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INVARIANT SUBSPACES FOR SUBQUASISCALAR OPERATORS

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Let Z be a Banach space over the complex field $\mathbf{C}$ and let $\mathscr{L}(Z)$ denote the algebra of all bounded linear operators acting in Z. Throughout this paper $S \in \mathscr{L}(Z)$ will be a fixed *quasiscalar* operator, i.e. there exists a continuous linear multiplicative extension

$$\mathscr{V} : \mathscr{C}(\sigma(S)) \to \mathscr{L}(Z)$$

of the Riesz-Dunford functional calculus of S with analytic functions ($\mathscr{C}(\sigma(S))$ denotes the supnorm algebra of all continuous complex functions defined on the spectrum of S). The symbol T will denote the restriction of S to an invariant subspace $X \subset Z$. In other words, $T \in \mathscr{L}(X)$ will be a *subquasiscalar* operator having $S \in \mathscr{L}(Z)$ as a quasiscalar extension. A rationally invariant subspace of T will mean a subspace in X invariant for $(\lambda - T)^{-1}$, $(\forall)$ $\lambda \notin \sigma(T)$ and in particular an invariant subspace of T.

We shall prove the following two theorems:

THEOREM 1. *T has a proper invariant subspace.*

THEOREM 2. *If* $\operatorname{int}\sigma(T) \neq \emptyset$ *then T has a proper rationally invariant subspace.*

The above Theorem 1 is a Banach space variant of Scott Brown's Theorem [4]. The proof of both theorems will be given at the end of the paper.

Let G be a bounded open connected subset of $\mathbf{C}$, $G \neq \emptyset$, and let $L^1(G)$, $L^\infty(G)$ be the integrable, resp. essentially bounded with respect to Lebesgue planar measure, classes of complex functions on G. The weak* topology of $L^\infty(G)$ is the $L^1(G)$-topology. The subspace $H^\infty(G) \subset L^\infty(G)$ of all bounded analytic functions defined in G is w^*-closed and can be canonically identified with the dual of the separable Banach space $M'(G) = L^1(G)/H^\infty(G)^\perp \cap L^1(G)$ (see [11], § 4). The weak* topology of $H^\infty(G)$ will be the $M'(G)$-topology. For any $\mu \in G$ the evaluation at μ in $H^\infty(G)$ will be denoted by $\mathscr{E}_\mu$. By the part (i) of the proof of [11], Theorem 4.1, we know that we have $\mathscr{E}_\mu \in M'(G)$. As in [11], § 4, we shall say that a subset σ of G is dominating in G if

$$\|f\|_\infty = \sup_{\lambda \in \sigma} |f(\lambda)|\ , \qquad (\forall) f \in H^\infty(G).$$

Let us put

$$\mathscr{C}_G(\sigma(S)) = \{\varphi \in \mathscr{C}(\sigma(S)) \colon \operatorname{supp} \varphi \subset G\}$$

and for any $f \in H^\infty(G)$, $\varphi \in \mathscr{C}_G(\sigma(S))$ define $f\varphi \in \mathscr{C}_G(\sigma(S))$ by the equation

$$(f\varphi)(\lambda) = \begin{cases} f(\lambda)\varphi(\lambda), & \lambda \in \operatorname{supp} \varphi, \\ 0 \quad , & \lambda \in \sigma(S) \setminus \operatorname{supp} \varphi. \end{cases}$$

Let Z^* denote the dual of Z and let $S^* \in \mathscr{L}(Z^*)$ be the conjugate of S. It is plain that S^* is a quasiscalar operator with the functional calculus

$$\mathscr{V}^* \colon \mathscr{C}(\sigma(S)) \to \mathscr{L}(Z^*), \quad \mathscr{V}^*(\varphi) = \mathscr{V}(\varphi)^*, \quad \varphi \in \mathscr{C}(\sigma(S)).$$

For any $\varphi \in \mathscr{C}_G(\sigma(S))$, $z \in Z$, $z^* \in Z^*$ define the functional φ^{z,z^*} in $H^\infty(G)$ by the equation

$$\varphi^{z,z^*}(f) = z^*(\mathscr{V}(f\varphi)z), \; f \in H^\infty(G).$$

Finally recall that a hyperinvariant subspace of T is a subspace in X invariant for the commutant of T.

The next four lemmas represent a simplified version of a part of the proof of Scott Brown's Theorem, [4], on invariant subspaces for subnormal operators.

LEMMA 1. *For any* $\varphi \in \mathscr{C}_G(\sigma(S))$, $z \in Z$, $z^* \in Z^*$ *the functional* φ^{z,z^*} *is* w^*-*continuous. If* $\lambda \in G \setminus \operatorname{supp}(1-\varphi)$ *is given and* $\{z_n\}_{n=1}^\infty \subset Z$ *is a bounded sequence such that* $\lim\limits_{n\to\infty} \|(S-\lambda)z_n\| = 0$, *then we have*

$$\lim_{n\to\infty} \|z_n - \mathscr{V}(\varphi)z_n\| = 0 \,, \quad \lim_{n\to\infty} \|z^*(z_n)\mathscr{E}_\lambda - \varphi^{z_n, z^*}\| = 0,$$

uniformly with respect to $z^* \in Z^*$ *in bounded sets.*

Proof. If $\{f_n\}_{n=1}^\infty \subset H^\infty(G)$ is w^*-convergent to 0, then it converges uniformly on compact subsets of G, thus $\lim\limits_{n\to\infty} \|f_n\varphi\| = 0$ and in particular $\lim\limits_{n\to\infty} \varphi^{z,z^*}(f_n) = 0$. Because $M'(G)$ is separable, the w^*-continuity of φ^{z,z^*} follows by [5], Theorem 2.3. Now choose $\psi \in \mathscr{C}(\sigma(S))$ such that $1 - \varphi(\mu) = \psi(\mu)(\mu - \lambda)$, $\mu \in \sigma(S)$ and for every $f \in H^\infty(G)$ define $f_\lambda \in H^\infty(G)$ by the equation

$$f(\mu) - f(\lambda) = f_\lambda(\mu)(\mu - \lambda), \qquad \mu \in G.$$

Since $\{f_\lambda\}_{\|f\|_\infty \leq 1}$ is a bounded set, we deduce

$$\lim_{n\to\infty} \|z_n - \mathscr{V}(\varphi)z_n\| = \lim_{n\to\infty} \| \mathscr{V}(\psi)(S-\lambda)z_n\| = 0,$$

$$\|z^*(z_n)\,\mathscr{E}_\lambda - \varphi^{z_n, z^*}\| \leq \|z^*\| \,(\sup_{\|f\|_\infty = 1} \|\mathscr{V}(f_\lambda\varphi)\| \; \|(S-\lambda)z_n\| + \|z_n - \mathscr{V}(\varphi)z_n\|),$$

which concludes the proof.

LEMMA 2. *Suppose* $\sigma_p(T^*)=\emptyset$ *and* $\{\varphi_n\}_{n=1}^{\infty}\subset\mathscr{C}_G(\sigma(S))$ *is a bounded sequence such that* $\lim\limits_{n\to\infty}\operatorname{diam}(\operatorname{supp}\varphi_n)=0$. *Then for every* $x\in X$ *we have* $\lim\limits_{n\to\infty}\|\mathscr{V}(\varphi_n)x\|=0$.

Proof. In the contrary case we may suppose that there exist $\lambda\in\sigma(S)$, $\{z_n^*\}_{n=1}^{\infty}\subset Z^*$ such that

$$\lim_{n\to\infty}\operatorname{dist}(\lambda,\operatorname{supp}\varphi_n)=0,\ \|z_n^*\|=1,\quad w^*\text{-}\lim_{n\to\infty}\mathscr{V}^*(\varphi_n)z_n^*=z^*,\ z^*(x)\neq 0.$$

Since the restriction x^* of z^* to X differs from 0 and for any $y\in X$ we have

$$|((T^*-\lambda)x^*)(y)|=\lim_{n\to\infty}|((S^*-\lambda)\mathscr{V}^*(\varphi_n)z_n^*)(y)|\leqslant$$
$$\leqslant\|y\|\,\|\mathscr{V}\|\,\overline{\lim_{n\to\infty}}\|\varphi_n\|\operatorname{dist}(\lambda,\operatorname{supp}\varphi_n)=0;$$

we contradict the assumption $\sigma_p(T^*)=\emptyset$.

LEMMA 3. *Suppose* $\sigma_p(T^*)=\emptyset$ *and* $\sigma(T)\cap G$ *is dominating in* G *and let* $\mu\in G$, $0<b<1$ *be given. Then there exist* $\{x_n\}_{n=0}^{\infty}\subset X$, $\{z_n^*\}_{n=0}^{\infty}\subset Z^*$, $\{\varphi_n\}_{n=0}^{\infty}\subset\mathscr{C}_G(\sigma(S))$ *such that*

$$\|x_{n+1}-x_n\|<b^{n-1},\ \|z_{n+1}^*-z_n^*\|<\|\mathscr{V}\|^2b^{n-1},$$

$$\mathscr{V}^*(\varphi_n)z_n^*=z_n^*,\quad\|\mathscr{E}_\mu-\varphi_n^{x_n,z_n^*}\|<b^{2(n-1)}.$$

Proof. Proceeding by induction, assume that $\{x_j\}_{j=0}^n$, $\{z_j^*\}_{j=0}^n$, $\{\varphi_j\}_{j=0}^n$ are determined, with $\|x_0\|=\|z_0^*\|=\|\varphi_0\|=0$. Since $\sigma_n=(\sigma(T)\cap G)\setminus\operatorname{supp}\varphi_n$ is still dominating in G, arguing as in [4], Lemma 4.4 or [5], Proposition 2.8, we can find $\{c_k\}_{k=1}^m\subset\mathbf{C}$, $\{\mu_k\}_{k=1}^m\subset\sigma_n$ such that

$$0<\sum_{k=1}^m|c_k|=c<b^{2(n-1)},\quad\|\mathscr{E}_\mu-\varphi_n^{x_n,z_n^*}-\sum_{k=1}^m c_k\mathscr{E}_{\mu_k}\|<b^{2n}.$$

Let $\varepsilon>0$ be given. Because $T-\lambda$, $\lambda\in\sigma(T)$ is not bounded from below (via $\sigma_p(T^*)=\emptyset$) we can find $\{y_k(\varepsilon)\}_{k=1}^m\subset X$, $\{\psi_k\}_{k=1}^m\subset\mathscr{C}_G(\sigma(S))$ such that

$$0\leqslant\psi_k\leqslant 1,\ \operatorname{supp}\varphi_n\cap\operatorname{supp}\psi_k=\operatorname{supp}\psi_j\cap\operatorname{supp}\psi_k=\emptyset,\quad j\neq k,$$

$$\mu_k\notin\operatorname{supp}(1-\psi_k),\ \|y_k(\varepsilon)\|=1,\ \|(S-\mu_k)y_k(\varepsilon)\|<\varepsilon.$$

Since obviously

$$\|\sum_{k=1}^m\alpha_k e^{i\theta_k}\mathscr{V}(\psi_k)y_k(\varepsilon)\|\leqslant\|\mathscr{V}\|\,\|\sum_{k=1}^m\alpha_k\mathscr{V}(\psi_k)y_k(\varepsilon)\|,\quad\alpha_k\in\mathbf{C},\quad\theta_k\in[0,2\pi),$$

applying [3], Lemma 4.3 we can find $\{\alpha_k(\varepsilon)\}_{k=1}^m \subset \mathbf{C}$, $z_\varepsilon^* \in Z^*$ such that

$$\left\|\sum_{k=1}^m \alpha_k(\varepsilon)\mathscr{V}(\psi_k)y_k(\varepsilon)\right\| \leqslant 1, \quad \|z_\varepsilon^*\| \leqslant \|\mathscr{V}\|, \quad \alpha_k(\varepsilon)z_\varepsilon^*(\mathscr{V}(\psi_k)y_k(\varepsilon)) = c^{-1}c_k .$$

If we choose $\varphi_{n+1} \in \mathscr{C}_G(\sigma(S))$ such that $\varphi_{n+1}\left(\varphi_n + \sum_{k=1}^m \psi_k\right) = \varphi_n + \sum_{k=1}^m \psi_k$ and we set

$$y(\varepsilon) = x_n + c^{1/2}\sum_{k=1}^m \alpha_k(\varepsilon)y_k(\varepsilon), \quad y_\varepsilon^* = z_n^* + c^{1/2}\mathscr{V}^*\left(\sum_{k=1}^m \psi_k\right)z_\varepsilon^*$$

we have $\mathscr{V}^*(\varphi_{n+1})y_\varepsilon^* = y_\varepsilon^*$ and by Lemma 1

$$\overline{\lim_{\varepsilon\to 0}}\, \|y_k(\varepsilon) - \mathscr{V}(\psi_k)y_k(\varepsilon)\| = 0, \quad \overline{\lim_{\varepsilon\to 0}}\left\|\sum_{k=1}^m c_k\mathscr{E}_{\mu_k} - c\sum_{k=1}^m \alpha_k(\varepsilon)\psi_k^{y_k(\varepsilon),\, z_\varepsilon^*}\right\| = 0,$$

$$\overline{\lim_{\varepsilon\to 0}}\, \|y(\varepsilon) - x_n\| < b^{n-1}, \quad \overline{\lim_{\varepsilon\to 0}}\, \|y_\varepsilon^* - z_n^*\| < \|\mathscr{V}\|^2 b^{n-1}.$$

Because by Lemma 2 we can make $c^{1/2}\sum_{k=1}^m \left(\overline{\lim_{\varepsilon\to 0}}\, \|\psi_k^{x_n,\, z_\varepsilon^*}\|\right)$ arbitrarily small and we have

$$\overline{\lim_{\varepsilon\to 0}}\, \|\mathscr{E}_\mu - \varphi_{n+1}^{y(\varepsilon),\, y_\varepsilon^*}\| \leqslant$$

$$\leqslant \|\mathscr{E}_\mu - \varphi_n^{x_n,\, z_n^*} - \sum_{k=1}^m c_k\mathscr{E}_{\mu_k}\| + c^{1/2}\sum_{k=1}^m \left(\lim_{\varepsilon\to 0}\|\psi_k^{x_n,\, z_\varepsilon^*}\|\right) <$$

$$< b^{2n} + c^{1/2}\sum_{k=1}^m \left(\lim_{\varepsilon\to 0}\|\psi_k^{x_n,\, z_\varepsilon^*}\|\right),$$

we can take $x_{n+1} = y(\varepsilon)$, $z_{n+1}^* = y_\varepsilon^*$ for ε small enough.

LEMMA 4. *If $\sigma(T) \cap G$ is dominating in G, then there exists a proper subspace of X, invariant for $(\lambda - T)^{-1}$, $(\forall)\ \lambda \notin \sigma(T) \cup \overline{G}$.*

Proof. We may assume that T has no proper hyperinvariant subspace, consequently $\sigma_p(T^*) = \emptyset$. Let $\mu \in G$, $0 < b < 1$ be given and let $\{x_n\}_{n=0}^\infty \subset X$, $\{z_n^*\}_{n=0}^\infty \subset Z^*$, $\{\varphi_n\}_{n=0}^\infty \subset \mathscr{C}_G(\sigma(S))$ be as in Lemma 3. Put $x = \lim_{n\to\infty} x_n$, $z = \lim_{n\to\infty} z_n$ and denote by Y the invariant subspace of T generated by $\{(\lambda - T)^{-1}x\}_{\lambda\notin\sigma(T)\cup\overline{G}}$. Since we have

$$1 = \mathscr{E}_\mu(1) = \lim_{n\to\infty} z_n^*(x_n) = z^*(x)$$

we derive $x \neq 0$, $z^* \neq 0$, $Y \neq \{0\}$, $X \notin \ker z^*$. For any rational function r with poles off $\sigma(T) \cup \bar{G}$ define $r^\mu \in H^\infty(G)$ by the equation

$$r^\mu(\zeta) = r(\zeta)(\zeta - \mu), \quad \zeta \in G.$$

Using the relations

$$z^*(r(T)(T - \mu)x) = \lim \varphi_n^{x_n, z_n^*}(r^\mu) = r^\mu(\mu) = 0,$$

we deduce $(T - \mu)Y \subset \ker z^*$ and because $\overline{(T - \mu)X} = X$ (via $\sigma_p(T^*) = \emptyset$) we also have $Y \neq X$.

Proof of Theorem 1. Let $\mathscr{F}$ denote the family of all compact sets $\sigma \subset \mathbf{C}$ such that

(i) σ is a union of $\sigma(T)$ with a union of bounded connected components of $\rho(T)$,

(ii) $R(\sigma)$, the closure in $\mathscr{C}(\sigma)$ of all rational functions with poles off σ, is a Dirichlet algebra (see [9], II, § 3 for a definition).

Then $\mathscr{F}$ is nonvoid (see [9], II, § 3) and inductively ordered by inclusion. Indeed, if $\{\sigma_\iota\}_\iota \subset \mathscr{F}$ is totally ordered, then obviously $\bigcap_\iota \sigma_\iota$ is the intersection of a decreasing sequence of σ_ι's, thus by [10], Corollary 9.6 we have $\bigcap_\iota \sigma_\iota \in \mathscr{F}$. Now by [8], I, § 2, Theorem 7, we can find a minimal $\delta \in \mathscr{F}$. If $\operatorname{int}\delta = \emptyset$, then $\delta = \sigma(T)$ and by [9], II, Corollary 9.2 we deduce $R(\sigma(T)) = \mathscr{C}(\sigma(T))$ and T will be a quasiscalar operator. Because any quasiscalar operator has proper rationally invariant subspaces (see [6], III, § 1 and IV, § 1) we assume further $\operatorname{int}\delta \neq \emptyset$. Let G be any connected component of $\operatorname{int}\delta$. If $\sigma(T) \cap G$ is not dominating in G we can find $f \in H^\infty(G)$ such that

$$\|f\|_\infty = \sup_{\lambda \in \rho(T) \cap G} |f(\lambda)| > \sup_{\lambda \in \sigma(T) \cap G} |f(\lambda)|.$$

But it is easy to see that this is possible only if there exists a connected component G' of $\rho(T)$, $G' \subset G$, such that $\partial G' \cap \partial G \neq \emptyset$ (via the Maximum Modulus Theorem), thus by [10], Corollary 9.7 we infer $\delta \setminus G' \in \mathscr{F}$, contradicting the minimality of δ. The conclusion is that $\sigma(T) \cap G$ is dominating in G and we terminate the proof applying Lemma 4.

Remark. The existence of G in the proof of Theorem 1, can be derived as Stampfli did when he proved [12], Lemma 1.

Proof of Theorem 2. Let G be any connected component of $\operatorname{int}\sigma(T)$. Since $\bar{G} \subset \sigma(T)$ the subspace produced by Lemma 4 is rationally invariant.

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