ n \in \mathbf{N}. \tag{8}$$

Proof. $n^2\|a_n(I - E_{k_n})\varphi\|^2 \underset{(6)}{\leqslant} \sum_{i=1}^{k_n} \beta_i\|a_i(I - E_{k_n})\varphi\|^2 \underset{(1)}{\leqslant}$

$$\underset{(1)}{\leqslant} \sum_{i=1}^{k_n} 4\alpha_i\|a_i\varphi\|^2 \leqslant \left(\sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|\right)^2,$$

i.e.

$$n\|a_n(I - E_{k_n})\varphi\| \leqslant \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|.$$

Since

$$\|\varphi\| \leqslant \|a_i\varphi\| \quad \forall\ \varphi \in \mathscr{D},\ i \in \mathbf{N},$$

the last inequality immediately implies (8). ▨

Now we consider an arbitrary element $x \in \mathscr{A}$. Then there is a constant C_x and an $r \in \mathbf{N}$ so that $\|x\varphi\| \leqslant C_x\|a_r\varphi\|$ and $\|x^+\varphi\| \leqslant C_x\|a_r\varphi\|$ $\forall\ \varphi \in \mathscr{D}$. Let $n \in \mathbf{N}$ such that $n \geqslant C_x$ and $n \geqslant r$.

LEMMA 6.

$$|\langle (I - E_{k_n})x(I - E_{k_n})\varphi, \varphi\rangle| \leqslant \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|^2, \tag{9}$$

$$|\langle (I - E_{k_n})xE_{k_n}\varphi, \varphi\rangle| \leqslant \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|^2, \tag{10}$$

$$|\langle E_{k_n}x(I - E_{k_n})\varphi, \varphi\rangle| \leqslant \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|^2 \quad \forall\ \varphi \in \mathscr{D}. \tag{11}$$

Proof. $|\langle (I - E_{k_n})x(I - E_{k_n})\varphi, \varphi\rangle| \equiv |\langle x(I - E_{k_n})\varphi, (I - E_{k_n})\varphi\rangle| \leqslant C_x\|a_r(I - E_{k_n})\varphi\|\ \|(I - E_{k_n})\varphi\| \leqslant n\|a_n(I - E_{k_n})\varphi\|\ \|\varphi\| \underset{(8)}{\leqslant} \sum_{i=1}^{k_n} 2\alpha_i\|a_i\varphi\|^2$. Similarly, (10) and (11) will be verified. ▨

Next we inductively choose a sequence $\gamma = \{\gamma_i, i \in \mathbf{N}\}$ of positive real numbers so that $\gamma_i^2 \leqslant \beta_i$ and

$$\gamma_i\|a_iE_k\| \leqslant \gamma_1 \quad \text{for } k = 1, \ldots, i;\ i \in \mathbf{N}. \tag{12}$$

Using this sequence we define

$$h_\gamma(k, \varphi) := \sup_{i \in \mathbf{N}} \gamma_i\|a_iE_k\varphi\|,\ k \in \mathbf{N},\ \varphi \in \mathscr{D}.$$

Let

$$\mathfrak{m} = \{\bigcup_{\substack{k \in \mathbf{N} \\ \varphi \in \mathscr{D}}} h_\gamma(k, \varphi)^{-1}E_k\varphi\}.$$

Since

$$h_\gamma(k, \varphi) \geqslant \gamma_i \|a_i E_k \varphi\| \quad \forall\ i, k \in \mathbf{N},\ \varphi \in \mathscr{D},$$

we have

$$\sup_{\psi \in \mathfrak{m}} \|a_i \psi\| \equiv \sup_{\substack{k \in \mathbf{N} \\ \varphi \in \mathscr{D}}} h_\gamma(k, \varphi)^{-1} \|a_i E_k \varphi\| \leqslant 1/\gamma_i.$$

Hence $\mathfrak{m}$ is a $t_{\mathscr{A}}$-bounded subset of the domain. Let

$$\mathscr{W}_{\mathfrak{m}} = \{a \in \mathscr{A} : p_{\mathfrak{m}}(a) \leqslant 1\}$$

be the corresponding 0-neighbourhood for $\tau_{\mathscr{D}}$. Assume that $3\alpha_i \leqslant 1 \quad \forall\, i \in \mathbf{N}$.

LEMMA 7. *If* $x \in \mathscr{W}_{\mathfrak{m}}$, *then*

$$|\langle x\varphi, \varphi\rangle| \leqslant \sum_{i=1}^{k_n} 8\alpha_i \|a_i \varphi\|^2 \quad \textit{for all } \varphi \in \mathscr{D}. \tag{13}$$

Proof. First we note that

$$h_\gamma(k, \varphi) = \sup_{i=1,\ldots,k} \gamma_i \|a_i E_k \varphi\|$$

because

$$\gamma_i \|a_i E_k \varphi\| \leqslant \gamma_i \|a_i E_k\| \|E_k \varphi\| \leqslant \gamma_1 \|a_i E_k \varphi\| \quad \text{for } k \leqslant i,\ \ i, k \in \mathbf{N}, \varphi \in \mathscr{D}.$$

From

$$1 \geqslant p_{\mathfrak{m}}(x) \geqslant |\langle x E_k \varphi, E_k \varphi\rangle| h_\gamma(k, \varphi)^{-2}$$

it follows that for $\varphi \in \mathscr{D}$ and $k \in \mathbf{N}$

$$|\langle E_k x E_k \varphi, \varphi\rangle| \leqslant h_\gamma(k, \varphi)^2 = \sup_{i=1,\ldots,k} \gamma_i^2 \| a_i E_k \varphi\|^2 \underset{(12)}{\leqslant}$$

$$\underset{(12)}{\leqslant} \sum_{i=1}^{k} \beta_i \|a_i E_k \varphi\|^2 \underset{(1)}{\leqslant} \sum_{i=1}^{k} (\beta_i + 4\alpha_i^2) \|a_i \varphi\|^2 \leqslant \sum_{i=1}^{k} 2\alpha_i \|a_i \varphi\|^2.$$

Here was applied $\beta_i \leqslant \alpha_i^2$ and $3\alpha_i \leqslant 1$. Since

$$x = (I - E_{k_n}) x (I - E_{k_n}) + (I - E_{k_n}) x E_{k_n} + E_{k_n} x (I - E_{k_n}) + E_{k_n} x E_{k_n},$$

the last inequality together with (9), (10), (11) imply (13). ▨

From Lemma 7 it is only a small step to complete the proof of theorem II. Let f be a strongly positive linear functional on the $Op*$-algebra $\mathscr{A}$. Since $t_{\mathscr{A}} = t_+$ by the closed graph theorem, the hermitian part $\mathscr{A}_h$ is cofinal in the ordered vector space $L^+(\mathscr{D})_h$. Hence, f can be extended to a strongly positive linear functional on $L^+(\mathscr{D})$ ([3], p. 82). Thus, we may assume that $\mathscr{A} = L^+(\mathscr{D})$.

Now we choose the positive sequence $\{\alpha_i, i \in \mathbf{N}\}$ such that $3\alpha_i \leqslant 1$ and $16\alpha_i f(a_i^+ a_i) \leqslant 2^{-i}$ $\forall$ $i \in \mathbf{N}$. Let $\mathscr{W}_m$ be the $\tau_{\mathscr{D}}$-neighbourhood constructed above. If $x \in \mathscr{W}_m$, then according to Lemma 7

$$|\langle x\varphi, \varphi\rangle| \leqslant \left\langle \sum_{i=1}^{k_n} 8\alpha_i a_i^+ a_i \varphi, \varphi \right\rangle \quad \forall \ \varphi \in \mathscr{D}.$$

By lemma 2, we get

$$| f(x)| \leqslant \sqrt{2} f\left(\sum_{i=1}^{k_n} 8\alpha_i a_i^+ a_i \right) \leqslant 1,$$

that is, f is $\tau_{\mathscr{D}}$-continuous on $L^+(\mathscr{D})$. Applying the proposition stated in Section 1 to $L^+(\mathscr{D})$, it follows that there is an $f \in \mathfrak{S}_1(\mathscr{D})$ so that $f(a) = \operatorname{Tr} ta$ $\forall$ $a \in L^+(\mathscr{D})$. Since $f(P_\varphi) = \operatorname{Tr} tP_\varphi \equiv \langle t\varphi, \varphi\rangle \geqslant 0$ for each one-dimensional projection P_φ, $\varphi \in \mathscr{D}$, t is a positive operator. This completes the proof of the theorem.

4. REMARKS

1) By using some arguments of the preceding section we can give a second proof of the following theorem which was shown in [4].

THEOREM. *Let $\mathscr{A}$ be a self-adjoint Op*-algebra on a domain $\mathscr{D}$. Suppose there is an operator $c \in \mathscr{A}$ such that the canonical embedding map of the domain $\mathscr{D}(c)$ (endowed with the norm $\|\varphi\|_c^2 = \|c\varphi\|^2 + \|\varphi\|^2$) in the Hilbert space $\mathscr{H}$ is compact. Then each strongly positive linear functional f on $\mathscr{A}$ is of the form $f(a) = \operatorname{Tr} ta$, $a \in \mathscr{A}$, where $t \in \mathfrak{S}_1(\mathscr{A})_+$.*

Proof. First we extend f to a strongly positive linear functional on the vector space $\mathscr{A} \dotplus \mathscr{F}(\mathscr{D})$ spanned by $\mathscr{A}$ and the vector space $\mathscr{F}(\mathscr{D})$ of all operators in $L^+(\mathscr{D})$ with finite-dimensional range. Now, by remark 3.2 in [4], it is sufficient to prove that for each $a \in \mathscr{A}$ there is a $b \in \mathscr{A}$ so that for every $\varepsilon > 0$ there exists an operator $x_\varepsilon \in \mathscr{F}(\mathscr{D})$ with

$$|\langle (a - x_\varepsilon)\varphi, \varphi\rangle| \leqslant 2\varepsilon\|b\varphi\|^2 \quad \forall \ \varphi \in \mathscr{D}.$$

Let $b = aa^+ + c^+c + I$. Putting $a_1 = b$, $a_k = 0$ $\forall$ $k \geqslant 2$, Lemma 3 yields

$$\|b(I - E_n)\varphi\|^2 \leqslant 4\|b\varphi\|^2 \quad \forall \ \varphi \in \mathscr{D}, n \in \mathbf{N}.$$

Further, for every $\varepsilon > 0$ there is a number $n_\varepsilon \in \mathbf{N}$ such that

$$\|(I - E_{n_\varepsilon})\, \varphi\| \leqslant \varepsilon\|b(I - E_n)\varphi\|.$$

Indeed, otherwise we would have

$$\|\psi_n\| > \varepsilon\|b\psi_n\|$$

for certain elements $\psi_n \in (I - E_n)\mathscr{D}$, $\|\psi_n\| = 1$. By the definition of the vectors φ_n (see the proof of Lemma 3) this implies

$$\|b\varphi_n\| \leqslant 2\|b\psi_n\| < 2/\varepsilon.$$

Since b has a compact inverse and the set $\{\varphi_n\}$ is orthonormal, this is a contradiction.

Finally, for $x_\varepsilon = aE_{n_\varepsilon}$ we obtain

$$|\langle (a - x_\varepsilon)\varphi, \varphi\rangle| \leqslant \|(I - E_{n_\varepsilon})\varphi\| \, \|a^+\varphi\| \leqslant$$

$$\leqslant \varepsilon\|b(I - E_{n_\varepsilon})\varphi\| \, \|b\varphi\| \leqslant 2\varepsilon\|b\varphi\|^2 \quad \forall \ \varphi \in \mathscr{D}$$

which completes the proof. ▨

2) The investigations in Section 3 and the proof of Corollary 2 in [5] suggest the following problem:

Suppose that $\mathscr{D}[t_+]$ is a Fréchet-Montel space. Does $\tau_\mathscr{D}$ coincide with the order topology on $L^+(\mathscr{D})$?

If in addition $\mathscr{D}[t_+]$ has an unconditional basis, this is true ([5], Cor. 4.2). In the preceding proof we only showed that all strongly positive linear functionals on an $Op*$-algebra $\mathscr{A}$ are $\tau_\mathscr{D}$-continuous provided $\mathscr{D}[t_\mathscr{A}]$ is a Fréchet-Montel space. We remark that for ordered vector spaces the order topology is always finer than any locally convex topology for which the positive cone is normal ([3], p. 118), in particular finer than $\tau_\mathscr{D}$. We note that the question has an affirmative answer if $L^+(\mathscr{D})$ contains an operator with compact inverse.

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