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ON THE WEAKLY CLOSED ALGEBRA GENERATED BY A WEAK CONTRACTION

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For a bounded linear operator T acting on a complex, separable Hilbert space, let $\{T\}'$, $\{T\}''$ and Alg T denote the commutant, double commutant and the weakly closed algebra generated by T and I, respectively. Let Lat T and Alg Lat T denote the lattice of invariant subspaces of T and the (weakly closed) algebra of operators which leave invariant all the subspaces in Lat T. It was shown in [5] and [9] that $C_0(N)$ contractions and completely non-unitary (c.n.u.) C_{11} contractions with finite defect indices satisfy $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T$. (Indeed, in the latter case we showed that they are even reflexive.) In this paper we generalize these to c.n.u. weak contractions with finite defect indices, that is, we show that $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T$ also holds for such more general contractions (Theorem 3). As an application a question raised in [8] is answered. If T_1 and T_2 are the contractions considered above, we obtain necessary and sufficient conditions for the splitting of Alg $(T_1 \oplus T_2)$ (Theorem 4), which generalize previous results for $C_0(N)$ and C_{11} contractions.

Let T be a c.n.u. weak contraction with finite defect indices defined on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$ by $T(f \oplus g) = P(e^{it} f \oplus e^{it} g)$ for $f \oplus g \in H$, where L_n^2 and H_n^2 denote the standard Lebesgue and Hardy spaces of $\mathbf{C}^n$-valued functions defined on the unit circle, Θ_T is the characteristic function of T, $\Delta = (I - \Theta_T^* \Theta_T)^{\frac{1}{2}}$ and P denotes the (orthogonal) projection onto H. (For the definition and properties of weak contractions, readers are referred to Chapter VIII of [3] or to our previous paper [8].) Recall that T is *multiplicity-free* if it admits a cyclic vector. For other equivalent conditions for T being multiplicity-free, compare Theorem 5 of [6]. We start by showing that $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T$ holds for multiplicity-free T.

LEMMA 1. *Assume that T is a c.n.u. multiplicity-free weak contraction with finite defect indices defined on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$. If $\Theta_T(t)$ is not isometric for almost all t, then*

$$\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{\varphi(T) : \varphi \in H^\infty\}.$$

Proof. Let H_0 and H_1 be the invariant subspaces for T such that $T_0 \equiv T|H_0$ and $T_1 \equiv T|H_1$ are the C_0 and C_{11} parts of T, respectively, and let m denote the minimal function of T_0. By virtue of [9], Theorem 3, we may assume that T_0 is not missing. Let $S \in \{T\}' \cap \operatorname{Alg\,Lat} T$ and let $S_0 = S|H_0$, $S_1 = S|H_1$. It is easily seen that $S_0 \in \{T_0\}' \cap \operatorname{Alg\,Lat} T_0 = \operatorname{Alg} T_0$ and $S_1 \in \{T_1\}' \cap \operatorname{Alg\,Lat} T_1 = \operatorname{Alg} T_1$. Hence there exist η, ψ and φ in H^∞ with $\eta \wedge m = 1$ such that $S_0 = \eta(T_0)^{-1}\psi(T_0)$ and $S_1 = \varphi(T_1)$, where $\eta \wedge m = 1$ denotes that η and m have no nontrivial common inner divisor (cf. [5] and [9], resp.). If $S' = \eta(T)S$, then $S' \in \{T\}'$ and hence $S' = P\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$, where A is a bounded analytic function while B and C are bounded measurable functions satisfying $A\Theta_T = \Theta_T A_0$ and $B\Theta_T + C\Delta = \Delta A_0$ for some bounded analytic function A_0 (cf. [4]).

Since H_0 corresponds to the $*$-canonical factorization $\Theta_T = \Theta_{*e}\Theta_{*i}$ of Θ_T, in the functional model

$$H_0 = \{\Theta_{*e}u \oplus \Delta\Theta_{*i}^{*}u : u \in H_n^2\} \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$$

(cf. [3], Theorem VII.1.1). Hence

$$S'\begin{bmatrix} \Theta_{*e}u \\ \Delta\Theta_{*i}^{*}u \end{bmatrix} = P\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\begin{bmatrix} \Theta_{*e}u \\ \Delta\Theta_{*i}^{*}u \end{bmatrix} = P\begin{bmatrix} A\Theta_{*e}u \\ B\Theta_{*e}u + C\Delta\Theta_{*i}^{*}u \end{bmatrix} =$$

$$= P\begin{bmatrix} A\Theta_{*e}u \\ \Delta A_0\Theta_{*i}^{*}u \end{bmatrix}.$$

On the other hand,

$$S'\begin{bmatrix} \Theta_{*e}u \\ \Delta\Theta_{*i}^{*}u \end{bmatrix} = \psi(T_0)\begin{bmatrix} \Theta_{*e}u \\ \Delta\Theta_{*i}^{*}u \end{bmatrix} = P\begin{bmatrix} \psi\Theta_{*e}u \\ \psi\Delta\Theta_{*i}^{*}u \end{bmatrix}.$$

Thus for any $u \in H_n^2$, there exists some $w \in H_n^2$ such that $\psi\Theta_{*e}u - A\Theta_{*e}u = \Theta_T w$ and $\psi\Delta\Theta_{*i}^{*}u - \Delta A_0\Theta_{*i}^{*}u = \Delta w$. If $u = 0$, then $\Theta_T w = 0$ and $\Delta w = 0$, whence $w = 0$. Therefore the linear mapping $u \to w$ on H_n^2 is well-defined. In a similar fashion it can be shown that it is continuous, hence bounded. That it commutes with the unilateral shift on H_n^2 is obvious. Thus there exists a bounded analytic function Φ such that $\Phi u = w$ for all $u \in H_n^2$ (cf. [3], Lemma V.3.2). Therefore $\psi\Theta_{*e} - A\Theta_{*e} = \Theta_T\Phi$. Let $\delta \not\equiv 0$ be an outer scalar multiple of Θ_{*e} and let Ω be a contractive analytic function such that $\Omega\Theta_{*e} = \Theta_{*e}\Omega = \delta I$. Multiplying both sides of the above equation by Ω from right, we obtain

$$\psi\delta - A\delta = \Theta_T\Phi\Omega. \tag{1}$$

On the other hand, corresponding to the canonical factorization $\Theta_T = \Theta_i \Theta_e$ of Θ_T,

$$H_1 = \{\Theta_i u \oplus v : u \in H_n^2, v \in \overline{\Delta L_n^2}\} \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$$

(cf. [3], Theorem VII.1.1). Note that $\{P(0 \oplus v) : v \in \overline{\Delta L_n^2}\}$ is dense in H_1 (cf. proof of Lemma 2 in [9]). Hence $S'(P(0 \oplus v)) = P(0 \oplus Cv)$ and $S'(P(0 \oplus v)) = \eta(T_1)\varphi(T_1)P(0 \oplus v) = P(0 \oplus \eta\varphi v)$ for $v \in \overline{\Delta L_n^2}$. It follows that $P(0 \oplus (C - \eta\varphi)v) = 0$, which implies that there exists some $w \in H_n^2$ such that $\Theta_T w = 0$ and $(C - \eta\varphi)v = \Delta w$. Since Θ_T admits a scalar multiple, $\Theta_T(t)$ is invertible a.e. on $\mathbf{C}^n$, and we deduce that $w = 0$. Hence $(C - \eta\varphi)v = 0$ for all $v \in \overline{\Delta L_n^2}$. Therefore we obtain

$$C = \eta\varphi. \tag{2}$$

Define the linear operator $X : H \to (H_n^2 \ominus \Theta_{*i} H_n^2) \oplus \overline{\Delta_* L_n^2}$ by

$$X(f \oplus g) = P_1(\Omega f) \oplus (-\Delta_* f + \Theta_T g) \quad \text{for } f \oplus g \in H,$$

where $\Delta_* = (I - \Theta_T \Theta_T^*)^{\frac{1}{2}}$ and P_1 denotes the (orthogonal) projection from H_n^2 onto $H_n^2 \ominus \Theta_{*i} H_n^2$. Let U be the operator on $H_n^2 \ominus \Theta_{*i} H_n^2$ defined by

$$Uh = P_1(e^{it}h) \quad \text{for} \quad h \in H_n^2 \ominus \Theta_{*i} H_n^2$$

and let V be the operator of multiplication by e^{it} on $\overline{\Delta_* L_n^2}$. It can be easily verified that X intertwines T and $U \oplus V$ (cf. [8], Lemma 3.4). Note that U is unitarily equivalent to T_0 (cf. [3], Prop. VII.2.1) and so, by the assumption that T is multiplicity-free, U is quasi-similar to $S(m)$, the operator defined on $H^2 \ominus mH^2$ by $S(m)h = P'(e^{it}h)$ for $h \in H^2 \ominus mH^2$, where P' denotes the (orthogonal) projection from H^2 onto $H^2 \ominus mH^2$ (cf. [6], Theorem 5). Let $Y_1 : H_n^2 \ominus \Theta_{*i} H_n^2 \to H^2 \ominus mH^2$ be a quasi-affinity intertwining U and $S(m)$. On the other hand, note that V is unitarily equivalent to the bilateral shift W on L^2. Indeed, since T is multiplicity-free and $\Theta_T(t)$ is not isometric for almost all t, rank $\Delta_*(t) =$ rank $\Delta(t) = 1$ a.e. (cf. [3], p. 273 and [6], Theorem 5). Let $\{\psi_k(t)\}_1^n$ be an orthonormal base of $\mathbf{C}^n$ composed of eigenvectors of $\Delta_*(t)$, that is, such that

$$\Delta_*(t)\psi_k(t) = \begin{cases} \delta_k(t)\psi_k(t) & \text{if } k = 1, \\ 0 & \text{if } k = 2, \ldots, n, \end{cases}$$

where $0 < \delta_1(t) \leqslant 1$ a.e. Then the linear transformation $\Delta_* v \to x_1\delta_1$, where $v \in L_n^2$ and $x_1(t) = (v(t), \psi_1(t))_{\mathbf{C}^n}$, extends to a unitary operator $Y_2 : \overline{\Delta_* L_n^2} \to L^2$ intertwining V and W (cf. [3], pp. 272—273). Let $Y = Y_1 \oplus Y_2$.

Consider $K = \{P'h \oplus h : h \in H^2\} \subseteq (H^2 \ominus mH^2) \oplus L^2$. Obviously, K is a (closed) invariant subspace for $S(m) \oplus W$. Let $K_1 = X^{-1}Y^{-1}K \in \operatorname{Lat} T$ and let

$S'' = \delta(T)\eta(T)S = \delta(T)S' = P\begin{bmatrix}\delta A & 0\\ \delta B & \delta C\end{bmatrix} \in \operatorname{Alg}\operatorname{Lat} T$. Then $S''K_1 \subseteq K_1$ and hence $YXS''(f \oplus g) \in K$ for any $f \oplus g \in K_1$. We have

$$(3) \qquad YXS''(f \oplus g) = YXP(\delta Af \oplus (\delta Bf + \delta Cg)) =$$

$$= Y[P_1(\Omega\delta Af) \oplus (-\Delta_*\delta Af + \Theta_T(\delta Bf + \delta Cg))] =$$

$$= Y[P_1(\Omega\psi\delta f - \Omega\Theta_T\Phi\Omega f) \oplus \delta\Theta_T C\Theta_T^{-1}(-\Delta_* f + \Theta_T g)] =$$

$$= Y[P_1(\Omega\psi\delta f - \delta\Theta_{*i}\Phi\Omega f) \oplus \delta\eta\varphi(-\Delta_* f + \Theta_T g)] =$$

$$= Y[P_1(\Omega\psi\delta f) \oplus \delta\eta\varphi(-\Delta_* f + \Theta_T g)] =$$

$$= Y[(\psi\delta)(U)P_1(\Omega f) \oplus (\delta\eta\varphi)(V)(-\Delta_* f + \Theta_T g)] =$$

$$= (\psi\delta)(S(m))Y_1(P_1(\Omega f)) \oplus (\delta\eta\varphi)(W)Y_2(-\Delta_* f + \Theta_T g),$$

where we make use of (1), (2) and the fact that $-\Delta_* Af + \Theta_T(Bf + Cg) = \Theta_T C\Theta_T^{-1}(-\Delta_* f + \Theta_T g)$ (cf. proof of Theorem 3.5 in [8]).

Next we want to show that there exists some $f \oplus g \in K_1$ such that $YX(f \oplus g) = P'h \oplus h \in K$ with $h \wedge m = 1$. Let $\xi \in H_n^2 \ominus \Theta_{*i}H_n^2$ be a cyclic vector for U and let $Y_1\xi = P'\rho \in H^2 \ominus mH^2$. Assume that

$$\rho(\lambda) = \rho_i(\lambda)\rho_e(\lambda) = \rho_i(\lambda)\exp\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+\lambda}{e^{it}-\lambda}k(t)\,dt\right] \qquad |\lambda| < 1$$

is the canonical factorization of ρ, where $k(t) = \log|\rho_e(t)|$ a.e. Fix $M > 0$ and let $\alpha = \{t : |\rho_e(t)| \geqslant M\}$. Let

$$\tau(\lambda) = \exp\left[\frac{1}{2\pi}\int_\alpha \frac{e^{it}+\lambda}{e^{it}-\lambda}(-k(t))\,dt\right] \quad \text{for } |\lambda| < 1.$$

Then it is easily seen that τ and $\tau\rho$ are both in H^∞, and $Y_1(P_1(\tau\xi)) = P'(\tau\rho)$. That ξ and $P'\rho$ are cyclic for U and $S(m)$, respectively, implies that the same hold for $P_1(\tau\xi)$ and $P'(\tau\rho)$ (since τ is outer). Without loss of generality, we may assume that ξ and $P'\rho$, where $\rho \in H^\infty$, are cyclic for U and $S(m)$, respectively. In particular, $\rho \wedge m = 1$. Consider the element $f \oplus g = P(\Theta_{*e}\xi \oplus (\Delta\Theta_{*i}^*\xi + \frac{1}{\delta_1}\Delta\Theta_{*i}^*\Omega\rho\psi_1))$ in H. Indeed, since for the self-adjoint operator $\Delta_*(t)$, $\|\Delta_*(t)\| = \delta_1(t)$ a.e., we have $\frac{1}{\delta_1}\Delta\Theta_{*i}^*\Omega\rho\psi_1 = \frac{1}{\delta_1}\Theta_{*i}^*\Omega\Delta_*\rho\psi_1 \in L_n^2$. If we let $E_j = \left\{t : \delta_1(t) \geqslant \frac{1}{j}\right\}$ for each

$j \geq 1$, then $\chi_{E_j} \frac{1}{\delta_1} \Delta\Theta^*_{*i}\Omega\rho\psi_1 \in \Delta L^2_n$ and $\chi_{E_j} \frac{1}{\delta_1} \Delta\Theta^*_{*i}\Omega\rho\psi_1 \to \frac{1}{\delta_1} \Delta\Theta^*_{*i}\Omega\rho\psi_1$ in norm as $j \to \infty$, which shows that $\frac{1}{\delta_1} \Delta\Theta^*_{*i}\Omega\rho\psi_1 \in \overline{\Delta L^2_n}$ and hence $f \oplus g \in H$, as asserted. We deduce that

$$X(f \oplus g) = P_1(\Omega\Theta_{*e}\xi) \oplus \left(-\Delta_*\Theta_{*e}\xi + \Theta_T\Delta\Theta^*_{*i}\xi + \Theta_T \frac{1}{\delta_1} \Delta\Theta^*_{*i}\Omega\rho\psi_1\right) =$$

$$= P_1(\delta\xi) \oplus \left(-\Delta_*\Theta_{*e}\xi + \Delta_*\Theta_T\Theta^*_{*i}\xi + \frac{1}{\delta_1} \Delta_*\Theta_T\Theta^*_{*i}\,\Omega\rho\psi_1\right) =$$

$$= P_1(\delta\xi) \oplus \left(\frac{1}{\delta_1} \Delta_*\delta\rho\psi_1\right) = (\delta(U)\xi) \oplus \delta(V)\left(\frac{1}{\delta_1} \Delta_*\rho\psi_1\right) =$$

$$= \delta(U \oplus V)\left(\xi \oplus \frac{1}{\delta_1} \Delta_*\rho\psi_1\right)$$

and hence

$$YX(f \oplus g) = \delta(S(m) \oplus W)\left(Y_1\xi \oplus Y_2\left(\frac{1}{\delta_1} \Delta_*\rho\psi_1\right)\right) =$$

$$= \delta(S(m) \oplus W)(P'\rho \oplus \rho) = P'(\delta\rho) \oplus (\delta\rho) \in K,$$

where $(\delta\rho) \wedge m = 1$. Thus $f \oplus g \in K_1$ is what we wanted.

Let $h = \delta\rho$. From (3), we infer that

$$YXS''(f \oplus g) = (\psi\delta)(S(m))(P'h) \oplus (\delta\eta\varphi)(W)h = \tag{4}$$

$$= P'(\psi\delta h) \oplus \delta\eta\varphi h.$$

On the other hand, since $YXS''(f \oplus g) \in K$,

$$YXS''(f \oplus g) = P'h' \oplus h' \quad \text{for some } h' \in H^2. \tag{5}$$

It follows from (4) and (5) that $m|\psi\delta h - h'$ and $\delta\eta\varphi h = h'$, and so $m|(\psi - \eta\varphi)\delta h$. Since δ is outer and $h \wedge m = 1$, $m|\psi - \eta\varphi$. Thus $S_0 = \eta(T_0)^{-1}\psi(T_0) = \eta(T_0)^{-1}(\eta\varphi)(T_0) = \varphi(T_0)$. From $S_1 = \varphi(T_1)$ and the fact that $H_0 \vee H_1 = H$, we conclude that $S = \varphi(T)$. It follows immediately that

$$\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{\varphi(T) : \varphi \in H^\infty\}.$$

LEMMA 2. *Let T be a c.n.u. weak contraction on H and let H_0, H_1 be the (hyper)invariant subspaces for T such that $T_0 = T|H_0$ and $T_1 = T|H_1$ are the C_0*

and C_{11} parts of T, respectively . For $S \in \{T\}'$, let $S_0 = S|H_0$ and $S_1 = S|H_1$. Then $S \in \{T\}''$ if and only if $S_0 \in \{T_0\}''$ and $S_1 \in \{T_1\}''$.

Proof. Assume that $S \in \{T\}''$. By Theorem 3.1 of [7], there exists an operator $W \in \{T\}''$ such that $H_0 = \overline{WH}$. For any $V \in \{T_0\}'$, consider VW as an operator on H. Then $VWT = VTW = TVW$, which shows that $VW \in \{T\}'$. Since $S \in \{T\}''$, we have $SVW = VWS = VSW$, that is, $SV = VS$ on $\overline{WH} = H_0$ or $S_0V = VS_0$. Hence $S_0 \in \{T_0\}''$. Since $H_1 = \overline{m(T)H}$ (cf. [3], Prop. VIII.2.4), where m denotes the minimal function of T_0, in a similar fashion we may show that $S_1 \in \{T_1\}''$.

Conversely, assume that $S_0 \in \{T_0\}''$ and $S_1 \in \{T_1\}''$. Let $V \in \{T\}'$ and $V_0 = V|H_0$, $V_1 = V|H_1$. Then $V_0 \in \{T_0\}'$ and $V_1 \in \{T_1\}'$. By assumption, $V_0S_0 = S_0V_0$ and $V_1S_1 = S_1V_1$, that is, $VS = SV$ on H_0 and H_1. Thus $VS = SV$ on $H_0 \vee H_1 = H$, whence $S \in \{T\}''$.

After these painstaking computations, we are now ready to prove our main result.

THEOREM 3. *Assume that T is a c.n.u. weak contraction with finite defect indices defined on $H \equiv [H_n^2 \oplus \overline{\Delta L_n^2}] \ominus \{\Theta_T w \oplus \Delta w : w \in H_n^2\}$ by*

$$T(f \oplus g) = P(e^{it}f \oplus e^{it}g) \quad \textit{for } f \oplus g \in H.$$

(1) *If $\Theta_T(t)$ is isometric for t in a set of positive Lebesgue measure, then $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{T\}''$.*

(2) *If $\Theta_T(t)$ is not isometric for almost all t, then $\{T\}' \cap \text{Alg Lat } T = \text{Alg } T = \{\varphi(T) : \varphi \in H^\infty\}$.*

Proof. Let $T_0 = T|H_0$, $T_1 = T|H_1$ and m be as in Lemma 1. For $S \in \{T\}' \cap \text{Alg Lat } T$, let $S_0 = S|H_0$ and $S_1 = S|H_1$. As before, $S_0 \in \{T_0\}' \cap \text{Alg Lat } T_0 = \text{Alg } T_0$ and $S_1 \in \{T_1\}' \cap \text{Alg Lat } T_1 = \text{Alg } T_1$.

(1) In this case, $\text{Alg } T = \{T\}''$ (cf. [8], Theorem 4.4). Since $S_0 \in \{T_0\}''$ and $S_1 \in \{T_1\}''$, Lemma 2 implies that $S \in \{T\}'' = \text{Alg } T$.

(2) Since T_0 is of class $C_0(N)$, there exists a (bi-) invariant subspace $K_0 \subseteq H_0$ for T_0 such that $T_0|K_0$ is quasi-similar to $S(m)$ on $H^2 \ominus mH^2$ (cf. [2]). On the other hand, it follows from [6] Theorem 2, [8] Corollary 3.6 and our assumption that for the C_{11} contraction T_1 there exists a bi-invariant subspace $K_1 \subseteq H_1$ for T_1 such that $T_1|K_1$ is quasi-similar to the bilateral shift M on L^2. Then $K \equiv K_0 \vee K_1$ is bi-invariant for T, whence $T' \equiv T|K$ is a weak contraction (cf. [8], Theorem 4.1). Since $S(m) \oplus M$ is a quasi-affine transform of T', they are quasi-similar to each other (cf. [6], p. 194). We conclude that T' is multiplicity-free and $\Theta_{T'}(t)$ is not isometric for almost all t (cf. [6], Theorem 5).

Assume that $S_0 = \eta(T_0)^{-1}\psi(T_0)$ for $\eta, \psi \in H^\infty$ with $\eta \wedge m = 1$, and $S_1 = \varphi(T_1)$ for $\varphi \in H^\infty$. Since $S' \equiv S|K \in \{T'\}' \cap \operatorname{Alg Lat} T'$, $S' = \rho(T')$ for some $\rho \in H^\infty$ by Lemma 1. Hence $\rho(T')|K_0 = \eta(T_0)^{-1}\psi(T_0)|K_0$ and $\rho(T')|K_1 = \varphi(T_1)|K_1$, which imply that $m|\eta\rho - \psi$ and $\rho = \varphi$ (since $T_1|K_1$ is not of class C_0). Thus $m|\eta\varphi - \psi$. We infer that $\eta(T_0)\varphi(T_0) = \psi(T_0)$ or $\varphi(T_0) = \eta(T_0)^{-1}\psi(T_0) = S_0$. This implies that $S = \varphi(T)$ on H_0 and H_1, whence on $H_0 \vee H_1 = H$. This completes the proof.

Now we can give necessary and sufficient conditions for the splitting of $\operatorname{Alg}(T_1 \oplus T_2)$ for weak contractions T_1 and T_2. These generalize previous results for $C_0(N)$ and C_{11} contractions (cf. [1], Theorem 3.1 and [8], Theorem 3.10, resp.).

THEOREM 4. *For $j = 1, 2$, let T_j be a c.n.u. weak contraction with finite defect indices. Let m_j be the minimal function of the C_0 part of T_j and let $E_j = \{t : \Theta_{T_j}(t)$ not isometric$\}$ for each j. Then the following are equivalent:*

(1) $\operatorname{Alg}(T_1 \oplus T_2) = \operatorname{Alg} T_1 \oplus \operatorname{Alg} T_2$;

(2) $\operatorname{Lat}(T_1 \oplus T_2) = \operatorname{Lat} T_1 \oplus \operatorname{Lat} T_2$;

(3) *$m_1 \wedge m_2 = 1$, $E_1 \cap E_2$ has Lebesgue measure zero and $\mathbf{T}\setminus(E_1 \cup E_2)$ has positive Lebesgue measure, where $\mathbf{T}$ denotes the unit circle in the complex plane.*

Proof. Note that $E_1 \cup E_2 = \{t : \Theta_{T_1 \oplus T_2}(t)$ not isometric$\}$.

(1) $\Rightarrow$ (2). This is proved in Proposition 1.3 of [1].

(2) $\Rightarrow$ (1). We have only to show that $\operatorname{Alg} T_1 \oplus \operatorname{Alg} T_2 \subseteq \operatorname{Alg}(T_1 \oplus T_2)$. Let $S_1 \in \operatorname{Alg} T_1$ and $S_2 \in \operatorname{Alg} T_2$. Then $S_1 \oplus S_2 \in \{T_1 \oplus T_2\}'$ and $S_1 \oplus S_2 \in$ $\in \operatorname{Alg Lat} T_1 \oplus \operatorname{Alg Lat} T_2 = \operatorname{Alg Lat}(T_1 \oplus T_2)$ (cf. [1], Prop. 1.3). We conclude from Theorem 3 that $S_1 \oplus S_2 \in \operatorname{Alg}(T_1 \oplus T_2)$.

(3) $\Rightarrow$ (1). (3) implies that $\mathbf{T}\setminus E_1$, $\mathbf{T}\setminus E_2$ and $\mathbf{T}\setminus(E_1 \cup E_2)$ all have positive Lebesgue measure. Hence $\operatorname{Alg} T_1 = \{T_1\}''$, $\operatorname{Alg} T_2 = \{T_2\}''$ and $\operatorname{Alg}(T_1 \oplus T_2) =$ $= \{T_1 \oplus T_2\}''$ (cf. [8], Theorem 4.4). Since $m_1 \wedge m_2 = 1$ and $E_1 \cap E_2$ has Lebesgue measure zero, $\{T_1 \oplus T_2\}'' = \{T_1\}'' \oplus \{T_2\}''$ by Theorem 4.7 of [8] and Proposition 1.3 of [1]. (1) follows immediately.

(1) $\Rightarrow$ (3). (1) implies that $\{T_1 \oplus T_2\}' = \{T_1\}' \oplus \{T_2\}'$ (cf. [1], Prop. 1.3). Thus $m_1 \wedge m_2 = 1$ and $E_1 \cap E_2$ has Lebesgue measure zero by Theorem 4.7 of [8]. Assume that $\mathbf{T}\setminus(E_1 \cup E_2)$ has Lebesgue measure zero. Then, by Theorem 3, $\operatorname{Alg}(T_1 \oplus T_2) = \{\varphi(T_1 \oplus T_2) : \varphi \in H^\infty\}$. We consider the following three cases separately:

(*i*) Assume that both E_1 and $\mathbf{T}\setminus E_2$ have Lebesgue measure zero. In this case T_1 is of class $C_0(N)$. Note that for any $\psi, \eta \in H^\infty$, $\psi(T_1) \in \operatorname{Alg} T_1$ and $\eta(T_2) \in \operatorname{Alg} T_2$. Hence (1) implies that $\psi(T_1) \oplus \eta(T_2) = \varphi(T_1 \oplus T_2)$ for some $\varphi \in H^\infty$. It follows that $m_1|\psi - \varphi$ and $\eta = \varphi$ (since T_2 is not of class C_0), or $m_1|\psi - \eta$, which is certainly absurd.

(*ii*) Assume that both E_2 and $\mathbf{T}\setminus E_1$ have Lebesgue measure zero. A similar argument as in (*i*) applies.

(*iii*) Assume that both E_1 and E_2 have positive Lebesgue measure. Then neither T_1 nor T_2 is of class C_0. As in (*i*), for any $\psi, \eta \in H^\infty$, there exists some $\varphi \in H^\infty$ such that $\psi(T_1) \oplus \eta(T_2') = \varphi(T_1 \oplus T_2)$. So $\psi = \varphi$ and $\eta = \varphi$, or $\psi = \eta$, which is again a contradiction.

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