J. OPERATOR THEORY
2(1979), 247–275

STABILITY OF THE INDEX OF A COMPLEX OF BANACH SPACES

F.-H. VASILESCU

1. PRELIMINARIES

Let X and Y be two Banach spaces over the complex field $\mathbf{C}$. We denote by $\mathscr{C}(X, Y)$ the set of all linear and closed operators, defined on linear submanifolds of X, assigning values in Y. The subset of those operators of $\mathscr{C}(X, Y)$ which are everywhere defined, hence continuous, will be denoted by $\mathscr{B}(X, Y)$. We write $\mathscr{C}(X)$ and $\mathscr{B}(X)$ for $\mathscr{C}(X, X)$ and $\mathscr{B}(X, X)$, respectively. We put also $X^* = \mathscr{B}(X, \mathbf{C})$, i.e. the *dual* space of X.

For every $S \in \mathscr{C}(X, Y)$ we denote by $D(S)$, $R(S)$ and $N(S)$ the *domain of definition*, the *range* and the *null-space* of S, respectively. We recall that the *index* of S is given by

$$\text{ind } S = \dim N(S) - \dim Y/R(S), \tag{1.1}$$

provided that $R(S)$ is closed in Y and at least one of the numbers $\dim N(S)$, $\dim Y/R(S)$ is finite. For every complex vector space M we denote by $\dim M$ the *algebraic dimension* of M. If we represent the action of S by the sequence

$$0 \to X \overset{S}{\to} Y \to 0, \tag{1.2}$$

not forgetting that S acts only on $D(S) \subset X$, then the number (1.1) may be interpreted as the *Euler characteristic* of the complex (1.2) (see [9] or [7]). This remark suggests a more general definition of the index, which will be presented in the sequel.

Consider a countable family of Banach spaces $\{X^p\}_{p=-\infty}^{+\infty}$ and a family of operators $\alpha^p \in \mathscr{C}(X^p, X^{p+1})$ such that $R(\alpha^p) \subset N(\alpha^{p+1})$, for each integer p. We represent them by the sequence

$$\cdots \xrightarrow{\alpha^{p-1}} X^p \xrightarrow{\alpha^p} X^{p+1} \xrightarrow{\alpha^{p+1}} \cdots \tag{1.3}$$

and we say that (1.3) is a *(cochain-) complex of Banach spaces*. The sequence $(X, \alpha) = (X^p, \alpha^p)_{p=-\infty}^{+\infty}$ can be associated with the cohomology sequence $H(X, \alpha) =$

$=(H^p(X,\alpha))_{p=-\infty}^{+\infty}$, where $H^p(X,\alpha)=N(\alpha^p)/R(\alpha^{p-1})$. Let us assume that $\dim H^p(X,\alpha)<\infty$ for every integer p and that $\dim H^p(X,\alpha)=0$ for all but a finite number of indices. Then we may define

$$\text{ind}\,(X,\alpha)=\sum_{p=-\infty}^{+\infty}(-1)^p\dim H^p(X,\alpha). \tag{1.4}$$

The number ind (X,α), which may be interpreted as the Euler characteristic of the complex (1.3), will be called shortly the *index* of the complex (X,α).

It is easy to imagine a trick which makes possible the reduction of the case of unbounded operators $\{\alpha^p\}$ to the case of bounded ones (see the proof of Lemma 2.5 below), and we use occasionally such a procedure. However, we do not generalize that procedure since it involves the transformation of the original topology into a rather artificial one and some estimations become less precise.

Let us discuss the significance of the number (1.4) in the finite-dimensional case. If $\dim X^p<\infty$ for every p, $\alpha^p\in\mathscr{B}(X^p,X^{p+1})$ and $\dim H^p(X,\alpha)=0$ if $p<0$ and $p>n$ then one can easily see that

$$\text{ind}\,(X,\alpha)=\sum_{p=0}^{n}(-1)^p\dim X^p-\dim R(\alpha^{-1})+(-1)^{n+1}\dim X^n/N(\alpha^n). \tag{1.5}$$

This remark shows that for arbitrary Banach spaces the number (1.4) cannot be, in general, invariant under compact perturbations, as a well-behaved index is expected to be. When $\alpha^{-1}=0$ and $\alpha^n=0$, the number (1.5) depends just on the geometry of the spaces, therefore only a certain type of complexes of Banach spaces, namely of finite length, is significant from the point of view of the classical stability theorems of the index [3], at least for compact perturbations. However, the number (1.4) makes sense and is stable under small perturbations for larger conditions (see Theorem 2.12).

When dealing with complexes of Banach spaces of the form $(X,\alpha)=(X^p,\alpha^p)_{p=-\infty}^{+\infty}$ with $X^p=0$ for $p<0$ and $p>n$ (i.e. complexes of finite length), we write them as $(X,\alpha)=(X^p,\alpha^p)_{p=0}^{n}$, using freely the assumptions $X^p=0$ for $p\leqslant -1$ or $p\geqslant n+1$ and $\alpha^p=0$ for $p\leqslant -1$ or $p\geqslant n$.

1.1. Definition. Let $(X,\alpha)=(X^p,\alpha^p)_{p=0}^{n}$ be a complex of Banach spaces. If $R(\alpha^{n-1})$ is closed in X^n, $\dim H^p(X,\alpha)<\infty$ for $1\leqslant p\leqslant n-1$ and at least one of the numbers $\dim H^0(X,\alpha)$, $\dim H^n(X,\alpha)$ is finite then (X,α) will be called a *semi-Fredholm complex of Banach spaces.*

When $\dim H^p(X,\alpha)<\infty$ for $p=0,1,\ldots,n$ then (X,α) is called a *Fredholm complex of Banach spaces.*

We specify that for a semi-Fredholm complex of Banach spaces (X,α) the number (1.4), possibly infinite, still makes sense and is called the *index* of (X,α).

Note that if $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ is a semi-Fredholm complex of Banach spaces then $R(\alpha^p)$ is closed for all $p = 0, 1, \ldots, n-1$. Indeed, $R(\alpha^{n-1})$ is closed by definition and $R(\alpha^p)$ is closed by the condition $\dim H^p(X, \alpha) < \infty$, for $1 \leqslant p \leqslant n-1$ (see [3] or [9]).

In the next two sections of this work we shall obtain extensions of the usual stability theorems of the index [3], valid for a semi-Fredholm complexes.

The fourth section contains some consequences of the stability theorems of the index for finite systems of closed operators, commuting in a sense which will be specified.

There is a consensus of the specialists (R. G. Douglas, D. Voiculescu etc.) that a suitable notion of index for commuting systems of bounded operators on Hilbert spaces must be connected with the Euler characteristic of an associated complex (this was one of the facts which inspired our Definition 1.1). An approach to the Fredholm theory in this context has been already developed in [2]. With these conditions, the index of a commuting system turns out to be the index of a certain operator, therefore the stability theorems can be reduced to the classical ones. As a matter of fact, the index of a Fredholm complex of Hilbert spaces is always equal to the index of a certain operator, as our Theorem 3.8 shows. However, it seems that the case of commuting operators acting in Banach spaces (and, in general, the case of complexes of Banach spaces) cannot be reduced to the case of one operator, while our methods still work.

Let us also mention that the Cauchy-Riemann complex of the $\bar{\partial}$-operator [4] is semi-Fredholm in certain conditions (this was another fact which led us to Definition 1.1) and an application related to this result ends the present work.

2. THE STABILITY UNDER SMALL PERTURBATIONS

In this section we investigate the stability under small perturbations of the index of a semi-Fredholm complex of Banach spaces.

Let X and Y be Banach spaces and $S \in \mathscr{C}(X, Y)$. We recall that the *reduced minimum modulus* of $S(\neq 0)$ is given by

$$\gamma(S) = \inf_{\substack{x \in D(S) \\ x \notin N(S)}} \frac{\|Sx\|}{\mathrm{d}(x, N(S))},$$

where "d" stands for the *distance*. It is known [3] that $R(S)$ is closed if and only if $\gamma(S) > 0$. In this case there is a continuous operator

$$\hat{S}^{-1} : Sx \mapsto x + N(S) \quad (x \in D(S))$$

which maps $R(S)$ into $X/N(S)$ and with $\|\hat{S}^{-1}\| = \gamma(S)^{-1}$.

When $S \subset 0$ then one defines $\gamma(S) = \infty$.

2.1. LEMMA. *Let X, Y and Z be Banach spaces, $S \in \mathscr{C}(X, Y)$, $T \in \mathscr{C}(Y, Z)$, with $R(S) = N(T)$ and $R(T)$ closed. Assume that $A : D(S) \to Y$, $B : D(T) \to Z$ are bounded operators and $R(\widetilde{S}) \subset N(\widetilde{T})$, where $\widetilde{S} = S + A$, $\widetilde{T} = T + B$. If*

$$\|A\| \gamma(S)^{-1} + \|B\| \gamma(T)^{-1} + \|A\| \|B\| \gamma(S)^{-1} \gamma(T)^{-1} < 1 \tag{2.1}$$

then $R(\widetilde{S}) = N(\widetilde{T})$.

Proof. Take $r_S > \gamma(S)^{-1}$ and $r_T > \gamma(T)^{-1}$ such that

$$\|A\| r_S + \|B\| r_T + \|A\| \|B\| r_S r_T < 1, \tag{2.2}$$

which is possible by (2.1). Consider then $y \in N(\widetilde{T})$ arbitrary. We shall construct an element $x \in X$ such that $\widetilde{S}x = y$. We shall use a closed graph type procedure inspired from [10, Lemma 2.1]. Choose first $y' \in Y$ such that

$$Ty' = Ty$$

and

$$\|y'\| \leqslant r_T \|Ty\| = r_T \|By\| \leqslant \|B\| r_T \|y\|.$$

Since $y - y' \in N(T)$, there is an $x_1 \in X$ such that $y - y' = Sx_1$; moreover, we may suppose that

$$\|x_1\| \leqslant r_S \|y - y'\| \leqslant r_S(1 + \|B\| r_T) \|y\|.$$

Let us define $y_1 = y - \widetilde{S}x_1$. Then we have

$$\|y_1\| \leqslant \|y - Sx_1\| + \|Ax_1\| \leqslant \|B\| r_T \|y\| + \|A\| \|x_1\| \leqslant$$
$$\leqslant (\|A\| r_S + \|B\| r_T + \|A\| \|B\| r_S r_T) \|y\|.$$

Note that $y_1 \in N(\widetilde{T})$, therefore we may apply the same construction for y_1 and find $y_2 \in N(T)$ and $x_2 \in X$ such that $y_2 = y_1 - \widetilde{S}x_2 = y - \widetilde{S}(x_1 + x_2)$. We obtain in general the sequences $\{y_k\}_k \subset N(\widetilde{T})$ and $\{x_k\}_k \subset D(\widetilde{S})$ such that $y_k = y - \widetilde{S}(x_1 + \ldots + x_k)$. Moreover,

$$\begin{aligned} &\|y_k\| \leqslant (\|A\| r_S + \|B\| r_T + \|A\| \|B\| r_S r_T)^k \|y\| \\ &\|x_k\| \leqslant r_S(1 + \|B\| r_T)(\|A\| r_S + \|B\| r_T + \|A\| \|B\| r_S r_T)^{k-1} \|y\|, \end{aligned} \tag{2.3}$$

for any natural k. By the relation (2.2) the series $\sum_k x_k$ is convergent in X and let x be its sum. As $y_k \to 0$ when $k \to \infty$, we obtain that $\sum_k \widetilde{S}x_k$ is also convergent, hence $x \in D(\widetilde{S})$ and $y = \widetilde{S}x$.

2.2. COROLLARY. *Consider* $S, T, A, B, \widetilde{S}, \widetilde{T}$ *as in Lemma* 2.1. *If* $r_S \geqslant \gamma(S)^{-1}$, $r_T \geqslant \gamma(T)^{-1}$, $\varepsilon_A \geqslant \|A\|$, $\varepsilon_B \geqslant \|B\|$ *and* $\varepsilon_A r_S + \varepsilon_B r_T + \varepsilon_A \varepsilon_B r_S r_T < 1$ *then*

$$\gamma(\widetilde{S})^{-1} \leqslant \frac{r_S(1 + \varepsilon_B r_T)}{2 - (1 + \varepsilon_A r_S)(1 + \varepsilon_B r_T)}. \tag{2.4}$$

Proof. Assuming momentarily $r_S > \gamma(S)^{-1}$ and $r_T > \gamma(T)^{-1}$, we obtain that the solution x of the equation $y = \widetilde{S}x$ constructed in the previous lemma satisfies the estimations

$$\|x\| \leqslant \sum_{k=1}^{\infty} \|x_k\| \leqslant \frac{r_S(1 + \varepsilon_B r_T)}{2 - (1 + \varepsilon_A r_S)(1 + \varepsilon_B r_T)} \|y\|,$$

obtained from (2.3). As r_S, r_T are arbitrarily close to $\gamma(S)^{-1}$, $\gamma(T)^{-1}$ respectively, we infer easily the relation (2.4).

2.3. COROLLARY. *Consider* $S, T, A, B, \widetilde{S}, \widetilde{T}$ *as in Lemma* 2.1. *Then there is a constant* $\varepsilon_0(S, T) > 0$ *such that if* $\|A\| < \varepsilon_0(S, T)$ *and* $\|B\| < \varepsilon_0(S, T)$ *then the inclusion* $R(\widetilde{S}) \subset N(\widetilde{T})$ *is equivalent to the equality* $R(\widetilde{S}) = N(\widetilde{T})$.

Proof. If at least one of the operators S, T is non-null then we can choose

$$\varepsilon_0(S, T) = (\sqrt{2} - 1) \min \{\gamma(S), \gamma(T)\}.$$

Indeed, if $\theta = \max \{\gamma(S)^{-1}, \gamma(T)^{-1}\}$ and $\varepsilon \geqslant \|A\|$, $\varepsilon \geqslant \|B\|$ then the condition $2\varepsilon\theta + \varepsilon^2\theta^2 < 1$ implies the condition (2.1), therefore we may take

$$\varepsilon_0(S, T) = \sup \{\varepsilon > 0;\ 2\varepsilon\theta + \varepsilon^2\theta^2 < 1\} = (\sqrt{2} - 1)\theta^{-1}.$$

If both S and T are null then $\varepsilon_0(S, T)$ may be any positive number.

The bounded perturbations from Lemma 2.1 may be replaced with *relatively bounded perturbations* in the sense of the following

2.4. DEFINITION. Consider $S \in \mathscr{C}(X, Y)$, $T \in \mathscr{C}(Y, Z)$ and A a linear operator with $D(A) \supset D(S)$ and $R(A) \subset D(T)$. We say that A is (S, T)*-bounded* if

$$\|Ax\| + \|TAx\| \leqslant a\|x\| + b\|Sx\|, \quad x \in D(S), \tag{2.5}$$

where a, b are nonnegative constants.

The operator A is $(S, 0)$-bounded if and only if A is S-bounded in the sense of [3, Ch. IV].

Let us also note that the operator A from Lemma 2.1 satisfies the evaluation

$$\|Ax\| + \|TAx\| \leqslant (\|A\| + \|BA\|)\,\|x\| + \|B\|\,\|Sx\|, \quad x \in D(S),$$

therefore A is (S, T)-bounded.

We shall obtain a variant of Lemma 2.1 for relatively bounded perturbations.

2.5. LEMMA. *Let $(X^p, \alpha^p)_{p=0}^3$ be a complex of Banach spaces with $R(\alpha^0) = N(\alpha^1)$ and $R(\alpha^1)$ closed. Assume that β^p is an (α^p, α^{p+1})-bounded operator ($p = 0, 1$) satisfying $R(\alpha^0 + \beta^0) \subset N(\alpha^1 + \beta^1)$ and*

$$\|\beta^p x\| + \|\alpha^{p+1}\beta^p x\| \leqslant a_p\|x\| + b_p\|\alpha^p x\|, \quad x \in D(\alpha^p). \tag{2.6}$$

If $c_p = \max\{a_p, b_p\}$ and

$$c_0(1 + \gamma(\alpha^0)^{-1}) + c_1(1 + \gamma(\alpha^1)^{-1}) + c_0c_1(1 + \gamma(\alpha^0)^{-1})(1 + \gamma(\alpha^1)^{-1}) < 1 \tag{2.7}$$

then $R(\alpha^0 + \beta^0) = N(\alpha^1 + \beta^1)$.

Proof. The present statement can be reduced to the case of Lemma 2.1 by a well-known procedure. Namely, consider $\hat{X}^p = D(\alpha^p)$ and define on $\hat{X}^p$ the norm

$$\|x\|_0 = \|x\| + \|\alpha^p x\|, \qquad x \in \hat{X}^p. \tag{2.8}$$

Then $\hat{X}^p$, endowed with the norm (2.8), becomes a Banach space ($p = 0, 1, 2$). Moreover, if $\hat{\alpha}^p : \hat{X}^p \to \hat{X}^{p+1}$ is the operator induced by α^p then $\|\hat{\alpha}^p\|_0 \leqslant 1$. Analogously, if $\hat{\beta}^p : \hat{X}^p \to \hat{X}^{p+1}$ is the operator induced by β^p then, by (2.6), we obtain that $\|\hat{\beta}^p\|_0 \leqslant$ $\leqslant c_p (p = 0, 1)$.

Note also the equalities

$$\gamma(\hat{\alpha}^p) = \inf_{\substack{x \in \hat{X}^p \\ x \notin N(\hat{\alpha}^p)}} \frac{\|\hat{\alpha}^p x\|_0}{d(x, N(\hat{\alpha}^p))} = \inf_{\substack{x \in D(\alpha^p) \\ x \notin N(\alpha^p)}} \frac{\|\alpha^p x\|}{\inf\limits_{y \in N(\alpha^p)} \|x - y\| + \|\alpha^p x\|} = \frac{\gamma(\alpha^p)}{1 + \gamma(\alpha^p)}.$$

Then the condition (2.7) implies the inequality

$$\|\hat{\beta}^0\|\gamma(\hat{\alpha}^0)^{-1} + \|\hat{\beta}^1\|\,\gamma(\hat{\alpha}^1)^{-1} + \|\hat{\beta}^0\|\,\|\hat{\beta}^1\|\,\gamma(\hat{\alpha}^0)^{-1}\,\gamma(\hat{\alpha}^1)^{-1} < 1,$$

which in turn implies, by Lemma 2.1, the equality

$$R(\alpha^0 + \beta^0) = N(\alpha^1 + \beta^1).$$

The proof of Lemma 2.5 shows that we can reduce the case of relatively bounded perturbations to the case of bounded perturbations. Moreover, actually the perturbed operators may be supposed bounded. Such a reduction will be applied in the next section, when the estimations of the norms are not interesting for the final results. However, we prefer in general the conditions of Lemma 2.1, which provide better estimations (compare, for instance, (2.1) and (2.7)) and a simplified language.

For any pair of closed subspaces M and N in the Banach space X we set

$$\delta(M, N) = \sup_{\substack{x \in M \\ \|x\| \leqslant 1}} \mathrm{d}(x, N)$$

and $\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$. When $\hat{\delta}(M, N) < 1$ then $\dim M = \dim N$ (see [5] or [3]).

2.6. LEMMA. *If $S \in \mathscr{C}(X, Y)$, $A : D(S) \to Y$ is a bounded operator, $\widetilde{S} = S + A$ and $R(\widetilde{S})$ is closed in Y then*

$$\delta(N(S), N(\widetilde{S})) \leqslant \|A\| \, \gamma(\widetilde{S})^{-1}.$$

Proof. Taking $r > \gamma(\widetilde{S})^{-1}$ and $x \in N(S)$ arbitrary then we can find $v \in N(\widetilde{S})$ such that

$$\|x + v\| \leqslant r\|\widetilde{S}x\| = r\|Ax\| \leqslant r\|A\| \, \|x\|,$$

therefore

$$\mathrm{d}(x, N(\widetilde{S})) \leqslant \|A\| \, \gamma(\widetilde{S})^{-1}\|x\|.$$

If X and Y are two Banach spaces then we denote by $X \oplus Y$ their *direct sum*, endowed with the norm $\|x \oplus y\|^2 = \|x\|^2 + \|y\|^2$ ($x \in X, y \in Y$). We identify sometimes X with $X \oplus 0$ and Y with $0 \oplus Y$.

2.7. LEMMA. *Consider $S \in \mathscr{C}(X, Y)$ and take a finite dimensional Banach space M and $A \in \mathscr{B}(M, Y)$. Define then $S_A \in \mathscr{C}(X \oplus M, Y)$ by the relation $S_A(x \oplus v) = Sx + Av$, for every $x \in D(S)$ and $v \in M$. Then we have*

$$\dim N(S_A)/N(S) + \dim R(S_A)/R(S) = \dim M.$$

Proof. Let us write $R(A) = N_1 + N_2$, where $N_1 = R(S) \cap R(A)$ and $N_1 \cap N_2 = 0$. Clearly, $R(S_A) = R(S) + N_2$, hence $\dim R(S_A)/R(S) = \dim N_2$.

Consider then $M_1 = A^{-1}(N_1)$, $M_2 \subset A^{-1}(N_2)$ such that $M_1 + M_2 = M$, $M_1 \cap M_2 = 0$ and with $A : M_2 \to N_2$ an isomorphism. Take $x \in D(S)$, $v_1 \in M_1$ and $v_2 \in M_2$ such that $S_A(x \oplus (v_1 + v_2)) = 0 = Sx + Av_1 + Av_2$. Then $Av_2 = 0$, thus $v_2 = 0$. We can write

$$N(S_A) = \{x \oplus v_1;\ x \in D(S),\ v_1 \in M_1,\ Sx + Av_1 = 0\}.$$

If we consider the space $X/N(S)$ and the linear operator

$$\hat{S}^{-1} : Sx \to x + N(S), \quad x \in D(S),$$

we infer the equality

$$N(S_A)/N(S) = \{(\hat{S}^{-1}Av_1 + N(S), v_1);\ v_1 \in M_1\},$$

showing that $N(S_A)/N(S)$ is isomorphic to M_1. We conclude that

$$\dim N(S_A)/N(S) + \dim R(S_A)/R(S) = \dim M_1 + \dim M_2 = \dim M.$$

Let us mention that a variant of this lemma can be found in [5], for S injective.

2.8. Definition. Let $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ be a complex of Banach spaces and $\{Y, \gamma\} = \{Y^p, \gamma^p\}_{p=0}^n$ a system with the following properties: Each Y^p is a finite dimensional Banach space and each $\gamma^p \in \mathscr{B}(Y^p, X^{p+1})$. Let us define $\beta^p(x \oplus y) = \alpha^p x + \gamma^p y$, where $x \in D(\alpha^p)$ and $y \in Y^p$, and assume that $(X \oplus Y, \beta) = (X^p \oplus Y^p, \beta^p)_{p=0}^n$ is a complex of Banach spaces. In this case we say that $(X \oplus Y, \beta)$ is an *extension* of (X, α) by the system $\{Y, \gamma\}$.

2.9. Proposition. *Let $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ be a semi-Fredholm complex of Banach spaces. If $(X \oplus Y, \beta)$ is an extension of (X, α) by the system $\{Y, \gamma\} = \{Y^p, \gamma^p\}_{p=0}^n$ then $(X \oplus Y, \beta)$ is also semi-Fredholm and*

$$\operatorname{ind}(X \oplus Y, \beta) = \operatorname{ind}(X, \alpha) + \sum_{p=0}^{n} (-1)^p \dim Y^p.$$

Proof. By Lemma 2.7, it will be enough to prove the assertion when (X, α) is actually Fredholm.

Note that for an arbitrary p we have the equalities

$$\dim N(\beta^p)/R(\beta^{p-1}) = \dim N(\beta^p)/R(\alpha^{p-1}) - \dim R(\beta^{p-1})/R(\alpha^{p-1}) =$$

$$= \dim N(\alpha^p)/R(\alpha^{p-1}) + \dim N(\beta^p)/N(\alpha^p) - \dim R(\beta^{p-1})/R(\alpha^{p-1}).$$

By Lemma 2.7 we have also

$$\dim N(\beta^p)/N(\alpha^p) + \dim R(\beta^p)/R(\alpha^p) = \dim Y^p.$$

By summing up these equalities multiplied with suitable powers of -1 we obtain

$$\operatorname{ind}(X \oplus Y, \beta) = \operatorname{ind}(X, \alpha) + \sum_p (-1)^p (\dim N(\beta^p)/N(\alpha^p) -$$

$$-\dim R(\beta^{p-1})/R(\alpha^{p-1})) = \operatorname{ind}(X, \alpha) + \sum_{p=0}^{n} (-1)^p \dim Y^p,$$

which completes the proof.

Consider a complex of Banach spaces $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$. If $D(\alpha^p)$ is dense in X^p then the adjoint α^{p*} is defined and belongs to $\mathscr{C}(X^{p+1*}, X^{p*})$. Moreover, $R(\alpha^{p+1*}) \subset N(\alpha^{p*})$, therefore

$$0 \to X^{n*} \xrightarrow{\alpha^{n-1*}} X^{n-1*} \xrightarrow{\alpha^{n-2*}} \cdots \xrightarrow{\alpha^{0*}} X^{0*} \to 0$$

is again a complex of Banach spaces; it will be denoted by (X^*, α^*) and called the *dual* of (X, α).

2.10. LEMMA. *Let* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ *be a complex of Banach spaces with* $D(\alpha^p)$ *dense in* X^p *for every* p. *Then* (X, α) *is semi-Fredholm if and only if the dual complex* (X^*, α^*) *is semi-Fredholm. In this case* $\operatorname{ind}(X^*, \alpha^*) = (-1)^n \operatorname{ind}(X, \alpha)$.

Proof. Assume first that (X, α) is semi-Fredholm. Then $R(\alpha^p)$ is closed for every p, therefore $R(\alpha^{p*}) = N(\alpha^p)^\perp$ and $N(\alpha^{p*}) = R(\alpha^p)^\perp$ (where "$\perp$" denotes, as usually, the *annihilator* of the corresponding subspace in the dual). From simple arguments of duality we have that the space

$$N(\alpha^{p-1*})/R(\alpha^{p*}) = R(\alpha^{p-1})^\perp/N(\alpha^p)^\perp$$

is isomorphic to the space $(N(\alpha^p)/R(\alpha^{p-1}))^*$, therefore we can write

$$\operatorname{ind}(X^*, \alpha^*) = \sum_{p=0}^{n} (-1)^p \dim N(\alpha^{n-p-1*})/R(\alpha^{n-p*}) = (-1)^n \operatorname{ind}(X, \alpha).$$

The converse implication is similar.

2.11. THEOREM. *Assume that* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ *is a semi-Fredholm complex of Banach spaces. Then there exists a positive number* $\varepsilon(X, \alpha)$ *such that if* $\gamma^p \colon D(\alpha^p) \to X^{p+1}$ *is bounded,* $\|\gamma^p\| < \varepsilon(X, \alpha)$, $\beta^p = \alpha^p + \gamma^p$ $(p = 0, 1, \ldots, n)$ *and* $(X, \beta) = (X^p, \beta^p)_{p=0}^n$ *is a complex of Banach spaces then* $\dim H^p(X, \beta) \leqslant \dim H^p(X, \alpha)$ *for every* p *and* $\operatorname{ind}(X, \beta) = \operatorname{ind}(X, \alpha)$.

Proof. Notice first that we may suppose $\dim H^n(X, \alpha) < \infty$. Indeed, there is no loss of generality in assuming that $D(\alpha^p)$ is dense in X^p for every p; if $\dim H^n(X, \alpha) = \infty$, by passing to the dual complex we obtain, by Lemma 2.10, the desired situation. We shall obtain our theorem from a more general statement.

2.12. THEOREM. *Assume that* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^\infty$ *is a complex of Banach spaces with* $\dim H^p(X, \alpha) < \infty$ *for every* $p \geqslant 1$. *Assume also that* $H^p(X, \alpha) = 0$ *for all but a finite number of indices. Then there exists a sequence of positive numbers* $\{\varepsilon_p\}_{p\geqslant 0}$ *such that if* $\gamma^p : D(\alpha^p) \to X^{p+1}$ *is bounded,* $\|\gamma^p\| < \varepsilon_p$, $\beta^p = \alpha^p + \gamma^p$ *and* $(X, \beta) = (X^p, \beta^p)_{p=0}^\infty$ *is a complex of Banach spaces then* $\dim H^p(X, \beta) \leqslant \dim H^p(X, \alpha)$ *for every* p *and* $\operatorname{ind}(X, \beta) = \operatorname{ind}(X, \alpha)$.

Proof. Let us define the number

$$m(X, \alpha) = \min \{m; H^p(X, \alpha) = 0, \ p \geqslant m\}.$$

We shall obtain the assertion by an inductive argument with respect to $m(X, \alpha)$.

Assume first that $m(X, \alpha) = 0$. Then we take

$$\varepsilon_p \leqslant \min \{\varepsilon_0(\alpha^{p-1}, \alpha^p), \ \varepsilon_0(\alpha^p, \alpha^{p+1})\}, \quad p = 0, 1, 2, \dots, \tag{2.9}$$

where $\varepsilon_0(\alpha^p, \alpha^{p+1})$ is given by Corollary 2.3. If we have $\|\gamma^p\| < \varepsilon_p$ for every p then by Corollary 2.3 we infer that $H^p(X, \beta) = 0$, hence ind $(X, \beta) =$ ind $(X, \alpha) = 0$.

The case $m(X, \alpha) = 1$ needs a special treatment. Take first $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ small enough in order to have

$$0 < \frac{\varepsilon_0 \gamma(\alpha^0)^{-1}(1 + \varepsilon_1 \gamma(\alpha^1)^{-1})}{2 - (1 + \varepsilon_0 \gamma(\alpha^0)^{-1})(1 + \varepsilon_1 \gamma(\alpha^1)^{-1})} < 1,$$

and $\varepsilon_0 \gamma(\alpha^0)^{-1} < 1$. Then from Lemma 2.6 and the relation (2.4) we obtain that $\hat{\delta}(N(\alpha^0), N(\beta^0)) < 1$, therefore $\dim N(\beta^0) = \dim N(\alpha^0)$ (see [6] or [3]). If we take ε_p satisfying (2.9) for $p \geqslant 1$ then we have $H^p(X, \beta) = 0$ by Lemma 2.3, hence the assertion is valid in this case.

Suppose now that the assertion is true for $m(X, \alpha) = m \geqslant 1$ and let us obtain it for $m(X, \alpha) = m + 1$. We have therefore $H^p(X, \alpha) = 0$ if $p \geqslant m + 1$ and $\dim H^m(X, \alpha) = n_m < \infty$. Let us write $R(\alpha^{m-1}) + M = N(\alpha^m)$, where $\dim M = n_m$. We define the space $\widetilde{X}^{m-1} = X^{m-1} \oplus M$ and the operator

$$\tilde{\alpha}^{m-1}(x \oplus v) = \alpha^{m-1}(x) + v, \quad x \in D(\alpha^{m-1}), \quad v \in M.$$

It is clear that $R(\tilde{\alpha}^{m-1}) = N(\alpha^m)$, hence if $\widetilde{X}^p = X^p$ and $\tilde{\alpha}^p = \alpha^p$ for $p \neq m - 1$ then $(\widetilde{X}, \tilde{\alpha}) = (\widetilde{X}^p, \tilde{\alpha}^p)_{p=0}^\infty$ has the property $m(\widetilde{X}, \tilde{\alpha}) = m$. Let $\{\tilde{\varepsilon}_p\}_{p \geqslant 0}$ be the sequence given by the induction hypothesis for $(\widetilde{X}, \tilde{\alpha})$. By changing, if necessary, $\tilde{\varepsilon}_m$, $\tilde{\varepsilon}_{m+1}$ with smaller positive numbers, we may assume that there exists $\delta > 0$ with the properties

$$\delta > \frac{\tilde{\varepsilon}_m \gamma(\alpha^m)^{-1}(1 + \tilde{\varepsilon}_{m+1} \gamma(\alpha^{m+1})^{-1})}{2 - (1 + \tilde{\varepsilon}_m \gamma(\alpha^m)^{-1})(1 + \tilde{\varepsilon}_{m+1} \gamma(\alpha^{m+1})^{-1})} > 0 \tag{2.10}$$

and $n_m \delta < \tilde{\varepsilon}_{m-1}$. We define then $\varepsilon_p = \tilde{\varepsilon}_p$ $(p \neq m - 1)$ and take $\varepsilon_{m-1} \leqslant (\tilde{\varepsilon}_{m-1}^2 - n_m^2 \delta^2)^{1/2}$.

Consider now $\gamma^p : D(\alpha^p) \to X^{p+1}$ with $\|\gamma^p\| < \varepsilon_p$ and $\beta^p = \alpha^p + \gamma^p$. We shall construct a map $\widetilde{\beta}^{m-1}$ on $D(\alpha^{m-1}) \oplus M$ such that if $\tilde{\gamma}^{m-1} = \widetilde{\beta}^{m-1} - \tilde{\alpha}^{m-1}$ then $\|\tilde{\gamma}^{m-1}\| < \tilde{\varepsilon}_{m-1}$. For, take a basis $\{v_1, \dots, v_{n_m}\}$ of M with the property that if $v = \sum_j \lambda_j v_j$ then $|\lambda_j| \leqslant \|v\|$ $(j = 1, \dots, n_m)$. The existence of such a basis follows from the well-known *lemma of Auerbach* (see, for instance, [1]). Take then $\tilde{v}_j \in N(\beta^m)$ such that

$\|v_j - \tilde{v}_j\| < \delta\ (j = 1, \ldots, n_m)$, which is possible according to (2.10), (2.4) and Lemma 2.6. Then for all $x \in D(\alpha^{m-1})$ and $v \in M$, $v = \sum_j \lambda_j v_j$, we define

$$\widetilde{\beta}^{m-1}(x \oplus v) = \beta^{m-1}(x) + \sum_{j=1}^{n_m} \lambda_j \tilde{v}_j.$$

Note that we can write

$$\|\tilde{\gamma}^{m-1}(x \oplus v)\| \leqslant \|\gamma^{m-1}x\| + \left\| \sum_{j=1}^{n_m} \lambda_j(v_j - \tilde{v}_j) \right\| <$$

$$< \varepsilon_{m-1}\|x\| + n_m\delta\|v\| \leqslant (\varepsilon_{m-1}^2 + n_m^2\delta^2)^{1/2} \|x \oplus v\| \leqslant \tilde{\varepsilon}_{m-1}\|x \oplus v\|.$$

If we put $\widetilde{\beta}^p = \beta^p$ for $p \neq m - 1$ then, by the induction hypothesis, the complex $\widetilde{X}, \widetilde{\beta}) = (\widetilde{X}^p, \widetilde{\beta}^p)_{p=0}^\infty$ satisfies $\dim H^p(\widetilde{X}, \widetilde{\beta}) \leqslant \dim H^p(\widetilde{X}, \tilde{\alpha})$ for every p and $\text{ind}(\widetilde{X}, \widetilde{\beta}) =$ $(= \text{ind}\,(\widetilde{X}, \tilde{\alpha})$. Since by Lemmas 2.1 and 2.7

$$\dim N(\beta^m)/R(\beta^{m-1}) = \dim R(\widetilde{\beta}^{m-1})/R(\beta^{m-1}) \leqslant n_m$$

and by the induction hypothesis

$$\dim N(\beta^{m-1})/R(\beta^{m-2}) \leqslant \dim N(\widetilde{\beta}^{m-1})/R(\widetilde{\beta}^{m-2}) \leqslant$$

$$\leqslant \dim N(\tilde{\alpha}^{m-1})/R(\tilde{\alpha}^{m-2}) = \dim N(\alpha^{m-1})/R(\alpha^{m-2}),$$

we obtain $\dim H^p(X, \beta) \leqslant \dim H^p(X, \alpha)$ for any $p \geqslant 0$.

From Proposition 2.9 we infer the relations

$$\text{ind}\,(\widetilde{X}, \tilde{\alpha}) = \text{ind}\,(X, \alpha) + (-1)^{m-1} n_m$$

and

$$\text{ind}\,(\widetilde{X}, \widetilde{\beta}) = \text{ind}\,(X, \beta) + (-1)^{m-1} n_m\,,$$

therefore $\text{ind}\,(X, \beta) = \text{ind}\,(X, \alpha)$ and the proof of Theorem 2.12 is complete.

Theorem 2.11 is a particular case of Theorem 2.12, with $X^p = 0$ for $p \geqslant n + 1$ In this case we may take

$$\varepsilon(X, \alpha) = \min\,\{\varepsilon_p; 0 \leqslant p \leqslant n\}.$$

By using a duality argument one can state and prove a variant of Theorem 2.12 for a complex of Banach spaces of the form $(X, \alpha) = (X^p, \alpha^p)_{p=-\infty}^0$ such that $\dim H^p(X, \alpha) < \infty$ for $p < 0$ and $H^p(X, \alpha) = 0$ for all but a finite number of indices, provided that $R(\alpha^{-1})$ is supposed closed in X^0.

3. THE STABILITY UNDER COMPACT PERTURBATIONS

A notion analogous to relative boundedness (Definition 2.4) is that of relative compactness.

3.1. DEFINITION. Consider $S \in \mathscr{C}(X, Y)$, $T \in \mathscr{C}(Y, Z)$ and A a linear operator with $D(A) \supset D(S)$ and $R(A) \subset D(T)$. We say that A is (S, T)-*compact* if for every sequence $\{x_k\}_k \subset D(S)$ with both $\{x_k\}_k$ and $\{Sx_k\}_k$ bounded, the sequences $\{Ax_k\}_k$ and $\{TAx_k\}_k$ contain convergent subsequences.

Note that A is $(S, 0)$-compact if and only if A is S-compact in the sense of [3, Ch. IV].

3.2. LEMMA. *If A is (S, T)-compact then A is (S, T)-bounded.*

Proof. Indeed, if A is not (S, T)-bounded then there is a sequence $\{x_k\}_k \subset D(S)$ such that $\|x_k\| + \|Sx_k\| \leqslant 1$ and $\|Ax_k\| + \|TAx_k\| \geqslant k$, therefore $\{Ax_k\}_k$ and $\{TAx_k\}_k$ cannot contain convergent subsequences.

Let us remark that if A is (S, T)-compact, $\hat{X} = D(S)$ is endowed with the norm $\|x\|_0 = \|x\| + \|Sx\|$ $(x \in \hat{X})$, $\hat{Y} = D(T)$ is endowed with the norm $\|y\|_0 = \|y\| + \|Ty\|$ $(y \in \hat{Y})$ and $\hat{A}$ is the operator from $\hat{X}$ into $\hat{Y}$ induced by A then $\hat{A} \in \mathscr{B}(\hat{X}, \hat{Y})$ and $\hat{A}$ is compact in the usual sense, as follows from Definition 3.1 and Lemma 3.2. Conversely, the compact operators that we work with are relatively compact in the sense of Definition 3.1 (see Lemma 3.4 below), hence it is enough, from our standpoint, to consider only compact perturbations.

3.3. LEMMA. *Consider $S \in \mathscr{C}(X, Y)$ and $T \in \mathscr{C}(Y, Z)$ with $R(S) \subset N(T)$ and $R(S)$ closed. We have* $\dim N(T)/R(S) < \infty$ *if and only if for every bounded sequence $\{y_k\}_k \subset N(T)$ there exists a sequence $\{x_k\}_k \subset D(S)$ with the property that $\{y_k - Sx_k\}_k$ contains a convergent subsequence.*

Proof. If $\dim N(T)/R(S) < \infty$ then we can write $N(T) = R(S) + M$, where $\dim M < \infty$ and $M \cap R(S) = 0$. Since both M and $R(S)$ are closed, the projection P of $N(T)$ onto M parallel to $R(S)$ is continuous. If $\{y_k\}_k \subset N(T)$ is a bounded sequence then $y_k = Sx_k + w_k$, with $\{w_k\}_k \subset M$. As $\|w_k\| \leqslant \|P\| \, \|y_k\|$, the sequence $\{w_k\}_k = \{y_k - Sx_k\}_k$ contains a convergent subsequence.

Conversely, let us assume that $\dim N(T)/R(S) = \infty$. Then we can construct a sequence $\{y_k\}_k \subset N(T)$ such that $\|y_k\| = 1$, $\mathrm{d}(y_1, R(S)) \geqslant 1/2$ and

$$\mathrm{d}(y_k, \mathrm{sp}\{R(S), y_1, \ldots, y_{k-1}\}) \geqslant \frac{1}{2}, \quad k \geqslant 2,$$

by a well-known *lemma of Riesz* [3], where "sp" stands for the expression "the linear space spanned by". In this case for each $\{x_k\}_k \subset D(S)$ the sequence $\{y_k - Sx_k\}_k$ cannot contain any convergent subsequence.

3.4. LEMMA. *Consider* $S \in \mathscr{C}(X, Y)$ *and* $T \in \mathscr{C}(Y, Z)$ *with* $R(S) \subset N(T)$, $R(T)$ *closed and* $\dim N(T)/R(S) < \infty$. *Take the compact operators* $A \in \mathscr{B}(X, Y)$ *and* $B \in \mathscr{B}(Y, Z)$ *with the properties* $R(\widetilde{S}) \subset N(\widetilde{T})$ *and* $R(\widetilde{S})$ *closed, where* $\widetilde{S} = S + A$ *and* $\widetilde{T} = T + B$. *Then* A *is* (S, T)*-compact,* $\dim N(\widetilde{T})/R(\widetilde{S}) < \infty$ *and* $R(\widetilde{T})$ *is closed.*

Proof. We show first that A is (S, T)-compact. Indeed, if $\{x_k\}_k \subset D(S)$ and $\{Sx_k\}_k$ are bounded sequence then, by the equality $TAx_k = -(BS + BA)x_k$ for all k, we infer that both $\{Ax_k\}_k$ and $\{TAx_k\}_k$ contain convergent subsequences.

The other assertions are consequences of the following fact: If $\{y_k\}_k \subset D(\widetilde{T})$ is a bounded sequence with $\widetilde{T}y_k \to 0$ as $k \to \infty$ then there exists a sequence $\{x_k\}_k \subset \subset D(S)$ such that $\{y_k + \widetilde{S}x_k\}_k$ contains a convergent subsequence. Let us prove this statement. Since $Ty_k + By_k \to 0$ as $k \to \infty$, we may suppose that $\{By_k\}_k$, hence $\{Ty_k\}_k$, is a convergent sequence. As $R(T)$ is closed, we can find $v \in D(T)$ and a sequence $\{v_k\}_k \subset N(T)$ with $y_k + v_k + v \to 0$ as $k \to \infty$.

Now, let us write $N(T) = R(S) + M$, where $M \cap R(S) = 0$ and $\dim M < \infty$. Denote by P the projection of $N(T)$ onto M parallel to $R(S)$. Then $v_k = Sx_k + w_k$ with $w_k \in M$ for all k. The vectors x_k can be chosen such that

$$\|x_k\| \leqslant r\|Sx_k\| \leqslant r\|1 - P\| \, \|v_k\|,$$

where $r > \gamma(S)^{-1}$ is fixed. Since $\{v_k\}_k$ is bounded, we may suppose that the sequences $\{w_k\}_k \subset M$ and $\{Ax_k\}_k$ are convergent. Then we have

$$y_k + v_k + v = y_k + \widetilde{S}x_k - Ax_k + w_k + v \to 0, \quad k \to \infty,$$

hence $\{y_k + \widetilde{S}x_k\}_k$ is convergent.

In particular, if $\{y_k\}_k \subset N(\widetilde{T})$ is a bounded sequence then we can find $\{x_k\}_k \subset \subset D(\widetilde{S})$ such that $\{y_k + \widetilde{S}x_k\}_k$ contains a convergent subsequence, hence $\dim N(\widetilde{T})/R(\widetilde{S}) < \infty$, by the previous lemma.

Assume now that $R(\widetilde{T})$ is not closed. Let $\widetilde{T}_0$ be the (closed) operator induced by $\widetilde{T}$ in $Y_0 = Y/N(\widetilde{T})$. Then $\widetilde{T}_0$ is injective and $R(\widetilde{T}_0) = R(\widetilde{T})$. Since $R(\widetilde{T}_0)$ is not closed, we can find a sequence $\{\eta_k\}_k \subset Y_0$ with $\|\eta_k\| = 1$ and $\widetilde{T}_0\eta_k \to 0$ as $k \to \infty$. Let us choose a bounded sequence $\{y_k\}_k$, with y_k representing η_k for each k. Then $\widetilde{T}y_k \to 0$ as $k \to \infty$, hence there exists a sequence $\{x_k\}_k \subset D(\widetilde{S})$ with $\{y_k + \widetilde{S}x_k\}_k$ containing a convergent subsequence. In this way the sequence $\{\eta_k\}_k$ may be supposed convergent to a certain η_0 and $\|\eta_0\| = 1$. Moreover, $\widetilde{T}_0\eta_0 = 0$, hence η_0 is an eigenvector of $\widetilde{T}_0$. This contradiction shows that $R(\widetilde{T})$ must be closed.

3.5. COROLLARY. *Let* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ *be a semi-Fredholm (Fredholm) complex of Banach spaces. If* $\gamma^p \in \mathscr{B}(X^p, X^{p+1})$ *is compact for every* p *and* $(X, \beta) = = (X^p, \beta^p)_{p=0}^n$ *is a complex of Banach spaces, where* $\beta^p = \alpha^p + \gamma^p$*, then* (X, β) *is semi-Fredholm (Fredholm).*

Proof. If (X, α) is semi-Fredholm, but not Fredholm, with no loss of generality we may suppose that $\dim H^n(X, \alpha) = \infty$. If $p < n$ and $R(\beta^{p-1})$ is supposed closed then we obtain that $\dim H^p(X, \beta) < \infty$ and $R(\beta^p)$ is closed, by Lemma 3.4. As $R(\beta^{-1}) = 0$ is closed, the property is true for every $p < n$, by induction. In particular, $R(\beta^{n-1})$ is closed. In this case we cannot have $\dim H^n(X, \beta) < \infty$, by the same Lemma 3.4.

From this argument, the case (X, α) Fredholm is clear.

Corollary 3.5 shows that in order to investigate the stability of the index under compact perturbations, only the case of Fredholm complexes must be took into consideration.

3.6. LEMMA. *Let* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$, $(Y, \beta) = (Y^p, \beta^p)_{p=0}^n$ and $(Z, \gamma) = (Z^p, \gamma^p)_{p=0}^n$ *be complexes of Banach spaces. Assume that the sequence*

$$0 \to D(\alpha^p) \overset{u^p}{\to} D(\beta^p) \overset{v^p}{\to} D(\gamma^p) \to 0$$

is exact and $u^{p+1}\alpha^p = \beta^p v^p$, $v^{p+1}\beta^p = \gamma^p v^p$, *for every p. If any two of the complexes* (X, α), (Y, β), (Z, γ) *are Fredholm then the third is also Fredholm and we have the equality*

$$\operatorname{ind}(Y, \beta) = \operatorname{ind}(X, \alpha) + \operatorname{ind}(Z, \gamma).$$

Proof. The hypothesis implies the existence of a long exact sequence of cohomology

$$\text{(3.1)} \qquad \ldots \to H^p(X, \alpha) \xrightarrow{\tilde{u}^p} H^p(Y, \beta) \xrightarrow{\tilde{v}^p} H^p(Z, \gamma) \xrightarrow{\tilde{w}^p} H^{p+1}(X, \alpha) \to \ldots$$

where $\tilde{u}^p$ and $\tilde{v}^p$ are induced by u^p and v^p respectively, while $\tilde{w}^p$ is a connecting homomorphism (see [7] for details). From the exactness of (3.1) it follows that if any two of the complexes (X, α), (Y, β), (Z, γ) are Fredholm then the third is Fredholm as well. In this case (3.1) is a complex of finite dimensional spaces, whose index must be zero on account of its exactness. On the other hand, by the formula (1.5),

$$\operatorname{ind}(X, \alpha) - \operatorname{ind}(Y, \beta) + \operatorname{ind}(Z, \gamma) = 0.$$

3.7. THEOREM. *Assume that* $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ *is a Fredholm complex of Banach spaces. Take* $\gamma^p \in \mathscr{B}(X^p, X^{p+1})$ *compact for each p, such that* $(X, \beta) = (X^p, \beta^p)_{p=0}^n$ *be a complex of Banach spaces, where* $\beta^p = \alpha^p + \gamma^p$. *If*

$$\text{(3.2)} \qquad \dim R(\gamma^{p+1}\gamma^p) < \infty, \quad p = 0, 1, \ldots n-2$$

then $\operatorname{ind}(X, \beta) = \operatorname{ind}(X, \alpha)$.

Proof. Let us denote by $\tilde{X}^p$ the finite dimensional space $R(\gamma^{p-1}\gamma^{p-2})$ for $0 \leqslant p \leqslant n$, where $\gamma^{-2} = 0$, $\gamma^{-1} = 0$. Plainly, $\tilde{X}^p \subset D(\alpha^p)$. If we consider in $D(\alpha^p)$ the norm given by (2.8), we may suppose with no loss of generality that α^p is continuous for every p. Note that both $N(\alpha^p)$ and $R(\alpha^p)$ remain unchanged in the new topology and that the index is preserved. Moreover, as γ^p is (α^p, α^{p+1})-compact by Lemma 3.4, the restriction of γ^p on $D(\alpha^p)$ will be still compact in the new topology.

Let $\tilde{\alpha}^p$ be the restriction of α^p on $\widetilde{X}^p$. We have $R(\tilde{\alpha}^p) \subset \widetilde{X}^{p+1}$. Indeed, if $x \in \widetilde{X}^p$, $x = \gamma^{p-1}\gamma^{p-2}v$, by the identity $\beta^p\beta^{p-1} = 0$ we can write

$$\alpha^p x = -(\gamma^p\alpha^{p-1} + \gamma^p\gamma^{p-1})\gamma^{p-2}v =$$

$$= \gamma^p(\gamma^{p-1}\alpha^{p-2} + \gamma^{p-1}\gamma^{p-2})v - \gamma^p\gamma^{p-1}\gamma^{p-2}v = \gamma^p\gamma^{p-1}\alpha^{p-2}v.$$

Note also that $R(\tilde{\gamma}^p) \subset \widetilde{X}^{p+1}$, where $\tilde{\gamma}^p$ is the restriction of γ^p on $\widetilde{X}^p$. In this way both $(\widetilde{X}, \tilde{\alpha}) = (\widetilde{X}^p, \tilde{\alpha}^p)_{p=0}^n$ and $(\widetilde{X}, \tilde{\beta}) = (\widetilde{X}^p, \tilde{\beta}^p)_{p=0}^n$, with $\tilde{\beta}^p = \tilde{\alpha}^p + \tilde{\gamma}^p$, are complexes of finite dimensional Banach spaces, therefore by the formula (1.5) we obtain

$$\operatorname{ind}(\widetilde{X}, \tilde{\alpha}) = \operatorname{ind}(\widetilde{X}, \tilde{\beta}). \tag{3.3}$$

Consider now the quotient space $\hat{X}^p = X^p/\widetilde{X}^p$ and denote by $\hat{\alpha}^p$ and $\hat{\gamma}^p$ the maps induced in $\hat{X}^p$ by α^p and γ^p, respectively, for all p. From the equality

$$(\alpha^{p+1} + \theta\gamma^{p+1})(\alpha^p + \theta\gamma^p) + \theta(1-\theta)\gamma^{p+1}\gamma^p = 0,$$

where $0 \leqslant \theta \leqslant 1$, we infer that $R(\hat{\alpha}^p + \theta\hat{\gamma}^p) \subset N(\hat{\alpha}^{p+1} + \theta\hat{\gamma}^{p+1})$, therefore $(\hat{X}, \hat{\alpha} + \theta\hat{\gamma}) = (\hat{X}^p, \hat{\alpha}^p + \theta\hat{\gamma}^p)_{p=0}^n$ is a complex of Banach spaces. As (X, α) and $(\widetilde{X}, \tilde{\alpha})$ are Fredholm, by Lemma 3.6 it follows that $(\hat{X}, \hat{\alpha})$ is also Fredholm and $\operatorname{ind}(X, \alpha) = \operatorname{ind}(\widetilde{X}, \tilde{\alpha}) + \operatorname{ind}(\hat{X}, \hat{\alpha})$. A similar property is also true for $(\hat{X}, \hat{\beta}) = (\hat{X}, \hat{\alpha} + \hat{\gamma})$, therefore if $\operatorname{ind}(\hat{X}, \hat{\alpha}) = \operatorname{ind}(\hat{X}, \hat{\beta})$ then, by (3.3), $\operatorname{ind}(X, \alpha) = \operatorname{ind}(X, \beta)$ as well. Indeed, by Theorem 2.11 we have that $\operatorname{ind}(\hat{X}, \hat{\alpha}) = \operatorname{ind}(\hat{X}, \hat{\alpha} + \theta\hat{\gamma})$ for small values of θ. By Corollary 3.5 $(\hat{X}, \hat{\alpha} + \theta\hat{\gamma})$ is Fredholm for each θ. Since the index is continuous by Theorem 2.11 and its values are integers it must be constant, and the proof is complete.

We think that Theorem 3.7 is true without the condition (3.2). Besides Corollary 3.5, one reason for this conjecture is a consequence of the following

3.8. THEOREM. *Assume that $(X, \alpha) = (X^p, \alpha^p)_{p=0}^n$ is a complex of Hilbert spaces. Then there exist two Hilbert spaces H_0 and H_1 and a closed operator T_α from H_0 into H_1 with the properties*:

(1) *$R(\alpha^p)$ is closed for all p if and only if $R(T_\alpha)$ is closed;*

(2) *(X, α) is Fredholm if and only if T_α is Fredholm and in this case* $\operatorname{ind} T_\alpha = \operatorname{ind}(X, \alpha)$;

(3) *$H^p(X, \alpha) = 0$ for all p if and only if T_α has a bounded inverse from H_1 into H_0*

Proof. With no loss of generality we may suppose that each α^p is densely defined, therefore the adjoint α^{p*} is also (densely) defined. Let us set

$$H_0 = \bigoplus_{k\geqslant 0} X^{2k}, \qquad H_1 = \bigoplus_{k\geqslant 0} X^{2k+1},$$

and define the operator

$$T_\alpha(\bigoplus_{k\geqslant 0} x_{2k}) = \bigoplus_{k\geqslant 0} (\alpha^{2k}x_{2k} + \alpha^{2k+1*}x_{2k+2}), \tag{3.4}$$

where $x_{2k} \in D(\alpha^{2k}) \cap D(\alpha^{2k-1*})$, for each $k \geqslant 0$. Plainly, T_α maps a subspace of H_0 into H_1.

Let us prove that T_α is closed. Note that $R(\alpha^{2k}) \subset N(\alpha^{2k+1})$ and $R(\alpha^{2k+1*}) \subset \subset N(\alpha^{2k+1})^\perp$, and take $\xi_m = \bigoplus_{k\geqslant 0} x_{2k}^m \in D(T_\alpha)$ with $\{\xi^m\}_m$ and $\{T_\alpha\xi^m\}_m$ convergent. By the above remark we obtain that both $\{\alpha^{2k}x_{2k}^m\}_m$ and $\{\alpha^{2k+1*}x_{2k+2}^m\}_m$ are convergent sequences. By using that α^{2k} and α^{2k+1*} are closed we infer easily that T_α itself is closed.

Let us prove the equality

$$N(T_\alpha) = \bigoplus_{k\geqslant 0} (N(\alpha^{2k}) \ominus R(\alpha^{2k-1})) \tag{3.5}$$

(where $H \ominus K$ denotes the orthocomplement of K in H). Indeed, take $\bigoplus_{k\geqslant 0} x_{2k} \in N(T_\alpha)$, hence $\alpha^{2k}x_{2k} = 0$ and $\alpha^{2k+1*}x_{2k+2} = 0$ by the orthogonality, for every $k \geqslant 0$. In this way we have also $x_{2k} \in N(\alpha^{2k-1*}) = R(\alpha^{2k-1})^\perp$, hence $x_{2k} \in N(\alpha^{2k}) \ominus R(\alpha^{2k-1})$. Conversely, if $x_{2k} \in N(\alpha^{2k}) \ominus R(\alpha^{2k-1})$ then $x_{2k} \in D(\alpha^{2k}) \cap D(\alpha^{2k-1*})$ and $\bigoplus_{k\geqslant 0} x_{2k} \in N(T_\alpha)$.

We have also the equality

$$H_1 \ominus R(T_\alpha) = \bigoplus_{k\geqslant 0} (N(\alpha^{2k+1}) \ominus R(\alpha^{2k})). \tag{3.6}$$

Indeed, if $\bigoplus_{k\geqslant 0} y_{2k+1} \in H_1 \ominus R(T_\alpha)$ then y_{2k+1} is orthogonal to both $R(\alpha^{2k})$ and $R(\alpha^{2k+1*})$, therefore $y_{2k+1} \in N(\alpha^{2k+1}) \ominus R(\alpha^{2k})$ for all k, which gives one inclusion. The other inclusion is similar.

One more equality is needed. Namely we have

$$R(T_\alpha) = \bigoplus_{k\geqslant 0} (R(\alpha^{2k}) \oplus R(\alpha^{2k+1*})) \tag{3.7}$$

(extending the meaning of the direct sum for orthogonal not necessarily closed linear manifolds). It is clear that $R(T_\alpha)$ is contained in the second side of (3.7). Conversely, take $\bigoplus_{k\geqslant 0} y_{2k+1}$ belonging to the second side of (3.7). Then we can write that $y_{2k+1} = \alpha^{2k}x_{2k}' + \alpha^{2k+1*}x_{2k+2}''$. Moreover, according to the decomposition $X^{2k} = N(\alpha^{2k}) \oplus$

$\oplus \overline{R(\alpha^{2k*})}$, we may suppose that $x'_{2k} \in \overline{R(\alpha^{2k*})}$ and $x''_{2k} \in N(\alpha^{2k})$, for all k. In this way, setting $x_{2k} = x'_{2k} + x''_{2k} \in D(\alpha^{2k}) \cap D(\alpha^{2k-1*})$, we have $\alpha^{2k}x_{2k} + \alpha^{2k+1*}x_{2k+2} = \alpha^{2k}x'_{2k} + \alpha^{2k+1*}x''_{2k+2} = y_{2k+1}$, therefore $\bigoplus_{k\geqslant 0} y_{2k+1} \in R(T_\alpha)$.

The assertion (1) results in the following way: If $R(\alpha^p)$ is closed for all p then $R(\alpha^{p*})$ is closed for all p, hence (3.7) implies that $R(T_\alpha)$ is closed. Conversely, $R(T_\alpha)$ closed implies that $R(\alpha^{2k})$ and $R(\alpha^{2k+1*})$ are closed for every k, hence $R(\alpha^p)$ is closed for all p.

From the equalities (3.5) and (3.6) we obtain that (X, α) is Fredholm if and only if T_α is Fredholm and in this case ind $(X, \alpha) =$ ind T_α, therefore (2) is true.

Finally, if $H^p(X, \alpha) = 0$ for all p then $N(T_\alpha) = 0$ by (3.5) and $R(T_\alpha) = H_1$ by (3.6), thus T_α^{-1} exists as a bounded operator from H_1 into H_0. The converse assertion is similar, showing that (3) is also true.

Let us mention that the consideration of the spaces H_0 and H_1 has been suggested by a similar construction in [9, Ch. IV].

3.9. COROLLARY. *If $(X, \alpha)=(X^p, \alpha^p)_{p=0}^n$ is a Fredholm complex of Hilbert spaces, $\gamma^p \in \mathscr{B}(X^p, X^{p+1})$ is compact, $\beta^p = \alpha^p+\gamma^p$ and $(X, \beta)=(X^p, \beta^p)_{p=0}^n$ is a complex of Hilbert spaces then (X, β) is Fredholm and* ind $(X, \beta) =$ ind (X, α).

Proof. Let T_α and T_β be the corresponding operators given by Theorem 3.8 for (X, α), (X, β) respectively. It is clear that $T_\beta - T_\alpha$ is compact, therefore ind $T_\beta =$ ind T_α by the classical theorem of stability of the index under compact perturbations [3] (which is also a consequence of our Theorem 3.7). By Theorem 3.8 we obtain that ind $(X, \beta) =$ ind (X, α).

Let us remark that by using Theorem 3.8 one can define a more extensive concept of semi-Fredholm complex of Hilbert spaces, in connection with the same notion for the corresponding operator.

One more remark. In the proof of Theorem 3.8 we can consider another operator $T_\alpha^\#$ from H_1 into H_0, defined by

$$T_\alpha^\#\Big(\bigoplus_{k\geqslant 0} x_{2k+1}\Big) = \bigoplus_{k\geqslant 0} (\alpha^{2k-1}x_{2k-1} + \alpha^{2k*}x_{2k+1}),$$

where $x_{2k+1} \in D(\alpha^{2k+1}) \cap D(\alpha^{2k*})$, for all $k \geqslant 0$. Then $N(T_\alpha^\#)$, $H_0 \ominus R(T_\alpha^\#)$ and $R(T_\alpha^\#)$ satisfy some variants of the formulas (3.5), (3.6) and (3.7), respectively. It is easy to see that $\langle T_\alpha\xi, \eta\rangle = \langle \xi, T_\alpha^\#\eta\rangle$ for all $\xi \in D(T_\alpha)$ and $\eta \in D(T_\alpha^\#)$. One can see that $T_\alpha^\#$ is the adjoint of T_α.

Finally, let us mention that the stability of the index of a Fredholm complex of Hilbert spaces is also pointed out by M. A. Šubin in his book *Pseudo-differential operators and spectral theory* (Russian), Nauka, Moskow, 1978. We have become recently aware of this work, containing an assertion that overlaps Corollary 3.9.

4. COMMUTING SYSTEMS OF LINEAR TRANSFORMATIONS

For finite commuting systems of linear continuous operators in Banach spaces there is an adequate concept of joint spectrum which is strongly related to the combined action of the operators on the space, introduced and studied by J. L. Taylor [10]. The purpose of this section is to present in this spirit some elements of spectral and Fredholm theory, valid for certain systems of linear transformations, not necessarily continuous (see also [11] for a slightly different approach to the spectral theory of the unbounded systems).

Let us recall some basic definitions and notations [10], [11], [12]. Consider a system of n indeterminates $\sigma = (\sigma_1, \ldots, \sigma_n)$ and let $\Lambda[\sigma]$ be the exterior algebra over $\mathbf{C}$ generated by $\sigma_1, \ldots, \sigma_n$. For any integer p, $0 \leqslant p \leqslant n$, we denote by $\Lambda^p[\sigma]$ the space of all homogeneous exterior forms of degree p in $\sigma_1, \ldots, \sigma_n$. The space $\Lambda[\sigma]$ has a natural structure of Hilbert space in which the elements

$$\sigma_{j_1} \wedge \ldots \wedge \sigma_{j_p}, \quad 1 \leqslant j_1 < \ldots < j_p \leqslant n; \quad p = 1, \ldots, n$$

and $1 \in \mathbf{C} = \Lambda^0[\sigma]$ form an orthonormal basis.

An important role will be played by the operators

$$S_j\xi = \sigma_j \wedge \xi, \quad \xi \in \Lambda[\sigma], \quad j = 1, \ldots, n \tag{4.1}$$

and by their adjoints

$$S_j^*(\xi_j' + \sigma_j \wedge \xi_j'') = \xi_j'', \quad j = 1, \ldots, n, \tag{4.2}$$

where $\xi_j' + \sigma_j \wedge \xi_j''$ is the canonical decomposition of an arbitrary element $\xi \in \Lambda[\sigma]$ with ξ_j' and ξ_j'' not containing σ_j. Note the anticommutation relations

$$\left.\begin{aligned} S_jS_k + S_kS_j &= 0 \\ S_jS_k^* + S_k^*S_j &= \varepsilon_{jk} \end{aligned}\right\} \quad j, k = 1, \ldots, n, \tag{4.3}$$

where ε_{jk} is the Kronecker symbol.

Consider now a complex linear space L. Then the tensor product $L \otimes \Lambda[\sigma]$ will be always denoted by $\Lambda[\sigma, L]$. Analogously, $\Lambda^p[\sigma, L]$ is $L \otimes \Lambda^p[\sigma]$, $0 \leqslant p \leqslant n$. If λ is any endomorphism of L then the action of λ is extended on $\Lambda[\sigma, L]$ by the endomorphism $\lambda \otimes 1$. The latter will be also denoted by λ (as a rule, we omit the symbol "$\otimes$" when representing elements and endomorphisms connected with $\Lambda[\sigma, L]$). Analogously, if θ is an endomorphism of $\Lambda[\sigma]$ then the endomorphism $1 \otimes \theta$, acting on $\Lambda[\sigma, L]$, will be also denoted by θ.

When L is a complex Banach (Hilbert) space X then $\Lambda[\sigma, X]$ is also a Banach (Hilbert) space, which can be identified with a direct sum of 2^n copies of X. The action of each $T \in \mathscr{C}(X)$ will be extended in $\Lambda[\sigma, X]$ by $T \otimes 1$, denoted simply by T, defined on $D(T) \otimes \Lambda[\sigma] = \Lambda[\sigma, D(T)]$, which is still a closed operator. Clearly, for any endomorphism θ of $\Lambda[\sigma]$ the endomorphism $T\theta$ extends θT.

Let X be a fixed complex Banach space and $\sigma = (\sigma_1, \ldots, \sigma_n)$ a fixed system of indeterminates.

4.1. DEFINITION. We say that $a = (a_1, \ldots, a_n) \subset \mathscr{C}(X)$ is a *D-commuting system* if there exists a dense subspace D of X in $\bigcap_{j=1}^{n} D(a_j)$ with the properties:

(1) The restriction $\hat{\delta}_a = (a_1 S_1 + \ldots + a_n S_n)|\Lambda[\sigma, D]$ is closable;

(2) If δ_a is the canonical closure of $\hat{\delta}_a$ then $R(\delta_a) \subset N(\delta_a)$.

Note that Definition 4.1 can be equivalently expressed in the following way:

(1′) The restriction $\hat{\delta}_a^p = (a_1 S_1 + \ldots + a_n S_n)|\Lambda^p[\sigma, D]$ is closable for every p, $0 \leqslant p \leqslant n$;

(2′) If δ_a^p is the canonical closure of $\hat{\delta}_a^p$ then $R(\delta_a^p) \subset N(\delta_a^{p+1})$, for $p = 0, 1, \ldots, n$, where $\delta_a^{n+1} = 0$.

From this equivalent form of Definition 4.1 we obtain that each D-commuting system can be associated with the complex of Banach spaces $(\Lambda^p[\sigma, X], \delta_a^p)_{p=0}^n$, which makes the connection with the previous sections. The cohomology corresponding to this complex will be denoted by $H(X, a; D) = (H^p(X, a; D))_{p=0}^n$.

Plainly, if $a = (a_1, \ldots, a_n) \subset \mathscr{B}(X)$ is a D-commuting system then δ_a is continuous and from $\delta_a \delta_a = 0$ we infer that $(a_1, \ldots, a_n)$ is a system of mutually commuting operators.

The concept given by Definition 4.1 is a notion of "strong commutativity". We can give also a concept of "weak commutativity".

4.2. DEFINITION. Consider a system $a = (a_1, \ldots, a_n)$ of densely defined operators in $\mathscr{C}(X)$. We say that $a = (a_1, \ldots, a_n)$ is a *D_*-weakly commuting system* if $a^* = (a_1^*, \ldots, a_n^*) \subset \mathscr{C}(X^*)$ is a D_*-commuting system.

In Definition 4.1 the basic operator $a_1 S_1 + \ldots + a_n S_n$ may be replaced with the operator $a_1 S_1^* + \ldots + a_n S_n^*$. In order to prove this assertion, let us introduce a "Hodge type" transformation of $\Lambda[\sigma, X]$ into itself [9].

Note that each $\xi \in \Lambda^p[\sigma, X]$ may be represented uniquely as

$$\xi = \sum_{1 \leqslant j_1 < \ldots < j_p \leqslant n} S_{j_1} \ldots S_{j_p} x_{j_1 \ldots j_p}, \qquad x_{j_1 \ldots j_p} \in X.$$

Let us define then in $\Lambda^{n-p}[\sigma, X]$ the element

$$\#\xi = \mathrm{i}^{n(n-1)/2} \sum_{1 \leqslant j_1 < \ldots < j_p \leqslant n} S_{j_1}^* \ldots S_{j_p}^* S_1 \ldots S_n x_{j_1 \ldots j_p}, \tag{4.4}$$

with $\mathrm{i}^2 = -1$, and extend the map $\#$ by linearity on the whole space $\Lambda[\sigma, X]$.

4.3. LEMMA. *The map* $\#$ *of* $\Lambda[\sigma, X]$ *into itself is an isomorphism whose square is the identity.*

When X *is a Hilbert space then* $\#$ *is a unitary transformation.*

Proof. If $1 \leqslant j_1 < \ldots < j_p \leqslant n$ is a fixed system of indices then for every $x \in X$ we have by (4.3)

$$\#(S_{j_1} \ldots S_{j_p} x) = \mathrm{i}^{n(n-1)/2}(-1)^{j_1-1} \ldots (-1)^{j_p-1} S_{k_1} \ldots S_{k_{n-p}} x,$$

where $1 \leqslant k_1 < \ldots < k_{n-p} \leqslant n$ and $\{j_1, \ldots, j_p\} \cup \{k_1, \ldots, k_{n-p}\} = \{1, 2, \ldots, n\}$. We can write then

$$\#(\#(S_{j_1} \ldots S_{j_p}x)) =$$

$$= \mathrm{i}^{n(n-1)}(-1)^{j_1-1} \ldots (-1)^{j_p-1}(-1)^{k_1-1} \ldots (-1)^{k_{n-p}-1} S_{j_1} \ldots S_{j_p} x =$$

$$= \mathrm{i}^{n(n-1)}(-1)^{n(n-1)/2} S_{j_1} \ldots S_{j_p}x = S_{j_1} \ldots S_{j_p}x.$$

From this calculation we obtain by linearity that the square of $\#$ is the identity on $\Lambda[\sigma, X]$, hence $\#$ is an isomorphism.

When X is a Hilbert space then $\#$ is an isometry, hence $\#$ is actually a unitary transformation.

4.4. LEMMA. *Consider a system* $a = (a_1, \ldots, a_n) \subset \mathscr{C}(X)$ *and assume that there exists a subspace* $D \subset \bigcap_{j=1}^{n} D(a_j)$, *$D$ dense in X. Let us denote by* $\hat{\delta}_a(\hat{\delta}_a^p)$ *the restriction of* $a_1S_1 + \ldots + a_nS_n$ *on* $\Lambda[\sigma, D]$ $(\Lambda^p[\sigma, D])$ *and by* $\hat{\gamma}_a(\hat{\gamma}_a^p)$ *the restriction of* $a_1S_1^* + \ldots + a_nS_n^*$ *on* $\Lambda[\sigma, D]$ $(\Lambda^p[\sigma, D])$. *Then we have the properties:*

(1) $\hat{\delta}_a(\hat{\delta}_a^p)$ *is closable if and only if* $\hat{\gamma}_a(\hat{\gamma}_a^{n-p})$ *is closable;*

(2) *The system* $a = (a_1, \ldots, a_n)$ *is D-commuting if and only if* $R(\gamma_a) \subset N(\gamma_a)$, *where* γ_a *is the canonical closure of* $\hat{\gamma}_a$.

Proof. Let $\#$ be the map given by (4.4). Obviously, $\#\Lambda^p[\sigma, D] = \Lambda^{n-p}[\sigma, D]$. Take now $x \in D$ and fix a system of indices $1 \leqslant j_1 < \ldots < j_p \leqslant n$. Then we can write

$$\#((a_1S_1 + \ldots + a_nS_n)S_{j_1} \ldots S_{j_p} x) =$$

$$= \#\left(\sum_{k=1}^{n} S_kS_{j_1} \ldots S_{j_p}a_kx\right) = \sum_{k=1}^{n} a_kS_k^* \#(S_{j_1} \ldots S_{j_p}x),$$

hence

$$\# \hat{\delta}_a^p\xi = \hat{\gamma}_a^{n-p}\#\xi, \qquad \xi \in \Lambda^p[\sigma, D]. \tag{4.5}$$

From (4.5) we obtain easily that $\hat{\delta}_a^p$ is closable if and only if $\hat{\gamma}_a^{n-p}$ is closable. In this case, if δ_a^p is the closure of $\hat{\delta}_a^p$ and γ_a^p is the closure of $\hat{\gamma}_a^p$ we have also $\#\delta_a^p = \gamma_a^{n-p}\#$. In particular, $a = (a_1, \ldots, a_n)$ is D-commuting if and only if $R(\gamma_a) \subset N(\gamma_a)$.

Let us remark that if $a = (a_1, \ldots, a_n) \subset \mathscr{C}(X)$ is a D_*-weakly commuting system and X is reflexive then we may define $\delta_a^w = (\gamma_{a*})^*$, where γ_{a*} is given by Lemma 4.4. Then δ_a^w has the property $R(\delta_a^w) \subset N(\delta_a^w)$, which follows from the corresponding property of γ_{a*}. It is easily seen that δ_a^w is an extension of the operator

$$(a_1S_1 + \ldots + a_nS_n)|\Lambda\left[\sigma, \bigcap_{j=1}^{n} D(a_j)\right].$$

In particular, if $a = (a_1, \ldots, a_n)$ is D-commuting for a certain D then δ_a^w always extends δ_a.

Let us illustrate the consistency of Definitions 4.1 and 4.2 with a significant particular case. Take again an arbitrary Banach space X.

4.5. PROPOSITION. *Assume that* $a_1, \ldots, a_n$ *from* $\mathscr{C}(X)$ *are densely defined,* $b_j = a_j^{-1} \in \mathscr{B}(X)$ $(j = 1, \ldots, n)$ *and that* $b_1, \ldots, b_n$ *mutually commute. Assume also that* $a_1^*, \ldots, a_n^*$ *are densely defined. Then we have the following properties:*

(1) $a = (a_1, \ldots, a_n)$ *is a D-commuting system, where* $D = b_1 \ldots b_nX$;

(2) $a^* = (a_1^*, \ldots, a_n^*)$ *is a* D_**-commuting system, where* $D_* = b_1^* \ldots b_n^*X^*$;

(3) $R(\delta_a) = N(\delta_a)$ *and* $R(\delta_{a*}) = N(\delta_{a*})$.

Proof. Since $a_1, \ldots, a_n$ are densely defined, the subspace $D = b_1 \ldots b_nX$ s dense in X. Analogously, $D_* = b_1^* \ldots b_n^*X^*$ is dense in X^*. Moreover, $D \subset \bigcap_{j=1}^{n} D(a_j)$ and $D_* \subset \bigcap_{j=1}^{n} D(a_j^*)$. Consider the restriction $\hat{\delta}_a$ of $a_1S_1 + \ldots + a_nS_n$ on $\Lambda[\sigma, D]$. Let us show first that $\hat{\delta}_a$ is closable. For, consider $\xi \in \Lambda[\sigma, D]$. Then for any $\theta \in \Lambda[\sigma, D_*]$ we can write $\langle \hat{\delta}_a\xi, \theta\rangle = \langle \xi, \hat{\gamma}_{a*}\theta\rangle$, where $\hat{\gamma}_{a*}$ is given by $a_1^*S_1^* + \ldots + a_n^*S_n^*$ restricted on $\Lambda[\sigma, D_*]$ and $\langle \eta, \xi\rangle$ is the form associated with the duality of $\Lambda[\sigma, X]$ and $\Lambda[\sigma, X^*]$ naturally induced by the duality of X and X^*. As $\Lambda[\sigma, D_*]$ is dense in $\Lambda[\sigma, X^*]$ we infer that $\hat{\delta}_a$ is closable.

A similar argument shows that $\hat{\delta}_{a*}$ is closable in $\Lambda[\sigma, X^*]$.

Let us prove the inclusion $R(\delta_a) \subset N(\delta_a)$. Notice that every $\eta \in \Lambda[\sigma, D]$ can be written as $\eta = b_1 \ldots b_n\xi$ and by the density of $\Lambda[\sigma, D]$ in $\Lambda[\sigma, X]$ we have that $\xi = \lim_{k\to\infty} b_1 \ldots b_n\zeta_k$, therefore

$$\delta_a\eta = \sum_{j=1}^{n} b_1 \ldots \hat{b}_j \ldots b_nS_j\xi =$$

$$= \lim_{k\to\infty} \sum_{j=1}^{n} b_1 \ldots \hat{b}_j \ldots b_nb_1 \ldots b_nS_j\zeta_k,$$

where the hat over b_j means its deletion. On the other hand, for every k we have

$$\delta_a\left(\sum_{j=1}^{n} b_1 \dots \hat{b}_j \dots [b_n b_1 \dots b_n S_j \zeta_k\right) =$$

$$= \sum_{j<q} (b_1 \dots \hat{b}_j \dots b_n b_1 \dots \hat{b}_q \dots b_n - b_1 \dots \hat{b}_q \dots b_n b_1 \dots \hat{b}_j \dots b_n) S_j S_q \zeta_k = 0,$$

hence $\hat{\delta}_a \eta \in D(\delta_a)$ and $\delta_a \hat{\delta}_a \eta = 0$. Since δ_a is the canonical closure of $\hat{\delta}_a$ we get actually $R(\delta_a) \subset N(\delta_a)$, therefore $a = (a_1, \dots, a_n)$ is a D-commuting system. Analogously, $a^* = (a_1^*, \dots, a_n^*)$ is a D_*-commuting system, hence the first and the second assertion are proved.

In order to prove the third assertion we need the relation

$$\delta_a \gamma_b \xi + \gamma_b \delta_a \xi = n\xi, \quad \xi \in D(\delta_a), \tag{4.6}$$

where γ_b is given by Lemma 4.4 for $(b_1, \dots, b_n)$. Indeed, if $\xi \in \Lambda[\sigma, D]$ then $\gamma_b \xi \in \Lambda[\sigma, D]$ and we have by (4.3)

$$\delta_a \gamma_b \xi + \gamma_b \delta_a \xi = \sum_{j=1}^{n} a_j b_j \xi = n\xi,$$

whence we derive (4.6). In particular, if $\eta \in N(\delta_a)$ then $\eta = \delta_a \xi$, where $\xi = n^{-1} \gamma_b \eta$, hence $R(\delta_a) = N(\delta_a)$. Similarly, $R(\delta_{a^*}) = N(\delta_{a^*})$, and the proof is complete.

Let us remark that Proposition 4.5 applies to the case of the operators $a_j = c_j - z_j$ $(j = 1, \dots, n)$, where $c_1, \dots, c_n$ are unbounded self-adjoint operators whose spectral measures mutually commute, and $z_1, \dots, z_n$ are complex numbers whose imaginary part is non-null.

Consider a D-commuting system $a = (a_1, \dots, a_n) \subset \mathscr{C}(X)$ and a system of complex numbers $z = (z_1, \dots, z_n) \in \mathbf{C}^n$. It is easily seen that $z - a = (z_1 - a_1, \dots, z_n - a_n)$ is also D-commuting.

4.6. Definition. Suppose that $a = (a_1, \dots, a_n) \subset \mathscr{C}(X)$ is a D-commuting system. Then it is called *nonsingular (singular)* if $R(\delta_a) = N(\delta_a)(R(\delta_a) \neq N(\delta_a))$.

The system $a = (a_1, \dots, a_n)$ is said to be *semi-Fredholm (Fredholm)* if the associated complex of Banach spaces $(\Lambda^p[\sigma, X], \delta_a^p)_{p=0}^n$ is semi-Fredholm (Fredholm).

For a D-commuting system $a = (a_1, \dots, a_n) \subset \mathscr{C}(X)$ we can introduce now a notion of *joint spectrum*, denoted by $\sigma_D(a, X)$, consisting of those points $z \in \mathbf{C}^n$ such that $z - a$ is singular. When $a = (a_1, \dots, a_n) \subset \mathscr{B}(X)$ this notion coincides with that of J. L. Taylor [10].

For $a = (a_1, \dots, a_n) \subset \mathscr{C}(X)$ semi-Fredholm we may define its *index* by the equality

$$\operatorname{ind}_D a = \operatorname{ind}\,(\Lambda^p[\sigma, X], \delta_a^p)_{p=0}^n.$$

Similarly, for a D_*-weakly commuting system $a = (a_1, \ldots, a_n)$ in $\mathscr{C}(X)$ we can introduce a notion of *weak joint spectrum* $\sigma^w_{D_*}(a, X)$ given by $\sigma_{D_*}(a^*, X^*)$, as well as a notion of *weak index* defined by $\mathrm{ind}^w_{D_*} a = \mathrm{ind}_{D_*} a^*$. Since both notions of weak joint spectrum and weak index are expressed in terms of the corresponding "strong" ones, their properties can be easily derived from the properties of the other, therefore we shall not deal with the "weak" concepts in the sequel.

Note that the system of operators $a = (a_1, \ldots, a_n)$ with the properties stated in Proposition 4.5 is nonsingular. In fact, a method used in Proposition 4.5 can be adapted in order to obtain a more general criterion of nonsingularity (see also [10, Lemma 1.1] for bounded systems).

4.7. LEMMA. *Assume that $a = (a_1, \ldots, a_n) \subset \mathscr{C}(X)$ is a D-commuting system. Assume also that there exists a system $b = (b_1, \ldots, b_n)$ in $\mathscr{B}(X)$ with the properties:*

(1) *$b_j D \subset D$ and $a_k b_j x = b_j a_k x$ for all $j, k = 1, \ldots, n$ and $x \in D$;*

(2) *$\sum_{j=1}^{n} a_j b_j x = x$ for every $x \in D$.*

Then $a = (a_1, \ldots, a_n)$ is nonsingular.

Proof. The assertion can be obtained by using an equality similar to (4.6). We omit the details.

It is beyond our scope to make an extensive study of the notions of joint spectrum and index for commuting systems. We shall restrict ourselves to some consequences of the previous sections.

4.8. THEOREM. *Consider a D-commuting system $a = (a_1, \ldots, a_n)$ in $\mathscr{C}(X)$ which is semi-Fredholm. There exists an $\varepsilon_a > 0$ such that for each system $(c_1, \ldots, c_n) \subset \subset \mathscr{B}(X)$ with $\|c_j\| < \varepsilon_a$, if $b_j = a_j + c_j$ for $j = 1, \ldots, n$ and $b = (b_1, \ldots, b_n)$ is a D-commuting system, then $b = (b_1, \ldots, b_n)$ is semi-Fredholm,* $\dim H^p(X, b; D) \leqslant \leqslant \dim H^p(X, a; D)$ *for all p and* $\mathrm{ind}_D\, b = \mathrm{ind}_D a$.

If $a = (a_1, \ldots, a_n)$ is nonsingular then $b = (b_1, \ldots, b_n)$ is also nonsingular.

Proof. Let $\tilde{\varepsilon}_a > 0$ be given by Theorem 2.11 applied to the complex $(\Lambda^p[\sigma, X], \delta^p_a)^n_{p=0}$. Take $\varepsilon_a = n^{-1}\tilde{\varepsilon}_a$. If b is as stated then $(\Lambda^p[\sigma, X], \delta^p_b)^n_{p=0}$ is a complex of Banach spaces with the property

$$\|\delta^p_b - \delta^p_a\| \leqslant \|\sum_{j=1}^{n} c_j S_j\| < \tilde{\varepsilon}_a, \qquad 0 \leqslant p \leqslant n,$$

since $\|S_j\| = 1$ for each j. By Theorem 2.11 we obtain

$$\dim H^p(X, b; D) \leqslant \dim H^p(X, a; D) \text{ and } \mathrm{ind}_D\, b = \mathrm{ind}_D\, a.$$

In particular, when $a = (a_1, \ldots, a_n)$ is nonsingular then $b = (b_1, \ldots, b_n)$ is also nonsingular.

The hypothesis $b = (b_1, \ldots, b_n)$ be a D-commuting system is redundant. As the operator δ_b exists, it is enough to ask $R(\delta_b) \subset N(\delta_b)$.

4.9. COROLLARY. *The joint spectrum of a D-commuting system* $a = (a_1, \ldots, a_n) \subset \subset \mathscr{C}(X)$ *is a closed set in* $\mathbf{C}^n$.

Similarly, the set of those points $z \in \mathbf{C}^n$ *such that* $z - a$ *is not semi-Fredholm (Fredholm) is closed.*

The corresponding result concerning the invariance of the index under compact perturbations is given by the following

4.10. THEOREM. *Consider a D-commuting system* $a = (a_1, \ldots, a_n)$ *in* $\mathscr{C}(X)$ *and a system of compact operators* $c = (c_1, \ldots, c_n) \subset \mathscr{B}(X)$. *Suppose that* $b = = (b_1, \ldots, b_n)$ *is also D-commuting, where* $b_j = a_j + c_j$, $j = 1, \ldots, n$. *If* $a = (a_1, \ldots, a_n)$ *is semi-Fredholm (Fredholm) and* $\dim R(c_jc_k - c_kc_j) < \infty$ *for all j and k then* $b = (b_1, \ldots, b_n)$ *is semi-Fredholm (Fredholm) and* $\mathrm{ind}_D\, b = \mathrm{ind}_D\, a$.

Proof. Note that $\delta_b - \delta_a = c_1S_1 + \ldots + c_nS_n$ is compact on $\Lambda^p[\sigma, X]$, for each p. Moreover,

$$(c_1S_1 + \ldots + c_nS_n)(c_1S_1 + \ldots + c_nS_n) = \sum_{j<k} (c_jc_k - c_kc_j)S_jS_k.$$

This equality shows that the condition $\dim R(c_jc_k - c_kc_j) < \infty$ for all j and k implies the condition (3.2), therefore the conclusion can be derived from Theorem 3.7.

4.11. THEOREM. *Consider a D-commuting system* $a = (a_1, \ldots, a_n)$ *in* $\mathscr{C}(X)$, *where X is a Hilbert space. Then there are two Hilbert spaces* H_0, H_1 *and* $T_a \in \mathscr{C}(H_0, H_1)$ *with the following properties*:

(1) *The system* $a = (a_1, \ldots, a_n)$ *is Fredholm if and only if* T_a *is a Fredholm operator and in this case* $\mathrm{ind}_D\, a = \mathrm{ind}\, T_a$;

(2) *If* $(c_1, \ldots, c_n) \subset \mathscr{B}(X)$ *is a system of compact operators and* $b = (b_1, \ldots, b_n)$ *is a D-commuting system, where* $b_j = a_j + c_j$ $(j = 1, \ldots, n)$, *when* $a = (a_1, \ldots, a_n)$ *is Fredholm then* $b = (b_1, \ldots, b_n)$ *is Fredholm and* $\mathrm{ind}_D\, b = \mathrm{ind}_D\, a$;

(3) *The system* $a = (a_1, \ldots, a_n)$ *is nonsingular if and only if* $T_a^{-1} \in \mathscr{B}(H_1, H_0)$.

Proof. We consider the associated complex of Hilbert spaces $(\Lambda^p[\sigma, X], \delta_a^p)_{p=0}^n$ and apply Theorem 3.8. The spaces H_0, H_1 and the operator T_a correspond to this complex in the quoted theorem. The assertion (1) follows from Thm. 3.8 (2), the assertion (2) is a consequence of Corollary 3.9 and the assertion (3) can be derived from Thm. 3.8 (3).

Note that Theorem 4.10 contains a characterization of the nonsingularity in terms of invertibility. Similar characterizations of the nonsingularity can be found in [11] and [12].

Some results concerning almost commuting (i.e. commuting modulo the compacts) Fredholm systems of bounded operators in Hilbert spaces (and actually an idea of Fredholm complex in this context) can be found in [2].

We end this section with the following question: Is there any reasonable connection between the "weak" and the "strong" nonsingularity of a system of closed operators in an arbitrary Banach space?

5. AN EXAMPLE

Let Ω be a bounded open set in $\mathbf{C}^n$ and H a finite dimensional Hilbert space (some of the assertions which follow can be obtained in an arbitrary Hilbert space). We denote by H_Ω the space $L^2(\Omega, H)$ of all (classes of) H-valued measurable functions on Ω, whose norm is a square integrable function with respect to the Lebesgue measure. Consider also $C_0^\infty(\Omega, H)$ $(C_0^\infty(\Omega))$, the space of all indefinitely differentiable H-valued (complex-valued) functions on Ω, whose support is compact.

Notice that every areolar differential operator $\partial/\partial\bar{z}_j$, defined on $C_0^\infty(\tilde{\Omega}, H)$, where $\tilde{\Omega}$ contains the closure $\bar{\Omega}$ of Ω, is preclosed in H_Ω and denote by a_j its canonical closure ($j = 1, \ldots, n$). We shall show that the system $a = (a_1, \ldots, a_n) \subset \mathscr{C}(H_\Omega)$ is a D_*-weakly commuting one, where $D_* = C_0^\infty(\Omega, H)$ and that the $\bar{\partial}$-operator [4], [12] is strongly connected with this property of $(a_1, \ldots, a_n)$.

Let us fix a system of indeterminates $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$, which play the role of the system of differentials $\mathrm{d}\bar{z} = (\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n)$, but with no special meaning related to the points of Ω. The operators given by (4.1) and (4.2), which correspond to $\bar{\zeta}_j$, will be denoted by $\bar{Z}_j$ and $\bar{Z}_j^*$, respectively.

We recall that the $\bar{\partial}$-operator may be defined, as a $\Lambda[\bar{\zeta}, H]$-valued distribution, in the following way:

We say that $\xi \in D(\bar{\partial}) \subset \Lambda[\bar{\zeta}, H_\Omega]$ if there exists $\eta \in \Lambda[\bar{\zeta}, H_\Omega]$ with the property

$$\int \varphi(z)\eta(z)\,\mathrm{d}\lambda(z) = -\int \left(\frac{\partial\varphi}{\partial\bar{z}_1}(z)\,\bar{\zeta}_1 + \ldots + \frac{\partial\varphi}{\partial\bar{z}_n}(z)\,\bar{\zeta}_n\right) \wedge \xi(z)\,\mathrm{d}\lambda(z), \tag{5.1}$$

for all $\varphi \in C_0^\infty(\Omega)$, where $\mathrm{d}\lambda$ is the Lebesgue measure. In this case we put $\bar{\partial}\xi = \eta$ (see [12] for details).

Each element $\xi \in \Lambda^p[\bar{\zeta}, H_\Omega]$ $(0 \leqslant p \leqslant n)$ will be written in what follows as

$$\xi(z) = \sum_{1 \leqslant j_1 < \ldots < j_p \leqslant n} \xi_{j_1 \ldots j_p}(z)\,\bar{\zeta}_{j_1} \wedge \ldots \wedge \bar{\zeta}_{j_p}, \qquad z \in \Omega.$$

We remark also that the scalar product $\langle x, y\rangle$ of H combined with the Lebesgue integral defines naturally a scalar product on $\Lambda^p[\bar{\zeta}, H]$ by the formula

$$\int \langle \xi(z), \eta(z)\rangle\,\mathrm{d}\lambda(z) = \sum_{j_1 < \ldots < j_p} \int \langle \xi_{j_1 \ldots j_p}(z), \eta_{j_1 \ldots j_p}(z)\rangle\,\mathrm{d}\lambda(z).$$

5.1. LEMMA. *$\xi \in D(\bar{\partial}) \cap \Lambda^p[\bar{\zeta}, H_\Omega]$ if and only if there exists $\eta \in \Lambda^{p+1}[\bar{\zeta}, H_\Omega]$ uch that*

$$\int \langle \eta(z), \theta(z) \rangle \, d\lambda(z) = \int \langle \xi(z), \vartheta\theta(z) \rangle \, d\lambda(z),$$

or all $\theta \in \Lambda^{p+1}[\bar{\zeta}, D_*]$, where

$$\vartheta = -\left(\frac{\partial}{\partial z_1} \bar{Z}_1^* + \ldots + \frac{\partial}{\partial z_n} \bar{Z}_n^*\right).$$

Proof. Take $\xi \in D(\bar{\partial}) \cap \Lambda^p[\bar{\zeta}, H_\Omega]$ and set $\eta = \bar{\partial}\xi$. Fix also $1 \leqslant j_1 < \ldots \ldots < j_{p+1} \leqslant n$ and consider $\theta(z) = \varphi(z) \bar{\zeta}_{j_1} \wedge \ldots \wedge \bar{\zeta}_{j_{p+1}}$, where $\varphi \in C_0^\infty(\Omega, H)$ Since φ is a finite linear combination of fixed vectors of H whose coefficients are functions from $C_0^\infty(\Omega)$, the formula (5.1) applies to φ and we can write, by identification, the following equalities:

$$\int \langle \eta(z), \theta(z) \rangle \, d\lambda(z) = \int \langle \eta_{j_1 \ldots j_{p+1}}(z), \varphi(z) \rangle \, d\lambda(z) =$$

$$= -\int \sum_{m=1}^{p+1} (-1)^{m-1} \left\langle \xi_{j_1 \ldots \hat{j}_m \ldots j_{p+1}}(z), \frac{\partial \varphi}{\partial z_{j_m}}(z) \right\rangle d\lambda(z) =$$

$$= -\int \left\langle \xi(z), \sum_{m=1}^{p+1} (-1)^{m-1} \frac{\partial \varphi}{\partial z_{j_m}}(z) \bar{\zeta}_{j_1} \wedge \ldots \wedge \hat{\bar{\zeta}}_{j_m} \wedge \ldots \wedge \bar{\zeta}_{j_{p+1}} \right\rangle d\lambda(z) =$$

$$= \int \langle \xi(z), \vartheta\theta(z) \rangle \, d\lambda(z),$$

where the hat means deletion. For an arbitrary $\theta \in \Lambda^{p+1}[\bar{\zeta}, D_*]$ we obtain the conclusion by linearity.

The converse implication is similar and we omit it.

5.2. LEMMA. *$a = (a_1, \ldots, a_n)$ is a D_*-weakly commuting system.*

Proof. For any $\varphi \in D_*$ we have $a_j^* \varphi = -(\partial\varphi/\partial z_j)$ $(j = 1, \ldots, n)$, therefore, with the notations from Lemma 4.4,

$$\hat{\gamma}_{a^*} = (a_1^* \bar{Z}_1^* + \ldots + a_n^* \bar{Z}_n^*)|\Lambda[\bar{\zeta}, D_*] = \vartheta|\Lambda[\bar{\zeta}, D_*]. \tag{5.2}$$

The operator $\hat{\gamma}_{a^*}$ is preclosed since $\bar{\partial}$, which is the formal adjoint of ϑ, is densely defined. Plainly, $\hat{\gamma}_{a^*}\hat{\gamma}_{a^*} = 0$, implying the same relation for its closure.

If γ_{a^*} is the canonical closure of $\hat{\gamma}_{a^*}$, we define the operator $\delta_a^w = (\gamma_{a^*})^*$ (see the comments following the proof of Lemma 4.4).

5.3. COROLLARY. *The operator* $\bar{\partial}$ *is equal to* δ_a^w.

Proof. The assertion follows from Lemma 5.1 and the equality (5.2).

From now on we assume that Ω is a *strongly pseudoconvex domain* in the sense of [4]. Denote also by $\tilde{\Omega}$ an open set containing $\bar{\Omega}$.

5.4. LEMMA. *For every* $\xi \in D(\bar{\partial})$ *there is a sequence* $\xi_k \in \Lambda[\bar{\zeta}, C_0(\tilde{\Omega}, H)]$ *such that* $\xi_k \to \xi$ *and* $\bar{\partial}\xi_k \to \bar{\partial}\xi$ $(k \to \infty)$ *in* $\Lambda[\bar{\zeta}, H_\Omega]$. *In particular,* $a = (a_1, \ldots, a_n)$ *is a D-commuting system, where* $D = C_0^\infty(\tilde{\Omega}, H)$.

Proof. Such an approximation result is known for Hilbert-Sobolev spaces on bounded domains with smooth boundary (see, for instance, [8, Ch. 3, §4]) and the methods can be adapted to this case too. However, we shall sketch the proof of this result on a somewhat different line. Assume first that $U \subset \tilde{\Omega}$ is an analytic coordinate neighbourhood which is *star-shaped*, i.e.

$$\{tz = (tz_1, \ldots, tz_n); \quad z \in \bar{\Omega} \cap U\} \subset \Omega \cap U$$

for every $t < 1$, $t \geqslant 0$. It is known that $\bar{\Omega}$ can be covered with a finite family of such coordinate neighbourhoods [4]. If $\theta \in \Lambda[\bar{\zeta}, H_\Omega]$ and supp $\theta \subset U$ (where "supp" stands for the support) then $\lim_{t \to 1} \theta_t = \theta$ in $\Lambda[\bar{\zeta}, H_\Omega]$, where $\theta_t(z) = \theta(tz)$ [4]. Moreover, if $\theta \in D(\bar{\partial})$ then $\theta_t \in D(\bar{\partial})$ for every t and $\bar{\partial}\theta_t = t(\bar{\partial}\theta)_t$, as a consequence of the formula (5.1).

Let us fix $\xi \in D(\bar{\partial})$ with supp $\xi \subset \Omega \cap U$ and $t < 1$. Then ξ_t can be naturally extended in the set $V_t = \{z \in U;\ tz \in \Omega\}$, and keep the same notation for this extension. We assume also $\xi_t = 0$ in $U \setminus V_t$. Consider then a function $\chi \in C_0^\infty(\mathbf{C}^n)$ such that

supp $\chi = \{z;\ |z_1|^2 + \ldots + |z_n|^2 \leqslant 1\}$, $0 \leqslant \chi \leqslant 1$, $\chi(-z) = \chi(z)$ and $\int \chi(z) \mathrm{d}\lambda(z) = 1$.

For $\varepsilon > 0$ we set $\chi_\varepsilon(z) = \varepsilon^{-2n}\chi(z/\varepsilon)$. If we define the convolution product

$$\xi_{t,\varepsilon}(z) = \int \chi_\varepsilon(z - w)\, \xi_t(w)\, \mathrm{d}\lambda(w),$$

then we have $\lim_{\varepsilon \to 0} \xi_{t,\varepsilon} = \xi_t$ in $\Lambda[\bar{\zeta}, H_U]$ (see, for instance, [8]). When $z \in V_s$, where $t < s < 1$ then by (5.1) and a change of variables we infer that

$$\bar{\partial}\xi_{t,\varepsilon}(z) = \int \chi_\varepsilon(z - w)\, \bar{\partial}\xi_t(w)\, \mathrm{d}\lambda(w),$$

for a sufficiently small $\varepsilon > 0$, therefore the assertion of the lemma can be obtained in star-shaped coordinate neighbourhoods. The general assertion follows by an argument of partition of unity type.

Lemma 5.4 shows that the definition (5.1) of $\bar{\partial}$ and the definition of $\bar{\partial}$ in [4] (which is actually the stated property) are equivalent in domains with smooth boundary.

Let us consider the Cauchy-Riemann complex

(5.3) $$0 \to \Lambda^0[\bar{\zeta}, H_\Omega] \xrightarrow{\bar{\partial}^0} \dots \xrightarrow{\bar{\partial}^{n-1}} \Lambda^n[\bar{\zeta}, H_\Omega] \to 0$$

which is semi-Fredholm and for which

$$\dim N(\bar{\partial}^p)/R(\bar{\partial}^{p-1}) < \infty, \quad p = 1, \dots, n,$$

where $\bar{\partial}^p$ is the restriction of $\bar{\partial}$ on $\Lambda^p[\bar{\zeta}, H_\Omega]$. This assertion is a consequence of the theory developed by J. J. Kohn in [4]. In fact, in this case we have actually $N(\bar{\partial}^p) = R(\bar{\partial}^{p-1})$ for $p \geq 1$, via the Grauert theorem asserting that strongly pseudoconvex domains are holomorphically convex (see also [4]) and the well-known Theorem B of Cartan. We shall combine this property of $\bar{\partial}$ with our statements from the previous section in order to obtain some significant results.

Consider the functions $c_j : \tilde{\Omega} \to \mathscr{B}(H)$, which are analytic in $\tilde{\Omega}$, such that $c_j(z)c_k(z) = c_k(z)c_j(z)$, for all $z \in \bar{\Omega}$ and $j, k = 1, \dots, n$. Define then on $\Lambda^p[\bar{\zeta}, H_\Omega]$ the continuous operator

$$(\delta_c^p \xi)(z) = (c_1(z)\bar{Z}_1 + \dots + c_n(z)\bar{Z}_n)\xi(z).$$

As $\bar{\partial}^{p+1}\delta_c^p\xi = -\delta_c^{p+1}\bar{\partial}^p\xi$ for $\xi \in D(\bar{\partial}^p)$, we have that

(5.4) $$(\Lambda^p[\bar{\zeta}, H_\Omega], \bar{\partial}^p + \delta_c^p)_{p=0}^n$$

is a complex of Hilbert spaces.

5.5. Proposition. *There is an $\varepsilon_0 > 0$ such that if*

$$\max_{1 \leq j \leq n} \sup_{z \in \Omega} \|c_j(z)\| < \varepsilon_0$$

then the complex (5.4) is semi-Fredholm. In particular, the equation $(\bar{\partial}^{p-1} + \delta_c^{p-1})\xi = \eta$ has a solution $\xi \in \Lambda^{p-1}[\bar{\zeta}, H_\Omega]$, for every $\eta \in N(\bar{\partial}^p + \delta_c^p)$ and $p \geq 1$.

Proof. The assertion follows from Lemma 5.4, Theorem 4.8 and from the mentioned properties of the complex (5.3).

Taking instead of $\Lambda^0[\bar{\zeta}, H_\Omega] = H_\Omega$ the orthocomplement in H_Ω of the space of analytic square integrable H-valued functions then the analogues of the complexes (5.3) and (5.4) are actually Fredholm and their indexes are null.

Added in proof. A variant of Lemma 2.1, stated for bounded operators and with a different proof, has been obtained also by V. Pták (*Commentationes Mathematicae,* **21**(1978), 343–348).

Gr. Segal has defined a concept of Fredholm complex in the context of vector-bundles (*Quart. J. Math. Oxford,* **21**(1970), 385–402). One can see that, locally, the concept given by our Definition 1.1 on Banach spaces is more general.

REFERENCES

1. BANACH, S., *Théorie des opérations linéaires*, Warsaw, 1932.
2. CURTO, R. E., Fredholm and invertible tuples of bounded linear operators, *Dissertation*, State Univ. of New York at Stony Brook, 1978.
3. KATO, T., *Perturbations Theory for Linear Operators*, Springer-Verlag, Berlin-Heidelberg-New York, 1966.
4. KOHN, J. J., Harmonic integrals on strongly pseudoconvex manifolds I, *Ann. of Math.*, **78** (1963), 112–148.
5. KREĬN, M. G.; KRASNOSEL'SKIĬ, M. A., Stability of the index of an unbounded operator (Russian), *Mat. Sb.*, **30** (1952), 219–224.
6. KREĬN, M. G.; KRASNOSEL'SKIĬ M. A.; MIL'MAN, D. C., On the defect numbers of linear operators in Banach space and on some geometric problems (Russian), *Sbornik Trud. Inst. Mat. Akad. Nauk Ukrain SSR*, **11** (1948), 97–112.
7. MACLANE, S., *Homology*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.
8. MIZOHATA, S., *The Theory of Partial Differential Equations* (Russian), The Publishing House "MIR", Moscow, 1977.
9. PALAIS, R. S., Seminar on the Atiyah-Singer Index Theorem, Princeton Univ. Press, Princeton, New Jersey, 1965.
10. TAYLOR, J. L., A joint spectrum for several commuting operators, *J. Functional Analysis*, **6** (1970), 172–191.
11. VASILESCU, F.-H., *Multi-Dimensional Analytic Functional Calculus* (Romanian), The Publishing House of the Academy, Bucharest, 1979.
12. VASILESCU, F.-H., Analytic perturbations of the $\bar{\partial}$-operator and integral representation formulas in Hilbert spaces, *J. Operator Theory*, **1** (1979), 187–205.

F.-H. VASILESCU
National Institute for Scientific and Technical Creation,
Department of Mathematics,
Bd. Păcii 220, 77538 Bucharest,
Romania

Received March 27, 1979.