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CHARACTERIZATION OF SOME HARNACK PARTS OF CONTRACTIONS

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In [1], Zoia Ceaușescu introduced a relation of domination between contractions acting on a Hilbert space, characterized by some simple norm inequalities. This relation is easily shown to be weaker than Harnack domination (cf. [4], [5] for the latter). Zoia Ceaușescu asked whether the equivalence classes of this relation do coincide with Harnack parts. In this paper we give a positive answer to this question in the case when the contractions have spectrum contained in the open unit disc of the complex plane. As a consequence, we prove that the Harnack part of any spectral norm strict contraction is not trivial. For the general case, we show that the answer is negative by exhibiting a class of counterexamples. The techniques used yield also some other results concerning Harnack domination and its connection to the relation introduced by Zoia Ceaușescu.

§ 1. NOTATION AND PRELIMINARIES.

In the sequel T, T' will be linear contractive operators acting on the Hilbert space $\mathscr{H}$; U acting on $\mathscr{K}$ and U' acting on $\mathscr{K}'$ will be the minimal isometric dilations of T, T' respectively. We shall denote by $r(T)$ the spectral radius of T, and by T_λ the operator defined by

$$T_\lambda = (T - \lambda)(I - \bar{\lambda}T)^{-1} \quad (|\lambda| < 1)$$

We say (cf. [4]) that T is Harnack dominated by T' (notation $T \overset{H}{\prec} T'$) if there exists a positive constant a such that for any analytic polynomial p verifying $\operatorname{Re} p(z) \geqslant 0$ for $|z| \leqslant 1$ we have

$$\operatorname{Re} p(T) \leqslant a \operatorname{Re} p(T'). \tag{1.1}$$

Obviously $\overset{H}{\prec}$ is a preorder relation, and we shall denote by $\overset{H}{\sim}$ the equivalence relation induced by it. The equivalence classes are called Harnack parts.

It was shown in [5] that $T \overset{H}{\prec} T'$ is equivalent to the following property:

There exists a bounded operator $S: \mathscr{K}' \to \mathscr{K}$ such that for any $h_0, h_1, \ldots, h_n \in \mathscr{H}$ we have

$$S\left(\sum_{j=0}^{n} U'^j h_j\right) = \sum_{j=0}^{n} U^j h_j.$$

Since $\mathscr{K} = \bigvee_{j=0}^{\infty} U^j \mathscr{H}$, $\mathscr{K}' = \bigvee_{j=0}^{\infty} U'^j \mathscr{H}$, it is obvious that in this case S is the unique bounded operator from $\mathscr{K}'$ into $\mathscr{K}$ which intertwines U' and U and whose restriction to $\mathscr{H}$ is the identity operator. Also, $T \overset{H}{\sim} T'$ if and only if S has bounded inverse.

In [1], Zoia Ceauşescu considered the following preorder relation, which we shall denote by $\overset{Z}{\prec}$:

$T \overset{Z}{\prec} T'$ if there exists a bounded operator $\widetilde{S}$: $\mathscr{H} \vee U'\mathscr{H} \to \mathscr{H} \vee U\mathscr{H}$ such that for any $h_0, h_1 \in \mathscr{H}$,

$$\widetilde{S}(h_0 + U'h_1) = h_0 + Uh_1.$$

The operator $\widetilde{S}$, if it exists, is the unique bounded operator from $\mathscr{H} \vee U'\mathscr{H}$ to $\mathscr{H} \vee U\mathscr{H}$ which intertwines U' and U, and whose restriction to $\mathscr{H}$ is the identity operator. We shall denote by $\overset{Z}{\sim}$ the equivalence relation induced by this preorder relation; $T \overset{Z}{\sim} T'$ if and only if $\widetilde{S}$ has a bounded inverse.

From these definitions it is obvious that $T \overset{H}{\prec} T'$ implies $T \overset{Z}{\prec} T'$; consequently $T \overset{H}{\sim} T'$ implies $T \overset{Z}{\sim} T'$.

We shall occasionally write $T \underset{c}{\overset{H}{\prec}} T' (T \underset{c}{\overset{Z}{\prec}} T')$ in order to indicate that $\|S\| \leqslant c$ $(\|\widetilde{S}\| \leqslant c)$. Since S and $\widetilde{S}$ restricted to $\mathscr{H}$ are equal to the identity, it follows that in both cases we must have $c \geqslant 1$. Moreover, it is proved in [5] that $\|S\| \leqslant c$ is equivalent to $a \leqslant c^2$ in relation (1.1).

§ 2. PROPERTIES OF THE RELATION $\overset{Z}{\prec}$.

LEMMA 1. *The following statements are equivalent:*

(i) $T \overset{Z}{\prec} T'$,

(ii) *there exists* $c' \geqslant 1$ *such that* $\|c'T' - T\| \leqslant c' - 1$,

(iii) *there exists* $c'' \geqslant 1$ *such that for any* $h \in \mathscr{H}$,

$$\|D_T h\| \leqslant c'' \| D_{T'} h\| \tag{2.1}$$

$$\|(T - T')h\| \leqslant c'' \|D_{T'} h\|. \tag{2.2}$$

Proof. (i) $\Leftrightarrow$ (ii) $T \overset{Z}{\prec} T'$ means that there exists $c \geqslant 1$ such that for any $h_0, h_1 \in \mathscr{H}$, we have

$$\|h_0 + Uh_1\|^2 \leqslant c^2\|h_0 + U'h_1\|^2 \tag{2.3}$$

or

$$\|h_0\|^2 + \|h_1\|^2 + 2\,\mathrm{Re}(h_0, Uh_1) \leqslant c^2[\|h_0\|^2 + \|h_1\|^2 + 2\mathrm{Re}(h_0, U'h_1)]$$

But U, U' being isometric dilations of T, T' the last inequality is equivalent to

$$\|h_0\|^2 + \|h_1\|^2 + 2\mathrm{Re}(h_0, Th_1) \leqslant c^2[\|h_0\|^2 + \|h_1\|^2 + 2\mathrm{Re}(h_0, T'h_1)]$$

or

$$(c^2 - 1)(\|h_0\|^2 + \|h_1\|^2) + 2\mathrm{Re}(h_0, (c^2T' - T)h_1) \geqslant 0. \tag{2.4}$$

We claim that the last inequality is equivalent to

$$\|c^2T' - T\| \leqslant c^2 - 1. \tag{2.5}$$

Indeed, if (2.4) is satisfied, then

$$\|c^2T' - T\| = \sup_{\substack{\|h_0\| \leqslant 1\\ \|h_1\| \leqslant 1}} - \mathrm{Re}(h_0, (c^2T' - T)\, h_1) \leqslant \sup_{\substack{\|h_0\| \leqslant 1\\ \|h_1\| \leqslant 1}} \frac{1}{2}(c^2 - 1)(\|h_0\|^2 + \|h_1\|^2) = c^2 - 1.$$

Conversely, if (2.5) is valid, then

$$|\,2\mathrm{Re}(h_0, c^2T' - Th_1)\,| \leqslant 2(c^2 - 1)\,\|h_0\|\cdot\|h_1\|$$

and (2.4) follows.

The equivalence (i) $\Leftrightarrow$ (ii) is therefore proved; moreover, we have in this case the exact relation $c' = c^2$.

(i) $\Leftrightarrow$ (iii) The relation (2.3) can be written

$$\|h_0 + Th_1 + (U - T)h_1\|^2 \leqslant c^2\|h_0 + T'h_1 + (U' - T')\, h_1\|^2$$

which is equivalent, by standard properties of the isometric dilation (see [7], II, § 1), to

$$\|h_0 + Th_1\|^2 + \|D_T h_1\|^2 \leqslant c^2(\|h_0 + T'h\|^2 + \|D_{T'}\, h_1\|^2). \tag{2.6}$$

If we put in this inequality $h_0 = -T'h_1$, relations (2.1), (2.2) follow at once (with $c'' = c$). Conversely, if (2.1), (2.2) are valid, then, since

$$\|h_0 + Th_1\| \leqslant \|h_0 + T'h_1\| + \|(T - T')h_1\| \leqslant \|h_0 + T'h_1\| + c''\|D_{T'}\, h_1\|$$

relation (2.6) follows (with $c^2 = 3c''^2$).

REMARK. The inequalities (2.1), (2.2) clearly make sense in the case T, T' are operators from a Hilbert space $\mathscr{H}_1$ to another Hilbert space $\mathscr{H}_2$. In [1] Zoia Ceauşescu used (2.1), (2.2) to define a preorder relation for $T, T' \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$. She proved also an analogue of the equivalence (i) $\Leftrightarrow$ (iii) for that case.

COROLLARY 1. (a) *If* $T \overset{Z}{\prec} T'$, *then* $T^* \overset{Z}{\prec} T'^*$.

(b) *If* $T_n \underset{c}{\overset{Z}{\prec}} T'_n$ $(n=0, 1, \ldots)$, *and* λ_n *are complex numbers such that* $\sum_{n=0}^{\infty} |\lambda_n| \leqslant 1$, *then*

$$\sum_{n=0}^{\infty} \lambda_n T_n \underset{c}{\overset{Z}{\prec}} \sum_{n=0}^{\infty} \lambda_n T'_n$$

(c) *If* $T_n \underset{c_n}{\overset{Z}{\prec}} T'_n$ $(n = 1, \ldots, N)$, *then*

$$T_1 \ldots T_N \underset{c}{\overset{Z}{\prec}} T'_1 \ldots T'_N, \text{ where } c = c_1 \ldots c_N.$$

Proof. (a) and (b) follow at once from the characterization (ii) in Lemma 1; so does (c) if we remark that by induction it is sufficient to consider $N = 2$, and that

$$c_1^2 c_2^2 T'_1 T'_2 - T_1 T_2 = c_2^2(c_1^2 T'_1 - T_1) T'_2 + T_1(c_2^2 T'_2 - T_2).$$

COROLLARY 2. *Every contraction* T *is Z-dominated by* 0; $0 \overset{Z}{\prec} T$ *if and only if* $\|T\| < 1$.

Proof. By Lemma 1 (ii) $T \underset{c}{\overset{Z}{\prec}} 0$ is equivalent to $\|T\| \leqslant c^2 - 1$; $0 \underset{c}{\overset{Z}{\prec}} T$ is equivalent to $\|T\| \leqslant \frac{c^2 - 1}{c^2}$.

COROLLARY 3. *If* $T \overset{Z}{\sim} T'$ *then* $\|T - T'\| < 2$.

Proof. If $T \underset{c}{\overset{Z}{\prec}} T'$ and $T' \underset{c}{\overset{Z}{\prec}} T$, by Lemma 1(ii) we have

$$\|(c^2 + 1)(T - T')\| \leqslant \|c^2 T - T'\| + \|c^2 T' - T\| \leqslant 2(c^2 - 1)$$

therefore

$$\|T - T'\| \leqslant 2\frac{c^2 - 1}{c^2 + 1}.$$

PROPOSITION 1. *If* $T \underset{c}{\overset{Z}{\prec}} T'$, *then for any* $|\lambda| < 1$

$$T_\lambda \underset{d_\lambda}{\overset{Z}{\prec}} T'_\lambda \quad \text{where } d_\lambda = c\frac{1 + |\lambda|}{1 - |\lambda|}$$

Proof. If U, U' are minimal isometric dilations of T, T', then, by [7], I, § 4, U_λ, U'_λ are minimal isometric dilations of T_λ, T'_λ respectively. Therefore

$$\|h_0 + U_\lambda h_1\| = \|(I - \bar{\lambda}U)^{-1}(h_0 - \lambda h_1 + U(h_1 - \bar{\lambda}h_0))\| \leqslant$$

$$\leqslant (1 - |\lambda|)^{-1}\|h_0 - \lambda h_1 + U(h_1 - \bar{\lambda}h_0)\| \leqslant$$

$$\leqslant c(1 - |\lambda|)^{-1}\|h_0 - \lambda h_1 + U'(h_1 - \bar{\lambda}h_0)\| =$$

$$= c(1 - |\lambda|)^{-1}\|(I - \bar{\lambda}U')(h_0 + U'_\lambda h_1)\| \leqslant$$

$$\leqslant c\frac{1 + |\lambda|}{1 - |\lambda|}\|h_0 + U'_\lambda h_1\|.$$

COROLLARY 4. *If $T \overset{Z}{\prec} T'$, then $b(T) \overset{Z}{\prec} b(T')$ for any finite Blaschke product $b(z)$.*

LEMMA 2. *Let P, Q be orthogonal projections, and T, T' such that*

$$PTQ \underset{c}{\overset{Z}{\prec}} PT'Q,\ (I - P)\,T(I - Q) \underset{c}{\overset{Z}{\prec}} (I - P)\,T'(I - Q).$$

Then

$$PTQ + (I - P)\,T(T - Q) \underset{c}{\overset{Z}{\prec}} PT'Q + (I - P)\,T'(I - Q).$$

Proof. The lemma is a consequence of Lemma 1 (ii) and the obvious relation

$$\|P(c'T' - T)\,Q + (I - P)(c'T' - T)(I - Q)\| =$$

$$= \max\{\|P(c'T' - T)\,Q\|,\ \|(I - P)(c'T' - T)(I - Q)\|\}.$$

PROPOSITION 2. *The Z-part (equivalence class) of T is equal to $\{T\}$ if and only if T is an isometry or a coisometry.*

Proof. If T is an isometry, it follows from Lemma 1(iii) that $T' \overset{Z}{\prec} T$ implies $T' = T$; therefore the Z-part of T is trivial. Then, from Corollary 1(a) we can deduce that it is trivial also for T coisometry.

Suppose T is neither isometry nor coisometry. Let $T = V|T|$ be the polar decomposition of T. If $|T|$ is not a projection, let E be a nontrivial spectral projection of $|T|$, such that

$$\varepsilon E \leqslant |T|E \leqslant (1 - \varepsilon)\,E \text{ for some } \varepsilon > 0.$$

Let $T' = T(I - E)$. Obviously $T' \neq T$. We claim $T \overset{Z}{\sim} T'$. Indeed, let $Q = E$, $P = VEV^*$. Then we have

$$(I - P)\,T(I - Q) = (I - P)\,T'(I - Q),$$

$PTQ = TE$ is a strict contraction, and $PT'Q = 0$. By Lemma 2 and Corollary 2, it follows that $T \overset{Z}{\sim} T'$.

If $|T|$ is a projection, neither the initial nor the final space of V are $\mathscr{H}$, since T is neither isometry nor coisometry. Let $P = VV^*$, $Q = V^*V$, and S an operator such that $0 < \|S\| < 1$, $SQ = PS = 0$. Then, if $T' = T + S$, we have $T \neq T'$, $PTQ = PT'Q$, $(I - P)T(I - Q) = 0$ and $(I - P)T'(I - Q) = S$. Again by Lemma 2 and Corollary 2, it follows that $T \overset{Z}{\sim} T'$.

§ 3. HARNACK DOMINATION AND Z-DOMINATION.

We have seen that Harnack domination implies Z-domination. In the sequel we shall try to find partial converses of this implication. A basic tool is the following lemma.

LEMMA 3. *The following statements are equivalent:*

(i) $T \underset{c}{\overset{H}{\prec}} T'$,

(ii) *for any* $|\lambda| < 1$ *we have*

$$(3.1)\qquad \begin{aligned}&(I - \lambda T^*)^{-1}(I - |\lambda|^2 T^*T)(I - \bar{\lambda}T)^{-1} \leqslant \\ &\leqslant c^2(I - \lambda T'^*)^{-1}(I - |\lambda|^2 T'^*T')(I - \bar{\lambda}T')^{-1}.\end{aligned}$$

Proof. (i) $\Rightarrow$ (ii). For any $|\lambda| < 1$, let

$$f(z, \lambda) = \frac{1 + \bar{\lambda}z}{1 - \bar{\lambda}z}.$$

We have $\operatorname{Re} f(z, \lambda) \geqslant 0$ for $|z| \leqslant 1$, and $f(z, \lambda)$ is, in $\{|z| \leqslant 1\}$, a uniform limit of polynomials with positive real part. It follows, from the definition of Harnack domination, that

$$\operatorname{Re} f(T, \lambda) \leqslant c^2 \operatorname{Re} f(T', \lambda)$$

which is exactly (3.1).

(ii) $\Rightarrow$ (i). Suppose conversely that (3.1) is valid. Let $p(z)$ be any analytic polynomial with positive real part. Since

$$p(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) + ia = \int_0^{2\pi} \frac{1 + e^{-i\theta} z}{1 - e^{-i\theta} z}\, d\mu(\theta) + ia$$

where μ is a positive measure and a is a real constant, we have, by (3.1), for $0 < r < 1$,

$$\operatorname{Re} p(rT) = \int_0^{2\pi} \operatorname{Re} f(T, re^{i\theta})\, d\mu(\theta) \leqslant c^2 \int_0^{2\pi} \operatorname{Re} f(T', re^{i\theta})\, d\mu(\theta) = c^2 \operatorname{Re} p(rT').$$

Letting $r \to 1$, we have

$$\operatorname{Re} p(T) \leqslant c^2 \operatorname{Re} p(T')$$

so $T \overset{H}{\underset{c}{\prec}} T'$.

THEOREM 1. *If* $r(T) < 1$ *and* $T \overset{Z}{\prec} T'$, *then* $r(T') < 1$ *and* $T \overset{H}{\prec} T'$.

Proof. $T \overset{Z}{\prec} T'$ implies by Lemma 1 (iii) that for some $c' \geqslant 1$ and all $|\lambda| \leqslant 1$, $h \in \mathscr{H}$, we have

$$\|(I - |\lambda|^2 T^*T)^{1/2}h\| \leqslant c'\|(I - |\lambda|^2 T'^*T')^{1/2}h\|, \tag{3.2}$$

$$\|(T - T')h\| \leqslant c'\|(I - |\lambda|^2 T'^*T')^{1/2}h\|. \tag{3.3}$$

Suppose $r(T') = 1$. Then some λ_0 with $|\lambda_0| = 1$ is an approximate eigenvalue of T': that is, there is a sequence $\{g_n\} \subset \mathscr{H}$ such that $\|g_n\| = 1$ and $\|T'g_n - \lambda_0 g_n\| \to 0$. Since then $\|T'g_n\| \to 1$, it follows from (3.3) (with $\lambda = \lambda_0$) that $\|Tg_n - \lambda_0 g_n\| \to 0$, which contradicts $r(T) < 1$.

Now for any $|\lambda| < 1$ we have

$$\|(I - |\lambda|^2 T^*T)^{1/2}(I - \bar{\lambda}T)^{-1}h\| =$$

$$= \|(I - |\lambda|^2 T^*T)^{1/2}[(I - \bar{\lambda}T')^{-1} + \bar{\lambda}(I - \bar{\lambda}T)^{-1}(T - T')(I - \bar{\lambda}T')^{-1}]h\| \leqslant$$

$$\leqslant \|(I - |\lambda|^2 T^*T)^{1/2}(I - \bar{\lambda}T')^{-1}h\| +$$

$$+ |\lambda|\|(I - |\lambda|^2 T^*T)^{1/2}(I - \bar{\lambda}T)^{-1}(T - T')(I - \bar{\lambda}T')^{-1}h\| \leqslant$$

$$\leqslant c'\|(I - |\lambda|^2 T'^*T')^{1/2}(I - \bar{\lambda}T')^{-1}h\| + d\|(T - T')(I - \bar{\lambda}T')^{-1}h\|$$

where $d = \sup_{|\lambda| \leqslant 1} \|(I - \bar{\lambda}T)^{-1}\| < \infty$; the last inequality is a consequence of (3.2). By (3.3), we have

$$\|(I - |\lambda|^2 T^*T)^{1/2}(I - \bar{\lambda}T)^{-1}h\| \leqslant c'(1 + d)\|(I - |\lambda|^2 T'^*T')^{1/2}(I - \bar{\lambda}T')^{-1}h\|$$

which is equivalent to (3.1), with $c = c'(1 + d)$. Therefore, by Lemma 3, $T \overset{H}{\prec} T'$.

COROLLARY 5. a) *If either* $r(T) < 1$ *or* T *is compact, then* $T \overset{Z}{\sim} T'$ *implies* $T \overset{H}{\sim} T'$.

b) *If* $r(T) < 1$ *or T is compact and* $\mathscr{H}$ *is infinite dimensional, the Harnack part of T is not reduced to* $\{T\}$.

Proof. For $r(T) < 1$, Part a) follows readily from Theorem 1. Suppose T is compact and $T \overset{Z}{\sim} T'$. From Lemma 1 (iii) T and T' must have the same eigenvalues of modulus one and the eigenspace corresponding to these eigenvalues reduces both T and T'. Let E be the orthogonal projection onto this eigenspace. Then

$$T = T|_{E\mathscr{H}} \oplus T|_{(I-E)\mathscr{H}}$$

$$T' = T'|_{E\mathscr{H}} \oplus T'|_{(I-E)\mathscr{H}}$$

and

$$T|_{E\mathscr{H}} = T'|_{E\mathscr{H}'} \quad T|_{(I-E)\mathscr{H}} \overset{Z}{\sim} T'|_{(I-E)\mathscr{H}}.$$

But $T|_{(I-E)\mathscr{H}}$ has spectral radius less than 1, so $T|_{(I-E)\mathscr{H}}$ and $T'|_{(I-E)\mathscr{H}}$ are also Harnack equivalent. It follows then from the definition that $T \overset{H}{\sim} T'$.

Part b) is a consequence of Part a) and Proposition 2.

COROLLARY 6. (Foiaş [2]). *$T \overset{H}{\sim} 0$ if and only if $\|T\| < 1$.*

Proof. We have to apply Corollaries 2 and 5.

THEOREM 2. *$T \overset{H}{\prec} 0$ if and only if $r(T) < 1$.*

Proof. By Lemma 3, $T \overset{H}{\prec} 0$ is equivalent to the existence of $c \geqslant 1$ such that

$$I - |\lambda|^2 T^*T \leqslant c^2(I - \lambda T^*)(I - \bar{\lambda} T). \tag{3.4}$$

Suppose (3.4) is valid, but $r(T) = 1$. Assume λ_0 ($|\lambda_0| = 1$) is an approximate eigenvalue of T. Take $h_n \in \mathscr{H}$ such that $\|h_n\| = 1$ and $\|Th_n - \lambda_0 h_n\| \leqslant \frac{1}{n}$. From (3.4) it follows, with $\lambda = \lambda_0 \left(1 - \frac{1}{n}\right)$, that

$$1 - \left(1 - \frac{1}{n}\right)^2 \|Th_n\|^2 \leqslant c^2 \|\lambda_0 h_n - \left(1 - \frac{1}{n}\right) Th_n\|^2 =$$

$$= c^2 \left\|\lambda_0 h_n - Th_n + \frac{1}{n} Th_n\right\|^2 \leqslant$$

$$\leqslant 2c^2(\|\lambda_0 h_n - Th_n\|^2 + \frac{1}{n^2} \|Th_n\|^2).$$

Since $\|Th_n\| \leqslant 1$, we have

$$1 - \left(1 - \frac{1}{n}\right)^2 \leqslant 4c^2 \cdot \frac{1}{n^2}$$

or

$$2n \leqslant 4c^2 + 1 \quad (n = 1, 2, \ldots)$$

which is a contradiction. Therefore (3.4) implies $r(T) < 1$.

Conversely, if $r(T) < 1$, let

$$\delta = \sup_{|\lambda| \leqslant 1} \|(I - T)^{-1}\|^2.$$

Then

$$(I - \lambda T^*)(I - \bar{\lambda} T) \geqslant \delta^{-1}$$

therefore (3.4) holds with $c^2 = \delta$.

N. Suciu remarked in [6] that $T \overset{H}{\prec} 0$ is equivalent to T boundedly absolutely continuous in the sense of Schreiber ([3]). Therefore Theorem 2 is a reformulation of [3], Theorem 3.

The next theorem gives the general relation between Z-domination and Harnack domination.

THEOREM 3. *The following statements are equivalent:*

(i) $T \overset{H}{\prec} T'$,

(ii) *there exists* $c \geqslant 1$ *such that for any* $|\lambda| < 1$ *we have* $T_\lambda \underset{c}{\overset{Z}{\prec}} T'_\lambda$.

Proof. (i) $\Rightarrow$ (ii). Suppose $T \underset{c}{\overset{H}{\prec}} T'$. Fix λ, with $|\lambda| < 1$. Then, since for any polynomial $p(z)$ with $\operatorname{Re} p(z) \geqslant 0$ for $|z| \leqslant 1$, the analytic function $p\left(\dfrac{z - \lambda}{1 - \bar{\lambda} z}\right)$ has positive real part for $|z| \leqslant 1$ and is therefore a uniform limit of polynomials having positive real part for $|z| \leqslant 1$, it follows that

$$\operatorname{Re} p(T_\lambda) \leqslant c^2 \operatorname{Re} p(T'_\lambda).$$

By von Neumann's theorem, T_λ and T'_λ are contractions, therefore $T_\lambda \underset{c}{\overset{H}{\prec}} T'_\lambda$, and hence $T_\lambda \underset{c}{\overset{Z}{\prec}} T'_\lambda$.

(ii) $\Rightarrow$ (i). Suppose conversely that the Z-domination holds for all $|\lambda| < 1$. Inequality (3.1) is equivalent to

$$(3.5) \qquad \begin{aligned} &(I - |\lambda|^2)(I - \lambda T^*)^{-1}(I - \bar{\lambda} T)^{-1} + \frac{|\lambda|^2}{1 - |\lambda|^2}(I - T_\lambda^* T) \leqslant \\ &\leqslant c^2 \left[(I - |\lambda|^2)(I - \lambda T'^*)^{-1}(I - \bar{\lambda} T')^{-1} + \frac{|\lambda|^2}{1 - |\lambda|^2}(I - T_\lambda'^* T_\lambda') \right]. \end{aligned}$$

But $T \overset{Z}{\underset{c}{\prec}} T'$ implies, by Lemma 1 (iii), that there exists c'', such that

$$I - T_\lambda^* T_\lambda \leqslant c''^2(I - T_\lambda'^* T_\lambda'),$$
$$(T_\lambda'^* - T_\lambda^*)(T_\lambda' - T_\lambda) \leqslant c''^2(I - T_\lambda'^* T_\lambda').$$

Since

$$[(I - \bar{\lambda}T')^{-1} - (I - \bar{\lambda}T)^{-1} = \frac{\bar{\lambda}}{1 - |\lambda|^2}(T_\lambda' - T_\lambda)$$

by the obvious inequality for any A and B

$$A^*A \leqslant 2[B^*B + (A - B)^*(A - B)]$$

we have

$$(I - \lambda T^*)^{-1}(I - \bar{\lambda}T)^{-1} \leqslant$$
$$\leqslant 2(I - \lambda T'^*)(I - \bar{\lambda}T')^{-1} + \frac{2|\lambda|^2}{(1 - |\lambda|^2)^2}(T_\lambda'^* - T_\lambda^*)(T_\lambda' - T_\lambda).$$

Therefore we obtain

$$(1 - |\lambda|^2)(I - \lambda T^*)^{-1}(I - \bar{\lambda}T)^{-1} + \frac{|\lambda|^2}{1 - |\lambda|^2}(I - T_\lambda^* T_\lambda) \leqslant$$
$$\leqslant 2(1 - |\lambda|^2)(I - \lambda T'^*)^{-1}(I - \bar{\lambda}T')^{-1} + \frac{2|\lambda|^2}{1 - |\lambda|^2}(T_\lambda'^* - T_\lambda^*)(T_\lambda' - T_\lambda) +$$
$$+ \frac{|\lambda|^2}{1 - |\lambda|^2}(I - T_\lambda^* T_\lambda) \leqslant$$
$$\leqslant 2(1 - |\lambda|^2)(I - \lambda T'^*)^{-1}(I - \bar{\lambda}T')^{-1} + \frac{3c''^2|\lambda|^2}{1 - |\lambda|^2}(I - T_\lambda'^* T_\lambda').$$

Therefore (3.5) is valid with $c^2 = 3c''^2$.

§ 4. HARNACK DOMINATION FOR COMMUTING NORMAL CONTRACTIONS

THEOREM 4. *Let T, T' be commuting normal contractions. Then the following statements are equivalent:*

(i) $T \overset{H}{\prec} T'$,

(ii) $T' \overset{H}{\prec} T$,

(iii) *there exists $c' \geqslant 1$ such that*

$$\frac{1}{c'}(I - |T'|) \leqslant I - |T| \leqslant c'(I - |T'|),$$

$$|T' - T| \leqslant c'(I - |T|).$$

Proof. In view of the essential symmetry of (iii) with respect to T and T', it suffices to prove the equivalence of (i) and (iii).

(i) $\Rightarrow$ (iii) Suppose $T \overset{H}{\underset{c}{\prec}} T'$. By the obvious inequalities

$$I - |\lambda T| \leqslant I - |\lambda|^2 T^* T \leqslant 2(I - |\lambda T|)$$

$$I - |\lambda T'| \leqslant I - |\lambda|^2 T'^* T' \leqslant 2(I - |\lambda T'|)$$

the inequality (3.1) is equivalent, via the commutativity, to the existence of $a \geqslant 1$ such that

$$(I - |\lambda T|)\,|I - \bar{\lambda} T'|^2 \leqslant a(I - |\bar{\lambda} T'|)\,|I - \bar{\lambda} T|^2. \tag{4.1}$$

Since, by the normality of T'

$$I - |\bar{\lambda} T'| \leqslant |I - \bar{\lambda} T'|$$

(4.1) implies

$$(I - |\bar{\lambda} T|)(I - |\bar{\lambda} T'|) \leqslant a|I - \bar{\lambda} T|^2.$$

Let $\lambda = re^{i\theta}$ and fix r. Then we have

$$(I - r|T|)(I - r|T'|) \leqslant a|I - re^{-i\theta} T|^2.$$

Since θ is arbitrary, we can replace, by spectral theory, $e^{i\theta}$ by any unitary operator that commutes with T' and T. Take the unitary operator V for which $T = V|T|$. Then the above inequality leads to

$$(I - r|T|)(I - r|T'|) \leqslant a(I - r|T|)^2$$

hence

$$I - r|T'| \leqslant a(I - r|T|). \tag{4.2}$$

From (4.2) and (4.1) we have

$$(I - r|T|)\,|I - \bar{\lambda} T'|^2 \leqslant a^2(I - r|T|)|I - \bar{\lambda} T|^2$$

hence

$$|I - \lambda T'| \leqslant a\,|I - \bar{\lambda} T|.$$

Then

$$r|T' - T| \leqslant |I - \bar{\lambda} T'| + |I - \bar{\lambda} T| \leqslant$$

$$\leqslant (1 + a)|I - \bar{\lambda} T| = (1 + a)|I - re^{-i\theta} T|.$$

Again by substituting $e^{i\theta}$ by the unitary operator V, we obtain

(4.3) $$r|T'-T|\leqslant(1+a)(I-r|T|).$$

Letting $r\to 1$ in (4.2) and (4.3), we have

$$I-|T'|\leqslant a(I-|T|),$$

$$|T'-T|\leqslant(1+a)(I-|T|).$$

Finally, $T\overset{Z}{\prec}T'$ implies by (2.1) the existence of $a'\geqslant 1$ such that

$$I-|T|\leqslant a'(I-|T'|).$$

The last inequalities prove (iii).

(iii) $\Rightarrow$ (i) If (iii) is satisfied, then

$$I-|\bar\lambda T|=-|\bar\lambda|+1+|\bar\lambda|(I-|T|)\leqslant c'(I-|\bar\lambda T'|)$$

and

$$|I-\bar\lambda T'|\leqslant|I-\bar\lambda T|+|\bar\lambda||T'-T|\leqslant$$

$$\leqslant|I-\bar\lambda T|+c'|\bar\lambda|(I-|T|)\leqslant(1+c')|I-\bar\lambda T|.$$

Therefore we have

$$(I-|\bar\lambda T|)|I-\bar\lambda T'|^2\leqslant c'(1+c')^2(I-|\bar\lambda T'|)|I-\bar\lambda T|^2$$

which proves (i).

COROLLARY 7. *If T, T' are commuting normal contractions, then $T\overset{H}{\prec}T'$ implies $T\overset{H}{\sim}T'$.*

COROLLARY 8. *Let T be a normal nonunitary contraction. Then:*

(a) *the Harnack part of T is not reduced to $\{T\}$.*

(b) *Suppose the spectral measure of each set $E_r=\{z\in\mathbf{C}\,|\,r<|z|<1\}$ $(0<r<1)$ is not trivial. Then the Z-part of T is strictly larger than its H-part.*

Proof. (a) We can suppose, by Corollary 5 (b), that $\|T\|=1$. By assumption, there exists $0<r<1$ such that the spectral projection of T corresponding to the disc $D_r=\{z\in\mathbf{C}\,|\,|z|<r\}$ is nontrivial. Then put

$$\varphi(z)=z+\frac{1-r}{2}\chi_{D_r}(z)$$

where χ_{D_r} is the characteristic function of D_r. Then $T' = \varphi(T) \neq T$, and we can easily check the conditions of Theorem 4 (iii) in order to obtain $T \overset{H}{\sim} T'$.

(b) If we put

$$\psi(z) = z[|z| + \mathrm{i}(1 - |z|^2)^{1/2}], \ T' = \psi(T)$$

then again we can easily check, by Lemma 1(iii) and Theorem 4 (iii), that $T' \overset{Z}{\sim} T$, $T' \overset{H}{\nsim} T$.

We have seen that if T is either isometry or coisometry then its Harnack part is trivial (i.e. equal to $\{T\}$). On the other hand, if $r(T) < 1$ or T is compact, or normal and nonunitary, then its Harnack part is not trivial. It seems interesting to give necessary and/or sufficient conditions for a contraction to have trivial Harnack part.

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