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REMARKS ON THE CONNES SPECTRUM FOR C^*-DYNAMICAL SYSTEMS

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In [2], 2.3.17 A. Connes showed that for a W^*-dynamical system $(\mathfrak{M}, G, \alpha)$, where $\mathfrak{M}$ is a factor, the Connes spectrum $\Gamma(\alpha)$ of the representation α is the intersection of all Arveson spectra $\mathrm{Sp}(\beta)$, where β ranges over the representations exterior equivalent to α. Regarding β as a perturbation of α (by a unitary one-cocycle) this result characterizes $\Gamma(\alpha)$ as the part of $\mathrm{Sp}(\alpha)$ which is invariant under all perturbations. In the case where $\hat{G}/\Gamma(\alpha)$ is compact, Connes shows in [2], 2.3.13 the existence of a perturbation such that $\mathrm{Sp}(\beta) = \Gamma(\alpha)$, thus β has the minimal spectrum within its exterior equivalence class. Connes also solves the problem of finding, for such a minimal β, a state (or weight) satisfying the KMS condition with respect to β (generalized traces on III_λ factors, see [2], 4.3).

We shall in this paper obtain a version of Connes' results for C^*-dynamical systems. The main tool will be a theorem of L. G. Brown [1] on stably isomorphic C^*-algebras. The author is indebted to A. Connes and D. Olesen for valuable discussions during a stay in April 1978 at the Institut des Hautes Etudes Scientifiques, France, where this work was carried out.

Consider a C^*-dynamical system (A, G, α), i.e., a C^*-algebra A, a locally compact abelian group G and a pointwise continuous automorphic representation $\alpha : G \to \mathrm{Aut}(A)$ of G on A. Recall from [5], 3.3 that A is G-simple if it contains no nontrivial closed, G-invariant ideals. Fix an infinite dimensional, separable Hilber space H and let $\mathcal{K}$ denote the C^*-algebra of compact operators on H.

PROPOSITION. *Let (A, G, α) be a C^*-dynamical system where A is separable and G-simple. For each non-zero, G-invariant, hereditary C^*-subalgebra B of A there is a system $(A \otimes \mathcal{K}, G, \beta)$ such that β is exterior equivalent to $\alpha \otimes \iota$ on $A \otimes \mathcal{K}$ and* $\mathrm{Sp}(\beta) = \mathrm{Sp}(\alpha|B)$.

Proof. Since A is G-simple, B is not contained in any proper, closed ideal of A. Since moreover A is separable, we conclude from [1], 2.8 that $B \otimes \mathcal{K}$ is isomorphic to $A \otimes \mathscr{K}$. The isomorphism is induced by a partial isometry v in $M(C \otimes \mathscr{K})$, where C is the C^*-subalgebra of $A \otimes \mathscr{M}_2$ of the form

$$C = \begin{pmatrix} B & L \\ L^* & A \end{pmatrix},$$

L being the closed left ideal generated by B (whence $B = L \cap L^*$), and A, B are considered as hereditary C^*-subalgebras of C. It follows from the construction of v (cf. Lemma 2.5 of [1]) that

$$v^*v = \begin{pmatrix} 0 & 0 \\ 0 & 1_A \end{pmatrix}, \quad vv^* = \begin{pmatrix} 1_B & 0 \\ 0 & 0 \end{pmatrix};$$

so that v has the form

$$v = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix},$$

where $w \in (A \otimes \mathscr{K})''$. Thus $w^*w = 1$ (in $M(A \otimes \mathscr{K})$) and ww^* is a unit for $B \otimes \mathscr{K}$ and a fixed-point for $\alpha \otimes \iota$ (extended to $(A \otimes \mathscr{K})''$).

Define $u_t = w^*(\alpha_t \otimes \iota(w))$ and note that

$$v^*\alpha_t \otimes \iota(v) = \begin{pmatrix} 0 & 0 \\ 0 & u_t \end{pmatrix}.$$

Routine calculations show that $t \to u_t$ is a strictly continuous unitary-valued function from G to $M(A \otimes \mathscr{K})$ (i.e. $t \to u_t x$ is norm continuous for each x in $A \otimes \mathscr{K}$). Moreover, $u_{st} = u_s \alpha_s \otimes \iota(u_t)$, so that u is a one-cocycle. Set

$$(*) \qquad \beta_t(x) = w^*(\alpha_t \otimes \iota(wxw^*))w = u_t(\alpha_t \otimes \iota(x))u_t^*$$

for x in $A \otimes \mathscr{K}$. From [5], 4 we see that $(A \otimes \mathscr{K}, G, \beta)$ is a C^*-dynamical system such that β is exterior equivalent to $\alpha \otimes \iota$. Moreover, from $(*)$

$$\mathrm{Sp}(\beta) = \mathrm{Sp}(\alpha \otimes \iota | w(A \otimes \mathscr{K})w^*) =$$
$$= \mathrm{Sp}(\alpha \otimes \iota | B \otimes \mathscr{K}) = \mathrm{Sp}(\alpha \mid B).$$

THEOREM 1. *Let (A, G, α) be a C^*-dynamical system where A is separable and G-simple. Then*

$$\Gamma(\alpha) = \cap \mathrm{Sp}(\beta),$$

where β ranges over all systems $(A \otimes \mathscr{K}, G, \beta)$ such that β is exterior equivalent to $\alpha \otimes \iota$ on $A \otimes \mathscr{K}$.

Proof. If $(A \otimes \mathscr{K}, G, \beta)$ is a system such that β is equivalent to $\alpha \otimes \iota$ then

$$\mathrm{Sp}(\beta) \supset \Gamma(\beta) = \Gamma(\alpha \otimes \iota) = \Gamma(\alpha)$$

by [5], 4.4. Therefore

$$\Gamma(\alpha) \subset \bigcap_{\beta \sim \alpha \otimes \iota} \mathrm{Sp}(\beta).$$

The converse inclusion follows from the Proposition, since $\Gamma(\alpha)$ is defined as $\cap \operatorname{Sp}(\alpha|B)$, where B ranges over the set of non-zero, G-invariant, hereditary C^*-subalgebras of A.

THEOREM 2. *Let (A, G, α) be a C^*-dynamical system where A is separable and simple. Assume further that $G = \mathbf{R}^n \oplus H$, H compact, and that $\operatorname{Sp}(\alpha)/\Gamma(\alpha)$ is compact (in $\hat{G}/\Gamma(\alpha)$). There is then a system $(A \otimes \mathcal{K}, G, \beta)$ such that β is exterior equivalen to $\alpha \otimes \iota$ and $\operatorname{Sp}(\beta) = \Gamma(\alpha)$.*

Proof. Denote by G_0 the annihilator of $\Gamma(\alpha)$ in G. For a suitable choice of basis in $\mathbf{R}^n$ we have

$$G_0 = \mathbf{R}^k \oplus \mathbf{Z}^m \oplus H_0,\ k + m \leqslant n, \quad H_0 \subset H.$$

Since $\operatorname{Sp}(\alpha|\mathbf{R}^k)$ is compact (being contained in $\operatorname{Sp}(\alpha)/\Gamma(\alpha)$) there is by [3], 5.3 or [8], 8.5.9 a uniformly continuous, unitary representation u of $\mathbf{R}^k$ into $M(A)^G$, such that $\alpha|\mathbf{R}^k = \operatorname{Ad}(u)$. Let u also denote the trivial extension to a unitary representation of G and put $\alpha' = \alpha \circ \operatorname{Ad}(u^{-1})$. Then α' is exterior equivalent to α and $\alpha'_t = \iota$ for each t in $\mathbf{R}^k$. Since we are looking for a representation β which is trivial on all of G_0 we may replace α with α' and assume that $k = 0$, i.e. $G_0 = \mathbf{Z}^m \oplus H_0$.

Let $t_1, \ldots, t_m$ be a set of generators for $\mathbf{Z}^m$ and choose a neighbourhood Ω of 0 in $\hat{G}$ such that $\Omega \cap \hat{H} = \{0\}$ and

$$|1 - (t_j, \Omega)| < \sqrt{3}, \quad 1 \leqslant j \leqslant m.$$

By [4], 3.4 or [8], 8.8.7 there is a non-zero, G-invariant, hereditary C^*-subalgebra B of A such that

$$\operatorname{Sp}(\alpha|B) \subset \Gamma(\alpha) + \Omega.$$

It follows that $\alpha_t = \iota$ on B for all t in H_0 and that

$$\nu(\iota - \alpha_{t_j}|B) < \sqrt{3}, \quad 1 \leqslant j \leqslant m$$

(ν denoting spectral radius). Thus the automorphisms α_{t_j} are derivable, and since B is simple there exist by [4], 4.1 or [8], 8.9.7 positive elements $h_1, \ldots, h_m$ in the center of $M(B)^G$ such that

$$\alpha_{t_j}|B = \operatorname{Ad}(\exp(ih_j)), \quad 1 \leqslant j \leqslant m.$$

Define a uniformly continuous, unitary representation v of $\mathbf{R}^m$ into the center of $M(B)^G$ by

$$v_t = \prod_{1 \leqslant j \leqslant m} \exp(it_j h_j), \quad t = (t_1, \ldots, t_m),$$

and extend it trivially to a unitary representation (again denoted by v) of G (which is possible, since $\mathbf{R}^m$ is a direct summand in $G = \mathbf{R}^m \oplus H$). Now take w as in the proof of the Proposition, i.e. $w^*w = 1$ and $wB \otimes \mathcal{K} w^* = A \otimes \mathcal{K}$, and define

$$u_t = w^*(v_{-t} \otimes 1)(\alpha_t \otimes \iota(w)), \qquad t \in G.$$

As before we show that u is a strictly continuous, unitary representation of G into $M(A \otimes \mathscr{K})$. Since moreover the v_t's are fixed-points, u is again a one-cocycle, so that the definition

$$\beta_t(x) = u_t(\alpha_t \otimes \iota(x))u_t^*, \quad x \in A \otimes \mathscr{K},$$

gives rise to a C^*-dynamical system $(A \otimes \mathscr{K}, G, \beta)$ with β exterior equivalent to $\alpha \otimes \iota$. Finally, for each t in G_0 we have

$$\beta_t(x) = w^*(v_{-t} \otimes 1(\alpha_t \otimes \iota(wxw^*))v_t \otimes 1)w =$$
$$= w^*(wxw^*)w = x$$

for all x in $A \otimes \mathscr{K}$. Consequently $\mathrm{Sp}(\beta) \subset G_0^\perp = \Gamma(\alpha)$. However, $\mathrm{Sp}(\beta) \supset \Gamma(\beta) = \Gamma(\alpha)$, and we conclude that $\mathrm{Sp}(\beta) = \Gamma(\alpha)$, as desired.

REMARK 1. Theorem 2 covers the case of a connected abelian group G, but leaves completely open the discrete case. The problem is, that for a discrete group G and a unitary representation u of a subgroup G_0 into the center of $M(A)^G$, there seems to be no C^*-analogue of [2], 2.3.12, extending u to a unitary representation of G into the center of $M(A)^G$.

REMARK 2. Under the assumption that $\Gamma(\alpha)$ is discrete (which is natural because the most interesting case is $G = \mathbf{R}$), Connes shows in [2], 2.4.1 that for a W^*-system $(\mathfrak{M}, G, \alpha)$ one has $\mathrm{Sp}(\alpha) = \Gamma(\alpha)$ if and only if $\mathfrak{M}^G$ is a factor. Part of this result can be recovered for a C^*-dynamical system (A, G, α).

If A is G-prime (every two non-zero, G-invariant ideals have a non-zero intersection) and $\Gamma(\alpha)$ is discrete, then $\mathrm{Sp}(\alpha) = \Gamma(\alpha)$ implies that $M(A)^G$ is prime. The argument is exactly as the proof of Theorem 1 of [6].

If A is simple and $\mathrm{Sp}(\alpha)/\Gamma(\alpha)$ is compact, the condition that the center of $M(A)^G$ is trivial (and a fortiori the condition that $M(A)^G$ is prime) implies that $\mathrm{Sp}(\alpha) = \Gamma(\alpha)$. Indeed, each automorphism α_t, where $t \in \Gamma(\alpha)^\perp$, is implemented by a unitary in the center of $M(A)^G$ by [4], 4.1 or [8], 8.9.7. Our assumption implies that $\alpha_t = \iota$ for all t in $\Gamma(\alpha)^\perp$ and consequently $\mathrm{Sp}(\alpha) \subset \Gamma(\alpha)$.

THEOREM 3. *Let $(A, \mathbf{R}, \alpha)$ be a C^*-dynamical system where A is separable and simple, and let φ be a state of A that satisfies the* KMS *condition with respect to α. If $0 \neq \Gamma(\alpha) \neq \mathbf{R}$ there is a system $(A \otimes \mathscr{K}, \mathbf{R}, \beta)$, with β exterior equivalent to $\alpha \otimes \iota$, and a state ψ of $A \otimes \mathscr{K}$ satisfying the* KMS *condition with respect to β, such that* $\mathrm{Sp}(\beta) = \Gamma(\alpha)$; i.e. *$\beta$ is periodic.*

Proof. Take λ in $\mathbf{R}$ such that the annihilator of $\Gamma(\alpha)$ in $\mathbf{R}$ equals $\lambda\mathbf{Z}$. Arguing exactly as in the proof of Theorem 2 there is a non-zero, α-invariant, hereditary C^*-subalgebra B of A, and a positive element k in the center of $M(B)^\alpha$, such that

$$\exp(itk)x\exp(-itk) = \alpha_t(x), \quad t \in \lambda\mathbf{Z},$$

for all x in B. Choose w as usual $w^*w = 1$ in $(A \otimes \mathcal{K})''$ and $w(A \otimes \mathcal{K})w^* = B \otimes \mathcal{K}$. Remember that the projection ww^* in $(A \otimes \mathcal{K})''$ is a fixed-point for α (extended to $(A \otimes \mathcal{K})''$).

Take a positive operator h_0 on a separable Hilbert space, with eigenvalues $2\pi n \lambda^{-1}$, $n = 0, 1, 2, \ldots$, all with multiplicity one. Set $v_t = \exp(ith_0)$ and $\varphi_0 = \mathrm{Tr}(\exp(-h_0)\cdot)$. Then φ_0 is a positive functional on $\mathcal{K}$ (because $\exp(-h_0)$ is of trace class) and φ_0 satisfies the KMS condition with respect to the automorphism group $\mathrm{Ad}(v)$ on $\mathcal{K}$. Note that $v_t = 1$ for all t in $\mathbf{Z}$ so that $\mathrm{Ad}(v)$ is a periodic representation. The representation $\alpha \otimes \mathrm{Ad}(v)$ is exterior equivalent to $\alpha \otimes \iota$ and the functional $\tilde{\varphi} = \varphi \otimes \varphi_0$ on $A \otimes \mathcal{K}$ satisfies the KMS condition with respect to the representation $\tilde{\alpha} = \alpha \otimes \mathrm{Ad}(v)$.

Set $h = k \otimes 1$ and define a positive functional ψ on $A \otimes \mathcal{K}$ by

$$\psi(x) = \tilde{\varphi}(\exp(h)wxw^*).$$

Since $A \otimes \mathcal{K}$ is simple and $\tilde{\varphi}$ is a KMS functional it is faithful and consequently ψ is faithful too. As in the proof of Theorem 2 define the unitary one-cocycle

$$u_t = w^*\exp(-ith)\,\tilde{\alpha}_t(w),$$

and put $\beta = (\mathrm{Ad}(u)) \circ \tilde{\alpha}$. The representation β is exterior equivalent to $\tilde{\alpha}$, hence to $\alpha \otimes \iota$; and $\beta_t = \iota$ for all t in $\lambda\mathbf{Z}$ so that $\overset{\circ}{\mathrm{Sp}}(\beta) = \Gamma(\alpha)$.

Take elements x, y in $A \otimes \mathcal{K}$ and assume that x is analytic for β (i.e. the function $t \to \beta_t(x)$ extends to an analytic function from $\mathbf{C}$ to $A \otimes \mathcal{K}$, see e.g. [8], 8.12.1). From the definition of u this is equivalent with the demand that wxw^* is analytic for $\tilde{\alpha}$. Using the facts that $\tilde{\varphi} \mid B \otimes \mathcal{K}$ is a KMS functional for $\tilde{\alpha}$ and that the elements ww^* and h commute with $\tilde{\varphi}$ we then compute

$$\begin{aligned}
\psi(\beta_t(x)y) &= \tilde{\varphi}(\exp(h)w(w^*\exp(-ith)\tilde{\alpha}_t(wxw^*)\exp(ith)w)yw^*) = \\
&= \tilde{\varphi}(\tilde{\alpha}_t(wxw^*)\exp(ith)wyw^*\exp(h)ww^*)\exp(-ith)) = \\
&= \tilde{\varphi}(\tilde{\alpha}_t(wxw^*)\exp(ith)wyw^*\exp(-\mathrm{i}(t+\mathrm{i})h)) = \\
&= \tilde{\varphi}(\exp(ith)wyw^*\exp(-\mathrm{i}(t+\mathrm{i})h)\tilde{\alpha}_{t+\mathrm{i}}(wxw^*)) = \\
&= \tilde{\varphi}(\exp(h)wyw^*\exp(-\mathrm{i}(t+\mathrm{i})h)\,\tilde{\alpha}_{t+\mathrm{i}}(wxw^*)\exp(\mathrm{i}(t+\mathrm{i})h)ww^*) = \\
&= \tilde{\varphi}(\exp(h)wy\beta_{t+\mathrm{i}}(x)w^*) = \\
&= \psi(y\beta_{t+\mathrm{i}}(x)).
\end{aligned}$$

Thus ψ satisfies the KMS condition with respect to β (cf. [8], 8.12.3) and the proof is complete.

REMARK 3. Note from the definition of ψ in the preceeding theorem that if π_φ and π_ψ denote the cyclic representations associated with φ and ψ on A and $A \otimes \mathcal{K}$, respectively, then $\pi_\varphi \otimes \iota$ is (quasi) equivalent to π_ψ.

REMARK 4. A simple change of scale allows us to restate Theorem 3 for KMS states corresponding to (inverse) temperatures different from 1. However, the limit cases where φ is either an invariant tracial state or a ground state demand separate treatment. If φ is an α-invariant, tracial state and if we can perturb α to a periodic representation β of $\mathbf{R}$ on A we obtain easily a β-invariant trace by setting

$$\psi = \int \varphi \circ \beta_t \, dt,$$

integrating over a period for β. But if we have to perturb $\alpha \otimes \iota$ to find a periodic representation there is of course no solution, since $A \otimes \mathcal{K}$ does not admit any finite traces. A perturbation process which from a ground state for α produces a ground state for $\beta \sim \alpha \otimes \iota$ probably exists, but is not obvious to find.

REMARK 5. It may be objected that our results are not the true analogue of Connes', since we have to pass to the system $(A \otimes \mathcal{K}, G, \alpha \otimes \iota)$ to obtain our perturbations. However, there is an increasing awareness (prompted by K-theory and duality theory) that passing from A to $A \otimes \mathcal{K}$ (stabilizing A in the language of [1]) is a natural step which should be taken without hezitation. In this connection it may also be worthwhile to recall the results from [7], 4.7: If A and B are arbitrary Glimm algebras (UHF-algebras), then $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are Borel isomorphic, i.e. their enveloping Borel $*$-algebras are isomorphic.

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