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MAXIMAL OPERATORS ON HYPERBOLOIDS

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INTRODUCTION

In a paper in 1976 [8], E. M. Stein introduced and studied the following maximal function:

$$\mathfrak{m}f(x) = \sup_{t>0} \left| \int_{\Sigma} f(x - ty')\,\mathrm{d}\sigma(y') \right|$$

where f is any Borel measurable function on $\mathbf{R}^n$, Σ the unit sphere in $\mathbf{R}^n$ and $\mathrm{d}\sigma$ the area measure on Σ. His positive result is the following: for $n \geqslant 3$ and $p > \frac{n}{n-1}$, $\mathfrak{m}$ is a bounded operator on $L^p(\mathbf{R}^n)$. One of the applications of this result is a Fatou's theorem for the following Cauchy problem:

$$\Box u = 0, \ u(0) = 0, \quad u_t(0) = f$$

where $\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$.

For the local version of Stein's result, say on compact Riemannian manifold, one can obtain similar theorems using the theory of Fourier integral operators. (This was carried out in collaboration with R. Coifman, Y. Meyer, A. Nagel, E. Stein and S. Wainger).

When studying the Cauchy problem it is natural to consider other initial hypersurfaces than $\mathbf{R}^n$. In particular, the hyperboloids

$$t^2 - x_1^2 - x_2^2 - \ldots - x_n^2 = c$$

are of special interest. In this paper we obtain global L^p estimates for various maximal functions. Here we cannot use the local version since the behavior at infinity is crucial and differs in essential ways with the Euclidean case. We, thus, will follow Stein's program using the intrinsic harmonic analysis of these symmetric spaces.

Finally, I would like to thank Professor R. Coifman for his guidance and advice in this work.

§ 1

In this paragraph, we want to give some aspects of harmonic analysis on the hyperboloids.

Let G be the group of hyperbolic rotations in $\mathbf{R}^{n+1}$. G leaves invariant the quadratic form: $t^2 - x_1^2 - \ldots - x_n^2$ and preserves the orientation of the space. Now, let K be the isotropy subgroup corresponding to the vector $\mathbf{1} = (1, 0, 0, \ldots, 0)$ in $\mathbf{R}^{n+1}$, clearly we have:

$$K = \left\{ u \in G,\ u = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \text{ with } R \in SO(n) \right\}.$$

Also, we have: $G = KAK$, the Cartan decomposition of G, where:

$$A = \left\{ u \in G, u = \alpha(s) = \begin{pmatrix} \operatorname{ch} s & \operatorname{sh} s & \\ \operatorname{sh} s & \operatorname{ch} s & 0 \\ 0 & 0 & \end{pmatrix},\ s \geqslant 0 \right\}.$$

On G we have the following integration formula. For an adequate normalization of the Haar measures and all sufficiently nice function f,

$$\int_G f(u)\,\mathrm{d}u = \int_K \int_0^\infty \int_K f(k_1\alpha(s)k_2)\operatorname{sh} s^{n-1} \mathrm{d}s\, \mathrm{d}k_1\, \mathrm{d}k_2.$$

It can be shown that the homogeneous space G/K can be identified with the hyperboloid:

$$H = \{(t, x_1, \ldots, x_n) \in \mathbf{R}^{n+1},\ t^2 - x_1^2 - \ldots - x_n^2 = 1,\ t \geqslant 1\}.$$

The corresponding G-invariant measure induced on H is the area measure $\mathrm{d}\sigma$ on H.

A function f on G is said to be bi-invariant if f is invariant under left and right translations by K. For such functions the spherical Fourier transform is defined by

$$\tilde{f}(\lambda) = \int_G f(u)\, \Phi_\lambda^n(u^{-1})\,\mathrm{d}u$$

where Φ_λ^n are the zonal spherical functions for G. The functions Φ_λ^n are bi-invariant and as function of s are given by:

$$\Phi_\lambda^n(s) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^\pi (\operatorname{ch} s - \cos\varphi \operatorname{sh} s)^\sigma \sin^{n-2}\varphi\, \mathrm{d}\varphi$$

where $\sigma = -\dfrac{n-1}{2} + i\lambda$. See [10].

For nice function f, we also have the inverse Fourier formula:

$$f(u) = \int_0^\infty \tilde{f}(\lambda)\Phi_\lambda^n(u)|c(\lambda)|^{-2}\,d\lambda$$

where $|c(\lambda)|^{-2}\,d\lambda$ is some measure on $\mathbf{R}_+$.

A Fourier multiplier is defined by:

$$Mf(u) = \int_0^\infty m(\lambda)\,\Phi_\lambda^n * f(u)|\,c(\lambda)|^{-2}\,d\lambda$$

where $m(\lambda)$ is an element of $L^\infty(\mathbf{R}_+)$.

§2

We, now, introduce the following analytic family of operators

$$\tilde{M}_\varepsilon^\alpha f(x) = \frac{1}{(e^\varepsilon - 1)^{2\alpha+n-2}}\,\frac{1}{\Gamma(\alpha)}\int_{S_\varepsilon^x} f[e^\varepsilon x - y]^{\alpha-1}\,d\sigma(y)$$

where f is in the Schwartz space on H, $[x] = t^2 - x_1^2 - x_2^2 - \ldots - x_n^2$ if $x = (t, x_1, x_2, \ldots, x_n)$ and $S_\varepsilon^x = \{y \in H,\ [e^\varepsilon x - y] \geqslant 0,\ (e^\varepsilon x - y)\cdot\mathbf{1} \leqslant 0\}$ is a geodesic ball in H with center at x.

The reason for multiplying by the factor $\dfrac{1}{(e^\varepsilon - 1)^{2\alpha+n-2}}$ in the definition of $\tilde{M}_\varepsilon^\alpha$ will appear later on.

We first prove:

THEOREM 1. *For each* $\varepsilon > 0$ *and each* α *with* $\operatorname{Re}\alpha > \dfrac{1-n}{2}$, M_ε^α *is a Fourier multiplier bounded on* $L^2(G/K)$ *and is given by:*

$$\tilde{M}_\varepsilon^\alpha f(u) = \int_0^\infty m_\varepsilon^\alpha(\lambda)\,\Phi_\lambda^n * f(u)|\,c(\lambda)|^{-2}\,d\lambda$$

with

$$m_\varepsilon^\alpha(\lambda) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{1}{2}}(e^\varepsilon - 1)^{2\alpha+n-2}}\,\frac{e^{\varepsilon(\alpha-1)}}{\Gamma\left(\alpha + \frac{n-1}{2}\right)}\int_0^\pi (\operatorname{ch}\varepsilon + \cos\theta\,\operatorname{sh}\varepsilon)^{\sigma-\alpha}\sin^{2\alpha+n-2}\theta\,d\theta$$

where $\sigma = \dfrac{n-1}{2} + i\lambda$.

Proof: It is easily seen that $\tilde{M}_\varepsilon^\alpha$ commutes with the action of G. Also for α, $\mathrm{Re}\alpha > 0$, $\tilde{M}_\varepsilon^\alpha$ is clearly bounded on $L^2(G/K)$. By the analogue of Theorem 10.9 [2] for G (the proof given for $SL(2, \mathbf{C})$ in [2] carries in the same way for G) one can write:

$$\tilde{M}_\varepsilon^\alpha f(u) = \int_0^\infty m_\varepsilon^\alpha(\lambda)\, \Phi_\lambda^n * f(u) |\, c(\lambda)\,|^{-2} \mathrm{d}\lambda$$

where m_ε^α is the spherical Fourier transform of the function:

$$\frac{1}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} \frac{1}{\Gamma(\alpha)} \chi_{S_\varepsilon^1}(y)[\mathrm{e}^\varepsilon \mathbf{1} - y]^{\alpha-1}$$

where $\chi_{S_\varepsilon^1}$ is the characteristic function of S_ε^1. A simple calculation gives:

$$m_\varepsilon^\alpha(\lambda) = \frac{2^{\alpha-1}\mathrm{e}^{\varepsilon(\alpha-1)}}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} \frac{1}{\Gamma(\alpha)} \int_0^\varepsilon (\mathrm{ch}\varepsilon - \mathrm{ch}s)^{\alpha-1} \Phi_\lambda^n(s)\, \mathrm{sh}s^{n-1}\, \mathrm{d}s.$$

Also, by (7) p. 156 [5],

$$\Phi_\lambda^n(s) = 2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \mathrm{sh}s^{-\frac{n-2}{2}} P^{\frac{2-n}{2}}_{-\frac{1}{2}+\mathrm{i}\lambda}(\mathrm{ch}s)$$

where $P_\nu^\mu(z)$ is the Legendre function of order μ and degree ν. Hence, by the change of variable $x = \mathrm{ch}s$, we have:

$$m_\varepsilon^\alpha(\lambda) = \frac{2^{\frac{n-2}{2}} \Gamma\left(\frac{n}{2}\right) \mathrm{e}^{\varepsilon(\alpha-1)}}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}\, \Gamma(\alpha)} \int_1^{\mathrm{ch}\varepsilon} (\mathrm{ch}\varepsilon - x)^{\alpha-1} (x^2 - 1)^{\frac{n-2}{4}} P^{\frac{2-n}{2}}_{-\frac{1}{2}+\mathrm{i}\lambda}(x)\mathrm{d}x.$$

By (29), p. 159 [5],

$$(x^2 - 1)^{\frac{n-2}{4}} P^{\frac{n-2}{2}}_{-\frac{1}{2}+\mathrm{i}\lambda}(x) = I^{\frac{n-2}{2}} P_{-\frac{1}{2}+\mathrm{i}\lambda}(x)$$

where

$$I^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_1^x f(t)(x - t)^{\gamma-1}\, \mathrm{d}t.$$

For $\gamma_1 \geqslant 0$, $\gamma_2 \geqslant 0$, it is easily seen that:

$$I^{\gamma_1} I^{\gamma_2} = I^{\gamma_1+\gamma_2}$$

Thus:

$$m_\varepsilon^\alpha(\lambda) = \frac{2^{\frac{n-2}{2}+\alpha-1}\, \mathrm{e}^{c(\alpha-1)}\, \Gamma\left(\frac{n}{2}\right)}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} I^{\alpha+\frac{n-2}{2}} P_{-\frac{1}{2}+\mathrm{i}\lambda}(\mathrm{ch}\varepsilon).$$

Now, again, by (29) p. 159 and (7) p. 156 [5], we have:

$$m_\varepsilon^\alpha(\lambda) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{-\frac{1}{2}}(e^\varepsilon - 1)^{2\alpha+n-2}} \frac{e^\varepsilon(\alpha-1)}{\Gamma\left(\alpha+\frac{n-1}{2}\right)}\int_0^\pi (\mathrm{ch}\varepsilon + \cos\theta\,\mathrm{sh}\varepsilon)^{\sigma-\alpha}\sin^{2\alpha+n-2}\theta\, d\theta.$$

The last integral is clearly absolutely convergent and uniformly bounded in λ whenever $\mathrm{Re}\alpha > \frac{1-n}{2}$ which finish the proof.

In the sequence, we need the following lemma,

LEMMA 1. *For* α, α' *with* $\mathrm{Re}\alpha > \mathrm{Re}\alpha' > \frac{1-n}{2}$, *we have:*

$$m_\varepsilon^\alpha(\lambda) = \frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 m_{\varepsilon'}^{\alpha'}(\lambda) e^{\frac{n}{2}(\varepsilon'-\varepsilon)} t^{2\alpha'+n-1}(1-t^2)^{\alpha-\alpha'-1}\, dt$$

with $\mathrm{sh}\frac{\varepsilon'}{2} = t\,\mathrm{sh}\frac{\varepsilon}{2}\cdot$

Proof: Since $-\mathrm{Re}\alpha - \frac{n-2}{2} \neq 1, 2, \ldots$, by (3) p. 122 [5] we have:

$$P^{-\alpha-\frac{n-2}{2}}_{-\frac{1}{2}+i\lambda}(x) = \frac{1}{\Gamma\left(\alpha+\frac{n}{2}\right)}\left(\frac{x+1}{x-1}\right)^{-\frac{\alpha}{2}-\frac{n-2}{4}} F\left(\frac{1}{2}+i\lambda, \frac{1}{2}-\right.$$
$$\left.-i\lambda, \alpha+\frac{n}{2}; \frac{1}{2}-\frac{1}{2}x\right)\cdot$$

Hence

$$m_\varepsilon^\alpha(\lambda) = \frac{1}{2}\frac{\Gamma\left(\frac{n}{2}\right)e^{-\frac{n}{2}\varepsilon}}{(e^\varepsilon-1)^{2\alpha+n-2}} F\left(\frac{1}{2}-i\lambda, \frac{1}{2}+i\lambda, \alpha+\frac{n}{2}; \frac{1}{2}-\frac{1}{2}\mathrm{ch}\varepsilon\right)\cdot$$

Also by (2) p. 78 [5], we have:

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(s)\Gamma(c-s)}\int_0^1 F(a, b, c; uz)u^{s-1}(1-u)^{c-s-1}du$$

for $\mathrm{Re}c > \mathrm{Re}s > 0$ $z \neq 1$ and $|\arg(1-z)| < \pi$. By using the expression of m_ε^α in term of F and the above identity, we obtain:

$$m_\varepsilon^\alpha(\lambda) = \frac{2^{\alpha-\alpha'}}{\Gamma(\alpha-\alpha')}\int_0^1 m_{\varepsilon'}^{\alpha'}(\lambda)e^{\frac{n}{2}(\varepsilon'-\varepsilon)} u^{\alpha'+\frac{n-2}{2}}(1-u)^{\alpha-\alpha'-1}du$$

with $\mathrm{sh}^2\frac{\varepsilon'}{2} = u\,\mathrm{sh}^2\frac{\varepsilon}{2}$. By letting $u = t^2$, we get the desired formula.

LEMMA 2. *For* α, α' *with* $\mathrm{Re}\alpha > \mathrm{Re}\alpha' + \frac{1}{2}$, *we have:*

$$|\tilde{M}_\varepsilon^\alpha f(x)|^2 \leqslant C \frac{1}{\mathrm{sh}\frac{\varepsilon}{2}} \int_0^\varepsilon |\tilde{M}_t^{\alpha'} f(x)|^2 \mathrm{ch}\frac{t}{2}\, \mathrm{d}t$$

whenever $\mathrm{Re}\alpha > \frac{2-n}{2}$ *and where C depends only on* α, α', n.

Proof: We use Lemma 1 and Cauchy-Schwarz inequality to obtain:

$$|\tilde{M}_\varepsilon^\alpha f(x)|^2 \leqslant (\mathrm{const}) \left(\int_0^1 |\tilde{M}_{\varepsilon'}^{\alpha'} f(x)|^2 \mathrm{d}t\right) \left(\int_0^1 \mathrm{e}^{n(\varepsilon'-\varepsilon)} t^{4\alpha'+2n-2}(1-t^2)^{2\alpha-2\alpha'-2} \mathrm{d}t\right).$$

Now, from the relation $\mathrm{sh}\frac{\varepsilon'}{2} = t\,\mathrm{sh}\frac{\varepsilon}{2}$ where $0 \leqslant t \leqslant 1$, we have: $\varepsilon' \leqslant \varepsilon$, therefore $\mathrm{e}^{n(\varepsilon'-\varepsilon)} \leqslant 1$. The only condition for the constant C to be finite is the existence of the integral $\int_0^1 t^{4\alpha'+2n-2}(1-t^2)^{2\alpha-2\alpha'-2}\,\mathrm{d}t$, that is whenever $\mathrm{Re}\alpha - \mathrm{Re}\alpha' > \frac{1}{2}$. Since, $\mathrm{Re}\alpha, \mathrm{Re}\alpha' > \frac{1-n}{2}$, the condition of Lemma 2 follows.

§3

We now define the maximal function:

$$\tilde{\mathfrak{m}}^\alpha f(x) = \sup_{\varepsilon > 0} |\tilde{M}_\varepsilon^\alpha f(x)|$$

and to study the properties of $\tilde{\mathfrak{m}}^\alpha$, we introduce the following g-function:

$$\tilde{g}_\alpha(f)(x) = \left(\int_0^\infty |\tilde{M}_\varepsilon^\alpha f(x) - c_\alpha \varphi_{\varepsilon^2} * f(x)|^2 \frac{\mathrm{d}\varepsilon}{\mathrm{th}\varepsilon}\right)^{\frac{1}{2}}$$

where $\varphi_\varepsilon *$ is the heat diffusion semi-group on the homogeneous space $G/K = H$ given by:

$$\varphi_\varepsilon * f(u) = \int_0^\infty \mathrm{e}^{-\varepsilon\left(\frac{1}{4}+\lambda^2\right)} \Phi_\lambda^n * f(u) |c(\lambda)|^{-2} \mathrm{d}\lambda$$

and where c_α is some constant to be precised later on.

To study the L^2-boundness of the operator $\tilde{g}_\alpha$ we need to estimate the function $m_\varepsilon^\alpha(\lambda)$. The result is given in the following lemma,

LEMMA 3. *For ε small, we have:*

1) $$m^{\alpha}(\lambda) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{1}{2}}\Gamma\left(\alpha + \frac{n-1}{2}\right)} \int_0^1 (1 - s^2)^{\alpha + \frac{n-3}{2}} \cos\lambda\varepsilon s \, ds + O(\varepsilon^2)$$

uniformly in $\lambda \leqslant 1$.

2) $$m_{\varepsilon}^{\alpha}(\lambda) = \frac{1}{2}\Gamma\left(\frac{n}{2}\right)\left(\frac{\varepsilon\lambda}{2}\right)^{-\alpha - \frac{n-2}{2}} J_{\alpha + \frac{n-2}{2}}(\lambda\varepsilon) + O\left((\lambda\varepsilon)^{-\alpha - \frac{n+1}{2}}\right)$$

uniformly in $\lambda \geqslant 1$.

Proof: 1) is an easy consequence of (8) p. 156 [5] and Taylor expansion of the function:

$$\frac{\mathrm{ch}\varepsilon - \mathrm{ch}s}{\frac{\varepsilon}{2} - \frac{s}{2}} .$$

To prove 2) we use the expansion for a Legendre function and the estimate for the remainder term given in [7].

We now define

$$c_{\alpha} = m_{\varepsilon}^{\alpha}(\lambda)\,|_{\varepsilon=0} = \frac{1}{4}\,\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha + \frac{n}{2}\right)}{\Gamma\left(\alpha + \frac{n-1}{2}\right)\Gamma\left(\alpha + \frac{n+1}{2}\right)} .$$

To estimate the L^2-norm of $\tilde{g}_{\alpha}(f)$ we need, first, to establish some formulas: For f right invariant function, and k a bi-invariant function we have:

(a) $$\int_G |f(u)|^2 du = \int_0^{\infty} |c(\lambda)|^{-2} \left(\int \Phi_{\lambda}^{n} * f(v) \overline{f}(v)\, dv\right) d\lambda$$

(b) $$\Phi_{\lambda} * k * f(v) = \tilde{k}(\lambda) \Phi_{\lambda} * f(v).$$

Both formulas can be proved in the same way as in [2] for $SL(2, \mathbf{C})$. We, now, have, using formula (a):

$$\int_G |\tilde{g}_{\alpha} f)|^2(u) du = \int_0^{\infty} \frac{d\varepsilon}{\mathrm{th}\varepsilon} \int_G |\tilde{M}_{\varepsilon}^{\alpha} f(u) - c_{\alpha}\varphi_{\varepsilon^2} * f(u)|^2 du =$$

$$= \int_0^{\infty} \frac{d\varepsilon}{\mathrm{th}\varepsilon} \int_0^{\infty} |c(\lambda)|^{-2} d\lambda \int_G \Phi_{\lambda}^{n} * k * f(v) \overline{k * f(v)}\, dv,$$

where

$$k(u)=\frac{1}{(e^{\varepsilon}-1)^{2\alpha+n-2}}\,\frac{1}{\Gamma(\alpha)}\,\chi_{S^1_\varepsilon}(u\mathbf{1})[e^{\varepsilon}\mathbf{1}-u\mathbf{1})^{\alpha-1}-c_\alpha\varphi_{\varepsilon^2}(u).$$

Using formula (b) we have:

$$\int_G \Phi^n_\lambda * k * f(v)\overline{k*f}(v)\,\mathrm{d}v=\tilde{k}(\lambda)\int_G \Phi^n_\lambda * f(v)\overline{k*f}(v)\,\mathrm{d}v=$$

$$=\tilde{k}(\lambda)\int_G \overline{k*f}(v)\mathrm{d}v\int_G \Phi^n_\lambda(vu^{-1})f(u)\,\mathrm{d}u=$$

$$=\tilde{k}(\lambda)\int_G f(u)\,\mathrm{d}u\int_G \Phi^n_\lambda(vu^{-1})\,\overline{k*f}(v)\,\mathrm{d}v.$$

Now by formula (a) and the fact that $\Phi^n_\lambda(u)=\Phi^n_\lambda(u^{-1})$ we obtain:

$$\int_G \Phi^n_\lambda * k * f(v)\,\overline{k*f}(v)\,\mathrm{d}v=|\tilde{k}(\lambda)|^2\int_G f(u)\Phi^n_\lambda*\overline{f}(u)\;\mathrm{d}u.$$

Hence:

$$\int_G \tilde{g}_\alpha(f)(u)^2\mathrm{d}u=\int_0^\infty |m^\alpha_\varepsilon(\lambda)-c_\alpha\,\mathrm{e}^{-\varepsilon^2\left(\frac14+\lambda^2\right)}|^2\frac{\mathrm{d}\varepsilon}{\mathrm{th}\varepsilon}\int_0^\infty |c(\lambda)|^{-2}\,\mathrm{d}\lambda\int_G f(u)\Phi^n_\lambda*\overline{f}(u)\mathrm{d}u\leqslant$$

$$\leqslant\sup_{\lambda\geqslant 0}\int_0^\infty |m^\alpha_\varepsilon(\lambda)-c_\alpha\mathrm{e}^{-\varepsilon^2\left(\frac14+\lambda^2\right)}\;\frac{\mathrm{d}\varepsilon}{\mathrm{th}\varepsilon}\int_0^\infty |c(\lambda)|^{-2}\,\mathrm{d}\lambda\int_G f(u)\,\Phi^n_\lambda*f(u)\,\mathrm{d}u=$$

$$=\sup_{\lambda\geqslant 0}\int_0^\infty |m^\alpha_\varepsilon(\lambda)-c_\alpha\,\mathrm{e}^{-\varepsilon^2\left(\frac14+\lambda^2\right)}|^2\frac{\mathrm{d}\varepsilon}{\mathrm{th}\varepsilon}\,\|f\|_2^2.$$

The L^2 boundness of $\tilde{g}_\alpha$ will then follow from:

LEMMA 4. *The integral:*

$$\int_0^\infty |m^\alpha_\varepsilon(\lambda)-c_\alpha\,\mathrm{e}^{-\varepsilon^2\left(\frac14+\lambda^2\right)}|^2\frac{\mathrm{d}\varepsilon}{\mathrm{th}\varepsilon}$$

is finite, uniformly in $\lambda\geqslant 0$ *whenever* $\mathrm{Re}\alpha>\dfrac{1-n}{2}$.

Proof: Near $\varepsilon=0$, we consider three cases:
Case 1: $\varepsilon\lambda$ small. By Lemma 3, we have:

$$m^\alpha_\varepsilon(\lambda)-c_\alpha\mathrm{e}^{-\varepsilon^2\left(\frac14+\lambda^2\right)}=O((\lambda\varepsilon)^2).$$

Also we have:

$$\int_0^{B/\lambda} (\lambda\varepsilon)^4 \frac{d\varepsilon}{\varepsilon} = \frac{B^4}{4}$$

which is finite for any finite B.

Case 2: $\varepsilon\lambda$ large. By Lemma 3, we have:

$$m_\varepsilon^\alpha(\lambda) - c_\alpha\, e^{-\varepsilon^2\left(\frac{1}{4}+\lambda^2\right)} = O\left(\frac{1}{(\varepsilon\lambda)^{\frac{n-1}{2}} + \alpha}\right)$$

uniformly in λ. Also

$$\int_{\frac{A}{\lambda}}^{\infty} \frac{1}{(\lambda\varepsilon)^{n-1+2\alpha}} \frac{d\varepsilon}{\varepsilon} = \frac{A^{1-n-2\alpha}}{2\alpha + n - 1}$$

which is finite for any $A > 0$ since $\alpha + \dfrac{n-1}{2} > 0$.

Case 3: $0 < B \leqslant \varepsilon\lambda \leqslant A < \infty$. In this case, we have

$$m_\varepsilon^\alpha(\lambda) - c_\alpha\, e^{-\varepsilon^2\left(\frac{1}{4}+\lambda^2\right)} = O(1)$$

and

$$\int_{B/\lambda}^{A/\lambda} \frac{d\varepsilon}{\varepsilon} = \log \frac{A}{B}$$

which is finite.

For ε large, one can easily see from the integral representation of $m_\varepsilon^\alpha(\lambda)$ that

$$m_\varepsilon^\alpha(\lambda) = O\left(\frac{\varepsilon}{e^{\frac{n+1}{2}\varepsilon}}\right)$$

uniformly in λ. We therefore have:

$$m_\varepsilon^\alpha(\lambda) - c_\alpha\, e^{-\varepsilon^2\left(\frac{1}{4}+\lambda^2\right)} = O\left(\frac{\varepsilon}{e^{\frac{n+1}{2}\varepsilon}}\right)$$

uniformly in λ. Also the integral $\displaystyle\int_A^\infty \frac{\varepsilon^2}{e^{(n+1)\varepsilon}} d\varepsilon$ is finite.

To prove the lemma, we write:

$$\int_0^\infty |m_\varepsilon^\alpha(\lambda) - c_\alpha e^{-\varepsilon^2\left(\frac{1}{4}+\lambda^2\right)}|^2 \frac{d\varepsilon}{\text{th}\varepsilon} = \int_0^1 (\varepsilon\lambda \sim 0) + \int_0^1 (\varepsilon\lambda \sim 1) + \int_1^\infty (\varepsilon\lambda \sim \infty) + \int_1^\infty$$

and use the estimates above.

Now, we have:

THEOREM 2. $\tilde{\mathfrak{m}}^\alpha$ *is a bounded operator on* $L^2(G/K)$ *whenever* $\operatorname{Re}\alpha > \frac{2-n}{2}$.

Proof: By Lemma 2, we have

$$|\tilde{M}_\varepsilon^\alpha f(x)|^2 \leqslant (\text{const}) \frac{1}{\operatorname{sh}\frac{\varepsilon}{2}} \int_0^\varepsilon |\tilde{M}_t^{\alpha'} f(x)|^2 \operatorname{ch}\frac{t}{2}\, \mathrm{d}t$$

whenever $\operatorname{Re}\alpha > \frac{2-n}{2}$. Hence:

$$|M_\varepsilon^\alpha f(x)|^2 \leqslant (\text{const}) \left(\int_0^\infty |\tilde{M}_t^{\alpha'} f(x) - c_{\alpha'}\, \varphi_{t^2} * f(x)|^2 \frac{\mathrm{d}t}{\operatorname{th} t} + \right.$$

$$\left. + \sup_{\varepsilon>0} \frac{1}{\operatorname{sh}\frac{\varepsilon}{2}} \int_0^\infty |\varphi_{t^2} * f(x)|^2 \operatorname{ch}\frac{t}{2}\, \mathrm{d}t \right) \leqslant$$

$$\leqslant (\text{const}) \left(\tilde{g}_{\alpha'}(f)^2(x) + \sup_{\varepsilon>0} |\varphi_\varepsilon * f(x)|^2\right).$$

Since $\varphi_\varepsilon *$ is the diffusion semi-group, a known result on semi-groups of operators [9] and the L^2 boundness of $\tilde{g}_{\alpha'}$ finish the proof.

Theorem 2 and an interpolation argument give:

THEOREM 3. $\tilde{\mathfrak{m}}^\alpha$ *is bounded operator on* $L^p(G/K)$ *in the following two cases:*

1) $1 < p \leqslant 2$ *and* $\operatorname{Re}\alpha > 1 - n + \frac{n}{p}$,

2) $2 \leqslant p \leqslant \infty$ and $\operatorname{Re}\alpha > \frac{1}{p}(2-n)$.

To prove Theorem 3, we need:

LEMMA 5. *Define:*

$$f^*(x) = \sup_{\varepsilon>0} \frac{1}{|S_\varepsilon^1|} \left| \int_{S_\varepsilon^x} f(y)\, \mathrm{d}\sigma(y) \right|.$$

The map $f \to f^*$ *is a bounded operator on* $L^p(G/K)$ *for all* $1 < p \leqslant \infty$.

Proof: See also [1]. Let:

$$f_0^*(x) = \sup_{1 \geqslant \varepsilon > 0} \frac{1}{|S_\varepsilon^1|} \left| \int_{S_\varepsilon^x} f(y)\, \mathrm{d}\sigma(y) \right|$$

$$f_\infty^*(x) = \sup_{\varepsilon \geqslant 1} \frac{1}{|S_\varepsilon^1|} \left| \int_{S_\varepsilon^x} f(y)\, \mathrm{d}\sigma(y) \right|.$$

We first show that $f \to f_0^*$ is a bounded operator. To do this, we cover H by a countable number of balls of the form $S_1^{x_i}$ such that $\sum \chi_{i,2}$ is bounded, where $\chi_{i,2}$ is the characteristic function of $S_2^{x_i}$. We then have:

$$\chi_{i,1}(x) f^*(x) \leqslant \chi_{i,1} \tilde{f}(x)$$

where

$$\tilde{f}_i(x) = \sup_{S_\varepsilon^x \subset S_2^{x_i}} \frac{1}{|S_\varepsilon^1|} \left| \int_{S_\varepsilon^x} f(y)\, d\sigma(y) \right| .$$

Hence:

$$\|\chi_{i,1} f^*\|_p \leqslant \|\chi_{i,1} \tilde{f}\|_p \leqslant C_{p,i} \|f \chi_{i,2}\|_p$$

for all $1 < p \leqslant \infty$, where $C_{p,i}$ is the norm of the Hardy-Littlewood maximal operator on the space of finite measure $S_2^{x_i}$. Since for each pair x_i, x_j there exists an element $g_{ij} \in G$ such that $x_i = g_{ij} x_j$, the geometric constant $C_{p,i}$ does not depend on i. Thus:

$$\|f_0^*\|_p^p \leqslant \|\sum \chi_{i,1} f_0^*\|_p^p \leqslant \|\sum \chi_{i,1} \tilde{f}_i\|_p^p \leqslant$$

$$\leqslant (C_p)^p \sum \|\chi_{i,2} f\|_p^p \leqslant$$

$$\leqslant B (C_p)^p \|f\|_p^p$$

where B is a finite constant such that $\sum \chi_{i,2} \leqslant B$, for all $1 < p \leqslant \infty$. This shows that $\to f_0^*$ is bounded on $L^p(G/K)$ for all $1 < p \leqslant \infty$. To show that $f \to f_\infty^*$, let m be the bi-invariant function given by $m(s) = |B_s^1|$. It is easily seen that

$$f_\infty^*(x) \leqslant c\,|f| * \frac{1}{1+m}(x)$$

for an appropriate constant $c > 1$. Since $\dfrac{1}{1+m}$ is in $L^p(G/K)$ for all $1 < p < \infty$, for fixed $1 > r > 0$, we have:

$$\frac{1}{(1+m)^{1+r}} * \text{ is bounded on } L^1(G/K),$$

$$\frac{1}{(1+m)^{\frac{1+r}{2}}} * \text{ is bounded on } L^2(G/K).$$

The last assertion is a consequence of Kuntz-Stein convolution theorem. Using complex interpolation, we obtain that $\dfrac{1}{1+m} *$ is bounded on $L^{1+r}(G/K)$ for all

$0 < r < 1$. A linear interpolation between $p = 1 + r$ and $p = \infty$ finishes the proof of Lemma 5.

To prove Theorem 3, we use complex interpolation. In the first case we interpolate between $p = 2$, $\mathrm{Re}\alpha > \frac{2-n}{2}$ (Theorem 2) and $p = 1 + r$, $r > 0$, $\mathrm{Re}\alpha \geqslant 1$ using Lemma 5 and the fact that for $\mathrm{Re}\alpha \geqslant 1$, $\widetilde{\mathfrak{m}}^\alpha f$ is dominated by f^*. In the second case we interpolate between $p = 2$, $\mathrm{Re}\alpha > \frac{2-n}{2}$ and $p = \infty$, $\mathrm{Re}\alpha > 0$.

To calculate $\widetilde{M}_\varepsilon^\alpha f(x)$ for $\alpha = 0$; let $x = \mathbf{1}$, then:

$$\widetilde{M}_\varepsilon^\alpha f(\mathbf{1}) = \frac{1}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} \frac{1}{\Gamma(\alpha)} \int_{S_\varepsilon^1} f(y)[\mathrm{e}^\varepsilon \mathbf{1} - y]^{\alpha-1} \, \mathrm{d}\sigma(y) =$$

$$= \frac{1}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} \frac{1}{\Gamma(\alpha)} \int_0^\infty \int_K f(k\alpha(s)\mathbf{1})[\mathrm{e}^\varepsilon \mathbf{1} - k\alpha(s)\mathbf{1}]^{\alpha-1} \mathrm{sh}^{n-1} s \, \mathrm{d}s \mathrm{d}k.$$

From the invariance of $[\cdot]$ with respect to K, we have:

$$[\mathrm{e}^\varepsilon \mathbf{1} - k\alpha(s)\mathbf{1}] = [\mathrm{e}^\varepsilon k^{-1}\mathbf{1} - \alpha(s)\mathbf{1}] = [\mathrm{e}^\varepsilon \mathbf{1} - \alpha(s)\mathbf{1}] =$$

$$= (\mathrm{e}^\varepsilon - \mathrm{ch} s)^2 - \mathrm{sh}^2 s = \mathrm{e}^{2\varepsilon} - 2\mathrm{e}^\varepsilon \mathrm{ch} s + 1 =$$

$$= 2\mathrm{e}^\varepsilon (\mathrm{ch}\varepsilon - \mathrm{ch} s).$$

Hence,

$$\widetilde{M}_\varepsilon^\alpha f(\mathbf{1}) = \frac{(2\mathrm{e}^\varepsilon)^{\alpha-1}}{(\mathrm{e}^\varepsilon - 1)^{2\alpha+n-2}} \frac{1}{\Gamma(\alpha)} \int_0^\varepsilon f^\#(s)(\mathrm{ch}\varepsilon - \mathrm{ch} s)^{\alpha-1} \mathrm{sh}^{n-1} s \, \mathrm{d}s$$

where $f^\#(s) = \int_K f(k\alpha(s)\mathbf{1}) \, \mathrm{d}k$. Now,

$$\frac{1}{\Gamma(\alpha)} \int_0^\varepsilon f^\#(s)(\mathrm{ch}\varepsilon - \mathrm{ch} s)^{\alpha-1} \mathrm{sh}^{n-1} s \, \mathrm{d}s =$$

$$= \frac{-1}{\Gamma(\alpha+1)} \int_0^\varepsilon f^\#(s) \frac{\mathrm{d}}{\mathrm{d}s} (\mathrm{ch}\varepsilon - \mathrm{ch} s)^\alpha \mathrm{sh}^{n-2} s \, \mathrm{d}s =$$

$$= \frac{1}{\Gamma(\alpha+1)} \int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}s} (f^\#(s) \mathrm{sh}^{n-2} s)(\mathrm{ch}\,\varepsilon - \mathrm{ch} s)^\alpha \, \mathrm{d}s \qquad (\alpha > 0).$$

The last integral converges for $\alpha > -1$ giving the analytic continuation of $\widetilde{M}_\varepsilon^\alpha f(\mathbf{1})$ for $\alpha > -1$. We therefore have:

$$\widetilde{M}^\alpha f(\mathbf{1})_{|\alpha=0} = \frac{(2\mathrm{e}^\varepsilon)^{-1}}{(\mathrm{e}^\varepsilon - 1)^{n-2}} f^\#(\varepsilon) \mathrm{sh}^{n-2} \varepsilon.$$

Hence, $\widetilde{M}_\varepsilon^0 f(\mathbf{1}) = \dfrac{\frac{1}{2}\,\mathrm{e}^{-\varepsilon}}{(\mathrm{e}^\varepsilon - 1)^{n-2}}\,\mathrm{sh}^{n-2}\varepsilon \int_K f(k\alpha(s)\mathbf{1})\mathrm{d}k$. By translation by elements of the group G, we have:

$$\widetilde{M}_\varepsilon^0 f(x) = \frac{1}{2}\,\mathrm{e}^{-\varepsilon}\left(\frac{\mathrm{sh}\,\varepsilon}{\mathrm{e}^\varepsilon - 1}\right)^{n-2} \int_K f(uk\alpha(s)\mathbf{1})\,\mathrm{d}k$$

where $x = u\mathbf{1}$. The integral

$$\int_K f(uk\alpha(s)\mathbf{1})\mathrm{d}k$$

is the spherical mean on the hyperboloid. From the proof of Theorem 3, it is clear that the same conclusions are valid for the family of operators,

$$\mathfrak{m}^\alpha f(x) = \sup_{\varepsilon>0} 2\mathrm{e}^\varepsilon \left(\frac{\mathrm{e}^\varepsilon - 1}{\mathrm{sh}\,\varepsilon}\right)^{n-2} |\widetilde{M}_\varepsilon^\alpha f(x)|$$

whenever $n \geqslant 3$. For $\alpha = 0$, we have:

$$\mathfrak{m} f(x) = \mathfrak{m}^0 f(x) = \sup_{\varepsilon>0}\left|\int_K f(uk\alpha(\varepsilon)\mathbf{1})\,\mathrm{d}k\right|\cdot$$

Thus,

THEOREM 4. *$\mathfrak{m}$ is a bounded operator on $L^p(G/K)$ whenever $\dfrac{n}{n-1} < p \leqslant \infty$ and $n \geqslant 3$.*

Theorem 4 is the analogue of Stein's result stated in the Introduction.

§ 4.

As a consequence of Theorem 3, we have a Fatou's theorem for the following Cauchy problem:

$$\Box\, u = 0, \quad u_{|H} = 0, \quad \left.\frac{\partial u}{\partial n}\right|_H = f$$

where $\dfrac{\partial}{\partial n}$ is the normal derivative to H. The solution to this problem is given by:

$$u_\varepsilon(x) = c_n(\mathrm{e}^\varepsilon - 1)\,\widetilde{M}_\varepsilon^\alpha f(x) \tag{1}$$

for $\alpha = \dfrac{3-n}{2}$ and $c_n = \dfrac{1}{2}\,\pi^{\frac{1-n}{2}}$. In formula (1) $x \in H$, $\varepsilon > 0$ and $u_\varepsilon(x) = u(\mathrm{e}^\varepsilon x)$. Formula (1) can be derived directly from the general formula given in [6]. In fact, for a general spacelike and sufficiently smooth hypersurface S in $\mathbf{R}^{n+1}$, the solution to the following Cauchy problem:

$$\Box u = 0, \quad u|_S = 0, \quad \frac{\partial u}{\partial n}\Big|_S = 0$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \ldots - \frac{\partial^2}{\partial x_n^2}$$

and $\dfrac{\partial}{\partial n}$ the normal derivative to S, is given by:

$$(*) \qquad u(P) = \frac{1}{\gamma(\alpha, n)} \int_{S^P} f(Q)[P-Q]^{\alpha-1}\, \mathrm{d}\sigma(Q)\Big|_{\alpha = \frac{3-n}{2}}$$

in the sense of the analytic continuation of the integral as function of α, where

$$[P] = t^2 - x_1^2 - \ldots - x_n^2, \quad P = (t, x_1, \ldots, x_n)$$

$$\gamma(\alpha, n) = \pi^{\frac{n-1}{2}}\, 2^{2(\alpha-1)+n}\, \Gamma\left(\alpha - 1 + \frac{n+1}{2}\right) \Gamma(\alpha)$$

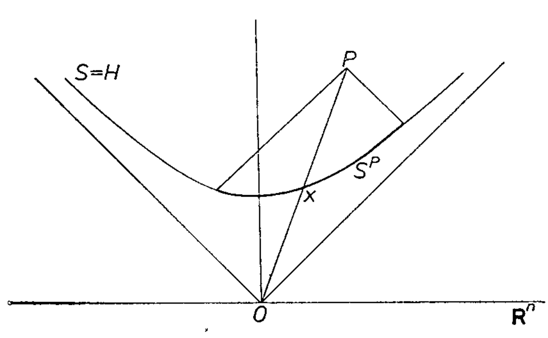
and

$$S^P = \{Q \in S,\ [P-Q] \geqslant 0 \text{ and } (P-Q)\cdot\mathbf{1} \leqslant 0\}.$$

Formula (*) is a straightforward application of Green's theorem (for details see [6]). When

$$S = H = \{t, x_1, \ldots, x_n) \in \mathbf{R}^{n+1}:\ t^2 - x_1^2 - \ldots - x_n^2 = 1,\ \ t \geqslant 1\},$$

we have the following picture:



where x is the intersection of the line through O and P with H. Since $[P] > 1$ and $[x] = 1$, one can write $P = e^{\varepsilon}x$, $\varepsilon > 0$. Formula (*), now, can be written,

$$u(P) = u_{\varepsilon}(x) = \frac{1}{\gamma(\alpha, n)} \int_{S_{\varepsilon}^{x}} f(y)[e^{\varepsilon}x - y]^{\alpha-1}\, d\sigma(y) \Big|_{\alpha = \frac{3-n}{2}}$$

where $P = e^{\varepsilon}x$ and $S_{\varepsilon}^{x} = S^{P}$.

Using the definition of $\tilde{M}_{\varepsilon}^{\alpha}$ given in § 2, we have:

$$u_{\varepsilon}(x) = \frac{\Gamma(\alpha)(e^{\varepsilon} - 1)^{2\alpha+n-2}}{\gamma(\alpha, n)} \tilde{M}_{\varepsilon}^{\alpha} f(x) \Big|_{\alpha = \frac{3-n}{2}}$$

Also:

$$\frac{\Gamma(\alpha)(e^{\varepsilon} - 1)^{2\alpha+n-2}}{\gamma(\alpha, n)} = \frac{(e^{\varepsilon} - 1)^{2\alpha+n-2}}{\pi^{\frac{n-1}{2}} 2^{2(\alpha-1)+n} \Gamma\left(\alpha - 1 + \frac{n+1}{2}\right)}$$

whose value at $\alpha = \frac{3-n}{2}$ is $\frac{1}{2} \pi^{\frac{1-n}{2}} (e^{\varepsilon} - 1)$. Hence:

$$u_{\varepsilon}(x) = \frac{1}{2} \pi^{\frac{1-n}{2}} (e^{\varepsilon} - 1) \tilde{M}_{\varepsilon}^{\alpha} f(x) \Big|_{\alpha = \frac{3-n}{2}}$$

which shows formula (1).

THEOREM 5. *For* $\frac{2n}{n+1} < p \leqslant \infty$, $f \in L^{p}$, *we have:*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} u_{\varepsilon}(x) = f(x)$$

a.e. When $n = 1$ *this holds for* $1 \leqslant p \leqslant \infty$. *The convergence is also dominated in the* L^{p} *norm if in addition* $p < \infty$, $n = 1, 2, 3$ *and* $p < \frac{2(n-2)}{n-3}$, $n \geqslant 4$.

Theorem 5 is an immediate consequence of Theorem 3.

REFERENCES

1. CLERK, J. L.; STEIN, E. M., L^{p}-multipliers for non-compact symmetric spaces, *Proc. Nat. Acad. Sci. U.S.A.*, **71** (1974), 3911–3912.

2. COIFMAN, R. R.; WEISS, G., Transference Methods in Analysis, *CBMS*, *Amer. Math. Soc.*, **31** (1977).

3. Coifman, R. R.; Weiss, G., *Analyse Harmonique Non-Commutative sur Certains Espaces Homogenes*, Lecture Notes in Math., **242** (1971), Springer-Verlag, Berlin and New York.

4. Coifman, R. R.; Rochberg, R.; Weiss, G., Application of transference: The L^p version of von Neumann's inequality and the Littlewood-Paley-Stein theory, preprint.

5. Erdélyi, A. et al., *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1953.

6. Riesz, M., L'intégrale de Riemann-Liouville et le problème de Cauchy, *Acta Math.*, **81** (1949), 1–223.

7. Schindler, S., Some transplantation theorems for the generalized Mehler transform and related asymptotic expensions, *Trans. Amer. Math. Soc.*, **155** (1971), 257–291.

8. Stein, E. M., Maximal functions: Spherical means, Part I, *Proc. Nat. Acad. Sci. U.S.A.*, **73** (1976), 2174–2175.

9. Stein, E. M., Topics in Harmonic Analysis, *Annals of Mathematical Studies*, **63** (1970), Princeton Univ. Press.

10. Vilenkin, N. Ja., Special functions and the theory of group representations, *Translation of Math. Monographs*, A.M.S., **22** (1968).

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