J. OPERATOR THEORY
2(1979), 169-209

OPERATOR SYSTEMS AND THEIR APPLICATION TO THE TOMITA-TAKESAKI THEORY

S. L. WORONOWICZ

0. INTRODUCTION

Let M be a von Neumann algebra and M_* be the predual of M.

Already in the elementary theory of von Neumann algebras one meets many natural operations involving elements of M and M_*. The best known example is the multiplication of normal functionals by elements of the algebra: $M \times M_* \ni \ni (A, \alpha) \to A\alpha \in M_*$ (for details see [3]). Another example is provided by the polar decomposition of normal functionals. Denoting by phase α the unitary appearing in the polar decomposition of $\alpha \in M_*$ we introduce a natural mapping phase: $M_* \to M$.

We shall prove in this paper that the basic operations investigated in the Tomita-Takesaki theory such as the action of the modular groups and the Radon-Nikodym derivatives can be expressed by the elementary mappings mentioned above. To this end we consider the similar problem in the purely algebraic setting where the faithful states are replaced by selfadjoint positive nondegenerate operators.

Let $\{\Delta_\alpha\}_{\alpha \in \Lambda}$ be a family of selfadjoint, positive nondegenerate operators acting on a Hilbert space H. We consider the following two families of mappings:

$$B(H) \ni A \to \Delta_\alpha^{it} A \Delta_\beta^{-it} \in B(H), \tag{0.1}$$

where $\alpha, \beta \in \Lambda$; $t \in \mathbf{R}$ and

$$B(H) \ni A \to \mathrm{Phase}\{A(a\Delta_\alpha \dotplus b\Delta_\beta)\} \in B(H), \tag{0.2}$$

where a, b are complex numbers with positive real parts; Phase X denotes the unitary factor appearing in the polar decomposition of X and the addition of operators (denoted by $\dotplus$) is understood in the sense explained in ([4], Ch VI, 2.5). The mappings (0, 1) are typical for the Tomita-Takesaki theory, where as (0, 2) are closely related to the elementary operations involving elements of M and M_* described in the begining.

The main result of this paper asserts that the families (0,1) and (0,2) are connected: each mapping belonging to the first family can be expressed by mappings of the second family and vice versa.

To make this statement precise we consider systems of the form $S=(M, \{\Delta_\alpha\}_{\alpha\in A})$ where M is a von Neumann algebra and $\{\Delta_\alpha\}_{\alpha\in A}$ is a family of selfadjoint, positive nondegenerate operators. Various properties of systems are the subject of our interest. For example S is called a modular system if M is invariant under the mappings (0, 1). Two modular systems $S=(M, \{\Delta_\alpha\}_{\alpha\in A})$ and $'S=(M, \{'\Delta_\alpha\}_{\alpha\in A})$ are called equivalent if the mappings (0, 1) restricted to M remain unchanged when one replaces Δ by $'\Delta$. Similarly the family of mappings (0,2) gives rise to the notion of a phase system.

It turns out that a system S is a modular system if and only if it is a phase system and then S is called an operator system. Moreover two operator systems S and S' are equivalent as modular system if and only if they are equivalent as phase systems (and then S and S' are called equivalent operator systems).

These statements, although of a very technical nature seem to be very profound (even more profound than the Tomita-Takesaki theory itself).

Applied to the Tomita-Takesaki theory they make the main results of this theory almost evident. By the way we gain the better (deeper) understanding of the foundations of the theory. For example we start to distinguish the properties of the modular operator relevant for the theory from those which are accidental and related to the particular definition of the modular operator.

Let us shortly describe the contents of the paper. In the first section we collect some definitions and results, which are indispensable for our theory, partly for the reader's convenience, partly in order to fix the notation. Some statements are proved, the others can be found in the book of Kato [4] (unless an explicite reference is given).

In Section 2 we introduce the fundamental concepts of the theory of operator systems and formulate the main results. Considering so called canonical phase systems associated with von Neumann algebras we are able to include the large part of Tomita-Takesaki theory into the operator system framework.

Section 3 is devoted to the theory of operator functions which constitutes the main tool of our investigations. The special attention is paid to homogeneous operator functions. We prove that any regular modular system is a phase system (and that equivalent modular systems are equivalent as phase systems).

In the next section we prove the converse, i.e. that any phase system is a modular system (and that equivalent phase systems are equivalent as modular systems). We use the differential calculus ideas (Gateau derivative) in order to relate nonlinear mappings of the form (0,2) to the linear maps (0,1).

In section 5 we show that for any von Neumann algebra a canonical phase system exists. For semifinite algebras this fact is almost obvious; for purely infinite algebras we give the explicite construction of a canonical phase system. Once the existence of a canonical phase system is proved, the fundamental concepts of the Tomita-Takesaki theory such as modular automorphism groups and Radon-Nikodym derivatives can be easily introduced.

In the last section we introduce the notion of a crossed product algebra associated with any modular system. For modular systems (M, Δ) with one operator Δ this algebra coincides with $W^*(M,\{\tau_t\}_{t\in R})$ introduced by Takesaki [10]. We prove that M coincides with the set of elements invariant under the action of the scaling group and that equivalent modular systems give rise to the isomorphic crossed product algebras.

Some technical details are collected in Appendices. Appendix I contains some informations on the interpolation theory; in Appendix II we give examples of operator functions.

1. PRELIMINARIES

In this section we collect some definitions and statements which will be used in this paper. By the way we fix the notation.

Let H be a Hilbert space. The set of all closed operators acting on H will be denoted by $\mathscr{C}(H)$. For any $A \in \mathscr{C}(H)$, $D(A)$ denotes the domain of A. An operator $A \in \mathscr{C}(H)$ is said to be *nondegenerate* if the kernel of A is trivial (i.e.: $x \in D(A)$ and $Ax = 0$ imply $x = 0$) and the image of A is dense in H. The algebra of all bounded operators acting on H will be denoted by $B(H)$. An operator $B \in B(H)$ is said to be *invertible* if $BC = CB = I$ (I denotes the unit of $B(H)$) for some $C \in B(H)$. Let $A \in \mathscr{C}(H)$ be nondegenerate and $B \in B(H)$ be invertible. Then AB and BA are closed and nondegenerate.

Any nondegenerate $A \in \mathscr{C}(H)$ admits the following polar decomposition

$$A = UK$$

where U is unitary and K is selfadjoint positive. The pair (U,K) is determined uniquely by A: $K = (A^*A)^{\frac{1}{2}}$; $U =$ the closure of AK^{-1}. In what follows, the unitary factor in the polar decomposition of a nondegenerate operator $A \in \mathscr{C}(H)$ will be denoted by Phase$A$:

$$\text{Phase}\, A = \text{the closure of } A(A^*A)^{-\frac{1}{2}}.$$

Let $\mathbf{C}_+$ denote the subset of the complex plane consisting of 0 and all numbers with positive real part:

$$\mathbf{C}_+ = \{z \in \mathbf{C} : z = 0 \text{ or } \operatorname{Re} z > 0\}.$$

Convex closed subsets of $\mathbf{C}$ invariant under the multiplication by positive numbers and contained in $\mathbf{C}_+$ will be called *sectors*. An operator $A \in \mathscr{C}(H)$ is said to be *sectorial* if there exists a sector S such that

$$(x|Ax) \in S$$

for all $x \in D(A)$. The smallest sector S satisfying the above relation will be denoted by $\mathrm{Sector}A$. A is called *m-sectorial* (maximal sectorial) if there is no proper sectorial extension of A i.e. if any sectorial operator $B \supset A$ coincides with A. It turns out that for any sectorial operator A the following three conditions are equivalent:

$$(1.1) \qquad \begin{cases} 1^\circ \ A \text{ is } m\text{-sectorial}, \\ 2^\circ \ (\mathrm{Spectrum}A) \cup (\mathrm{Sector}A) \neq \mathbf{C}, \\ 3^\circ \ (\mathrm{Spectrum}A) \subset (\mathrm{Sector}A). \end{cases}$$

Let φ be a sesquilinear form defined on a dense subset $D(\varphi)$ of H. φ is called *sectorial* if there exists a sector S such that

$$\varphi(x, x) \in S$$

for all $x \in D(\varphi)$. If φ is sectorial then we consider a new norm $\|x\|_\varphi = (\|x\|^2 + \mathrm{Re}\,\varphi(x, x))^{\frac{1}{2}}$ defined on $D(\varphi)$. φ is called *closed* if $D(\varphi)$ is complete with respect to $\|\cdot\|_\varphi$. φ is called *closable* if it can be extended to a closed sectorial form. The smallest closed extension of φ is called the *closure* of φ. We say that a linear subset $D \subset D(\varphi)$ is a *core* of a closable sectorial form φ if φ and the restriction of φ to D have the same closure.

There exists an $1 - 1$ correspondence between closed sectorial forms and m-sectorial operators. This correspondence is established by the following formula:

$$(1.2) \qquad \varphi(x, y) = (x|Ay)$$

More precisely, for a given m-sectorial operator A we consider the sesquilinear form defined on $D(A)$ introduced by the right hand side of (1.2). It turns out that this form is sectorial, closable and its closure φ is the closed sectorial form associated with A. Conversely, for a given closed sectorial form φ we consider the set $D(A)$ of all vectors $y \in D(\varphi)$ such that the left hand side of (1.2) is continuous functional of x. Then for any vector $y \in D(A)$ there exists unique $Ay \in H$ such that (1.2) holds for all $x \in D(\varphi)$. It turns out that the operator A introduced in this way is closed, densely defined and m-sectorial.

If Δ is a positive selfadjoint operator, then the domain of the sectorial form φ associated with Δ coincides with $D(\Delta^{\frac{1}{2}})$. Moreover a linear subset $D \subset D(\Delta^{\frac{1}{2}})$ is a core of φ if and only if D is a core of $\Delta^{\frac{1}{2}}$.

Let A_1 and A_2 be two m-sectorial operators acting on H. We denote by φ_1 and φ_2 the corresponding closed sectorial forms. If $D(\varphi_1) \cap D(\varphi_2)$ is dense in H then we may consider $\varphi_1 + \varphi_2$ as a sectorial form defined on $D(\varphi_1 + \varphi_2) = D(\varphi_1) \cap \cap D(\varphi_2)$. It turns out that $\varphi_1 + \varphi_2$ is automaticaly closed and therefore it is associated with some m-sectorial operator A. We write

$$A = A_1 \dotplus A_2. \tag{1.3}$$

The addition of m-sectorial operators introduced in this way has many pathological properties. For example if $D(\varphi_1) \cap D(\varphi_2)$ is not a core of φ_1, then for $\varepsilon \to 0$, $A_1 \dotplus \varepsilon A_2$ does not converge to A_1 in any reasonable sense (of. [5]).

This unexpected and unpleasant feature of the operation (1.3) can be removed if one assumes that $D(\varphi_1) \cap D(\varphi_2)$ is a core for φ_1 and φ_2 and only in this case we shall use the notation (1.3).

In particular we have

$$\underset{\varepsilon \to 0}{s\text{-lim}}\ (\Delta_0 \dotplus \varepsilon\Delta_1)^{it} = \Delta_0^{it} \tag{1.4}$$

for any $t \in \mathbf{R}$ and any two selfadjoint positive nondegenerate operators Δ_0 and Δ_1 such that $\Delta_0^{\frac{1}{2}}$ and $\Delta_1^{\frac{1}{2}}$ have a common core.

To prove (1.4) we set $\Delta_\varepsilon = \Delta_0 \dotplus \varepsilon\Delta_1$. Then according to ([4], Ch. VIII, §3, Theorem 3.6):

$$\underset{\varepsilon \to 0}{s\text{-lim}}\,(I + \Delta_\varepsilon)^{-1} = (I + \Delta_0)^{-1}.$$

Now, using ([8], Ch. II, Proposition 2.3.2) one can prove

$$\underset{\varepsilon \to 0}{s\text{-lim}}\,(\log \Delta_\varepsilon - \lambda I)^{-1} = (\log \Delta_0 - \lambda I)^{-1}$$

for any $\lambda \in \mathbf{C}$ such that $\operatorname{Im} \lambda \neq 0$ and (1.4) follows immediately from ([4], Ch. IX, § 2, Theorem 2.16).

Let A be a nondegenerate m-sectorial operator acting on H. In [12] we proved that

$$\text{Spectrum (Phase } A) \subset \text{Sector } A. \tag{1.5}$$

We shall use this result to prove the following interesting proposition revealing differential properties of the Phase mapping.

PROPOSITION 1.1. *Let Δ and Δ_1 be positive selfadjoint operators acting on a Hilbert space H. Assume that $\Delta_1 \leq N\Delta$ (where $N \in \mathbf{R}_+$) and that Δ is nondegenerate.*

Then

1° *There exists the weak limit*

$$\text{(1.6)} \qquad \underset{\substack{\varepsilon\to 0\\ \varepsilon \text{ real}}}{w\text{-lim}} \frac{1}{2i\varepsilon}\{\text{Phase}(\Delta + i\varepsilon\Delta_1) - I\} = Q.$$

The operator Q is selfadjoint and $\|Q\| \leqq N/2$.

2° *For any* $x, y \in D(\Delta)$ *we have*

$$\text{(1.7)} \qquad (\Delta y|Qx) + (y|Q\Delta x) = (\Delta_1^{\frac{1}{2}} y|\Delta_1^{\frac{1}{2}} x).$$

The operator $Q \in B(H)$ *is determined uniquely by* (1.7).

Proof. It follows directly from the estimate $\Delta_1 \leqq N\Delta$ that $T = \Delta_1^{\frac{1}{2}}(\Delta+I)^{-\frac{1}{2}}$ is a bounded operator defined on the whole H. For any $\varepsilon \in \mathbf{R}$, $\varepsilon \neq 0$ we set

$$\text{(1.8)} \qquad W_\varepsilon = \Delta \dotplus i\varepsilon\Delta_1,$$

$$\text{(1.9)} \qquad Q_\varepsilon = \frac{1}{2i\varepsilon}[(\text{Phase } W_\varepsilon) - I].$$

One can easily check that $I + W_\varepsilon$ is invertible and

$$\text{(1.10)} \qquad (I + W_\varepsilon)^{-1} = (I+\Delta)^{-\frac{1}{2}}(I + i\varepsilon T^*T)^{-1}(I+\Delta)^{-\frac{1}{2}}.$$

It is clear that W_ε is maximal sectorial and that

$$\text{Sector } W_\varepsilon \subset \{z \in \mathbf{C} : 0 \leqq \text{Im } z \leqq N\varepsilon \text{Re } z\}.$$

Therefore, according to (1.5)

$$\|(\text{Phase } W_\varepsilon) - I\| \leqq N\varepsilon$$

and

$$\text{(1.11)} \qquad \|Q_\varepsilon\| \leqq N/2.$$

We have to show that Q_ε is weakly convergent as $\varepsilon \to 0$. We know that any closed ball in $B(H)$ is weakly compact. Therefore (cf. (1.11)) it is sufficient to show that Q_ε has at most one accumulation point as $\varepsilon \to 0$.

In virtue of (1.9)

$$\text{(1.12)} \qquad \text{Phase } W_\varepsilon = I + 2i\varepsilon Q_\varepsilon.$$

Therefore $(I + 2i\varepsilon Q_\varepsilon)^* W_\varepsilon$ is selfadjoint positive and for any $x_\varepsilon \in D(W_\varepsilon)$ we have

$$\text{Im}((I + 2i\varepsilon Q_\varepsilon)x_\varepsilon | W_\varepsilon x_\varepsilon) = 0$$

and, after a simple computation

$$(Q_\varepsilon x_\varepsilon | W_\varepsilon x_\varepsilon) + (W_\varepsilon x_\varepsilon | Q_\varepsilon x_\varepsilon) = \frac{1}{\varepsilon} \operatorname{Im}(x_\varepsilon | W_\varepsilon x_\varepsilon).$$

Now, using (1.8) we get

$$(1.13) \qquad (Q_\varepsilon x_\varepsilon | W_\varepsilon x_\varepsilon) + (W_\varepsilon x_\varepsilon | Q_\varepsilon x_\varepsilon) = (\Delta_1^{\frac{1}{2}} x_\varepsilon | \Delta_1^{\frac{1}{2}} x_\varepsilon)$$

for any $x_\varepsilon \in D(W_\varepsilon)$. We put $x_\varepsilon = (I + W_\varepsilon)^{-1} y$ and $x = (I + \Delta)^{-1} y$. In virtue of (1.10) $x_\varepsilon \to x$, $\Delta_1^{\frac{1}{2}} x_\varepsilon \to \Delta_1^{\frac{1}{2}} x$ and $W_\varepsilon x_\varepsilon = y - x_\varepsilon \to y - x = \Delta x$ as $\varepsilon \to 0$. Therefore, denoting by Q an accumulation point of Q_ε (when $\varepsilon \to 0$) and using (1.13) we obtain

$$(Qx | \Delta x) + (\Delta x | Qx) = (\Delta_1^{\frac{1}{2}} x | \Delta_1^{\frac{1}{2}} x)$$

for any $x \in D(\Delta)$. To obtain (1.7) we use the polarisation formula and note that $Q^* = Q$. This fact follows easily from (1.12) since Phase W_ε is unitary.

This way we proved that any accumulation point Q of Q_ε (when $\varepsilon \to 0$) satisfies (1.7). To conclude the proof we have to show that (1.7) determines Q uniquely.

Assume that we have two operators Q and Q' satisfying (1.7). Then denoting by R their difference we have

$$(\Delta y | Rx) + (y | R\Delta x) = 0$$

for all $x, y \in D(\Delta)$. Since $\Delta^* = \Delta$, we get $Rx \in D(\Delta)$ and $\Delta Rx = -R\Delta x$ for all $x \in D(\Delta)$. In other words $-\Delta R \supset R\Delta$. Now, using the interpolation theory (see Appendix 1, Cor A3) we get $R = 0$ and $Q = Q'$. Q.E.D.

REMARK. Using the Van Daele method [2] one can find the following explicite expression for the operator Q satisfying (1.7):

$$Q = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\Delta^{it} C \Delta^{-it}}{\operatorname{ch} \pi t} \, dt$$

where $C \in B(H)$ is introduced by

$$(\Delta^{\frac{1}{2}} y | C \Delta^{\frac{1}{2}} x) = (\Delta_1^{\frac{1}{2}} y | \Delta_1^{\frac{1}{2}} x), \qquad x, y \in D(\Delta^{\frac{1}{2}}).$$

We shall not use this expression.

For any $A \in \mathscr{C}(H)$ we set

$$(1.14) \qquad Q_A = A(A^* A + I)^{-\frac{1}{2}}.$$

One can easily check that Q_A is a bounded operator and that A is uniquely determined by Q_A: the domain $D(A)$ coincides with the range of $(I - Q_A^*Q_A)^{\frac{1}{2}}$ and

$$A = Q_A(I - Q_A^*Q_A)^{-\frac{1}{2}}. \tag{1.15}$$

In other words, for any $x, y \in H$:

$$(x \in D(A) \text{ and } y = Ax) \Leftrightarrow (\text{there exists } z \in H \text{ such that } x = (I - Q_A^*Q_A)^{\frac{1}{2}}z \text{ and } y = Q_A z). \tag{1.16}$$

Let M be a von Neumann algebra contained in $B(H)$ and $A \in \mathscr{C}(H)$. We say that A is affiliated to M if $Q_A \in M$. Then we write $A \eta M$. This is the case if and only if $U^*AU = A$ for any unitary $U \in M'$.

Let M be a von Neumann algebra and σ be an automorphism of M. We extend the action of σ onto elements affiliated to M denoting by $\sigma(A)$ the unique operator such that

$$Q_{\sigma(A)} = \sigma(Q_A). \tag{1.17}$$

In Section 3 we shall use the direct integrals of measurable fields of closed operators. We refer to ([3], Ch. II) for the theory of the direct integral decomposition (see also [6]). Since only measurable fields of bounded operators are considered in these references, we have to say what we understand by the direct integral of a measurable field of closed operators.

Let (Λ, μ) be a measure space and $(H(\lambda))_{\lambda \in \Lambda}$ be a measurable field of Hilbert spaces. We say that a field of closed operators

$$\Lambda \ni \lambda \to A(\lambda) \in \mathscr{C}(H(\lambda)) \tag{1.18}$$

is *measurable* if

$$\Lambda \ni \lambda \to Q_{A(\lambda)} \in B(H(\lambda)) \tag{1.19}$$

is a measurable field of bounded operators. In this case there exists the unique closed operator $A \in \mathscr{C}\left(\int_\Lambda^\oplus H(\lambda)\mathrm{d}\mu(\lambda)\right)$ such that

$$Q_A = \int_\Lambda^\oplus Q_{A(\lambda)}\,\mathrm{d}\mu(\lambda). \tag{1.20}$$

The operator A is called the *direct integral* of the field (1.18) and denoted by $\int_\Lambda^\oplus A(\lambda)\mathrm{d}\mu(\lambda)$.

Let $x, y \in \int_\Lambda^\oplus H(\lambda)\mathrm{d}\mu(\lambda)$. Using (1.16) one can easily show that $x \in D\left(\int_\Lambda^\oplus A(\lambda)\mathrm{d}\mu(\lambda)\right)$ and $y = \int_\Lambda^\oplus A(\lambda)\mathrm{d}\mu(\lambda)x$ if and only if $x(\lambda) \in D(A(\lambda))$ and $y(\lambda) = A(\lambda)x(\lambda)$ for μ-almost all λ.

In particular this fact shows that for bounded fields of bounded operators, the direct integral introduced by (1.20) coincides with that considered in ([3], Ch. II).

2. BASIC DEFINITIONS AND RESULTS

The main objects investigated in this paper will be called systems. Each *system* S consists of a von Neumann algebra M of operators acting on a Hilbert space H and a family $\{\Delta_\alpha\}_{\alpha\in\Lambda}$ of positive selfadjoint operators acting on the same Hilbert space H:

$$S = (M, \{\Delta_\alpha\}_{\alpha\in\Lambda}).$$

We shall always assume that the operators Δ_α are nondegenerate and that thef index set Λ is not empty.

DEFINITION 2.1. We say that $S = (M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ is a *modular system* if

$$\Delta_\alpha^{it} A \Delta_\beta^{-it} \in M \tag{2.1}$$

for any $\alpha, \beta \in \Lambda$, $A \in M$ and $t \in \mathbf{R}$.

Two modular systems $S=(M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ and $S' = (M, \{\Delta'_\alpha\}_{\alpha\in\Lambda})$ with the same von Neumann algebra M and the same index set Λ are said to be *equivalent* if

$$\Delta_\alpha^{it} A \Delta_\beta^{-it} = \Delta'^{it}_\alpha A\, \Delta'^{-it}_\beta \tag{2.2}$$

for any $\alpha, \beta \in \Lambda$, $A \in M$ and $t \in \mathbf{R}$.

Assume that $S=(M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ is a modular system. Then for any $\alpha \in \Lambda$, $t \in \mathbf{R}$ and $A \in M$ we have $\Delta_\alpha^{it} A \Delta_\alpha^{-it} \in M$. Therefore the formula

$$\sigma_t^\alpha(A) = \Delta_\alpha^{it} A \Delta_\alpha^{-it} \tag{2.3}$$

introduces a family $\{\sigma_t^\alpha\}_{\alpha\in\Lambda}$ of one parameter groups of automorphisms of M. These groups are called *modular groups* of the modular system S.

Moreover, for any $\alpha, \beta \in \Lambda$ and any $t \in \mathbf{R}$ we have $\Delta_\alpha^{it}\Delta_\beta^{-it} \in M$. Therefore to every pair of indices $(\alpha, \beta) \in \Lambda^2$ we associated a one parameter family unitary elements of M. This family will be denoted by $\{(\alpha:\beta)_t\}_{t\in\mathbf{R}}$:

$$(\alpha:\beta)_t = \Delta_\alpha^{it}\Delta_\beta^{-it}. \tag{2.4}$$

One can easily check by direct computation that for any modular system, the quantities introduced above satisfy the following relations

$$\sigma_t^\alpha(A) = (\alpha : \beta)_t \, \sigma_t^\beta(A)(\alpha : \beta)_t^{-1} \tag{2.5}$$

$$(\alpha : \alpha)_t = I \tag{2.6}$$

$$(\alpha : \beta)_t \, (\beta : \gamma)_t = (\alpha : \gamma)_t \tag{2.7}$$

$$(\alpha : \beta)_{t+\tau} = (\alpha : \beta)_t \, \sigma_t^\beta((\alpha : \beta)_\tau) \tag{2.8}$$

for any $\alpha, \beta, \gamma \in \Lambda$; $t, \tau \in \mathbf{R}$ and $A \in M$. The relation (2.5) shows that any two modular groups are related by inner automorphisms. The relations (2.6) and (2.7) justify the notation (2.4). (2.8) is the well known cochain relation (cf. [1]). Following A. Connes we call $\{(\alpha:\beta)_t\}_{t \in \mathbf{R}}$ *the Radon-Nikodym derivatives* of the system S.

The second basic notion considered in this paper is that of a phase system. It plays an essential role in our approach to the Tomita-Takesaki theory. To introduce this notion we have to restrict ourselves to regular systems: A system $(M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ is said to be *regular* iff for any $\alpha, \beta \in \Lambda$ the operators $\Delta_\alpha^{\frac{1}{2}}$ and $\Delta_\beta^{\frac{1}{2}}$ have a common core.

DEFINITION 2.2. We say that a regular system $S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ is a *phase system* if

$$\text{Phase } [A(a\Delta_\alpha \dotplus b\Delta_\beta)] \in M \tag{2.9}$$

for any $\alpha, \beta \in \Lambda$, $a, b \in \mathbf{C}_+$, $a + b \neq 0$, and any invertible $A \in M$.

Two phase systems $S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ and $S' = (M, \{\Delta'_\alpha\}_{\alpha \in \Lambda})$ with the same von Neumann algebra M and the same index set Λ are said to be *equivalent* if

$$\text{Phase } [A(a\Delta_\alpha \dotplus b\Delta_\beta)] = \text{Phase } [A(a\Delta'_\alpha \dotplus b\Delta'_\beta)] \tag{2.10}$$

for any $\alpha, \beta \in \Lambda$; $a, b \in \mathbf{C}_+$, $a + b \neq 0$, and any invertible operator $A \in M$.

REMARK. We consider only invertible operators A and exclude the case $a = b = 0$ in order to have operators $A(a\Delta_\alpha \dotplus b\Delta_\beta)$ automaticly closed and non-degenerate.

The main results of the paper are contained in the following two theorems.

THEOREM 2.3. *Let $S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ be a regular system. Then the following two conditions are equivalent:*

$\mathrm{I_A}$. *S is a modular system.*

$\mathrm{II_A}$. *S is a phase system.*

This result shows that the notions of a modular system and a phase system coincide. We say that S is an *operator system* if one of the above conditions is satisfied. We shall also in the future use terms "modular" and "phase" in order to stress the particular properties of operator systems described in the Definitions 2.1 and 2.2.

THEOREM 2.4. *Let* $S=(M,\{\Delta_\alpha\}_{\alpha\in\Lambda})$ *and* $S'=(M,\{\Delta'_\alpha\}_{\alpha\in\Lambda})$ *be regular operator systems with the same von Neumann algebra and the same index set* Λ. *Then the following two conditions are equivalent:*

$\mathrm{I_B}$. *S and S′ are equivalent as modular systems.*

$\mathrm{II_B}$. *S and S′ are equivalent as phase systems.*

This theorem allows us to introduce the equivalence relation for operator systems. Two operator systems S and S' are called *equivalent* if one of the above conditions is satisfied.

For the moment the main application of the theory of operator systems is that to the Tomita-Takesaki theory. To present it we have to remind some natural operations involving normal functionals and elements of the von Neumann algebras. For simplicity we assume that the underlying Hilbert spaces of the von Neumann algebras, we deal with, are separable.

Let M be a von Neumann algebra. We denote by M_* the *predual* of M: M_* is the set of all normal (i.e. σ-weakly continuous) linear functionals on M endowed with the well known Banach space structure. The set of all normal positive functionals will be denoted by M_*^+.

Let $A\in M$ and $\alpha\in M_*$. For any $B\in M$ we put (cf. [3], Ch. I, § 4)

$$(A\alpha)(B)=\alpha(BA) \quad\text{and}\quad (\alpha A)(B)=\alpha(AB).$$

Clearly $A\alpha, \alpha A\in M_*$. The multiplication of normal functionals by elements of the algebra introduced in this way plays an important role in our approach to the Tomita-Takesaki theory.

Let $\alpha\in M_*$. The smallest projection $E\in M$ such that $E\alpha=\alpha$ is called the *left support* of α. Similarly one introduces the *right support* of α. α is called *nondegenerate* iff the left and the right supports of α are equal to I. For example a state is nondegenerate iff it is faithful.

We shall use the following result ([3], Ch. I, § 4): for any nondegenerate $\alpha\in M_*$ there exists an unitary operator $U\in M$ and a functional $\omega\in M_*^+$ such that

$$\alpha=U\omega. \tag{2.11}$$

The pair (U,ω) is determined uniquely by α.

(2.11) is called the *polar decomposition* of the functional α; the unitary factor of the polar decomposition (2.11) will be denoted by phase α.

The polar decomposition of normal functionals is in some sense isomorphic to the same operation for operators. This sense is made precise in the following definition.

Definition 2.5. Let M be a von Neumann algebra. A regular system $(M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ is called a *canonical phase system* associated with M if:

$$(2.12)\begin{cases} 1^\circ \text{ The index set } \Lambda \text{ is the set of all positive normal faithful functionals on } M. \\ 2^\circ \ \mathrm{Phase}\{A(a\Delta_\alpha \dotplus b\Delta_\beta)\} = \mathrm{phase}\{A(a\alpha + b\beta)\} \end{cases}$$

for all $a, b \in \mathbf{C}_+$, $a + b \neq 0$, $\alpha, \beta \in \Lambda$ and all invertible elements $A \in M$.

It is clear that any canonical phase system is a phase system in the sense of Definition 2.2. Moreover any two canonical phase systems associated with the same von Neumann algebra are equivalent as phase systems.

According to the results formulated above the canonical phase system is also a modular system. Therefore one may use formula (2.3) and (2.4) in order to introduce the modular group associated with a given faithful normal state and the Radon-Nikodym derivative of two faithful states. Since all canonical phase systems associated with a given von Neumann algebra are equivalent, the left hand sides of (2.3) and (2.4) are independent of the particular canonical phase system used.

Clearly all what has been said above applies only to these von Neumann algebras for which the canonical phase system exists. It turns out that this assumption is in no case restrictive.

Theorem 2.6. *Let M be a von Neumann algebra, H be the underlying Hilbert space and Λ be the set of all positive normal faithful functionals on M. We assume that H is separable. Then there exists a family $\{\Delta_\alpha\}_{\alpha\in\Lambda}$ of selfadjoint positive nondegenerate operators acting on H such that $(M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ is a canonical phase system.*

The proofs of the theorems stated in this section will be given in the forthcoming sections: Thm. 2.3 and 2.4 in Sec. 3 and Sec. 4; Thm. 2.6 in Sec. 5. In the last section we remind the Takesaki notion of crossed product which seems to be related to operator systems in a very natural way.

3. HOMOGENEOUS OPERATOR FUNCTIONS

In [11] we investigated an operator of the form

$$(3.1) \qquad F(A, \Delta) = ((A \circ \Delta^{\frac{1}{2}})^* (A \circ \Delta^{\frac{1}{2}}))^{\frac{1}{2}} \circ \Delta^{-\frac{1}{2}}$$

where Δ is a positive selfadjoint, nondegenerate operator acting on a Hilbert space H, $A \in B(H)$ is such that $A\Delta^{\frac{1}{2}}$ is closable and $X \circ Y$ denotes the closure of XY. We proved that

$$(3.2) \qquad F(A, \Delta) \in W^*(\{\Delta^{it} A \Delta^{-it} : t \in \mathbf{R}\}).$$

It turns out that the essential property of the operator function F which implies (3.2) is that of the homogeneity (of degree 0) with respect to the second variable: $F(A, \lambda\Delta) = F(A, \Delta)$ for any positive number λ. In this section we generalize this result for operator functions of many operator variables, homogeneous with respect to a group of the variables.

At first we have to explain what we understand by an operator function. Intuitively an operator function F of N variables is a kind of instruction saying what one has to do with any given sequence $(A_1, A_2, \ldots, A_N)$ of closed operators acting on a Hilbert space in order to obtain another closed operator $F(A_1, A_2, \ldots, A_N)$ acting on the same Hilbert space.

Such an instruction should not be related to any particular Hilbert space. It should be applicable to N-element sequences of closed operators acting on any other Hilbert space H giving as the result a closed operator acting on H.

On the other hand for a given Hilbert space H, the instruction need not be applicable to all sequences $(A_1, A_2, \ldots, A_N) \in \mathscr{C}(H)^N$. The set of all sequences belonging to $\mathscr{C}(H)^N$ for which the instruction is meaningful will be denoted by $D_F(H)$. For example, if F is introduced by (3.1), then $D_F(H)$ consists of all pairs (A, Δ), where Δ is a positive selfadjoint operator acting on H and $A \in \mathscr{C}(H)$ such that $A\Delta^{\frac{1}{2}}$ is closable and densely defined.

To conclude our analysis of the concept of an operator function we have to say what kind of instructions we are going to consider. For example we could set a list of "elementary orders" and say that any instruction should be composed of these orders. Instead we prefer another, more "axiomatic" approach. Briefly speaking we accept only these instructions which do not destroy the Hilbert space symmetry and are compatible with direct integral decomposition.

DEFINITION 3.1. Assume that for any Hilbert space H we have a distinguished subset $D_F(H) \subset \mathscr{C}(H)^N$ and a mapping

$$F\colon D_F(H) \to \mathscr{C}(H).$$

We say that F is an *operator function of N variables* if the following two conditions are satisfied:

1° For any unitary map $U: H \to K$ and any $(A_1, A_2, \ldots, A_N) \in D_F(H)$ we have

$$(UA_1U^*, UA_2U^*, \ldots, UA_NU^*) \in D_F(K)$$

and

$$F(UA_1U^*, UA_2U^*, \ldots, UA_NU^*) = UF(A_1, A_2, \ldots, A_N)\, U^*. \tag{3.3}$$

2° For any measure space (Λ, μ), any measurable field of Hilbert spaces $H(\lambda)$ and any N measurable fields of closed operators $A_n(\lambda) \in \mathscr{C}(H(\lambda))$ $(n = 1, 2, \ldots, N)$ we have

$$\left(\int_\Lambda^\oplus A_1(\lambda)\, d\mu(\lambda), \int_\Lambda^\oplus A_2(\lambda)\, d\mu(\lambda), \ldots, \int_\Lambda^\oplus A_N(\lambda)\, d\mu(\lambda)\right) \in D_F\left(\int_\Lambda^\oplus H(\lambda)\, d\mu(\lambda)\right)$$

if and only if

$$(A_1(\lambda), A_2(\lambda), \ldots, A_N(\lambda)) \in D_F(H(\lambda))$$

for μ-almost all $\lambda \in \Lambda$. Moreover in this case

$$F\left(\int_\Lambda^\oplus A_1(\lambda)\, d\mu(\lambda), \int_\Lambda^\oplus A_2(\lambda)\, d\mu(\lambda), \ldots, \int_\Lambda^\oplus A_N(\lambda)\, d\mu(\lambda)\right) =$$

$$= \int_\Lambda^\oplus F(A_1(\lambda), A_2(\lambda), \ldots, A_N(\lambda))\, d\mu(\lambda).$$

REMARK 1. It follows immediately from (3.3) that $F(A_1, A_2, \ldots, A_N)$ commutes with any unitary operator commuting with all A_n $(n = 1, 2, \ldots, N)$. Therefore $F(A_1, A_2, \ldots, A_N)$ is affiliated to the von Neumann algebra generated by $A_1, A_2, \ldots$ $\ldots, A_N$:

$$F(A_1, A_2, \ldots, A_N)\ \eta\ W^*(A_1, A_2, \ldots, A_N). \tag{3.4}$$

REMARK 2. Look at the Section 1 for the meaning of the direct integral of a measurable field of closed (unbounded) operators.

REMARK 3. One has to realize that Definition 3.1, although very natural, is very restrictive. For example an operator function of one selfadjoint variable is completely determined by the values it takes on the real multiples of the identity $I_K \in B(K)$, where K is a one dimensional Hilbert space. This fact follows from the spectral theory of selfadjoint operators ([6], Ch. IX). The similar result holds for operator functions of several strongly commuting normal variables.

REMARK 4. Examples of operator functions are given in Appendix B. Here we only note that for any fixed $a, b \in \mathbf{C}_+$ and $N > 1$

$$F_N(A, \Delta_1, \Delta_2) = \text{Phase}\,[A(a\Delta_1 \dot{+} b\Delta_2)] \tag{3.5}$$

defined on the domain $D_{F_N}(H)$ consisting of all $(A, \Delta_1, \Delta_2) \in \mathscr{C}(H)^3$ such that A is bounded invertible, $\|A\| \leq N$, $\|A^{-1}\| \leq N$, Δ_1 and Δ_2 are positive selfadjoint, $D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ is dense in H, and $a\Delta_1 + b\Delta_2$ is nondegenerate, is an operator function of three variables.

In this section we are particularly interested in operator functions of two groups of variables homogeneous with respect to the second group of variables.

DEFINITION 3.2. Let F be an operator function of $N+P$ variables: $A_1, \ldots, A_N$, $\Delta_1, \ldots, \Delta_P$. We say that F is *homogeneous with respect to the last P variables* if for any Hilbert space H, any $(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) \in D_F(H)$ and any positive number $\lambda > 0$ we have

$$(A_1, \ldots, A_N, \lambda\Delta_1, \ldots, \lambda\Delta_P) \in D_F(H)$$

and

$$F(A_1, \ldots, A_N, \lambda\Delta_1, \ldots, \lambda\Delta_P) = F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P). \tag{3.6}$$

The main result of this section is contained in the following theorem:

THEOREM 3.3. *Let F be an operator function of $N+P$ variables, homogeneous with respect to the group of the last P variables, H be a Hilbert space and $(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) \in D_F(H)$. We assume that all $\Delta_p (p = 1, 2, \ldots, P)$ are positive selfadjoint and nondegenerate. Then*

$$F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P)\, \eta M, \tag{3.7}$$

where M is the von Neumann algebra generated by all $\Delta_1^{it} A_n \Delta_1^{-it}$ and $\Delta_p^{it}\, \Delta_1^{-it}$ $(n = 1, 2, \ldots, N;\ p = 1, 2, \ldots, P;\ t \in \mathbf{R})$. Moreover, if $\Delta'_1, \ldots, \Delta'_P$ is another sequence of positive selfadjoint nondegenerate operators acting on H such that

$$\Delta_1'^{it}\, A_n\, \Delta_1'^{-it} = \Delta_1^{it}\, A_n\, \Delta_1^{-it} \tag{3.8}$$

and

$$\Delta_p'^{it}\, \Delta_1'^{-it} = \Delta_p^{it}\, \Delta_1^{-it} \tag{3.9}$$

for all $n = 1, 2, \ldots, N$; $p = 1, 2, \ldots, P$ and $t \in \mathbf{R}$, then

$$(A_1, \ldots, A_N, \Delta'_1, \ldots, \Delta'_P) \in D_F(H) \tag{3.10}$$

and

$$F(A_1, \ldots, A_N, \Delta'_1, \ldots, \Delta'_P) = F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P). \tag{3.11}$$

Before the proof let us notice that the operator function $F_N(A, \Delta_1, \Delta_2)$ introduced by (3.5) is homogeneous with respect to the last two variables. Applying Theorem 3.3 to this particular function we get immediately the following

COROLLARY 3.4. *Let $(M, \{\Delta_\alpha\}_{\alpha\in A})$ be a regular modular system. Then $(M, \{\Delta_\alpha\}_{\alpha\in A})$ is a phase system. If $(M, \{\Delta_\alpha\}_{\alpha\in A})$ and $(M, \{\Delta'_\alpha\}_{\alpha\in A})$ are equivalent modular systems then they are equivalent as phase systems.*

In other words, condition I implies II in Theorem 2.3 and Theorem 2.4.

Proof of Theorem 3.3. At first we note that (3.7) follows from the second part of the theorem. Indeed if W is a unitary operator commuting with $\Delta_1^{it} A_n \Delta_1^{-it}$ and $\Delta_p^{it} \Delta_1^{-it}$ ($n = 1, 2, \ldots, N$; $p = 1, 2, \ldots, P$, $t \in \mathbf{R}$) then $A_n = W^* A_n W$ and setting $\Delta'_p = W^* \Delta_p W$ (n and p as above) we get (3.8) and (3.9). Now, according to (3.11) and (3.3) we have:

$$F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) = F(A_1, \ldots, A_N, \Delta'_1, \ldots, \Delta'_P) =$$
$$= W^* F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) W.$$

It shows that $F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P)$ commutes with W and (3.7) follows.

To prove (3.10) and (3.11) we consider an auxiliary separable Hilbert space K and a selfadjoint positive operator L acting on K. We assume that L has simple (i.e. multiplicity free) spectrum and that the spectral measure of L is equivalent to the Lebesque measure on $\mathbf{R}_+$. It is known that these requirements define the operator L uniquely (up to unitary equivalence). In practice we set

$$K = L^2(\mathbf{R}, \mathrm{d}E) \tag{3.12}$$

and L equal to the multiplication operator:

$$(Lf)(E) = \mathrm{e}^E f(E) \tag{3.13}$$

for $f \in L^2(\mathbf{R}, \mathrm{d}E)$. We shall also use another realization:

$$K = L^2(\mathbf{R}, \mathrm{d}s) \tag{3.14}$$

and

$$L = \mathrm{e}^D, \tag{3.15}$$

where D is the generator of the translation group:

$$(\mathrm{e}^{itD} f)(s) = f(s - t) \tag{3.16}$$

for $f \in L^2(\mathbf{R}, \mathrm{d}s)$ and $t \in \mathbf{R}$. $\mathrm{d}E$ and $\mathrm{d}s$ denote the usual (translation invariant) Lebesgue measure on $\mathbf{R}$. Clearly these two realisations are related by the Fourier transformation.

We shall use the following two Lemmas:

LEMMA 3.5. *Assume that the operators $A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P, \Delta'_1, \ldots, \Delta'_P$ acting on H satisfy the relations* (3.8) *and* (3.9). *Then there exists a unitary operator U acting on $K \otimes H$ such that*

$$1^\circ \quad U(I \otimes A_n) U^* = I \otimes A_n \tag{3.17}$$

$$2^\circ \quad U(L \otimes \Delta_p) U^* = L \otimes \Delta'_p \tag{3.18}$$

where $n = 1, 2, \ldots, N$, $p = 1, 2, \ldots, P$, *and*

3° If $U(I \otimes B)U^* = I \otimes C$, *where* $B, C \in \mathscr{C}(H)$, *then* $B = C$.

LEMMA 3.6. *Let F be an operator function of $N + P$ variables homogeneous with respect to the last P variables and* $A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P \in \mathscr{C}(H)$. *Then*

$$(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) \in D_F(H) \tag{3.19}$$

if and only if

$$(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta_1, \ldots, L \otimes \Delta_P) \in D_F(K \otimes H). \tag{3.20}$$

Moreover in this case

$$F(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta_1, \ldots, L \otimes \Delta_P) = \\ = I \otimes F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) \tag{3.21}$$

The proof of Theorem 3.3 is now almost trivial. Assume that $A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P, \Delta'_1, \ldots, \Delta'_P$ satisfy the assumptions of the theorem. Then using Lemma 3.6 we get

$$(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta_1, \ldots, L \otimes \Delta_P) \in D_F(K \otimes H).$$

Now, using Lemma 3.5 and the condition 1° of Definition 3.1 we see that

$$(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta'_1, \ldots, L \otimes \Delta'_P) \in D_F(K \otimes H)$$

and applying again Lemma 3.6 we obtain (3.10). Moreover, using the basic properties of operator functions and taking into account formulae (3.17), (3.18) and (3.21) we have

$$U(I \otimes F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P))U^* =$$

$$= UF(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta_1, \ldots, L \otimes \Delta_P)U^* =$$

$$= F(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta'_1, \ldots, L \otimes \Delta'_P) =$$

$$= I \otimes F(A_1, \ldots, A_N, \Delta'_1, \ldots, \Delta'_P).$$

Now, applying the statement 3° of Lemma 3.5 we get (3.11). Q.E.D.

We finish this section with the proofs of Lemma 3.5 and 3.6.

Proof of Lemma 3.5 We use the second realization of K and L (formulae (3.14), (3.15) and (3.16)). In this case $K \otimes H$ can be identified with the space $L^2(\mathbf{R}, H)$ of all square integrable functions of one real variable s with values in H. Let $A \in \mathscr{C}(H)$,

Δ be a selfadjoint positive nondegenerate operator acting on H and $t \in \mathbf{R}$. Then for every $f \in L^2(\mathbf{R}, H)$ we have

$$((I \otimes A)f)(s) = Af(s), \tag{3.22}$$

$$((L \otimes \Delta)^{it} f)(s) = \Delta^{it} f(s-t). \tag{3.23}$$

The first formula is obvious, the second follows easily from (3.15) and (3.16).

Let U be the unitary operator acting on $K \otimes H = L^2(\mathbf{R}, H)$ according to the formula

$$(Uf)(s) = \Delta_1'^{is} \Delta_1^{-is} f(s). \tag{3.24}$$

Now, using (3.22), for any operator A_n satisfying (3.8) we obtain:

$$(U(I \otimes A_n)U^*f)(s) = \Delta_1'^{is}\Delta_1^{-is}((I \otimes A_n)U^*f)(s) =$$

$$= \Delta_1'^{is}\Delta_1^{-is} A_n(U^*f)(s) = \Delta_1'^{is}\Delta_1^{-is} A_n \Delta_1^{is}\Delta_1'^{-is} f(s) = ((I \otimes A_n)f)(s)$$

and 3.17) follows. Similarly, using (3.23) we calculate:

$$(U(L \otimes \Delta_p)^{it}\, U^*f)(s) = \Delta'^{is}\Delta_1^{-is}((L \otimes \Delta_p)^{it}\, U^*f)(s) =$$

$$= \Delta_1'^{is}\Delta_1^{-is}\Delta_p^{it}(U^*f)(s-t) =$$

$$= \Delta_1'^{is}\Delta_1^{-is}\Delta_p^{it}\Delta_1^{i(s-t)}\Delta_1'^{-i(s-t)} f(s-t).$$

If Δ_p and Δ_p' satisfy (3.9) then we may replace Δ_1 and Δ_1' by Δ_p and Δ_p' respectively and after trivial computation we get

$$(U(L \otimes \Delta_p)^{it}\, U^*f)(s) = \Delta_p'^{it} f(s-t) =$$

$$= ((L \otimes \Delta')^{it} f)(s)$$

and (3.18) follows.

If $U(I \otimes B)U^* = I \otimes C$ then $U(I \otimes B) = (I \otimes C)U$ and, using (3.22) and (3.24) we get

$$\Delta_1'^{is}\Delta_1^{-is} Bf(s) = C\Delta_1'^{is}\Delta_1^{-is} f(s)$$

for any $f \in L^2(\mathbf{R}, H)$. Therefore

$$\Delta_1'^{is}\Delta_1^{-is} B = C\Delta_1'^{is}\Delta_1^{-is}$$

for almost all $s \in \mathbf{R}$. Since both sides of the last formula are weakly continuous with respect to s, the equality holds for all $s \in \mathbf{R}$. Setting $s = 0$ we get $B = C$. Q.E.D.

Proof of Lemma 3.6. Now it is convenient to assume that K and L are introduced by (3.12) and (3.13). It is well known that the tensor product

$$K \otimes H = L^2 (\mathbf{R}, \mathrm{d}E) \otimes H$$

can be identified with the direct integral $\int_{\mathbf{R}}^{\oplus} H \mathrm{d}E$ of the constant field of Hilbert spaces (at each point $E \in \mathbf{R}$ we have the same Hilbert space $H(E) = H$). After this identification

$$I \otimes A_n = \int_{\mathbf{R}}^{\oplus} A_n \, \mathrm{d}E\,,$$

$$L \otimes \Delta_p = \int_{\mathbf{R}}^{\oplus} e^E \Delta_p \, \mathrm{d}E.$$

Assume that (3.19) holds. Then according to Definition 3.2, $(A_1, \ldots, A_N, \mathrm{e}^E \Delta_1, \ldots, \mathrm{e}^E \Delta_P) \in D_F(H)$ for all $E \in \mathbf{R}$ and using the second condition of Definition 3.1 we get (3.20). Moreover in this case (cf. (3.6))

$$F(I \otimes A_1, \ldots, I \otimes A_N, L \otimes \Delta_1, \ldots, L \otimes \Delta_P) =$$

$$= \int_{\mathbf{R}}^{\oplus} F(A_1, \ldots, A_N, \mathrm{e}^E \Delta_1, \ldots, e^E \Delta_P) \, \mathrm{d}E =$$

$$= \int_{\mathbf{R}}^{\oplus} F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P) \, \mathrm{d}E =$$

$$= I \otimes F(A_1, \ldots, A_N, \Delta_1, \ldots, \Delta_P).$$

If conversely (3.20) holds, then according to the second condition of Definition 3.1, $(A_1, \ldots, A_N, \mathrm{e}^E \Delta_1, \ldots, \mathrm{e}^E \Delta_P) \in D_F(H)$ for almost all $E \in \mathbf{R}$ and using again Definition 3.2 we get (3.19). Q.E.D.

4. PHASE SYSTEMS

This section is devoted to phase systems. We shall prove that any phase system is a modular system and two equivalent phase systems are equivalent as modular systems. In this context it is sufficient to consider systems containing at most two operators Δ. Indeed, modular systems are characterized by the set of relations (2.12), each involving only two operators Δ. It means that a system $(M, \{\Delta_\alpha\}_{\alpha \in A})$ is modular iff for all $\alpha, \beta \in A$, the systems $(M, \{\Delta_\alpha, \Delta_\beta\})$ are modular. Similarly, the equivalence of two modular systems can be expressed in terms of their "subsystems" containing only two operators Δ.

This section can be divided into two parts. In the first part we investigate phase systems containing only one operator Δ. We prove that the operations $A \to \Delta^{it} A \Delta^{-it}$ characteristic to the modular systems can be expressed in terms of mapping $A \to$ $\to$ Phase $(A\Delta)$. In the second part we deal with systems containing two operators Δ and complete the proof of Theorems 2.3 and 2.4 of Section 2.

We shall start with the following two lemmas:

LEMMA 4.1. *Let (M, Δ) be a phase system and M_Δ be the linear span of $\{A \in M$: A-invertible and $Phase(A\Delta) = I\}$. Then*

1° *For any $A \in M_\Delta$ there exists one and only one operator $Q \in B(H)$ such that*

$$\Delta Q \supset (A - Q)\Delta. \tag{4.1}$$

In what follows, this operator Q will be denoted by $K_\Delta A$.

2° *$K_\Delta A \in M_\Delta$ for any $A \in M_\Delta$ and*

$$K_\Delta : M_\Delta \to M_\Delta$$

is a linear map.

3° *If (M, Δ') is a phase system equivalent to (M, Δ), then $M_\Delta = M_{\Delta'}$ and $K_\Delta = K_{\Delta'}$.*

Proof. The uniqueness of Q follows immediately from the interpolation theory (cf. Appendix A, Corollary A.3).

To show that Q exists, we may assume without loss of generality that A is invertible and that Phase $(A\Delta) = I$. Then $\Delta_1 = A\Delta$ is positive selfadjoint. According to the interpolation theory (see Appendix A) there exists a bounded operator $A_{\frac{1}{2}}$ such that $\Delta_1 = \Delta^{\frac{1}{2}} A_{\frac{1}{2}} \Delta^{\frac{1}{2}}$ and we have the estimate $\Delta_1 \leqslant N\Delta$, where $N = \|A_{\frac{1}{2}}\|$. A moment of reflection shows that

$$(I + i\varepsilon A)\Delta = \Delta \dotplus i\varepsilon\Delta_1.$$

Now, using Proposition 1.1, we see that there exists the weak limit

$$Q = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \text{ real}}} \frac{1}{2i\varepsilon} \{\text{Phase}\,[(I + i\varepsilon A)\Delta] - I\} \tag{4.2}$$

We shall prove that Q satisfies (4.1). To this end we note that $D(\Delta_1) = D(\Delta)$, so $\Delta_1^{\frac{1}{2}} x \in D(\Delta_1^{\frac{1}{2}})$ for any $x \in D(\Delta)$. Therefore the equation (1.6) determining Q takes the following form:

$$(\Delta y | Qx) + (y | Q\Delta x) = (y | \Delta_1 x), \quad x, y \in D(\Delta).$$

Inserting here $\Delta_1 = A\Delta$ we get

$$(\Delta y|Qx) = (y|(A - Q)\Delta x), \quad x, y \in D(\Delta).$$

It shows that $Qx \in D(\Delta)$ and $\Delta Qx = (A - Q)\Delta x$ for every $x \in D(\Delta)$, i.e.(4.1) follows.

If (M, Δ) is a phase system and $A \in M$, then in virtue of (4.2) $Q \in M$. We know that $A\Delta = \Delta A^*$. Using (4.1), we get $\Delta(2Q - A^*) \supset (A - 2Q)\Delta$. It means that $\mathrm{i}(A-2Q)\Delta$ is symmetric. Therefore (cf. [4], Ch. V, Thm. 4.3, p.287) $[rI + \mathrm{i}(A - 2Q)]\Delta$ is positive selfadjoint for $r > \|A - 2Q\|$ and Phase $[rI + \mathrm{i}(A - 2Q)]\Delta = I$. It shows that $rI + \mathrm{i}(A - 2Q) \in M_\Delta$ and consequently $Q \in M_\Delta$. This ends the proof of the first two statements of the lemma. On the way we showed that

$$K_\Delta A = w\text{-}\lim_{\substack{\varepsilon\to 0 \\ \varepsilon \text{ real}}} \frac{1}{2\mathrm{i}\varepsilon} \{\mathrm{Phase}[(I + \mathrm{i}\varepsilon A)\Delta] - I\}$$

for any invertible $A \in M$ such that Phase $A\Delta = I$. To conclude the proof we note that the last statement follows directly from the definition of M_Δ and the above formula. Q.E.D.

LEMMA 4.2. *Let (M, Δ) be a phase system. Then for any $A \in M_\Delta$ the series*

$$F_\Delta(\mu, A) = \sum_{n=0}^{\infty} (1-\mu)^n K_\Delta^{n+1} A \tag{4.3}$$

is convergent in $\{\mu : |1 - \mu| < 1\}$ and the function $F_\Delta(\mu, A)$ introduced in this way has analytical continuation onto the region $\{\mu \in C: \mu$ is not real negative$\}$. Moreover,

$$\Delta^{\mathrm{i}t} A \Delta^{-\mathrm{i}t} = w\text{-}\lim_{\substack{z\to -\mathrm{i}t \\ \mathrm{Re}\, z > 0}} \frac{\sin \pi z}{\pi} \int_0^\infty \mu^{-z} F_\Delta(\mu, A)\, d\mu. \tag{4.4}$$

Proof. We may assume that A is invertible and that Phase $A\Delta = I$. Then $A\Delta$ is selfadjoint, i.e. $A\Delta = \Delta A^*$. Let Σ be the strip $\{z \in \mathbf{C} : 0 \leqslant \mathrm{Re} z \leqslant 1\}$. According to the interpolation theory (see Appendix A) there exists a weakly continuous function

$$\Sigma \ni z \to A_z \in B(H)$$

holomorphic inside Σ such that

$$A\Delta^z \subset \Delta^z A_z \tag{4.5}$$

for all $z \in \Sigma$. We set

$$F(\mu) = \frac{1}{2\mathrm{i}} \int_l \frac{A_z}{\sin \pi z} \mu^{z-1} dz, \tag{4.6}$$

where l is a contour inside Σ going from $-i\infty$ to $+i\infty$ (all contours involved in this proof are shown on the attached figure), $\mu \in \mathbf{C} \setminus \mathbf{R}_-$, $\mathbf{R}_-$ is the set of all real non-positive numbers, $\mu^\alpha = e^{\alpha \operatorname{Log} \mu}$ and Log denotes the principal value of the logarithm (i.e. $|\operatorname{Im} \operatorname{Log} \mu| < \pi$).

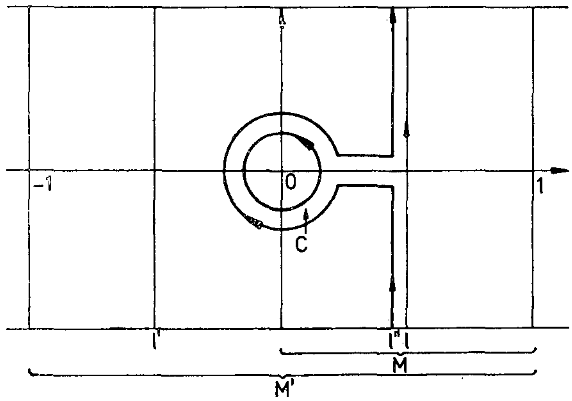

Figure 1.— Contours involved in the proof of Lemma 4.2.

One can easily check that the above integral is convergent and that F is an operator valued function holomorphic on $\mathbf{C} \setminus \mathbf{R}_-$. We shall prove that

$$(\Delta x|F(\mu)y) = (x|(A - \mu F(\mu))\Delta y) \tag{4.7}$$

for all $x, y \in D(\Delta)$. To this end we note that for such vectors

$$(\Delta x|A_{1+it}y) = (x|A_{it}\,\Delta y).$$

This fact follows easily from (4.5). Therefore there exists a function $G(z)$ continuous on the strip $\Sigma' = \{z \in \mathbf{C}: -1 \leqslant \operatorname{Re} z \leqslant 1)$ and holomorphic inside Σ' such that

$$G(z) = \begin{cases} (x|A_z\Delta y) & \text{for } \quad 0 \leqslant \operatorname{Re} z \leqslant 1 \\ (\Delta x|A_{1+z}y) & \text{for} - 1 \leqslant \operatorname{Re} z \leqslant 0. \end{cases} \tag{4.8}$$

Then $(\Delta x|A_z y) = G(z-1)$ for $z \in \Sigma$ and using (4.6) we get

$$(\Delta x|F(\mu)y) = \frac{1}{2i}\int_l \frac{G(z-1)}{\sin \pi z}\mu^{z-1}dz =$$

$$= -\frac{1}{2i}\int_l \frac{G(z-1)}{\sin \pi(z-1)}\mu^{z-1}dz = -\frac{1}{2i}\int_{l'} \frac{G(z)}{\sin \pi z}\mu^z dz,$$

where $l' = l^{-1}$ is the shifted contour. Since the subintegral function is holomorphic inside the region Σ' excluding the point $z = 0$ and tends to 0 when $|\operatorname{Im} z| \to \infty$, we may deform the contour as it is shown on the figure. Therefore

$$(\Delta x|F(\mu)y) = -\frac{1}{2i}\int_{l'} \frac{G(z)}{\sin \pi z}\mu^z\,dz =$$

$$= -\frac{1}{2i}\int_{l} \frac{G(z)}{\sin \pi z}\mu^z\,dz + \frac{1}{2i}\int_{C} \frac{G(z)}{\sin \pi z}\mu^z dz,$$

where C is a small closed contour surrounding the point $z = 0$ in the right direction. According to (4.6) and (4.8) the first term on the right hand side of the above equation coincides with $-(x|\mu F(\mu)\Delta y)$. The second term can be computed with the aid of the residuum theory. As the result we get

$$\frac{1}{2i}\int_{C} \frac{G(z)}{\sin \pi z}\mu^z dz = G(0) = (x|A\Delta y)$$

and (4.7) follows.

Let

$$F(\mu) = \sum_{n=0}^{\infty} (1-\mu)^n Q_n \tag{4.9}$$

be the Taylor series expansion of (4.6). Remembering that F is holomorphic on $\mathbf{C} \setminus \mathbf{R}_-$ we see that the radius of convergence of this equals to 1. Rewriting (4.7) in the following form:

$$(\Delta x|F(\mu)y) = (x|\{(1-\mu)F(\mu) + A - F(\mu)\}\Delta y)$$

and using (4.9) we get

$$(\Delta x|Q_0 y) = (x|(A - Q_0)\Delta y)$$

$$(\Delta x|Q_n y) = (x|(Q_{n-1} - Q_n)\Delta y), \quad n = 1, 2, \dots$$

for all $x, y \in D(\Delta)$. These equalities mean that

$$\Delta Q_0 \supset (A - Q_0)\Delta$$

$$\Delta Q_n \supset (Q_{n-1} - Q_n)\Delta, \qquad n = 1, 2, \dots\ .$$

Now, using Lemma 4.1 repeatedly we see that $Q_n = K_\Delta^{n+1} A$. Therefore, the function $F(\mu)$ introduced by (4.6) coincides with (4.3):

$$F_\Delta(\mu, A) = \frac{1}{2i}\int_{l} \frac{A_z}{\sin \pi z}\mu^{z-1}dz.$$

By using the Fourier integral techniques one can solve this equation with respect to A_z:

$$A_z = \frac{\sin \pi z}{\pi} \int_0^\infty \mu^{-z} F_\Delta(\mu, A) \mathrm{d}\mu$$

for z inside Σ. According to (4.5) $A_{it} = \Delta^{it} A \Delta^{-it}$ and (4.4) follows. This ends the proof of the lemma. Q.E.D.

Now we can prove that "phase" $\Rightarrow$ "modular" for systems with one operator Δ.

PROPOSITION 4.3 *Let (M, Δ) be a phase system. Then (M, Δ) is a modular system. If (M, Δ) and (M, Δ') are equivalent phase systems, then they are equivalent as modular systems.*

Proof. It follows immediately from Lemmas 4.1 and 4.2 that $\Delta^{it} A \Delta^{-it} \in M$ for any $A \in M_\Delta$ and $t \in \mathbf{R}$. If (M, Δ) and (M, Δ') are equivalent phase systems, then for the same reasons $\Delta^{it} A \Delta^{-it} = \Delta'^{it} A \Delta'^{-it}$ for any $A \in M_\Delta$ and $t \in \mathbf{R}$. Therefore we have only to show that M is generated by M_Δ. To this end we notice that any element of M is a linear combination of invertible positive $X \in M$ and that any such $X = A^* A$, where $A = \text{Phase}\,(X^{\frac{1}{2}} \Delta)^* X^{\frac{1}{2}}$. A belongs to M_Δ because Phase $(A\Delta) = $ $= \text{Phase}\,(X^{\frac{1}{2}} \Delta)^*\ \text{Phase}\,(X^{\frac{1}{2}} \Delta) = I.$ Q.E.D.

This proof ends the first part of this section. Now, we shall consider systems with two operators Δ. To complete the proof of Theorem 2.3 and Theorem 2.4 we have to demonstrate the following:

PROPOSITION 4.4. *Let $S = (M, \{\Delta_1, \Delta_2\})$ be a phase system. Then S is a modular system. If $S' = (M, \{\Delta_1', \Delta_2'\})$ is a phase system equivalent to S, then S and S' are equivalent as modular systems.*

Proof. To prove the first statement, it is sufficient to show that

$$\Delta_1^{it}\, \Delta_2^{-it} \in M \tag{4.10}$$

for all $t \in \mathbf{R}$. Indeed (M, Δ_1) is a phase system and on this basis of Proposition 4.3, we already know that

$$\Delta_1^{it}\, A \Delta_1^{-it} \in M \tag{4.11}$$

for all $t \in \mathbf{R}$ and $A \in M$. Now, combining (4.10) and (4.11) we see that $\Delta_\alpha^{it}\, A \Delta_\beta^{-it} \in M$ for all $t \in \mathbf{R}$, $A \in M$, and $\alpha, \beta = 1, 2$, i.e. that S is a modular system.

For any Hilbert space H and any positive number N, we denote by $D_{G_N}(H)$ the set of all pairs (Q, Δ), where Δ is a selfadjoint positive operator acting on H and $Q \in B(H)$, $\|Q\| \leqslant N/2$ is such that the sesquilinear form

$$\Phi(x, y) = (\Delta x | Q y) + (x | Q \Delta y)$$

defined on $D(\Delta)$ is positive and closeable. The positive selfadjoint operator associated with the closure of Φ will be denoted by $G_N(Q, \Delta)$.

It turns out that G_N is an operator function (cf. Appendix B). We also set

$$D_{G_N,t}(H) = \{(Q, \Delta) \in D_{G_N}(H)\colon \Delta \text{ and } G_N(Q, \Delta) \text{ are non-degenerate}\}$$

and

$$G_{N,t}(Q, \Delta) = \Delta^{it} G_N(Q, \Delta)^{-it}.$$

It follows immediately from the definition of G_N that $(Q, \lambda\Delta) \in D_{G_N}(H)$ and $G_N(Q, \lambda\Delta) = \lambda G_N(Q, \Delta)$ for any $\lambda > 0$ and $(Q, \Delta) \in D_{G_N}(H)$. It shows that $G_{N,t}(Q, \Delta)$ is homogeneous with respect to Δ.

Let $\lambda = \exp(-\pi i/4)$ and $\varepsilon > 0$. Then λ and $i\varepsilon\lambda$ belong to $\mathbf{C}_+$ and according to Definition 2.2, we have

$$\text{(4.12)} \qquad \text{Phase}\,(\Delta_1 \dotplus i\varepsilon\Delta_2) = \frac{1}{\lambda}\,\text{Phase}\,(\lambda\Delta_1 \dotplus i\varepsilon\lambda\Delta_2) \in M.$$

Let us assume for the moment that for some positive constant N we have

$$\text{(4.13)} \qquad \Delta_2 \leqslant N\Delta_1.$$

Then, according to Proposition 1.1 there exists the weak limit

$$Q = w\text{-}\lim_{\substack{\varepsilon\to 0\\ \varepsilon \text{ real}}} \frac{1}{2i\varepsilon}\{\text{Phase}(\Delta_1 \dotplus i\varepsilon\Delta_2) - I\},$$

$Q \in M$ and $\|Q\| \leqslant N/2$. Remembering that $\Delta_1^{\frac{1}{2}}$ and $\Delta_2^{\frac{1}{2}}$ have a common core (cf. Definition 2.2) and using (4.13) one can easily show that $D(\Delta_1)$ is a core of $\Delta_2^{\frac{1}{2}}$. Comparing (1.7) with the definition of G_N we see that

$$\Delta_2 = G_N(Q, \Delta_1)$$

and

$$\Delta_1^{it}\,\Delta_2^{-it} = G_{N,t}(Q, \Delta_1).$$

According to Theorem 3.3, $G_{N,t}(Q, \Delta_1)$ belongs to the von Neumann algebra generated by $\{\Delta_1^{it} Q \Delta_1^{-it}\colon\ t \in \mathbf{R}\}$ and using (4.11), we get (4.10).

If Δ_1 and Δ_2 do not satisfy (4.13), then we replace Δ_1 by $\Delta_{1\varepsilon} = \Delta_1 \dotplus \varepsilon\Delta_2$ (where $\varepsilon > 0$). Evidently $(M, \{\Delta_{1\varepsilon}, \Delta_2\})$ is a phase system satisfying the estimate of the form (4.13). Therefore, $\Delta_{1\varepsilon}^{it}\,\Delta_2^{-it} \in M$ for all $t \in \mathbf{R}$ and setting $\varepsilon \to 0$ we obtain (4.10) in full generality (cf. (1.4)).

Now, let $S = (M, \{\Delta_1, \Delta_2\})$ and $S' = (M, \{\Delta'_1, \Delta'_2\})$ be equivalent phase systems. We have to show that S and S' are equivalent as modular systems. To this end it is sufficient to show that

$$\text{(4.14)} \qquad \Delta_1^{it}\Delta_2^{-it} = \Delta_1'^{it}\,\Delta_2'^{-it}$$

for all $t \in \mathbf{R}$. Indeed, $S_1 = (M, \Delta_1)$ and $S_1' = (M, \Delta_1')$ are equivalent phase systems and on the basis of Proposition 4.3 we already know that

$$\Delta_1^{it} A \Delta_1^{-it} = \Delta_1'^{it} A \Delta_1'^{-it} \tag{4.15}$$

for all $t \in \mathbf{R}$ and $A \in M$. Now, combining (4.14) and (4.15), we see that $\Delta_\alpha^{it} A \Delta_\beta^{-it} = \Delta_\alpha'^{it} A \Delta_\beta'^{-it}$ for all $t \in \mathbf{R}$, $A \in M$ and $\alpha, \beta = 1, 2$, i.e. S and S' are equivalent as modular systems.

As it is shown above, we may assume without loss of generality that the operators Δ satisfy the estimates

$$\Delta_2 \leqslant N\Delta_1,$$

$$\Delta_2' \leqslant N\Delta_1'.$$

Then

$$\Delta_1^{it} \Delta_2^{-it} = G_{N,t}(Q, \Delta_1)$$

$$\Delta_1'^{it} \Delta_2'^{-it} = G_{N,t}(Q', \Delta_1'),$$

where

$$Q = w\text{-}\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \text{ real}}} \frac{1}{2i\varepsilon} \{\text{Phase}(\Delta_1 + i\varepsilon\Delta_2) - I\}$$

$$Q' = w\text{-}\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \text{ real}}} \frac{1}{2i\varepsilon} \{\text{Phase}(\Delta_1' + i\varepsilon\Delta_2') - I).$$

Using (4.12) and (2.9) we get $Q = Q'$. Now (4.14) follows directly from (4.15) and Theorem 3.3. Q.E.D.

5. TOMITA-TAKESAKI THEORY

In this section, for any von Neumann algebra M we give an explicit construction of a canonical phase system associated with M proving in the same way Theorem 2.6.

Since any von Neumann algebra can be decomposed into the direct sum of the semifinite and purely-infinite parts, it is sufficient to consider two cases:

A) M is semifinite. Then any normal functional can be represented by a density operator. More precisely, for any $\alpha \in M_*$ there exists an operator $\Delta_\alpha \eta M$ such that $\alpha(A) = \text{Tr}(A\Delta_\alpha)$ for all $A \in M$. One can easily check that the family $\{\Delta_\alpha\}_{\alpha \in \Lambda}$ introduced in this way satisfies (2.12).

B) M is purely infinite. Then M is standard and possesses a cyclic and separating vector (we have assumed that the carrier Hilbert space is separable). In this case the construction of a canonical phase system is described in the following theorem:

THEOREM 5.1. *Let M be a von Neumann algebra possessing cyclic and separating vector y. The set of all normal positive functionals on M will be denoted by M_*^+ and $\Lambda = \{\alpha \in M_*^+ : \alpha \text{ is faithful}\}$.*

I. *For any $\alpha \in M_*^+$, there exists a closed positive sesquilinear form Φ_α such that*

$$(5.1) \qquad \begin{cases} 1^\circ.\ D = \{Ay : A \in M\} \text{ is a core of } \Phi_\alpha \\ 2^\circ.\ \Phi_\alpha(Ay, By) = \alpha(BA^*) \end{cases}$$

for all $A, B \in M$. Clearly Φ_α is determined uniquely by these conditions. Moreover, denoting by Δ_α the positive selfadjoint operator associated with Φ_α we have

II. *$S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ is a canonical phase system associated with M.*

Proof. Ad I. It is sufficient to show that the positive sesquilinear form $\mathring{\Phi}_\alpha$ defined on D with the values given by (5.1) is closeable, i.e. for any sequence $A_n \in M$ such that

$$(5.2) \qquad A_n y \to 0$$

$$(5.3) \qquad \alpha((A_n - A_m)(A_n - A_m)^*) \to 0$$

as $n, m \to \infty$, we have

$$(5.4) \qquad \alpha(A_n A_n^*) \to 0$$

when $n \to \infty$. For standard von Neumann algebras, any positive normal functional can be represented by a vector. Therefore there exists $x \in H$ such that

$$(5.5) \qquad \alpha(Q) = (x|Qx)$$

for any $Q \in M$. Now, taking into account (5.3), we see that the sequence $(A_n^* x)_{n=1,2...}$ is convergent. Let

$$(5.6) \qquad z = \lim_{n\to\infty} A_n^* x.$$

For any $A' \in M'$ we have (cf. (5.2)):

$$(A'y|z) = \lim(A'y|A_n^* z) = \lim(A'A_n y|z) = 0.$$

Therefore $z = 0$ and (5.4) follows directly from (5.5) and (5.6). This ends the proof of the first part of Theorem 5.1. To prove the second part we need the following Lemmas:

LEMMA 5.2. *Let $\alpha(Q) = (x|Qx)$ be a faithful functional on M and $u \in D(\Phi_\alpha)$. Then there exists a unique closed $A \eta M$ such that $D' = \{A'y : A' \in M\}$ is a core of A and $Ay = u$. Moreover $x \in D(A^*)$ and*

$$(5.7) \qquad \Phi_\alpha(By, u) = (A^*x|B^*x)$$

for any $B \in M$.

Proof of Lemma 5.2. If $A \eta M$ and $Ay = u$, then $AA'y = A'u$ for any $A' \in M'$. Therefore A (whenever it exists) must coincide with the closure of the operator

$$\mathring{A}: A'y \to A'u \quad A' \in M$$

defined on the domain $D(\mathring{A}) = D'$. This proves the uniqueness of A.

Since $u \in D(\Phi_\alpha)$, there exists a sequence $A_n \in M$ such that

$$A_n y \to u$$

$$\alpha((A_n - A_m)(A_n - A_m)^*) \to 0$$

as $n, m \to \infty$. Remembering that α is related to the vector x by (5.5), we see that the sequence $(A_n^* x)_{n=1,2,\ldots}$ is convergent. We put

$$z = \lim_{n \to \infty} A_n^* x. \tag{5.8}$$

Let us note that

$$(\mathring{A}A'y|B'x) = (A'u|B'x) = \lim(A'A_n y|B'x) =$$

$$= \lim(A'y|B'A_n^* x) = (A'y|B'z)$$

for all $A', B' \in M'$. It means that $B'x \in D(\mathring{A}^*)$ and $\mathring{A}^* B'x = B'z$ for all $B' \in M'$. Since α is faithful, x is separating for M and cyclic for M'. Therefore $D(\mathring{A}^*)$ is dense in H and A is closeable. One can easily check that the closure of $\mathring{A}$ denoted by A possesses the properties stated in Lemma 5.2.

Setting in the previous relations $B' = I$ and remembering that $\mathring{A}^* = A^*$ we get $x \in D(A^*)$ and $A^* x = z$. Now, using (5.5) and (5.8) we have

$$\Phi_\alpha(By, u) = \lim \Phi_\alpha(By, A_n y) = \lim \alpha(A_n B^*) =$$

$$= \lim(A_n^* x|B^* x) = (z|B^* x) = (A^* x|B^* x)$$

for any $B \in M$, so the formula (5.7) holds.

LEMMA 5.3. *Let* $\alpha, \beta \in A$, $a, b \in \mathbf{C}_+$, $a + b \neq 0$ *and* $C \in M$. *Assume that* C *is invertible and that* $\gamma = C(a\alpha + b\beta)$ *is a positive functional on* M. *Then* $G = C(a\Delta_\alpha + b\Delta_\beta)$ *is a nondegenerate positive selfadjoint operator.*

Proof of Lemma 5.3. We may assume without loss of generality that $a \neq 0 \neq b$. Indeed, if for instance $b = 0$, then we reduce the problem to the previous one by replacing α, β, a, b by $\alpha, \alpha, a/2, a/2$ respectively.

At first we shall prove that G is positive. Let $u \in D(G)$. Then $u \in D(a\Delta_\alpha \dotplus b\Delta_\beta)$ and therefore $u \in D(\Phi_\alpha) \cap D(\Phi_\beta)$. Assume that

$$\alpha(Q) = (x|Qx), \tag{5.9}$$
$$\beta(Q) = (x'|Qx').$$

According to Lemma 5.2 there exists an operator $A\eta M$ such that $u = Ay$, $x, x' \in D(A^*)$ and

$$\Phi_\alpha(By, u) = (A^*x|B^*x), \tag{5.10}$$
$$\Phi_\beta(By, u) = (A^*x'|B^*x'),$$

for any $B \in M$. Let

$$A_N = A\chi_{[0,N]}(A^*A), \tag{5.11}$$

where $\chi_{[0,N]}$ denotes the characteristic function of the interval $[0, N]$: $\chi_{[0,N]}(\lambda)$ equals 1 for $\lambda \in [0, N]$ and 0 otherwise. Then $A_N \in M$ and $\lim_{N\to\infty} A_N z = Az$ for any $z \in D(A)$. In particular $\lim_{N\to\infty} A_N y = u$ and using (5.10) and (5.9) we have:

$$(u|Gu) = \lim(A_N y|Gu) = \lim(C^*A_N y|(a\Delta_\alpha \dotplus b\Delta_\beta)u) =$$
$$= \lim(a\Phi_\alpha + b\Phi_\beta)(C^*A_N y, u) =$$
$$= \lim\{a(A^*x|A_N^*Cx) + b(A^*x'|A_N^*Cx')\}.$$

It follows immediately from (5.11) that $A_N^*z \in D(A)$ and $AA_N^*z = A_N A_N^*z$ for any $z \in H$. Therefore

$$(u|Gu) = \lim\{a(x|A_N A_N^*Cx) + b(x'|A_N A_N^*Cx')\} =$$
$$= \lim\{a\alpha(A_N A_N^*C) + b\beta(A_N A_N^*C)\} =$$
$$= \lim(C(a\alpha + b\beta))(A_N A_N^*) = \tag{5.12}$$
$$= \lim \gamma(A_N A_N^*)$$

and G is positive since γ is positive.

To prove selfadjointness of G, it is sufficient to show that G^* is symmetric. We shall prove that G^* is positive.

To this end we note that $\gamma = C(a\alpha + b\beta) = (\bar{a}\alpha + \bar{b}\beta)C^*$ and that $C^{-1}(\bar{a}\alpha + \bar{b}\beta) = C^{-1}\gamma C^{-1*}$ is positive because γ is positive (see Section 2 or [3], Ch. 1, § 4 for the meaning of the multiplication above). Therefore on the basis of the results obtained so far we know that $C^{-1}(\bar{a}\Delta_\alpha \dotplus \bar{b}\Delta_\beta)$ is positive. Multiplying this operator by C from the left and by C^* from the right hand side we prove that $(\bar{a}\Delta_\alpha \dotplus \bar{b}\Delta_\beta)C^*$ is positive. The last operator coincides with G^*.

If $Gu = 0$, then according to (5.12) $\lim_{N\to\infty} \gamma(A_N A_N^*) = 0$. It follows immediately from (5.11) that $A_N A_N^*$ is an increasing function of N, so $\gamma(A_N A_N^*) = 0$ for all N. Since γ is faithful, $A_N = 0$ for all N and consequently $A = 0$ and $u = Ay = 0$. It shows that G is nondegenerate. Q.E.D.

We continue the proof of Theorem 5.1.
Setting in Lemma 5.3: $C = I$, $a = 1, b = 0$, we see that Δ_α is nondegenerate. Moreover, it follows immediately from the definition of Δ_α that for all $\alpha \in \Lambda$ the domains of $\Delta_\alpha^{\frac{1}{2}}$ contain common dense subset $D = \{Ay : A \in M\}$. Therefore $S = (M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ is a regular system.

Now assume that $\alpha, \beta \in \Lambda$, $a, b \in \mathbf{C}_+$, $a + b \neq 0$ and A is an invertible element of M. Let

$$A(a\alpha + b\beta) = U\gamma \tag{5.13}$$

be the polar decomposition of the functional $A(a\alpha + b\beta)$. Then $\gamma = U^*A(a\alpha + b\beta)$ is positive and using Lemma 5.3 we see that $G = U^*A(a\Delta_\alpha \dotplus b\Delta_\beta)$ is positive selfadjoint. Therefore

$$A(a\Delta_\alpha \dotplus b\Delta_\beta) = UG \tag{5.14}$$

is the polar decomposition of the operator $A(a\Delta_\alpha \dotplus b\Delta_\beta)$ and comparing the last two formulas ((5.13) and (5.14)) we get

$$\text{phase } A(a\alpha + b\beta) = \text{Phase } A(a\Delta_\alpha \dotplus b\Delta_\beta).$$

This ends the proof of Theorem 5.1. Q.E.D.

REMARK. The reader familiar with the Tomita-Takesaki theory certainly noticed that the operators Δ_α constructed in Theorem 5.1 coincide with $S_{yx}^* S_{yx}$ where x is a vector representing the functional $\alpha \in M_+^*$ (i.e. $\alpha(A) = (x|Ax)$ for all $A \in M$) and S_{yx} is the closure of the mapping

$$Ay \to A^*x, \quad (A \in M).$$

6. CROSSED PRODUCT ALGEBRA

In this section we show how the Takesaki notion of crossed product [10] works in the framework of the theory of operator systems.

We shall use the auxiliary Hilbert space K and the positive selfadjoint operator L introduced in Section 3. The operator L acting on K is defined uniquely (up to unitary equivalence) by saying that it has simple (i.e. multiplicity free) spectrum and that the spectral measure of L is equivalent to the Lebesque measure on $\mathbf{R}_+$.

Let $S = (M, \{\Delta_\alpha\}_{\alpha\in\Lambda})$ be a modular system and H be the carrier Hilbert space of S. We denote by $W^*(S)$ the von Neumann algebra of operators acting on $K \otimes H$ generated by $I \otimes A$ and $L \otimes \Delta_\alpha$ $(A \in M, \alpha \in \Lambda)$. $W^*(S)$ will be called *the crossed product algebra associated with* S. Let us notice that $W^*(S)$ contains the distinguished ubalgebra $\tilde{M} = \{I \otimes A : A \in M\}$ isomorphic to M. We often identify $\tilde{M}$ with M. We shall prove the following theorems:

THEOREM 6.1. *Let $S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ be a modular system. Then there exists niq ue one parameter group $\{\rho_E\}_{E \in \mathbf{R}}$ of automorphisms of $W^*(S)$ such that*

(6.1) $$\rho_E(I \otimes A) = I \otimes A$$

(6.2) $$\rho_E(L \otimes \Delta_\alpha) = e^E L \otimes \Delta_\alpha$$

for any $A \in M$, $\alpha \in \Lambda$ and $E \in \mathbf{R}$. This group is pointwise weakly continuous and

$$\tilde{M} = \{X \in W^*(S) : \rho_E(X) = X \text{ for all } E \in \mathbf{R}\}.$$

The group $\{\rho_E\}_{E \in \mathbf{R}}$will be called *the scaling group.*

It turns out that equivalent modular systems give rise to isomorphic crossed product algebras. More precisely we have

THEOREM 6.2. *Let $S = (M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ and $S' = (M, \{\Delta'_\alpha\}_{\alpha \in \Lambda})$ be equivalent modular systems. Then there exists unique isomorphism*

$$\sigma : W^*(S) \to W^*(S')$$

such that

(6.3) $$\sigma(I \otimes A) = I \otimes A$$

(6.4) $$\sigma(L \otimes \Delta_\alpha) = L \otimes \Delta'_\alpha$$

for all $A \in M$ and $\alpha \in \Lambda$. This isomorphism intertwines the scaling groups: for any $E \in \mathbf{R}$

(6.5) $$\sigma \circ \rho_E = \rho'_E \circ \sigma$$

where ρ_E and ρ'_E denote the scalling automorphisms of $W^*(S)$ and $W^*(S')$ respectively.

To prove these theorems we have to use concrete realization of K and L. We shall assume that $K = L^2(\mathbf{R})$ and $L = e^D$ where D is generator of the translation group: $(e^{itD}A(s) = f(s-t)$ for all $f \in L^2(\mathbf{R})$ (cf. formulae (3.14), (3.15) and (3.16)). Then $K \otimes H$ can be identified with $L^2(\mathbf{R}, H)$ and

(6.6) $$((I \otimes A)f)(s) = Af(s)$$

(6.7) $$((L \otimes \Delta_\alpha)^{it} f)(s) = \Delta_\alpha^{it} f(s-t)$$

(6.8) $$((L \otimes \Delta'_\alpha)^{it} f)(s) = \Delta'^{it}_\alpha f(s-t)$$

for all $f \in L^2(\mathbf{R}, H)$, $A \in M$ and $\alpha \in \Lambda$. Let us pick out an element $\gamma \in \Lambda$. For any operator X acting on $L^2(\mathbf{R}, H)$ we set

(6.9) $$\sigma(X) = U X U^*,$$

where U is the unitary operator introduced by the formula

(6.10) $$(Uf)(s) = \Delta'^{is}_\gamma \Delta_\gamma^{-is} f(s)$$

for all $f \in L^2(\mathbf{R}, H)$.

If $(M, \{\Delta_\alpha\}_{\alpha \in A})$ and $(M, \Delta'_\alpha\}_{\alpha \in A})$ are equivalent modular systems then

$$\Delta_\gamma'^{is} \Delta_\gamma^{-is} A \Delta_\gamma^{is} \Delta'^{-is}_\gamma = \Delta_\gamma'^{is} \Delta_\gamma'^{-is} A \Delta_\gamma'^{is} \Delta_\gamma'^{-is} = A$$

and

$$\Delta_\gamma'^{is} \Delta_\gamma^{-is} \Delta_\alpha^{it} \Delta_\gamma^{i(s-t)} \Delta_\gamma'^{-i(s-t)} = \Delta_\gamma'^{is} \Delta_\gamma^{-is} \Delta_\alpha^{is} \Delta_\alpha^{-i(s-t)} \Delta_\gamma^{i(s-t)} \Delta_\gamma'^{-i(s-t)} =$$

$$= \Delta'^{is}_\gamma \Delta_\gamma'^{-is} \Delta_\alpha'^{is} \Delta_\alpha'^{-i(s-t)} \Delta_\gamma'^{i(s-t)} \Delta_\gamma'^{-i(s-t)} = \Delta_\alpha'^{it}$$

for all $A \in M$, $\alpha \in A$, $t, s \in \mathbf{R}$. Therefore, taking into account (6.6), (6.7) and (6.8), for any $f \in L^2(\mathbf{R}, H)$ we get:

$$(U(I \otimes A)U^* f)(s) = \Delta_\gamma'^{is} \Delta_\gamma^{-is} A \Delta_\gamma^{is} \Delta_\gamma'^{-is} f(s) = Af(s) = ((I \otimes A)f)(s)$$

and

$$(U(L \otimes \Delta_\alpha)^{it} U^* f)(s) = \Delta_\gamma'^{is} \Delta_\gamma^{-is} \Delta_\alpha^{it} \Delta_\alpha^{i(s-t)} \Delta_\alpha'^{-i(s-t)} f(s-t) =$$

$$= \Delta_\alpha'^{it} f(s-t) = ((L \otimes \Delta'_\alpha)^{it} f)(s).$$

It shows that σ introduced by (6.9) satisfies (6.3) and (6.4). Since $\{I \otimes A, L \otimes \Delta_\alpha : A \in M, \alpha \in A\}$ and $\{I \otimes A, L \otimes \Delta'_\alpha : A \in M, \alpha \in A\}$ generate $W^*(S)$ and $W^*(S')$ respectively, σ is an isomorphism of $W^*(S)$ onto $W^*(S')$. This ends the proof of the first part of Theorem 6.2.

To prove Theorem 6.1 we note that for any modular system $S = (M, \{\Delta_\alpha\}_{\alpha \in A})$ and any $E \in \mathbf{R}$, $S_E = (M, \{e^E \Delta_\alpha\}_{\alpha \in A})$ is a modular system equivalent to S and that $W^*(S_E) = W^*(S)$. Therefore using the result obtained so far we see that there exists an isomorphism ρ_E of $W^*(S)$ satisfying (6.1) and (6.2). Clearly for any $X \in W^*(S)$:

(6.11) $$\rho_E(X) = U_E X U_E^*$$

where U_E is the unitary operator acting on $L^2(\mathbf{R}, H)$ introduced by the following formula obtained from (6.10) by setting $\Delta'_\alpha = e^E \Delta_\alpha$:

(6.12) $$(U_E f)(s) = e^{iEs} f(s).$$

Since operators (6.12) form a strongly continuous one parameter unitary group, $\{\rho_E\}_{E \in \mathbf{R}}$ is a one parameter automorphism group having the desired continuity property.

Now assume that an element $X \in W^*(S)$ is invariant under the action of ρ_E for all $E \in \mathbf{R}$. Then X commute with all U_E. Therefore X is decomposable: there exists a measurable field of operators

(6.13) $$\mathbf{R} \ni s \to X(s) \in B(H)$$

such that

(6.14) $$(Xf)(s) = X(s)f(s)$$

for all $f \in L^2(\mathbf{R}, H)$.

One can easily check that the translation operator $L^{it} \otimes I$ acting on $L^2(\mathbf{R}, H)$ commutes with the generators of $W^*(S)$. Therefore $L^{it} \otimes I \in W^*(S)'$ and $(L^{it} \otimes I) X(L^{-it} \otimes I) = X$. Now, using (6.14) we obtain $X(s-t) = X(s)$ for any $t \in \mathbf{R}$ and almost all $s \in \mathbf{R}$ and (6.13) must be a constant field of operators: there exists $B \in B(H)$ such that $X(s) = B$ for almost all $s \in \mathbf{R}$. In other words (cf. (6.6) and (6.14))

$$X = I \otimes B.$$

Let $Q \in M'$. We consider the operator $\tilde{Q}$ acting on $L^2(\mathbf{R}, H)$ such that

$$(\tilde{Q}f)(s) = \Delta_\alpha^{is} Q \Delta_\alpha^{-is} f(s) \tag{6.15}$$

Remembering that $(M, \{\Delta_\alpha\}_{\alpha \in \Lambda})$ is a modular system and using (6.6) and (6.7) one can easily check that $\tilde{Q}$ commutes with $I \otimes A$ and $(L \otimes \Delta_\alpha)^{it}$ for all $A \in M$, $\alpha \in \Lambda$ and $t \in \mathbf{R}$. Therefore $\tilde{Q} \in W^*(S)'$ and $[\tilde{Q}, X] = 0$. It means that

$$[\Delta_\alpha^{is} Q \Delta_\alpha^{-is}, B] = 0$$

for almost all $s \in \mathbf{R}$. Since the left hand side of the above formula is a weakly continuous function of s, this equality holds for all $s \in \mathbf{R}$ and setting $s=0$ we get $[Q, B] = 0$. This fact holds for any $Q \in M'$, so $B \in M'' = M$ and $X = I \otimes B \in \tilde{M}$. We proved that $\tilde{M}$ contains $\{X \in W^*(S) : \rho_E(X) = X \text{ for all } E \in \mathbf{R}\}$. Since the converse inclusion is obvious (cf. (6.1)), this result ends the proof of Theorem 6.1.

To end the proof of Theorem 6.2 we note that (6.5) follows directly from (6.1), (6.2), (6.3) and (6.4): both sides of (6.5) acting on any generator of $W^*(S)$ give the same result. Q.E.D.

APPENDIX A.

INTERPOLATION THEORY

In this section Δ is a nondegenerate positive selfadjoint operator acting on a Hilbert space H. Σ denotes the strip

$$\Sigma = \{z \in \mathbf{C} : 0 \leqq \operatorname{Re} z \leqq 1\}.$$

PROPOSITION A. 1. *Let $A, B \in B(H)$ and $A\Delta \subset \Delta B$. Then for any $z \in \Sigma$ there exists unique $A_z \in B(H)$ such that*

$$A\Delta^z \subset \Delta^z A_z. \tag{A.1}$$

The mapping $z \to A_z$ is weakly continuous on Σ and holomorphic inside Σ.

Proof. For any $x, y \in D(\Delta) \cap D(\Delta^{-1})$ the function

$$f(z) = (\Delta^{-\bar{z}} y | A\Delta^z x)$$

is continuous and bounded on Σ and holomorphic inside Σ. Moreover on the boundaries of Σ:

$$|f(i\tau)| = |(\Delta^{i\tau} y|A\Delta^{i\tau} x)| \leqq \|A\| \, \|y\| \, \|x\|$$

$$|f(1+i\tau)| = |(\Delta^{-1+i\tau}y|A\Delta^{1+i\tau}x)| = |(\Delta^{-1+i\tau}y|\Delta B\Delta^{i\tau}x)| =$$

$$= |(\Delta^{i\tau}y|B\Delta^{i\tau}x)| \leqq \|B\| \, \|y\| \, \|x\|,$$

for all $\tau \in \mathbf{R}$. Let $C = \max\{\|A\|, \|B\|\}$. In virtue of the maximum principle for holomorphic functions we get

$$|(\Delta^{-\bar{z}}y|A\,\Delta^z x)| \leqq C\|y\| \, \|x\|$$

for all $z \in \Sigma$ and $x, y \in D(\Delta) \cap D(\Delta^{-1})$. Remembering that $D(\Delta) \cap D(\Delta^{-1})$ is a core for Δ^{-z} and Δ^z one can easily deduce that $A\,\Delta^z x \in D(\Delta^{-z})$ and $\|\Delta^{-z}A\Delta^z x\| \leqq \leqq C\|x\|$ for any $x \in D(\Delta^z)$. It means that $D(\Delta^{-z}A\,\Delta^z) = D(\Delta^z)$ and that the closure of $\Delta^{-z}A\,\Delta^z$ belongs to $B(H)$. Denoting this closure by A_z we have: $\Delta^{-z}A\Delta^z \subset A_z$ and (A.1) follows immediately.

Let us note that for any $x, y \in D(\Delta) \cap D(\Delta^{-1})$, $(x|A_z y) = (\Delta^{-\bar{z}}x|A\Delta^z y)$. Therefore the function $z \to (x|A_z y)$ is continous on Σ and holomorphic inside Σ. Since $D(\Delta) \cap D(\Delta^{-1})$ is dense in H and A_z are uniformly bounded ($\|A_z\| \leqq C$ for all $z \in \Sigma$), this fact holds for all $x, y \in H$ and the last statement of Proposition A.1 is proved.

Q.E.D.

COROLLARY A.2. *Let $A \in B(H)$. Assume that $A\Delta$ is selfadjoint. Then $A\Delta = = \Delta^{\frac{1}{2}} A_{\frac{1}{2}} \Delta^{\frac{1}{2}}$ and $A\Delta \leqq \|A\|\Delta$.*

Proof. In this case $A\Delta = \Delta A^*$ and Proposition A.1 can be applied. We have $A\Delta^{\frac{1}{2}} \subset \subset \Delta^{\frac{1}{2}} A_{\frac{1}{2}}$ and $A\Delta \subset \Delta^{\frac{1}{2}} A_{\frac{1}{2}} \Delta^{\frac{1}{2}}$. One can easily check that $A_{\frac{1}{2}}$ is hermitian. Therefore $\Delta^{\frac{1}{2}} A_{\frac{1}{2}} \Delta^{\frac{1}{2}}$ is a symmetric extension of $A\Delta$ and since $A\Delta$ is maximal symmetric we get $A\Delta = \Delta^{\frac{1}{2}} A_{\frac{1}{2}} \Delta^{\frac{1}{2}}$. Moreover for any $x \in D(\Delta)$:

$$(x|A\Delta x) = (\Delta^{\frac{1}{2}} x|A_{\frac{1}{2}} \Delta^{\frac{1}{2}} x) \leqq \|A_{\frac{1}{2}}\| \, \|\Delta^{\frac{1}{2}} x\|^2 \leqq$$

$$\leqq \|A\| \, (x|\Delta x).$$

Q.E.D.

COROLLARY A.3. *Let $A \in B(H)$. Assume that $A\Delta \subset \Delta(-A)$. Then $A = 0$.*

Proof. In this case $A_{it+1} = -A_{it}$ and setting

$$A_{n+z} = (-1)^n A_z \qquad n\text{-integer}, \; z \in \Sigma$$

we define A_s for all $s \in \mathbf{C}$. Clearly A_s is bounded entire function and using the Liouville theorem we get $A_s = \text{const}$. In particular $A = A_0 = A_1 = -A$ and $A = 0$. Q.E.D.

APPENDIX B.

EXAMPLES OF OPERATOR FUNCTIONS

In this appendix we shall show that many operator expressions are in fact operator functions in the sense of Definition 3.1 provided one fixes the domains in a suitable way. We start with the following obvious statement.

PROPOSITION B.1. 1° *A^* defined on $\mathscr{C}(H)$ is an operator function of A.*

2° *A^{-1} defined on $\{A \in \mathscr{C}(H) : A$ is nondegenerate$\}$ is an operator function of A.*

3° *Phase A defined on $\{A \in \mathscr{C}(H) : A$ is nondegenerate$\}$ is an operator function of A.*

4° *A^*A defined on $\mathscr{C}(H)$ is an operator function of A.*

5° *Let f be a continuous complex function defined on a closed subset $K \subset \mathbf{C}$. Then, using the operator calculus notation, $f(A)$ defined on $\{A \in \mathscr{C}(H) : A$ is normal and Spectrum $A \subset K\}$ is an operator function of A.*

Unfortunately the usual algebraic operations such as the addition and the multiplication of bounded operators are not operator functions in the sense of Definition 3.1, because the direct integral of bounded operators need not be bounded. On the other hand for any fixed N, $\left\|\int_\Lambda^\oplus A(\lambda)\mathrm{d}\mu(\lambda)\right\| \leqq N$ if and only if $\|A(\lambda)\| \leqq N$ for μ-almost all λ. Therefore restricting the algebraic operations to the operators of the norm less than N we obtain further examples of operator functions.

PROPOSITION B.2. *Let $N > 0$. Then $A + B$ defines on $\{(A, B) \in B(H)^2 : \|A\| \leqq N$ and $\|B\| \leqq N\}$ is an operator function of (A, B). The same statement holds for AB.*

The restrictions assumed in Proposition B.2 are in many cases going too far. For example it is known that $A + B$ is a well defined closed operator if $A \in \mathscr{C}(H)$ and $B \in B(H)$. Similarly $A\ B$ is closed for any $B \in \mathscr{C}(H)$ and any invertible $A \in B(H)$. Therefore we have

PROPOSITION B.3. *Let $N > 1$. Then*

1° *$A + B$ defined on $\{(A, B) \in \mathscr{C}(H) \times B(H) : \|B\| \leqq N\}$ is an operator function of (A, B).*

2° *AB defined on $\{(A, B) \in B(H) \times \mathscr{C}(H) : A$-invertible and $\|A\|, \|A^{-1}\| \leqq N\}$ is an operator function of (A, B).*

It is rather clear that starting from a given family of operator functions one can build new operator functions by suitable compositions. We already used this method in Section 4. New operator functions can also be obtained by the strong limit procedure. More precisely we have

PROPOSITION B.4. *Let $F_1, F_2, \ldots$ be a sequence of operator functions of N variables. We assume that:*

1° $D_{F_n}(H) = D(H)$ *is independent of n.*

2° *Values of all F_n are bounded operators and there exists a constant K such that*

$$\|F_n(A_1, \ldots, A_N)\| \leqq K$$

for all n and $(A_1, \ldots, A_N) \in D(H)$.

3° *For any $(A_1, \ldots, A_N) \in D(H)$ there exists the strong limit*

$$F_\infty(A_1, \ldots, A_N) = s\text{-}\lim_{n\to\infty} F_n(A_1, \ldots, A_N).$$

Then $F_\infty(A_1, \ldots, A_N)$ defined on $D(H)$ is an operator function of $(A_1, \ldots, A_N)$.

Proof. This fact follows directly from ([3], Prop. 4, Ch. II, § 2, p. 160). Q.E.D.

Throughout the paper we used expressions of the form $a\Delta_1 \dotplus b\Delta_2$, where $a, b \in \mathbf{C}_+$ and Δ_1, Δ_2 were positive selfadjoint operators such that $D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ was dense in the Hilbert space. We would like to show that $a\Delta_1 \dotplus b\Delta_2$ is an operator function of (Δ_1, Δ_2). This is not quite obvious because the definition of $a\Delta_1 \dotplus b\Delta_2$ is based on the notion of sectorial sesquilinear form and we have no results on direct integral decomposition of such forms. To pass over this difficulty we have to find some other definitions of $a\Delta_1 \dotplus b\Delta_2$ making no use of sectorial forms. To this end we introduce the following notation:

For any positive selfadjoint operator Δ and any $\varepsilon > 0$ we put

$$\Delta_\varepsilon = \Delta(I + \varepsilon\Delta)^{-1}.$$

Clearly Δ_ε is bounded positive and

(B.1) $$\lim_{\varepsilon\to 0} \Delta_\varepsilon^{\frac{1}{2}} z = \Delta^{\frac{1}{2}} z$$

for any $z \in D(\Delta^{\frac{1}{2}})$. We shall prove

LEMMA B.5. *Let $a, b \in \mathbf{C}_+$, $a \neq 0 \neq b$. Then for any positive selfadjoint $\Delta_1, \Delta_2 \in \mathscr{C}(H)$ there exists the strong limit:*

(B.2) $$R = s\text{-}\lim_{\varepsilon\to 0}(a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I)^{-1}.$$

Moreover R is nondegenerate if and only if $D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ is dense in H and in this case

(B.3) $$a\Delta_1 \dotplus b\Delta_2 = R^{-1} - I$$

Proof. Let $y \in H$ and

$$x_\varepsilon = a(a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I)^{-1}y \tag{B.4}$$

Then $y = (a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I)x_\varepsilon$ and

$$(x_\varepsilon|y) = a\|\Delta_{1\varepsilon}^{\frac{1}{2}} x_\varepsilon\|^2 + b\|\Delta_{2\varepsilon}^{\frac{1}{2}} x_\varepsilon\|^2 + \|x_\varepsilon\|^2. \tag{B.5}$$

Remembering that the real parts of a and b are positive we get $\|x_\varepsilon\|\,\|y\| \geqq$ $\geqq \mathrm{Re}(x_\varepsilon|y) \geqq \|x_\varepsilon\|^2$ and $\|x_\varepsilon\| \leqq \|y\|$. Using (B.5) once more we see that x_ε, $\Delta_{1\varepsilon}^{\frac{1}{2}}x_\varepsilon$ and $\Delta_{2\varepsilon}^{\frac{1}{2}}x_\varepsilon$ remain bounded as $\varepsilon \to 0$.

At first we shall prove that x_ε is weakly convergent as $\varepsilon \to 0$. Since the bounded closed sets in H are weakly compact, it is sufficient to show that x_ε has at most one weak accumulation point for $\varepsilon \to 0$.

Let x be such a point. Then there exists a sequence $\varepsilon(n) \to 0$ such that

$$x = w\text{-}\lim_{n\to\infty} x_{\varepsilon(n)}. \tag{B.6}$$

It turns out that $x \in D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ and that

$$w\text{-}\lim_{n\to\infty} \Delta_{1\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)} = \Delta_1^{\frac{1}{2}}x \tag{B.7}$$

$$w\text{-}\lim_{n\to\infty} \Delta_{2\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)} = \Delta_2^{\frac{1}{2}}x. \tag{B.8}$$

Indeed, for any $z \in D(\Delta_1^{\frac{1}{2}})$ we have

$$\lim_{n\to\infty}(z|\Delta_{1\varepsilon(n)}^{\frac{1}{2}}x_{\varepsilon(n)}) = \lim_{n\to\infty}(\Delta_{1\varepsilon(n)}^{\frac{1}{2}}z|x_{\varepsilon(n)}) = (\Delta_1^{\frac{1}{2}}z|x)$$

(cf. (B.1) and (B.5)). Remembering that $\Delta_{1\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)}$ is uniformly bounded and that $D(\Delta_1^{\frac{1}{2}})$ is dense in H we see that $(\Delta_{1\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)})$ is weakly convergent and that

$$(z|w\text{-}\lim_{n\to\infty} \Delta_{1\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)}) = (\Delta_1^{\frac{1}{2}}z|x).$$

It shows that $x \in D(\Delta_1^{\frac{1}{2}})$ and that (B.7) holds. For the proof of (B.8) we procede in the same way.

Assume now that $z \in D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$. Then, using (B.1), (B.6), (B.7) and (B.8) we have

$$a(\Delta_1^{\frac{1}{2}} z|\Delta_1^{\frac{1}{2}} x) + b(\Delta_2^{\frac{1}{2}} z|\Delta_2^{\frac{1}{2}} x) + (z|x) =$$

$$= \lim_{n\to\infty}\{a(\Delta_{1\varepsilon(n)}^{\frac{1}{2}} z|\Delta_{1\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)}) + b(\Delta_{2\varepsilon(n)}^{\frac{1}{2}} z|\Delta_{2\varepsilon(n)}^{\frac{1}{2}} x_{\varepsilon(n)}) + (z|x_{\varepsilon(n)}) =$$

$$= \lim_{n\to\infty}(z|(a\Delta_{1\varepsilon(n)} + b\Delta_{2\varepsilon(n)} + I)x_{\varepsilon(n)})$$

and taking into account (B.4) we get

$$a(\Delta_1^{\frac{1}{2}} z|\Delta_1^{\frac{1}{2}} x) + b(\Delta_2^{\frac{1}{2}} z|\Delta_2^{\frac{1}{2}} x) + (z|x) = (z|y) \tag{B.9}$$

for any $z \in D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$.

If besides x, some other $x' \in D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ satisfies (B.9), then

$$a(\Delta_1^{\frac{1}{2}} z|\Delta_1^{\frac{1}{2}}(x'-x)) + b(\Delta_2^{\frac{1}{2}} z|\Delta_2^{\frac{1}{2}}(x'-x)) + (z|x'-x) = 0$$

and setting $z = x' - x$ we get $x' = x$. It means that x is uniquely determined by (B.9) i.e. that x is the only weak accumulation point of x_ε for $\varepsilon \to 0$. Therefore

$$x = w\text{-}\lim_{\varepsilon\to 0} x_\varepsilon \tag{B.10}$$

and consequently (cf. derivation of (B.7) and (B.8))

$$\Delta_1^{\frac{1}{2}} x = w\text{-}\lim_{\varepsilon\to\infty} \Delta_{1\varepsilon}^{\frac{1}{2}} x_\varepsilon, \tag{B.11}$$

$$\Delta_2^{\frac{1}{2}} = w\text{-}\lim_{\varepsilon\to 0} \Delta_{2\varepsilon}^{\frac{1}{2}} x_\varepsilon. \tag{B.12}$$

Setting $z = x$ in (B.9) and using (B.5) we get

$$a\|\Delta_1^{\frac{1}{2}} x\|^2 + b\|\Delta_2^{\frac{1}{2}} x\|^2 + \|x\|^2 = (x|y) = \lim_{\varepsilon\to 0}(x_\varepsilon|y) =$$

$$= \lim_{\varepsilon\to 0}(a\|\Delta_{1\varepsilon}^{\frac{1}{2}} x_\varepsilon\|^2 + b\|\Delta_{2\varepsilon}^{\frac{1}{2}} x_\varepsilon\|^2 + \|x_\varepsilon\|^2).$$

Now, using (B.10), (B.11), (B.12) and the above relation one can easily check that

$$\lim_{\varepsilon\to 0}(a\|\Delta_{1\varepsilon}^{\frac{1}{2}}x_\varepsilon - \Delta_1^{\frac{1}{2}}x\|^2 + b\,\|\Delta_{2\varepsilon}^{\frac{1}{2}}x_\varepsilon - \Delta_2^{\frac{1}{2}}x\|^2 + \|x_\varepsilon - x\|^2) = 0$$

and remembering that real parts of a and b are positive we see that the limits (B.10), (B.11) and (B.12) are in fact the strong limits. This proves the first of the Lemma.

It follows immediately from (B.9) that y is orthogonal to $D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ if and only if $x = 0$. Therefore

$$(D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}}))^{\perp} = \operatorname{Ker} R$$

and

$$D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$$

is dense if R is nondegenerate.

Conversely if

$$D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$$

is dense then according to (B.9)

$$x \in D(a\Delta_1 \dotplus b\Delta_2) \text{ and}$$

$$y = (a\Delta_1 + b\Delta_2 + I)x.$$

Therefore

$$R = (a\Delta_1 + b\Delta_2 + I)^{-1}$$

and (B.3) follows.

Now we are able to show

PROPOSITION B.6. *Let $a, b \in \mathbf{C}_+$. Then $a\Delta_1 \dotplus b\Delta_2$ defined on $\{(\Delta_1, \Delta_2) \in \mathscr{C}(H)^2 : \Delta_1$ and Δ_2 are positive selfadjoint and $D(\Delta_1^{\frac{1}{2}}) \cap D(\Delta_2^{\frac{1}{2}})$ is dense in $H\}$ is an operator function of (Δ_1, Δ_2).*

Proof. Using Proposition B.1.5°; Proposition B.2; Proposition B.1.2°; Lemma B.5 and Proposition B.4; Proposition B.1.2° and Proposition B.3.1° one proves step by step that: $\Delta_{1\varepsilon}$, $\Delta_{2\varepsilon}$, $a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I$, $(a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I)^{-1}$, $s\text{-}\lim_{\varepsilon\to 0}(a\Delta_{1\varepsilon} + b\Delta_{2\varepsilon} + I)^{-1}$ and $a\Delta_1 \dotplus b\Delta_2$ are operator functions of (Δ_1, Δ_2). Q.E.D.

Combining Proposition B.1.3°, Proposition B.3.2° and B.6 we see that

$$F_N(A, \Delta_1, \Delta_2) = \text{Phase } [A(a\Delta_1 \dotplus b\Delta_2)]$$

defined on the domain described in Section 3 is an operator function of (A, Δ_1, Δ_2).

In Section 4 we considered selfadjoint operators Q, Δ, where Q was bounded, $\|Q\| \leqq N/2$ and Δ was positive, such that the form

$$\psi(x, y) = (Qx \mid \Delta y) + (\Delta x \mid Qy)$$

defined on $D(\Delta)$ was positive and closeable. The positive selfadjoint operator associated with the closure of ψ was denoted by $G_N(Q, \Delta)$. We shall prove

PROPOSITION B.7. *$G_N(Q, \Delta)$ is an operator function of (Q, Δ).*

Proof. For any $\varepsilon > 0$ we consider the form ψ_ε associated with the positive selfadjoint operator

$$G_{N\varepsilon}(Q, \Delta) = \left(\varepsilon\Delta + \frac{1}{\varepsilon} Q\right)^2 - \left(\frac{1}{\varepsilon} Q\right)^2.$$

Obviously $D(\psi_\varepsilon) = D(\Delta)$ and

$$\psi_\varepsilon(x, y) = \psi(x, y) + \varepsilon(\Delta x|\Delta y)$$

for any $x, y \in D(\Delta)$. Take $\varepsilon \to 0$. Then ψ_ε is converging to ψ from above. Therefore, using ([4], Ch. VIII, Thm. 3.11, p. 457) we get

$$(G_N(Q, \Delta) + I)^{-1} = s'\text{-}\lim_{\varepsilon \to 0} (G_{N\varepsilon}(Q, \Delta) + I)^{-1}.$$

It is obvious that $G_{N\varepsilon}(Q, \Delta)$ is an operator function of (Q, Δ) and our statement follows from Proposition B.4.

ACKNOWLEDGEMENT

This research was originated during a visit to Zentrum für Interdisziplinäre Forschung at the University of Bielefeld. The author thanks Professor L. Streit for his kind hospitality in Bielefeld and Professors H. Araki, O. Bratelli, G. Elliott, R. Herman, R. Kadison and D. Kastler for stimulating discussions. The research was completed at the authors home institution, where it was supported by the Maria Skłodowska-Curie Fundation established by contributions of the United States and Polish Goverments. The author would like to thank Professor K. Maurin for his interest and encouragement. He also thanks dr. K. Napiórkowski and dr. W. Pusz for interesting remarks.

The present version of the paper has been written down in the Mathematical Institute of Aarhus University. The author is very grateful to all colleagues in this Institute for cresting the excelent atmosphere for the work.

REFERENCES

1. CONNES, A., Une classification des facteurs de type III, *Ann. Sci. École Norm. Sup.*, **6** (1973), 133–252.
2. VAN DAELE, A., A new approach to Tomita-Takesaki theory of generalized Hilbert algebras, *J. Functional Analysis*, **15** (1974), 378—VAN DAELE, A.; RIEFFEL, M., A bounded operator approach to Tomita-Takesaki theory, (Preprint).
3. DIXMIER, J., *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.
4. KATO, T., *Perturbation theory for linear operators*, Springer Verlag Berlin, Heidelberg, New York, 1966.
5. KLAUDER, J. R., Field structure through model studies, *Acta Physica Austriaca*, Suppl. XI (1973), 341. SIMON, B., Quadratic forms and Klauder's phemomenon, *J. Functional Analysis*, **14** (1973), 295.
6. MAURIN, K., *Methods of Hilbert space*, PWN, Warszawa, 1967.
7. NAIMARK, M. A., *Normed rings*, Noordhoff, Groningen, 1964.
8. PEDERSEN, G. K., An introduction to C^*-algebra theory, Preprint Københavns Universitet, Matematisk Institut, May 1974.
9. TAKESAKI, M. *Tomita's theory of modular Hilbert algebras and its applications*, Lecture Notes in Math., no. **128**, Springer Verlag, Berlin, 1970.
10. TAKESAKI, M., The structure of a von Neumann Algebra of type III and crossed products, *C^*-Algebras and their applications to Statistical Mechanics and Quantum Field Theory* 1975, LX Corso, Editrice Compositori Bologna-Italy.
11. WORONOWICZ, S. L., On the purification of factor states, *Comm. Math. Phys.*, **28** (1972), 221–235.
12. WORONOWICZ, S. L., A remark on the polar decomposition of m-sectorial operators, *Letters on Math. Physics I*, No. **5** (1977).

S. L. WORONOWICZ
Department of Mathematical Methods in Physics,
University of Warsaw,
Hoza 74, 00–682 Warsaw,
Poland

Received February 12, 1979.