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AN EXTENSION OF SCOTT BROWN'S INVARIANT SUBSPACE THEOREM: K-SPECTRAL SETS

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Recently, Scott Brown showed that every subnormal operator has an invariant subspace. Similar techniques can be used to show that every operator in $\mathscr{L}(\mathscr{H})$, the algebra of bounded linear operators on a separable Hilbert space $\mathscr{H}$, for which $\sigma(T)$ is a K-spectral set also has an invariant subspace. This result has been proved by J. Agler [30], when $\sigma(T)$ is a spectral set for T. There are more differences between spectral sets and K-spectral sets than might be apparent at first glance. First, dilation theory is available in the former case but not in the latter. Second, orthogonality disappears in several places as one moves from spectral to K-spectral sets. Third, there are several interesting special cases such as polynomially bounded operators and unitary ρ-dilations which are not covered by spectral sets.

DEFINITION. The compact set $M \supset \sigma(T)$ is a *K-spectral set for* $T \in \mathscr{L}(\mathscr{H})$ if

$$\|f(T)\| \leqslant K\|f\|_{\infty}^{M}$$

for all $f \in R(M)$ where

$$\|f\|_{\infty}^{M} = \sup\{|f(z)|: \ z \in M\}.$$

($R(M)$ denotes the uniform closure of the rational functions with poles off M.)

To begin with, we need an extension of the orthogonal direct sum decomposition for operators proved independently by Mlak [21] and Lautzenheiser [17] in the spectral set case. It should be mentioned that although K-spectral sets are never mentioned in [17] and only very briefly in [21], still many of the techniques carry over from their work. (See also [23]).

THEOREM 1. *Let $T \in \mathscr{L}(\mathscr{H})$. Assume M is a K-spectral set for T. Let $G_1, G_2, \ldots$ be the nontrivial Gleason parts of $R(M)$. Then there exists an invertible operator Q such that*

$$QTQ^{-1} = S = \bigoplus_{i=0}^{\infty} S_i$$

where S_0 *is normal and* $\sigma(S_i) \subset \bar{G}_i$ *for* $i = 1, 2, \ldots$. *Thus* $T = \dotplus T_i$ *(direct sum).* (Note: Some of the terms may be absent.)

Because of its length we have relegated the proof of Theorem 1 to the Appendix. However, we will invoke the notation and techniques in Proposition 5.

From now on we will be using K-spectral sets M where $R(M)$ is a Dirichlet algebra (see [10] for a definition). It is well known (see [24], Lemma 4.1 that the non-trivial Gleason parts of $R(M)$ in this case are $G_1, G_2, \ldots$ where the G_i's are the components of int$M$. (The G_i's must be simply connected). Combining these facts with Theorem 1 we obtain the following:

COROLLARY. *Let M be a K-spectral set for $T \in \mathscr{L}(\mathscr{H})$ where $R(M)$ is Dirichlet. Let $G_1, G_2, \ldots$ be the nontrivial Gleason parts for $R(M)$. If $\sigma(T) \cap G_j \neq \emptyset$ for two distinct j's then T has a complemented invariant subspace.*

Let $\sigma(T)$ be a K-spectral set for T. We wish to develop a functional calculus for T, and to do so, we must choose a more tractable spectral set than $\sigma(T)$ itself. The next lemma selects such a set which is both topologically nice and analytically minimal. Parts of the proof are taken from [4].

LEMMA 1. *Let $\sigma(T)$ be a K-spectral set for T and assume T has no complemented invariant subspaces. Then there exists an open connected, simply connected set G, such that $\bar{G} \supset \sigma(T)$, $R(\bar{G})$ is Dirichlet and*

$$\|h\|_\infty = \sup\{|h(\lambda)| : \lambda \in \sigma(T) \cap G\} \qquad \text{for all } h \in H^\infty(G).$$

Proof. To facilitate matters, we begin by listing the results we shall need. Throughout, M is a compact set in $\mathbf{C}$.

PROPOSITION 1. *Let M be a K-spectral set for T where $R(M)$ is a Dirichlet algebra. If $\sigma(T) \not\subset [\overline{\text{int } M}]$ then T has a complemented invariant subspace. In particular, if* int $M = \emptyset$ *then T is similar to a normal operator.*

Proof. $R(M)$ Dirichlet implies $R(\partial M) = C(\partial M)$. Thus, if $\sigma(T) \not\subset [\overline{\text{int } M}]$, then there exists a disc D_0 such that

$$R(\bar{D}_0 \cap \sigma(T)) = C(\bar{D}_0 \cap \sigma(T)) \neq \emptyset.$$

Thus the techniques of [7] apply.

PROPOSITION 2. ([11], Cor. 9.6) *Let M_k be compact, and $M_k \supset M_{k+1}$ for $k = 1, 2, \ldots$. If $R(M_k)$ is Dirichlet, then $R\left(\bigcap_{k=1}^{\infty} M_k\right)$ is Dirichlet.*

PROPOSITION 3. ([11], Theorem 9.3) *$R(M)$ is a Dirichlet algebra if and only if:*
i) *the components of* int M *are simply connected*

and

$$\text{ii) } \liminf \frac{\gamma(\Delta(z,\delta)\setminus M)}{\delta} > 0 \qquad \textit{for all } z \in \partial M.$$

(γ is the analytic capacity).

We have included Proposition 3, because it makes transparent why the next proposition is true.

PROPOSITION 4. *Let $N_1, N_2, \ldots$ be the components of* int M *where $R(M)$ is Dirichlet. Then $R(M\setminus N_1)$ is Dirichlet.*

Proof. It follows immediately from Proposition 3 if one notes that $\gamma(E) \geqslant (\operatorname{diam} E)/4$ for E connected. See also Cor. 9.7 of [11].

Proof of Lemma 1. Suppose we begin with a compact K-spectral set M for T where $R(M)$ is Dirichlet. For example, take $M = \widehat{\sigma(T)}$, the polynomially convex hull of $\sigma(T)$. We may assume that $\sigma(T) \subset \overline{\operatorname{int} M}$ by Proposition 1. Let $N_1, N_2, \ldots$ be the components of int M. By Theorem 1, we may assume that at most one component, say N_1, meets $\sigma(T)$. Set $M_1 = M\setminus N_2$, $M_2 = M_1\setminus N_3$, and so on. Then $R(M_j)$ is Dirichlet for each j, and hence $R(\cap M_j)$ is Dirichlet. Let $E = \cap M_j$. Then we are reduced to the following situation: E is a K-spectral set for T, $R(E)$ is Dirichlet, $\sigma(T) \subset \overline{\operatorname{int} E}$ and int E has just one component, say U.

We now ask whether

$$\|h\|_\infty = \sup\{|h(z)| : z \in \sigma(T) \cap U\}$$

for all $h \in H^\infty(U)$. If yes, we are done and set $G = U$. If no, we proceed as follows. Let φ be a conformal map of the unit disc D onto U. For $h \in H^\infty(U) = H^\infty(m_U)$ set $\hat{h}(z) = h(\varphi(z))$ for $z \in D$. Then $\hat{h} \in H^\infty(D)$ and $\|\hat{h}\|_\infty = \|h\|_\infty$. By assumption,

$$\sup\{|\hat{h}(z)| : \varphi(z) \in \sigma(T) \cap U\} < \|\hat{h}\|_\infty$$

for some $\hat{h} \in H^\infty(D)$. Thus, by well known properties of radial limits, there exists a set $\Theta \subset [0,2\pi]$ of positive measure such that for each $\theta \in \Theta$, there is a segment $L_\theta = [r_\theta e^{i\theta}, e^{i\theta})$ where $\varphi(L_\theta) \subset U\setminus\sigma(T)$. Choose a θ where $\lim_{r\to 1} \varphi(re^{i\theta})$ exists and call the limit $a \in \partial E$. Note that $\varphi(L_\theta)$ is contained in a component C of $E\setminus\sigma(T)$, which extends to ∂E. The set C must be simply connected else $\sigma(T)$ is disconnected. Since U was simply connected to begin with and C extends to ∂E, the components of int $E\setminus C$ must be simply connected. It follows from Proposition 3, by arguing as in Proposition 4, that $R(E\setminus C)$ is Dirichlet. (See also Cor. 9.7 of [11].) Note that int $E\setminus C$ need not be connected.

We may now start our reduction all over again with M replaced by $E\setminus C$, and argue by transfinite induction. We may assume $\overline{\operatorname{int} E\setminus C} \supset \sigma(T)$, by Proposition 1. Note that at each step, we obtain a compact set M_δ, where $\sigma(T) \subset M_\delta$ and $R(M_\delta)$ is Dirichlet. Note that $\sigma(T) \subset \overline{\text{int } M_\delta}$ at every stage, by Proposition 1. By construction $M_\delta \supset M_\beta$ for $\delta < \beta$. If $M_{\delta+1}$ has an immediate predecessor M_δ we show $R(M_{\delta+1})$ is Dirichlet by Proposition 4. If M_β is a limit ordinal ($M_\beta = \bigcap_{\delta<\beta} M_\delta$) then we show $R(M_\beta)$ is Dirichlet by Proposition 2. Note that at every step $M_\delta \setminus M_{\delta+1}$ is a nonempty open set. Thus, the process must terminate at some countable ordinal α and $M_{\alpha+1} = M_\alpha$ for that α. We then set $G = \text{int } M_\alpha$. For this choice of G, it follows that

$$\|h\|_\infty = \sup\{|h(z)| : z \in \sigma(T) \cap \text{int } M_\alpha\}$$

for all $h \in H^\infty(\text{int } M_\alpha)$. The proof is complete.

The proof obviously follows that of Theorem 2 of [24] in shape and form. Unfortunately, there seems to be no way to apply that theorem directly.

In Theorem 1 we developed a functional calculus for functions in $R(M)$. We next extend this to H^∞ functions. Let G be as in Lemma 1. Since $R(\bar{G})$ is Dirichlet, $H^\infty(G) = H^\infty(m_G)$ where m_G is harmonic measure on ∂G and

$$H^\infty(m_G) = H^2(m_G) \cap L^\infty(m_G)$$

(see [28], Section 8 for details). Observe that $R(M)$ is pointwise boundedly (sequentially) dense in $H^\infty(G)$ in the weak-* topology by [11], Theorem 5.1. (Actually, more is true—see [24], Lemma 4.3.) The functional calculus begins with:

Proposition 5. *Let G be an open, connected simply connected set where $\bar{G}$ is a K-spectral set for T, and $R(\bar{G})$ is Dirichlet. Assume T has no complemented invariant subspaces. Then there exists a homomorphism Γ of $H^\infty(G)$ into $\mathscr{L}(\mathscr{H})$ where $\Gamma: h \to h(T)$ and $\|h(T)\| \leqslant K\|h\|_\infty$ for all $h \in H^\infty(G)$.*

Proof. It follows from Theorem 1 that our decomposition for T contains only a single term. Thus if $\mu(x, y)$ is an elementary measure then $\mu(x, y) << m_G$ for all $x, y \in \mathscr{H}$. For $h \in H^\infty(G)$, choose rational functions $q_n \in R(\bar{G})$ such that $q_n \to h$ weak-* (that is pointwise boundedly in G).

Set

$$(h(T)x, y) = \int h \, d\mu(x, y) = \lim \int q_n \, d\mu(x, y) \qquad \text{for } x, y \in \mathscr{H}.$$

It is easy to see that $(h(T)x, y)$ is well defined and the resulting operator is linear. Since

$$|(h(T)x, y)| \leqslant \int |h| \, d|\mu(x, y)| \leqslant \|h\|_\infty K\|x\| \|y\|$$

it follows that

$$\|h(T)\| \leqslant K\|h\|_\infty .$$

For the multiplicativity of Γ, first show that $\Gamma(fq) = \Gamma(f)\Gamma(q)$ for $f \in H^\infty(G)$ and q a rational function. Then handle the general case by approximating $g \in H^\infty(G)$ weak-* by rational functions q_n.

From now on we assume G has the properties assigned it in the last proposition.

LEMMA 2. ([4], weak form of Lemma 4.2). *Let T be as in Proposition 5. Let $\lambda \in \sigma(T) \cap G$. Let $h \in H^\infty(G)$. Assume $\|T - \lambda)x\| < \varepsilon$ where $x \in \mathscr{H}$ and $\|x\| = 1$. Then,*

$$|(h(T)x, x) - h(\lambda)| < 2\varepsilon d^{-1}K\|h\|_\infty$$

where $d = \operatorname{dist}[\lambda, \partial G]$. In particular, if $\|(T - \lambda)x_m\| \to 0$ for a sequence of unit vectors, then

$$\lim_n (h(T)x_n, x_n) = h(\lambda).$$

Proof. We may write $h(z) - h(\lambda) = (z - \lambda)g(z)$ where $g \in H^\infty(G)$. By the maximum modulus principle, it follows that

$$\|g\|_\infty \leqslant 2\|h\|_\infty d^{-1}.$$

Thus

$$|((h(T) - h(\lambda))x, x| = |(g(T)(T - \lambda)x, x)| \leqslant$$

$$\leqslant \|(T - \lambda)x\| \, \|g(T)^*x\| \leqslant 2\varepsilon d^{-1}K\|h\| \, \|h\|_\infty .$$

The second statement is obvious.

COROLLARY. *Let $h \in H^\infty(G)$. Assume $\sigma(T)$ is all approximate point spectrum. Then*

$$\|h\|_\infty \leqslant \|h(T)\| \leqslant K\|h\|_\infty .$$

Thus, Γ is bounded above and below.

Proof. We already know $\|h(T)\| \leqslant K\|h\|_\infty$. Let $h \in H^\infty(G)$ and choose $\lambda \in \sigma(T) \cap G$ such that

$$\|h\|_\infty \leqslant |h(\lambda)| + \varepsilon.$$

Then, choose a unit vector $x \in \mathscr{H}$ such that

$$|h(\lambda)| \leqslant |(h(T)x, x)| + \varepsilon.$$

Hence

$$\|h\|_\infty \leqslant |(h(T)x, x)| + 2\varepsilon \leqslant \|h(T)\| + 2\varepsilon.$$

Since ε is arbitrary, we are done.

If $\sigma(T)$ is not all approximate point spectrum, then standard arguments show that T has an invariant subspace; so we can make the following:

Assumption. From now on we will assume $\sigma(T)$ is all approximate point spectrum.

Notation. Let $\mathscr{R}_T$ denote the weak-$*$closure of the rational functions in T with poles off $\bar{G} = M$. It is easy to see that Γ maps $H^\infty(G)$ into $\mathscr{R}_T$.

LEMMA 3. *Let T be as above. Then Γ maps $H^\infty(G)$ onto $\mathscr{R}_T$.*

Proof. We will need the following theorem of Banach ([2], page 213). Let M be a linear manifold in a separable Banach space. Let $M^1 = M$ and for each countable ordinal α let M^α be the set of weak-$*$ limits of convergent sequences in $\bigcup_{\beta<\alpha} M^\beta$. Then the weak-$*$ closure of M equals $\bigcup_\alpha M^\alpha$ and the M^α are all equal from some countable ordinal on.

To apply the theorem, we set M^1 = rational functions in T with poles off $\bar{G}$. Let $B = \lim q_n(T)$, q_n rational. Since $\|q_n(T)\| \leqslant c$ (c constant) it follows that $\|q_n\|_\infty \leqslant c$ for all n. Choose a weak-$*$convergent subsequence of $\{q_n\}$ (still denoted by $\{q_n\}$) which converges to $h \in H^\infty(G)$. Then for every $x, y \in \mathscr{H}$,

$$(h(T)x, y) = \int h \, \mathrm{d}\mu(x, y) = \lim_n \int q_n \mathrm{d}\mu(x, y) =$$

$$= \lim (q_n(T)x, y) = (Bx, y).$$

Thus, $B = h(T)$ so $M^2 \subset \Gamma(H^\infty(G))$. The rest of the transfinite induction proceeds in the same way if we appeal to the Corollary to Lemma 2.

We summarize the preceding lemmas.

COROLLARY. *Let T be as in Proposition 1. Then Γ maps $H^\infty(G)$ onto $\mathscr{R}_T$ and the norms are equivalent.*

Notation. For $\lambda \in G$ set $C_\lambda(h(T)) = h(\lambda)$ for $h(T) \in \mathscr{R}_T$. Then C_λ is clearly a weak-* continuous linear functional on $\mathscr{R}_T$. By $(x \otimes y)$ we denote the linear functional defined by $(x \otimes y)(B) = (Bx, y)$ for $B \in \mathscr{L}(\mathscr{H})$. By $\| \quad \|_*$ we denote the norm of a linear functional in $\mathscr{C}_1$ restricted to $\mathscr{R}_T$ ($\mathscr{C}_1$ = trace class). An excellent discussion of the duality between $\mathscr{C}_1$ and $\mathscr{L}(\mathscr{H})$ and the norm $\| \quad \|_*$ can be found in [4].

We may now go back and strengthen Lemma 2 as follows.

LEMMA 2′. ([4], Lemma 4.2). *Let $\lambda \in \sigma(T) \cap G$. Let $\|(T-\lambda)x_n\| \to 0$ for a sequence of unit vectors $x_n \in \mathscr{H}$. Then*

$$\|C_\lambda - (x_n \otimes x_n)\|_* \to 0.$$

Indeed,

$$\|C_\lambda - (x \otimes x)\|_* \leqslant 2d^{-1}K\|(T-\lambda)x\|.$$

The rest of the argument for the main result follows [4] very closely. However, to overcome one difficulty we will have to switch back to the disc and appeal to a clever result from [5].

To this end we first note that $H^\infty(G)$ and $H^\infty(D)$ are clearly isometrically isomorphic. Let φ be a conformal map of D onto G and set $\varphi^{-1} = \psi$. Set $S = \psi(T)$. Since $\psi \in H^\infty(G)$, $\psi(T)$ is well defined.

LEMMA 4. *Let S, T, ψ, φ be as above. Then*

1) *S has the disc $\bar{D}$ as a K-spectral set.*

2) *If $\lambda \in \sigma(T) \cap G$ then $\psi(\lambda) \in \sigma(S)$.*

3) *$\varphi(S) = T$ and S and T have the same invariant subspaces.*

4) *For every $h \in H^\infty(D)$*

$$\|h\|_\infty^D = \sup\{|h(z)| : z \in \sigma(S) \cap D\}.$$

5) *$\mathscr{R}_S$ is isomorphic to $H^\infty(D)$ and the norms are equivalent.*

6) $\mathscr{R}_S = \mathscr{R}_T$.

Proof. 1) Let p be a polynomial. Since $\Gamma: h \to h(T)$ is a homomorphism for $h \in H^\infty(G)$ it follows that

$$\|p(S)\| = \|p \circ \psi(T)\| \leqslant K\|p \circ \psi\|_\infty^G = K\|p\|_\infty^D$$

proving 1).

2) Let $\lambda_0 \in \sigma(T) \cap G$ and assume $\|(T - \lambda_0)x_n\| \to 0$ for a sequence of unit vectors $\{x_n\}$ (not necessarily orthogonal). Set

$$\psi(\lambda) - \psi(\lambda_0) = (\lambda - \lambda_0) g(\lambda).$$

Then

$$\|(S - \psi(\lambda_0))x_n\| = \|g(T)(T - \lambda_0)x_n\| \leqslant$$

$$\leqslant \|g(T)\| \, \|(T - \lambda_0)x_n\| \to 0.$$

This result does not depend on the Assumption. Indeed, if $(T - \lambda_0)$ is bounded below we turn our attention to $(T - \lambda_0)^*$ which *can not* be bounded below. Since the conjugate set $\overline{G}$ is a K-spectral set for T^* we may define S^* in analogous fashion and repeat the proof above.

3) If $f \in H^\infty(D)$, then the composition law $f(S) = f \circ \psi(T)$ is valid. This is easy to check if f is a polynomial and therefore holds in general by weak-$*$continuity (see [23], page 298). In particular, $\varphi(S) = T$, which implies that S and T have the same invariant subspaces.

4) Follows immediately from 2), the fact that

$$\|h\|_\infty^G = \sup\{|h(\lambda)|: \lambda \in \sigma(T) \cap G\} \qquad \text{for all } h \in H^\infty(G)$$

and the isometric isomorphism between $H^\infty(G)$ and $H^\infty(D)$ induced by φ (or ψ).

5) Follows from 2), 4) and the first part of this paper.

6) Follows from the relations $\psi(T) = S$, $\varphi(S) = T$.

LEMMA 5. ([4], Lemma 4.3.) *Let $\alpha, \beta \in \sigma(T)$. Let $\{x_n\}$, $\{y_n\}$ be mutually orthogonal orthonormal sequences where $\|(T - \alpha)x_n\| \to 0$ and $\|(T - \beta)y_n\| \to 0$. Assume T has no invariant subspaces. Then*

a) $\|x_n \otimes y_n\|_* \to 0$

b) $\|x_n \otimes w\|_* \to 0$ *for all $w \in \mathscr{H}$*

c) $\|w \otimes x_n\|_* \to 0$ *for all $w \in \mathscr{H}$.*

Proof. Parts a) and b) follow directly from [4] or one may imitate the proof in Lemma 2.

Part c) is considerably more difficult in the present context and we appeal to an ingenious argument from [5]. Since $\mathscr{R}_S = \mathscr{R}_T$ we may assume S is our primary operator if we wish. Part c) simply asserts that the linear functional $(w \otimes x_n) \to 0$ on $\mathscr{R}_S$. Our operator S has no invariant subspaces by 3) of Lemma 4. Thus the proof of Lemma 4.5 of [5] applies verbatim to prove c). We observe that the only property of the sequence $\{x_n\}$ used in the proof of c) is weak convergence to 0. We also remark that although the authors of [5] consider contractions S rather than power bounded operators; their proof handles the latter case.

REMARK. We introduced S and returned to the disc solely to facilitate the proof of part c) above. With that out of the way we focus on T.

LEMMA 6. ([4], Lemma 4.4.) *Let*

$$B' = \left\{ \sum_1^n \alpha_j C_{\lambda_j} : \sum |\alpha_j| = K \text{ and } \lambda_j \in \sigma(T) \cap G \right\}^{-\text{weak-}*}$$

Then $B' \supset$ unit ball $(\mathscr{R}_T)_*$.

Proof. Same as [4]. It is here that the comparability of $\|h\|_\infty$ and $\|h(T)\|$ is important. Note the K in the definition of B'.

LEMMA 7. ([4], Lemma 4.5) *Let* $s_n, s'_n \in \mathscr{H}$ *where* $\|C_0 - (s_n \otimes s'_n)\|_* < s^{-2n}$ ($C_0 = C_\lambda$ *for some* $\lambda \in G$). *Then there exist* $s_{n+1}, s'_{n+1} \in \mathscr{H}$ *such that*

1) $\|s_n - s_{n+1}\| < K2^{-n}$ *and* $\|s'_n - s'_{n+1}\| < K2^{-n}$

2) $\|C_0 - (s_{n+1} \otimes s'_{n+1})\|_* < 2^{-2(n+1)}$.

Proof. Same as [4]. The proof involves a choice of many orthonormal sequences $\{x_{ij}\}$ where $\lim_i \|(T - \lambda_j)x_{ij}\| \to 0$. We choose them mutually orthogonal in view of Lemma 5.

THEOREM 2. ([4], Theorem 4.6). *There exist vectors* $u, v \in \mathscr{H}$ *such that* $C_0 = u \otimes v$.

Proof. Let $u = \lim s_n$ and $v = \lim s'_n$.

We are now in a position to prove the main result.

THEOREM 3. *Let* $\sigma(T)$ *be a K-spectral set for T. Then T has an invariant subspace.*

Proof. Let

$$\mathscr{M} = \text{clm}\{(T - \lambda)^k u : k = 1, 2, \ldots\}$$

for the u of Theorem 2. Then

$$((T - \lambda)^k u, v) = C_0((z - \lambda)^k) = 0$$

for $k \geqslant 1$.
Since $\mathscr{M} = 0$ implies $(T - \lambda)u = 0$, the proof is complete.

COROLLARY 1. *Let T be a polynomially bounded operator with* $\sigma(T) \supset \overline{D}$. *Then T has an invariant subspace.*

REMARK. T is polynomially bounded if

$$\|p(T)\| \leqslant K\|p\|_\infty$$

for all polynomials p, where $\|p\|_\infty = \sup\{|p(z)| : |z| \leqslant 1\}$. Note that the conclusion of Lemma 1 is automatically satisfied.

COROLLARY 2. *Let $T \in \mathscr{L}(\mathscr{H})$ satisfy the conditions*

1) $\partial D \subset \sigma(T) \subset \bar{D}$

2) $\|(T-\lambda)^{-1}\| \leqslant \dfrac{1}{\operatorname{dist}[\lambda, \sigma(T)]}$ *for* $\lambda \notin \sigma(T)$.

Then T has an invariant subspace.

Proof. It follows from condition 2) that $T \in C_p$ (in particular, in C_2), and thus T is similar to a contraction S ([27]). Thus, $\partial D \subset \sigma(S) \subset \bar{D}$ and 2) becomes

$$\|(S-\lambda)^{-1}\| \leqslant \frac{K_0}{\operatorname{dist}[\lambda, \sigma(S)]} \quad \text{for } \lambda \notin \sigma(S).$$

In the framework of Lemma 1 we take $G = D$. If

$$\|h\|_\infty = \sup\{|h(z)|: z \in \sigma(S) \cap D\} \text{ for all } h \in H^\infty(D),$$

then we are done since the remaining lemmas and theorems are valid in this context.

Assume therefore there exists an $h \in H^\infty(D)$ such that

$$\sup\{|h(z)|: z \in \sigma(S) \cap D\} < \|h\|_\infty - \varepsilon.$$

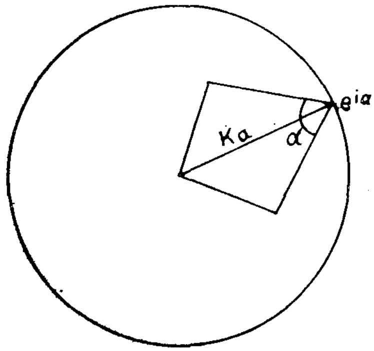
Then there exists a set $\Theta \subset [0,2\pi]$ of positive measure such that for each $\theta \in \Theta$, there is a segment $L_\theta = [r_\theta e^{i\theta}, e^{i\theta})$ where $L_\theta = D \diagdown \sigma(S)$ and

$$\lim_{r \to 1} |h(re^{i\theta})| \geqslant \|h\|_\infty - \frac{\varepsilon}{2}.$$

Since $D \diagdown \sigma(S)$ has only countably many components, there is some component C which contains two $L_\theta's$; say L_a and L_b. (This part of the proof follows [31], Theorem 3.) We connect $r_a e^{ia}$ to $r_b e^{ib}$ by a Jordan arc τ lying in C. Let $\gamma = L_a \cup L_b \cup \tau$. Thus, $\bar{\gamma}$ separates $\bar{D}$ into two components each of which intersects $\sigma(S)$. Let K_a and K_b denote the kite shaped regions at e^{ia} and e^{ib}, respectively, where $\alpha = \pi/2$. Then it follows from Theorem 1.3 of [9], that

$$\sigma(S) \cap [K_a \cup K_b] = \emptyset$$

near ∂D. Given the separation of the spectrum induced by K_a and K_b it is possible to integrate around the spectrum on γ plus anything reasonable outside $\bar{D}$ to produce invariant subspaces. The details of the integration are carried out in Examples 1 and 2 following Theorem 1 of [25]. (In that paper one integrates across the spectrum at one point rather than two as here, but the details are the same.)



COROLLARY 3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator and assume $\partial D \subset \sigma(T) \subset \subset \bar{D}$. Then T has an invariant subspace.*

DEFINITION. Let Γ be a Jordan curve. For any two points $z_1, z_2 \in \Gamma$ set

$$Q(z_1, z_2) = \frac{\operatorname{diam} \widehat{z_1 z_2}}{|z_1 - z_2|}$$

when $\widehat{z_1 z_2}$ is that arc of $\Gamma \setminus \{z_1 \cup z_2\}$ of the smaller Euclidean diameter. If $Q(z_1, z_2) \leqslant C$ for some constant C and all $z_1, z_2 \in \Gamma$, then Γ is of *bounded turning* (or cusp free).

I would like to thank Glenn Schober for pointing out the role of quasi-conformal mappings in problems such as this and for suggesting the argument in the next.

COROLLARY 4. *Let $T \in \mathcal{L}(\mathcal{H})$. Assume*

1) *$p(T)$ is hyponormal for all polynomials p and*

2) *$\partial \widehat{\sigma(T)}$ is a cusp free (or bounded turning) Jordan curve Γ.*

Then T has an invariant subspace.

Proof. It follows immediately from 1) and ([16], p. 106) that $M = \widehat{\sigma(T)}$ is a spectral set for T. Let $G = \operatorname{int} M$. We consider two cases.

Case 1.

$$\|h\|_\infty = \sup\{|h(z)| : z \in \sigma(T) \cap G\} \text{ for all } h \in H^\infty(G).$$

Since the conclusion of Lemma 1 holds, the rest of the proof goes through as before.

Case 2.

$$\sup\{|h_0(z)| : z \in \sigma(T) \cap G\} < \|h_0\|_\infty$$

for some $h_0 \in H^\infty(G)$. Let φ be the conformal map of D onto G. Let $\hat{h}(z) = h(\varphi(z))$ for $z \in D$ and $h \in H^\infty(G)$. Then $\hat{h} \in H^\infty(D)$ and $\|\hat{h}\|_\infty = \|h\|_\infty$. Arguing as in Corollary

2 and Lemma 1 we see there exists segments L_a and L_b such that

$$[\varphi(L_a) \cup \varphi(L_b)] \cap \sigma(T) = \emptyset,$$

and $\varphi(L_a)$ and $\varphi(L_b)$ are in the same component of $G \setminus \sigma(T)$. Thus, if K_a and K_b are the kite shaped regions at e^{ia} and e^{ib}, respectively, then $\varphi^{-1}(\sigma(T)) \cap [K_a \cup K_b] = \emptyset$ near ∂D, and hence $[\varphi(K_a) \cup \varphi(K_b)] \cap \sigma(T) = \emptyset$ near ∂G. We must show that $\varphi(K_a)$ and $\varphi(K_b)$ induce a separation of $\sigma(T)$.

Since Γ is of bounded turning, it is quasi-conformal by [19], Theorem II 8.6. Since Γ is quasi-conformal, there exists a quasi-conformal extension $\varphi^{\#}$ of φ to a domain $D_1 \supset \bar{D}$ by Theorem II 8.2 of [19]. Since $\varphi^{\#}$ is quasi-conformal, it follows from Theorem 4, part i) of [1] that $\varphi^{\#}(K_a)$ and $\varphi^{\#}(K_b)$ both subtend positive angles at $\varphi(e^{ia})$ and $\varphi(e^{ib})$, respectively. Thus it is possible to join $\alpha = \varphi(e^{ia})$ and $\beta = \varphi(e^{ib})$ by a polygonal arc γ which lies in $G \cup \{\alpha, \beta\}$ and which satisfies the condition

$$\frac{\text{dist}[\lambda, \partial G]}{\text{dist}[\lambda, \sigma(T)]} \leqslant C_1 \text{ (constant)}$$

for all $\lambda \in \gamma$. Since Γ is of bounded turning, it is easy to join α and β by a polygonal arc γ' which lies in $\complement \bar{G}$ and satisfies a similar condition. It is now possible to cut across (or integrate through) the spectrum of T on $\gamma \cup \gamma'$ precisely as in Example 1 to Theorem 1 of [25] to produce an invariant subspace for T.

REMARK 1. While "bounded turning" prohibits cusps on the curve Γ, such curves need not be C^1 or rectifiable. Indeed, examples of nowhere locally rectifiable curves of bounded turning may be found in [19], p. 104.

COROLLARY 4′ *Let* $T \in \mathscr{L}(\mathscr{H})$. *Assume*

1) $p(T)$ *is hyponormal for all polynomials* p

and

2) $\partial \widehat{\sigma(T)}$ *is a rectifiable Jordan curve* Γ.
Then T *has an invariant subspace.*

Proof: Follows the lines of Corollary 4. Let φ be the conformal map of D onto $G = \text{int} M$ where $M = \widehat{\sigma(T)}$. Then φ extends to a homeomorphism of ∂D onto ∂G and $\varphi'(e^{it}) \neq 0$ a.e. on ∂D (see [14] X, 1.1 and X, 1.3).

Since we can omit the set where $\varphi'(e^{it}) = 0$ (it has measure zero) and φ is conformal on ∂D on the complement of this set, the proof in Corollary 4 goes through as before.

COROLLARY 5. *Let* $T \in C_\rho$ *for* $\rho \geqslant 1$ *and assume* $\sigma(T) \supset \bar{D}$. *Then* T *has an invariant subspace.*

Proof: If $T \in C_\rho$(i.e., $T^n = \rho\, PU^n|\mathscr{H}$ for $n = 1, 2, \ldots$) then $\|p(T)\| \leqslant (2\rho - 1)$ $\|p\|_\infty$, for all polynomials p. One could also appeal to the Sz.-Nagy-Foiaş [27] result which says any C_ρ operator is similar to a contraction.

COROLLARY 6. *Let $T \in \mathscr{L}(\mathscr{H})$ be an essentially normal operator (that is, $T^*T - TT^* \in \mathscr{K}$; $\mathscr{K}$ denotes the compact operators). Assume*

$$\operatorname{dist}[R(T), B_1(\mathscr{K})] \geqslant \delta > 0.$$

Then T has an invariant subspace. ($R(T)$ denotes the rational functions in T with poles off $\sigma(T)$. $B_1(\mathscr{K})$ denotes the compact operators of norm 1.)

Proof. We may assume $\sigma(T) = \sigma_e(T)$, the spectrum of the coset $\overset{\circ}{T}$ in the Calkin algebra. Let f be a rational function with poles off $\sigma(T)$, and assume $\|f(T)\| = 1$. Then it is easy to see that

$$\|f(T)\| \leqslant \delta^{-1}\|f(\overset{\circ}{T})\| = \delta^{-1}\|f\|_\infty^{\sigma_e(T)} = \delta^{-1}\|f\|_\infty^{\sigma(T)}.$$

Hence $\sigma(T)$ is a K-spectral set for T.

COROLLARY 7. *Let $T \in \mathscr{L}(\mathscr{H})$, where $\sigma(T)$ does not separate the plane. Assume* $\operatorname{Re}\sigma(p(T)) = \sigma(\operatorname{Re} p(T))$ *for all polynomials p. Then T has an invariant subspace.*

Proof. Clearly we may assume $\sigma(T)$ is connected. Note that $\sigma(T) = \widehat{\sigma(T)}$. Given a polynomial p

$$\begin{aligned}\|p(T)\| &\leqslant \|\operatorname{Re} p(T)\| + \|\operatorname{Im} p(T)\| \leqslant \\ &\leqslant 2\max\{|\sigma(\operatorname{Re} p(T))|, |\sigma(\operatorname{Im} p(T))|\} = \\ &= 2\max\{|\operatorname{Re}\sigma(p(T))|, |\operatorname{Im}\sigma(p(T))|\} \leqslant \\ &\leqslant 2\,|\sigma((p(T))| = \\ &= 2\sup\{|p(\lambda)| : \lambda \in \sigma(T)\}.\end{aligned}$$

Since $P(\sigma(T)) = R(\sigma(T))$, it follows that $\sigma(T)$ is a K-spectral set ($K = 2$) for T.

REMARK. A number of different conditions on T which imply that $\operatorname{Re}\sigma(T) = \sigma(\operatorname{Re} T)$ may be found in [3].

APPENDIX

Proof of Theorem 1. Parts of the proof are standard arguments by now and certain parts follow either [17] or [21]. However, since the former may not be readily available to some and since the latter is written in the language of representation theory it seems best to at least sketch such features. Furthermore, working in the context of representations, Mlak does not consider the normality of T_0.

Step 1. (Elementary measures) It follows from standard techniques that for each $x, y \in \mathscr{H}$ there exists a (finite regular Borel) measure $\mu(x, y)$ supported on ∂M such that

$$(f(T)x, y) = \int f \, d\mu(x, y)$$

for all $x, y \in \mathscr{H}$ and

$$\|\mu(x, y)\| \leqslant K \|x\| \, \|y\|.$$

(These are termed elementary measures by Mlak.) For a fixed $a_i \in G_i$, let m_i be a representing measure for $i = 1, 2, \ldots$. As is well known the m_i's are mutually singular. Fix $x, y \in \mathscr{H}$. Following Mlak, we next decompose $\mu(x, y) = \mu$ as $\mu = \sum_{i=0} \mu_i$ where $\mu_i << m_i$ for $i = 1, 2, \ldots$ and μ_0 is singular with respect to all m_i's. (See [10], VI 2.3) Since $f(T) \in \mathscr{L}(\mathscr{H})$ it follows that

1) $[\alpha\mu(x, y) - \mu(\alpha x, y)] \perp R(M)$ for all $x, y \in \mathscr{H}$; $\alpha \in \mathbf{C}$
2) $\{\mu(x + y, w) - [\mu(x, w) + \mu(y, w)]\} \perp R(M)$
3) $\{\mu(x, y + w) - [\mu(x, y) + \mu(x, w)]\} \perp R(M)$.

A. Lebow has coined the term conjugate linear modulo $R(M)$ to describe any measure satisfying 1), 2) and 3).

We next show that the individual measures $\mu_i(x, y)$ in the decomposition are also conjugate linear modulo $R(M)$. To prove 2) note that

$$\sum \{\mu_i(x + y, w) - [\mu_i(x, w) + \mu_i(y, w)]\} \perp R(M).$$

Thus it follows from the abstract F. and M. Riesz Theorem [13] that

$$\{\mu_i(x + y, w) - [\mu_i(x, w) + \mu_i(y, w)]\} \perp R(M) \qquad \text{for } i = 0, 1, 2 \ldots .$$

The proofs of 1) and 3) are left to the reader. Because each μ_i is conjugate linear modulo $R(M)$, if we set

$$(f_i(T)x, y) = \int f \, d\mu_i(x, y)$$

for $x, y \in \mathscr{H}$, $f \in R(M)$ and $i = 0, 1, 2 \ldots$, we obtain a well defined operator $f_i(T)$ where $\|f_i(T)\| \leqslant K\|f\|_\infty$. It should be mentioned that the measures $\mu(x, y)$ are not unique in general; but all choices lead to the same definition.

Step 2. (Multiplicativity) We wish to show that the map $\Gamma_i : f \to f_i(T)$ is multiplicative. Observe first that

$$\int f \cdot g \, d\mu(x, y) = (f(T)g(T)x, y) = (g(T)x, f(T)^* y) =$$

$$= \int g \, d\mu(x, f(T)^* y) = \int f \, d\mu(g(T)x, y)$$

for all $x, y \in \mathscr{H}$ and $f, g \in R(M)$ as is well known. Thus $f\mu(x, y) - \mu(x, f(T)^*y) \perp$ $\perp R(M)$. But this implies the decomposed measures

$$[f\mu(x, y) - \mu(x, f(T)^*y)]_i \perp R(M) \qquad \text{for each } i = 0, 1, \ldots$$

and thus

$$f\mu_i(x, y) - \mu_i(x, f(T)^*y) \perp R(M).$$

The same argument shows

$$f\mu_i(x, y) - \mu_i(f(T)x, y) \perp R(M).$$

Let $g \in R(M)$. Then

$$\int f \cdot g \mathrm{d}\mu_i(x, y) = \int f \,\mathrm{d}\mu_i(x, g(T)^*y) = (f_i(T)x, g(T)^*y) =$$

$$= (g(T)f_i(T)x, y) = \int g \,\mathrm{d}\mu(f_i(T)x, y)$$

whence

$$(*) \qquad f\mu_i(x, y) - \mu(f_i(T)x, y) \perp R(M).$$

From (*) it follows that the i^{th} measure in its decomposition, namely $f\mu_i(x, y) -$ $- \mu_i(f_i(T)x, y) \perp R(M)$. The essential elements are at hand. Thus,

$$((fg)_i(T)x, y) = \int fg \,\mathrm{d}\mu_i(x, y) = \int f \,\mathrm{d}\mu_i(g_i(T)x, y) = (f_i(T)g_i(T)x, y)$$

and hence Γ_i is multiplicative. By repeating this argument, one shows

$$(f_i(T)\, g_j(T)x, y) = \int f \,\mathrm{d}\mu_i(g_j(T)x, y) = 0 \qquad \text{for } i \neq j.$$

Step 3. (The decomposition) Following Mlak, we next define operators F_i where

$$(F_i x, y) = \int 1 \,\mathrm{d}\mu_i(x, y) \qquad \text{for } x, y \in \mathscr{H} \text{ and } i = 0, 1, 2, \ldots .$$

From the multiplicative properties proved above, it follows that $F_i \cdot F_j = \delta_{ij} F_i$. Thus, the F_i's are pairwise disjoint idempotents. Clearly, $\sum F_i$ converges to I. The F_i are not selfadjoint in general. However, for any subset Q of the positive integers observe that

$$|(\sum_Q F_i x, y)| = |\sum_Q \int 1 \,\mathrm{d}\mu_i(x, y)| \leqslant \|\mu(x, y)\| \leqslant K\|x\|\|y\|.$$

Thus $\|\sum_{Q} F_i\| \leqslant K$, from which it follows that $\{F_i\}$ is a uniformly bounded Boolean algebra of projections or spectral measure in the language of Dunford. It follows from a result of Mackey ([20], Theorem 55) that there exists an invertible operator Q such that $QF_kQ^{-1} = E_k$ is selfadjoint for each k. Set $S = QTQ^{-1}$. Using the conjugate linear relations from Step 2, we see that

$$(F_k f(T)x, y) = \int 1 \, \mathrm{d}\mu_k(f(T)x, y) = \int f \, \mathrm{d}\mu_k(x, y) = \int 1 \, \mathrm{d}\mu_k(x, f(T)^* y) =$$

$$= (F_k x, f(T)^* y) = (f(T)F_k x, y).$$

In particular, T commutes with F_k and hence S commutes with the E_k. (It follows also that $f_k(T) = f(T)F_k$.) If we set $\mathscr{H}_k = E_k\mathscr{H}$ and $S_k = S|\mathscr{H}_k$ then $S = \oplus S_k$ on $\oplus \mathscr{H}_k$. Define $T_k = f(T)F_k | F_k\mathscr{H}$.

Step 4. (The spectrum). To see that $\sigma(T_k) \subset \overline{G_k}$, fix a $k \geqslant 0$. For $\lambda \notin \overline{G_k}$, set

$$(B_\lambda x, y) = \int (z - \lambda)^{-1} \mathrm{d}\mu_k(x, y)$$

for all $x, y \in F_k\mathscr{H}$. It is easy to see that

$$((T_k - \lambda)B_\lambda x, y) = \int \mathrm{d}\mu_k(x, y) = (F_k x, y) = (x, y).$$

Similarly, $B_\lambda(T_k - \lambda) = I_k$, the identity on $F_k\mathscr{H}$, hence $\lambda \notin \sigma(T_k)$. Actually, with some additional effort, it can be shown that $\overline{G_k}$ is a K-spectral set for T_k. The details are carried out in [17].

Step 5. (Normality of S_0) Consider the measure $\mu_0(x, y)$ for $x, y \in F_0\mathscr{H}$. As shown above μ_0 is conjugate linear modulo $R(M)$. But $\mu_0(x, y)$ is singular with respect to the G_i's (or m_i's) and thus, by Wilken's Theorem ([10], II 8.5), $\mu_0(x, y)$ *is* conjugate linear. This fact enables us to integrate something beyond rational functions.

Let γ be a Borel set. Following Lautzenheiser, we set

$$B_\gamma(x, y) = \int \chi_\gamma \, \mathrm{d}\mu_0(x, y) = \mu_0(x, y)(\gamma).$$

Since $\{\mu_0(x, y)\}$ is conjugate linear, B_γ is a bilinear functional on $F_0\mathscr{H} \times F_0\mathscr{H}$. Thus, there exists an operator $F(\gamma) \in \mathscr{L}(F_0)$ such that

$$(F(\gamma)x, y) = B_\gamma(x, y) = \mu_0(x, y)(\gamma) \quad \text{for all } x, y \in F_0\mathscr{H}.$$

(See [15], Section 22.)

Since

$$(T_0x, y) = \int z \,\mathrm{d}\mu_0(x, y) = \int z \,\mathrm{d}(F(\cdot)x, y),$$

to complete the proof, we need only to show that $F(\cdot)$ is a spectral measure in the sense of Dunford.

We know that Γ_0 is multiplicative on $R(M)$, but that is not enough since we wish to integrate characteristic functions. We reconsider our earlier estimates. Recall that

$$\nu_0 = [f\mu_0(x, y) - \mu_0(f(T)x, y)] \perp R(M) \qquad \text{for } f \in R(M).$$

But ν_0 is singular and thus again by Wilken's Theorem

$$f\mu_0(x, y) = \mu_0(f(T)x, y).$$

Let γ be a Borel set. Then for $x, y \in F_0\mathscr{H}$

$$\int \chi_\gamma f \,\mathrm{d}\mu_0(x, y) = \int \chi_\gamma \,\mathrm{d}\mu_0(f(T)x, y) = (F(\gamma)f(T)x, y) =$$

$$= (f(T)x, F(\gamma)^*y) = \int f \,\mathrm{d}\mu(x, F(\gamma)^*y).$$

Thus, $\chi_\gamma \,\mu_0(x, y) - \mu(x, F(\gamma)^*y) \perp R(M)$ and by the abstract F. and M. Riesz Theorem and Wilken's Theorem, we see that

$$\chi_\gamma\mu_0(x, y) = \mu_0(x, F(\gamma)^*y).$$

Let γ, δ be Borel sets. Then

$$\int \chi_\delta \cdot \chi_\gamma \,\mathrm{d}\mu_0(x, y) = \int \chi_{\delta\cap\gamma} \,\mathrm{d}\mu_0(x, y) = (F(\delta \cap \gamma)x, y).$$

On the other hand,

$$\int \chi_\delta\chi_\gamma \,\mathrm{d}\mu_0(x, y) = \int \chi_\delta \,\mathrm{d}\mu_0(x, F(\gamma)^*y) =$$

$$= (F(\delta)x, F(\gamma)^*y) = (F(\gamma)F(\delta)x, y).$$

Thus, $F(\gamma)F(\delta) = F(\gamma \cap \delta)$ and it follows that $\{F(\cdot)\}$ is a spectral measure or uniformly bounded Boolean algebra of projections since

$$|(F(\gamma)x, y)| = \left|\int \chi_\gamma \mathrm{d}\mu_0(x, y)\right| \leqslant K \,\|x\|\,\|y\|.$$

Thus, T_0 is a scalar type operator (see [8]) and hence is similar to a normal operator. (See [20], Theorem 55.) Consequently by modifying Q we can assure the normality of S_0. This completes the proof.

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Added in proof: It has come to our attention that W. Mlak proved Theorem 1 of this paper in the case when M is the unit disc in "Algebraic polynomially bounded operators", *Ann. Polon. Math.*, **24** (1974), 133–139.