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POSITIVE DIAGONAL AND TRIANGULAR OPERATORS

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INTRODUCTION

In this paper we study two classes of order bounded operators on a Dedekind complete Riesz space. In Section 1 we consider order bounded operators with a strong local property, the so called orthomorphisms, which have been studied by several authors (see e.g. [3], [11], [22] and [23]). For a Dedekind complete Riesz space L the collection of all orthomorphisms is equal to $\{I\}^{dd}$, the band generated by the identity operator in the Riesz space of all order bounded operators on L. Hence every positive linear operator $T: L \to L$ has a unique decomposition $T = T_1 + T_2$ with $0 \leqslant T_1 \in \{I\}^{dd}$ and $0 \leqslant T_2 \in \{I\}^{d}$. One can now consider T_1 as the diagonal component of T and some of the results in Section 1 have been motivated by this point of view. In Section 2 we prove a continuity theorem for the spectral radius of a certain class of positive operators on a Banach lattice. In Section 3 we study a class of operators, called triangular here, which generalize the classical Volterra integral operators. An operator is called triangular if it has a maximal chain of invariant bands. An important result is then, that every positive order continuous triangular compact operator with diagonal component zero is quasinilpotent. We also study the case that the diagonal component is not zero and prove that in that case the spectrum of the triangular operator is equal to the spectrum of its diagonal component. The author wishes to express his gratefulness to the referee for supplying shorter proofs for Lemma 2.3 and Theorem 2.4.

1. ORTHOMORPHISMS

Let L be an Archimedean Riesz space (for terminology not explained here, see [14] and [19]). A positive linear operator T from L into L is called a positive orthomorphism if $0 < u, v \in L$ and $u \wedge v = 0$ implies $Tu \wedge v = 0$. A linear map from L into L is now called an orthomorphism, if it is the difference of two positive orthomorphisms. The set of orthomorphisms from L into L shall be denoted by $\mathrm{Orth}(L)$. The main result about $\mathrm{Orth}(L)$ is that it is a commutative f-algebra with respect to pointwise defined product, supremum and infimum (see [3], [13] and [22]). An important subalgebra of Orth(L) is the ideal centre $Z(L)$ of L, consisting of all $T \in$ Orth(L) for which there exists a real number λ such that $-\lambda I \leqslant T \leqslant \lambda I$. In case L is a Dedekind complete Riesz space Orth(L) is equal to $\{I\}^{dd}$, the band generated by I in the space $\mathscr{L}_b(L, L)$ of all order bounded operators from L into L. From this it follows that every positive linear operator $T: L \to L$ has a unique decomposition $T = T_1 + T_2$ with $0 \leqslant T_1 \in \{I\}^{dd} =$ Orth(L) and $0 \leqslant T_2 \in \{I\}^d$. The next theorem gives a new formula for T_1.

THEOREM 1.1. *Let L be a Dedekind complete Riesz space and* $0 \leqslant T \in \mathscr{L}_b(L, L)$. *Then the component* $T_1 \in \{I\}^{dd}$ *of T is given by*

$$\inf \left(\sum_{i=1}^{n} P_i T P_i : 0 \leqslant P_i \leqslant I,\ P_i^2 = P_i,\ \sum_{i=1}^{n} P_i = I \right).$$

Proof. For each $0 \leqslant T \in \mathscr{L}_b$ we denote by $\mathfrak{Q}(T)$ the infimum of the set $\left(\sum_{i=1}^{n} P_i T P_i : 0 \leqslant P_i \leqslant I,\ P_i^2 = P_i,\ \sum_{i=1}^{n} P_i = I \right)$. We shall prove that $\mathfrak{Q}$ is the projection of $\mathscr{L}_b^+$ onto Orth$^+(L)$. One easily verifies that the set of which $\mathfrak{Q}(T)$ is the infimum, is directed downwards. It follows from this observation that $\mathfrak{Q}$ is additive on $\mathscr{L}_b^+$. We now show that $\mathfrak{Q}(T) \in$ Orth$^+(L)$, i.e., that $\mathfrak{Q}(T)$ leaves every band in L invariant. Let therefore B be a band in L with bandprojection P and let $0 \leqslant u \in B$. Then

$$PTPu + (I - P)T(I - P)u = PTPu$$

implies that

$$\mathfrak{Q}(T)u \leqslant PTPu \in B, \quad \text{so} \quad \mathfrak{Q}(T)(B) \subset B.$$

For $0 \leqslant T \in$ Orth(L) and $\sum_i P_i = I$ we have

$$\sum_i P_i T P_i = \sum_i P_i^2 T = \sum_i P_i T = T,$$

so that $\mathfrak{Q}(T) = T$ for $0 \leqslant T \in$ Orth(L). It follows that for $0 \leqslant T \in \mathscr{L}_b$ we have that $\mathfrak{Q}^2(T) = \mathfrak{Q}(\mathfrak{Q}(T)) = \mathfrak{Q}(T)$, i.e., $\mathfrak{Q}^2 = \mathfrak{Q}$ on $\mathscr{L}_b^+$. It is also clear, that $0 \leqslant \mathfrak{Q}(T) \leqslant T$ for all $T \in \mathscr{L}_b^+$ and we conclude that $\mathfrak{Q}$ is the projection of $\mathscr{L}_b^+$ onto Orth$^+(L)$ and the theorem is proved.

We present an application of the above theorem. First we recall, that if L is a Dedekind complete Banach lattice and if $T \in \mathscr{L}_b$, then $\|T\|_r$ denotes the operatornorm of $|T|$.

THEOREM 1.2. *Let L be a Dedekind complete Banach lattice and let* $T \in \mathscr{L}_b(L, L)$. *Then* $e^{tT} \geqslant 0$ *for all* $t \geqslant 0$ *if and only if* $T + \|T\|_r I \geqslant 0$.

Proof. Assume first that $T + \|T\|_r I \geqslant 0$. Then

$$e^{t(T+\|T\|_r I)} = I + \frac{t}{1!}(T + \|T\|_r I) + \dots \geqslant 0$$

for all $t \geqslant 0$. Since T and I commute we have $e^{t(T+\|T\|_r I)} = e^{t\|T\|_r} e^{tT}$, so $e^{tT} = e^{-t\|T\|_r} \cdot e^{t(T+\|T\|_r I)} \geqslant 0$ for all $t \geqslant 0$. Assume now that $e^{tT} \geqslant 0$ for all $t \geqslant 0$. Let $0 \leqslant \varphi \in L^*$ and $0 \leqslant u \in L$. Define $g(t) = \varphi(e^{tT}u)$ for $t \in \mathbf{R}$. Then $g(t)$ is differentiable and $g'(t) = \varphi(Te^{tT}u)$, in particular $g'(0) = \varphi(Tu)$. We have now $g(t) \geqslant 0$ for $t \geqslant 0$ and $g(0) = \varphi(u)$. We conclude from this that if $\varphi(u) = g(0) = 0$, then $\varphi(Tu) = g'(0) \geqslant 0$. Let now $0 \leqslant P \leqslant I$ be a bandprojection in L. Then for $0 \leqslant Q \leqslant I - P$, $Q^2 = Q$, we have $Q^*\varphi(Pu) = 0$, so $Q^*\varphi(TPu) \geqslant 0$ by the above argument applied to $Q^*\varphi$ and Pu. It follows that $\varphi(QTPu) \geqslant 0$ for all $\varphi > 0$ and $u \geqslant 0$, so $QTP \geqslant 0$ for all bandprojections P and Q with $0 \leqslant Q \leqslant I - P$. Decompose $T = T_1 + T_2$ with $T_1 \in \{I\}^d$ and $T_2 \in \{I\}^{dd}$. Then $QT_2P = 0$ for all bandprojections P and Q with $0 \leqslant Q \leqslant I - P$, so also $QT_1P \geqslant 0$ for all such P and Q. Let now $I = \sum_{i=1}^{n} P_i$, $P_i^2 = P_i$ and $0 \leqslant P_i \leqslant I$. Then it follows that $\sum_{i \neq j} P_i T_1 P_j \geqslant 0$, so

$$T_1 = \sum_{i \neq j} P_i T_1 P_j + \sum_{i=1}^{n} P_i T_1^+ P_i - \sum_{i=1}^{n} P_i T_1^- P_i \geqslant - \sum_{i=1}^{n} P_i T_1^- P_i.$$

Hence

$$T_1 \geqslant \sup\left(- \sum_{i=1}^{n} P_i T_1^- P_i : \sum_i P_i = I \right) = - \mathfrak{D}(T_1^-) = 0,$$

by Theorem 1.1. We conclude that $T = T_1 + T_2$, with $T_2 \in \mathrm{Orth}(L)$ and $0 \leqslant T_1 \in \{I\}^d$, so $T + \|T_2\| I = T_1 + (T_2 + \|T_2\| I) \geqslant 0$. From $|T| = T_1 + |T_2|$ it follows that $\|T_2\| \leqslant \|T\|_r$ and so also $T + \|T\|_r I \geqslant 0$.

Remarks: (i) The above result is related to a recent result of D. E. Evans and H. Hanche-Olsen [9]. Part of their results can be described as follows. Let E be a partially ordered real Banach space with the property that for all $x \in E$ there exists $y \in E^+$ such that $\mathrm{dist}(x, E^+) = \|x - y\|$. Let $T : E \to E$ be a norm bounded operator. Then $e^{tT} \geqslant 0$ for $t \geqslant 0$ if and only if $(\lambda I - T)^{-1} \geqslant 0$ for all $\lambda > \|T\|$.

It follows easily from $T + \|T\| I \geqslant 0$ that $(\lambda I - T)^{-1} \geqslant 0$ for all $\lambda > \|T\|$, but the converse seems less obvious and might be false in general.

(ii) The referee included in his report a better result than Theorem 1.2. He proved that if T is a *norm* bounded operator from L into L, then $e^{tT} \geqslant 0$ for all $t \geqslant 0$ if and only if $T + \|T\| I \geqslant 0$.

Corollary 1.3. *Let $L = \mathbf{R}^n$ with the canonical basis and order and let $T : L \to L$ be a linear operator with matrix $[t_{ij}]$. Then $e^{tT} \geqslant 0$ for all $t \geqslant 0$ if and only if $t_{ij} \geqslant 0$ for all $i \neq j$.*

We shall now present a number of properties of orthomorphisms on Banach lattices. The following theorem is essentially known (see e.g. [22]), but the technique used in the present simple proof will be used further on.

THEOREM 1.4. *Let L be a normed Riesz space and let* $0 \leqslant T \in \mathrm{Orth}(L)$ *be norm continuous. Then there is a positive number* λ *such that* $T \leqslant \lambda I$. *Moreover* $\|T\| = \inf(\lambda : T \leqslant \lambda I)$.

Proof. Assume $(T - \lambda_0 I)^+ > 0$ for some $\lambda_0 > 0$. Then there exists $0 \leqslant u_0 \in L$ such that $v_0 = (T - \lambda_0 I)^+ u_0 > 0$. Since $\mathrm{Orth}(L)$ is an f-algebra, this implies that

$$(T - \lambda_0 I)v_0 = \{(T - \lambda_0 I)^+\}^2 u_0 \geqslant 0,$$

so $Tv_0 \geqslant \lambda_0 v_0$. Hence $\|T\| \geqslant \lambda_0$. It follows that $(T - \lambda I)^+ = 0$ for all $\lambda > \|T\|$, i.e., $T \leqslant \lambda I$ for all $\lambda > \|T\|$. This implies that also $T \leqslant \|T\| I$ and so $\inf(\lambda : T \leqslant \lambda I) \leqslant \|T\|$; since the converse inequality is obvious, it follows that $\inf(\lambda : T \leqslant \lambda I) = \|T\|$.

COROLLARY 1.5. *If L is a Banach lattice, then*
$\mathrm{Orth}(L) = Z(L) = \{T \in \mathscr{L}_b(L, L) : -\lambda I \leqslant T \leqslant \lambda I$ *for some* $\lambda\}$.

Recall now that $u \in L^+$ is called an atom if $0 \leqslant v \leqslant u$ implies that $v = \lambda u$ for some $\lambda \geqslant 0$. The main step in the following theorem is due to T. Ando ([2]).

THEOREM 1.6. *Let L be a Banach lattice and let* $T: L \to L$ *be a positive compact operator. Then* $S \in \mathrm{Orth}(L)$ *and* $0 \leqslant S \leqslant T$ *imply that S is compact.*

Proof. Assume first that L is Dedekind complete and assume $S > 0$. Then S is the uniform limit of sums of the form $\sum_1^n \alpha_i P_i$ with $0 \leqslant \sum_i^n \alpha_i P_i \leqslant S \leqslant T$, where the P_i's are bandprojections. The proof for this case will therefore be complete if we show that each such P_i is compact. Consider therefore a bandprojection P with $0 \leqslant P \leqslant T$. If P is not compact, then there exist $0 \leqslant x_n \in P(L)$ with $x_n \wedge x_m = 0$ for $n \neq m$ and $\|x_n\| = 1$ for all n. Denote by P_n the projection on $\{x_n\}^{dd}$. Then $0 < P_n \leqslant P \leqslant T$ for all n. The compactness of T implies that we may assume that $Tx_n \to y$ in norm for some $y \geqslant 0$. From $0 \leqslant \sum_{k=1}^n P_n y \leqslant Py$ it follows that $P_n y \to 0$ for $\sigma(L, L^*)$. It follows that $TP_n(y) \to 0$ in norm and then the inequalities $0 \leqslant P_n y = P_n^2(y) \leqslant TP_n y$ imply that also $P_n y \to 0$ in norm. Hence

$$1 = \|x_n\| = \|P_n P x_n\| \leqslant \|P_n T x_n\| \leqslant \|P_n y\| + \|P_n(Tx_n - y)\| \leqslant$$
$$\leqslant \|P_n y\| + \|Tx_n - y\| \to 0$$

as $n \to \infty$ and we have a contradiction. This completes the proof in case L is Dedekind complete. The general case follows from $0 \leqslant S^* \leqslant T^*$ on L^* and the fact that $S^* \in Z(L^*)$, T^* is compact and that L^* is Dedekind complete.

COROLLARY 1.7. *Let L be a Banach lattice without atoms and let $T: L \to L$ be a positive compact operator. Then $S \in$ Orth(L) and $0 \leqslant S \leqslant T$ imply that $S = 0$.*

Proof. Follows immediately from the above theorem and Theorem 7.2. of [16].

REMARK. If in the above theorem S is not required to be an orthomorphism, then S need not be compact. It is however proved in [8], that S is compact, whenever the norms on L and L^* are order continuous, in particular when L is reflexive.

We shall now look at spectral properties of orthomorphisms. To this end we consider a complex Banach lattice L and complex orthomorphisms. Then Orth(L) is isometrically Riesz isomorph with $C_C(K)$, the space of complex valued continuous functions on some compact Hausdorff space K. Let $\mathscr{L} = \mathscr{L}(L, L)$ denote the (complex) linear space of continuous linear operators on L. Then the following theorem was stated without proof in [21].

THEOREM 1.8. Orth(L) *is a full sub-algebra of* $\mathscr{L}$, i.e., *if T^{-1} exists in $\mathscr{L}$, then* $T^{-1} \in$ Orth (L).

Proof. Let $T \in$ Orth(L) such that T^{-1} exists in $\mathscr{L}$. We shall then prove that there exists a constant $c > 0$ such that $|T| \geqslant cI$. If this is not the case, then for all $n \geqslant 1$ we have $(I - n|T|)^+ > 0$. It follows that there exist $u_n \in L^+$ such that $v_n = (I - n|T|)^+ u_n > 0$. Since $|T| \in$ Orth(L), it follows that

$$(I - n|T|)(v_n) = \{(I - n|T|)^+\}^2 v_n \geqslant 0,$$

so

$$v_n \geqslant n|T|v_n \geqslant 0.$$

From this we conclude that

$$\|v_n\| = \|T^{-1}Tv_n\| \leqslant \|T^{-1}\| \, \|Tv_n\| \leqslant \|T^{-1}\| \, \||T|v_n\| \leqslant \|T^{-1}\| \frac{1}{n} \|v_n\|.$$

Hence $\|T^{-1}\| \geqslant n$ for all $n \geqslant 1$, which contradicts $T^{-1} \in \mathscr{L}$. It follows that $|T| \geqslant cI$ for some $c > 0$. Since Orth(L) $\cong C_C(K)$ we conclude that T is invertible in Orth(L), which implies that $T^{-1} \in$ Orth(L).

LEMMA 1.9. *Let L be a Banach lattice with the property that $0 \leqslant u_n \leqslant u$, $u_n \to 0$ $\sigma(L, L^*)$, implies that $\|u_n\| \to 0$. Then L is atomic,* i.e., *there exists a maximal disjoint system in L, consisting only of atoms.*

Proof. Let $0 \leqslant u_n \leqslant u \in L$ with $u_n \wedge u_m = 0$ if $n \neq m$. Then $u_n \to 0$ $\sigma(L, L^*)$, so by hypothesis $\|u_n\| \to 0$. It follows from Meyer-Nieberg's Theorem (see [19], II Lemma 5.13), that L has order continuous norm. From Theorem II 5.10 of [19], it follows that $[0, u]$ is weakly-compact for every $u \in L^+$. By hypothesis it follows then that $[0, u]$ is norm compact for every $u \in L^+$. From Theorem 4.9 of [16] it follows that L is atomic.

LEMMA 1.10. *Let L be a Banach lattice without atoms and B is non-zero band in L. Then there exist $0 \leqslant u_n \in B$, $\|u_n\| = 1$ such that $u_n \to 0$ $\sigma(L, L^*)$.*

Proof. From the above lemma it follows that there exist $0 \leqslant u_n \in B$ such that $u_n \to 0$ $\sigma(L, L^*)$, but with $0 < \limsup \|u_n\| \leqslant 1$. By passing to a subsequence we can assume, after normalization, that $\|u_n\| = 1$.

THEOREM 1.11. *Let L be a (complex) Banach lattice without atoms and let $T \in \mathrm{Orth}(L)$. Then $\sigma(T) = \sigma_e(T)$, where σ_e denotes the essential spectrum of T.*

Proof. It is sufficient to prove that $\sigma(T) \subset \sigma_e(T)$. To show this we have to prove that if there exist a bounded operator S and a compact operator K such that $ST = I + K$, then T is invertible. If T is not invertible, then for all $\varepsilon > 0$ we have $(\varepsilon I - |T|)^- > 0$, by the same argument as in the proof of Theorem 1.8. Hence, if we set $B_\varepsilon = \{(\varepsilon I - |T|)^-(L)\}^{dd}$, then $|T|u \leqslant \varepsilon u$ for all $0 \leqslant u \in B_\varepsilon$. Let now $\varepsilon = \frac{1}{2} \cdot \|S\|^{-1}$. By the above lemma there exist $0 \leqslant u_n \in B_\varepsilon$, $\|u_n\| = 1$ such that $u_n \to 0$ $\sigma(L, L^*)$. Now

$$1 = \|u_n\| \leqslant \|STu_n\| + \|Ku_n\| \leqslant \|S\| \varepsilon + \|Ku_n\| \leqslant \frac{1}{2} + \|Ku_n\|$$

for all n, which is a contradiction, since $\|Ku_n\| \to 0$.

We remark that the above theorem supplies an alternative proof for Corollary 1.7.

2. A CONTINUITY THEOREM FOR THE SPECTRAL RADIUS

Throughout this section we shall denote by L a complex Banach lattice. Then this section is devoted to prove the following result: If $T_0, T_\tau : L \to L$ are linear operators and if $0 \leqslant T_\tau \uparrow T_0$ on L, i.e., for all $u \in L^+$, $T_\tau u \uparrow Tu$ on L and if T_0 is order continuous and compact, then $r(T_\tau) \uparrow r(T_0)$. With the additional hypothesis $\|T_\tau - T_0\| \to 0$ this would be an easy consequence of some simple facts of the operator calculus (see [15]). If L and L^* have order continuous norm, in particular, if L is reflexive, then $0 \leqslant T_\tau \uparrow T_0$ and T_0 compact, will imply that $\|T_0 - T_\tau\| \to 0$ ([8]). The complications in the present general case arise from the fact that in general without additional hypotheses $\|T_0 - T_\tau\|$ does not tend to zero. We start with a few simple lemmata.

LEMMA 2.1. *Let L be a Banach lattice and T_0, T_τ positive linear operators from L into L such that $T_\tau \uparrow T_0$ and such that T_0 is order continuous and compact. Then for every $u \in L^+$ we have that*

$$\|T_0(T_0 - T_\tau)u\| \downarrow 0.$$

Proof. Let $v_\tau = (T_0 - T_\tau)u$. Then $v_\tau \downarrow 0$ in L, so $T_0 v_\tau \downarrow 0$. From the precompactness of $\{T_0 v_\tau\}$ and the fact that $T_0 v_\tau \downarrow$ it follows that $\{T_0 v_\tau\}$ is a norm Cauchy net. From this we conclude that $\|T_0 v_\tau\| \downarrow 0$, i.e., $\|T_0(T_0 - T_\tau)u\| \downarrow 0$.

LEMMA 2.2. *Let T_0, T_τ be as in Lemma 2.1. Then we have that $\|T_0(T_0-T_\tau)T_0\| \downarrow 0$.*

Proof. Follows immediately from Dini's Theorem and Lemma 2.1.

LEMMA 2.3. *Let L be a Banach lattice and T_0, T_τ positive linear operators from L into L such that $T_\tau \uparrow T_0$ and such that T_0 is order continuous and compact. If $\lambda_0 \notin \sigma(T_0)$ and $\lambda_0 > \sup_\tau r(T_\tau)$, then $\sup_\tau \|R(\lambda_0, T_\tau)\| < \infty$.*

Proof.

$$T_0\{R(\lambda_0, T_0) - R(\lambda_0, T_\tau)\} =$$

$$= R(\lambda_0, T_0)\cdot T_0(T_0 - T_\tau)\cdot R(\lambda_0, T_\tau) =$$

$$= \lambda_0^{-1}R(\lambda_0, T_0)\, T_0(T_0 - T_\tau)T_\tau\cdot R(\lambda_0, T_\tau) +$$

$$+ \lambda_0^{-1}R(\lambda_0, T_0)\, T_0(T_0 - T_\tau).$$

Suppose that $\sup_\tau \|R(\lambda_0, T_\tau)\| = \infty$. Then

$$\|R(\lambda_0, T_\tau)\|^{-1} \|T_0(T_0 - T_\tau)T_\tau\cdot R(\lambda_0, T_\tau)\| \leqslant$$

$$\leqslant \|T_0(T_0 - T_\tau)T_0\| \to 0 \quad \text{(by Lemma 2.2)}$$

and

$$\|R(\lambda_0, T_\tau)\|^{-1}\|T_0(T_0 - T_\tau)\| \to 0.$$

Therefore

$$\|R(\lambda_0, T_\tau)\|^{-1} \|T_0\{R(\lambda_0, T_0) - R(\lambda_0, T_\tau)\}\| \to 0\,,$$

hence

$$\|R(\lambda_0, T_\tau)\|^{-1} \|T_0\cdot R(\lambda_0, T_\tau)\| \to 0\,,$$

which implies

$$\|R(\lambda_0, T_\tau)\|^{-1} \|T_\tau\cdot R(\lambda_0, T_\tau)\| \to 0\,.$$

But since

$$T_\tau R(\lambda_0, T_\tau) = -I + \lambda_0 R(\lambda_0, T_\tau),$$

this leads to a contradiction $\lambda_0 = 0$.

THEOREM 2.4. *Let $T_0, T_\tau\colon L \to L$ be linear operators such that $0 \leqslant T_\tau \uparrow T_0$ on L and such that T_0 is order continuous and compact. Then $r(T_\tau) \uparrow r(T_0)$.*

Proof. Suppose $\sup_\tau r(T_\tau) < r(T_0)$. Since T_0 is compact, there exists $\lambda_0 \notin \sigma(T_0)$ such that $\sup_\tau r(T_\tau) < \lambda_0 < r(T_0)$. By Lemma 2.3 $\sup_\tau \|R(\lambda_0, T_\tau)\| \equiv M_0 < \infty$. For any $m \geqslant 1$

$$\left\| \sum_{n=0}^{m-1} \lambda_0^{-n-1} T_\tau^n \right\| \leqslant \|R(\lambda_0, T_\tau)\| \leqslant M_0$$

and for each $n \geqslant 1$ $T_\tau^n \uparrow T_0^n$ so that

$$T_0 \left(\sum_{n=0}^{m-1} \lambda_0^{-n-1} T_0^n \right) u = \lim_\tau T_0 \left(\sum_{n=0}^{m-1} \lambda_0^{-n-1} T_\tau^n \right) u$$

because T_0 is order continuous and compact. This implies

$$\left\| \sum_{n=0}^{m-1} \lambda_0^{-n-1} T_0^n \right\| \leqslant \lambda_0^{-1} + \lambda_0^{-1} M_0 \|T_0\| \equiv M_1.$$

Since

$$I - \lambda_0^{-m} T_0^m = \left(\sum_{n=0}^{m-1} \lambda_0^{-n-1} T_0^n \right) (\lambda_0 I - T_0),$$

we have

$$\lambda_0^{-m} \|T_0^m\| \leqslant 1 + M_1(\lambda_0 + \|T_0\|)$$

hence

$$r(T_0) = \lim \|T_0^m\|^{1/m} \leqslant \lambda_0,$$

which is a contradiction.

COROLLARY 2.5. *Let T_0, T_τ be as in Theorem 2.4. If $\lambda_0 \notin \sigma(T_0)$ and $\lambda_0 > \sup_\tau r(T_\tau)$ then $R(\lambda_0, T_\tau) \uparrow R(\lambda_0, T_0)$.*

Proof. The proof of Theorem 2.4 shows that $\lambda_0 \geqslant r(T_0)$ hence $\lambda_0 > r(T_0)$. Then

$$R(\lambda_0, T_0) = \sum_{n=0}^{\infty} \lambda_0^{-n-1} T_0^n$$

hence

$$R(\lambda_0, T_\tau) \uparrow R(\lambda_0, T_0).$$

We give two examples to show that neither the order continuity nor the compactness of T_0 can be dropped from the assumptions in the above theorem.

EXAMPLE 2.6. Let $L = \ell_2$ and let T_0 be the shift operator $T_0(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$. Denote by $T_n\,(n \geqslant 1)$ the operator $T_n(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots, \xi_{n-1}, 0, 0, \ldots)$. Then $0 \leqslant T_n \uparrow T_0$ and T_0 is order continuous, but $r(T_n) = 0$ for all $n \geqslant 1$ and $r(T_0) = 1$. Hence the compactness condition can not be omitted.

EXAMPLE 2.7. Let $L = \ell_\infty$ and let $T_0 = \varphi \otimes e$, where φ is a Banach-Mazur limit on ℓ_∞ and $e = (1, 1, 1, \ldots)$. Let $e_n = (\overbrace{1, 1, \ldots, 1}^{n\text{-times}}, 0, \ldots)$ and let $T_n = \varphi \otimes e_n$. Then $0 \leqslant T_n \uparrow T_0$ and T_0 is compact, but $r(T_n) = 0$ for all n and $r(T_0) = 1$. Hence the order continuity condition can not be omitted.

We conclude this section by remarking that a special version of Theorem 2.4 has been proven by H. Krieger [12]. The proofs and theorems in this section constitute a generalization and clarification of his results.

3. TRIANGULAR OPERATORS

Let L be a Dedekind complete Riesz space and let $\mathscr{B}(L)$ be the Boolean algebra of (band) projections in L. A totally ordered subset of $\mathscr{B}(L)$ is called a chain and the set of all chains in $\mathscr{B}(L)$ is then partially ordered by refinement. Zorn's lemma guarantees then the existence of maximal chains in $\mathscr{B}(L)$. We fix from now on one such maximal chain Π_0 and we shall write $\Pi_0 = \{P_\tau\}$.

DEFINITION. A linear operator from L into L is called *triangular* (with respect to Π_0) if $P_\tau T P_\tau = T P_\tau$ for all $P_\tau \in \Pi_0$.

Let us look at two examples, which have motivated this definition.

EXAMPLE 1. Let L be an (order) ideal in the Riesz space of all sequences on $\mathbf{N}$. Denote by P_n the projection on the first n components. Then $\Pi_0 = \{P_n\} \cup \{0\} \cup \{I\}$ is a maximal chain in $\mathscr{B}(L)$. If $T: L \to L$ is an order bounded order continuous linear operator, then we can associate with T an infinite matrix $[t_{ij}]$ such that T is triangular with respect to Π_0 if and only if $[t_{ij}]$ is upper triangular. By choosing a different maximal chain we get the lower triangular matrices or some mixture of the two.

EXAMPLE 2. Let $L = L_\rho[0, 1]$ be a Banach function space on $[0, 1]$ with respect to Lebesgue measure. Let $P_t\,(0 \leqslant t \leqslant 1)$ denote the bandprojection corresponding to multiplication by $\chi_{[0,t]}$. Then $\Pi_0 = \{P_t\}$ is a maximal chain in $\mathscr{B}(L)$. An order bounded integral operator T with kernel $T(x, y)$ is triangular with respect to Π_0 if and only if $T(x, y) = 0$ a.e. on $\{(x, y) \in [0, 1]^2 : y > x\}$.

We return to discuss the general theory of triangular operators. If we restrict ourselves to order bounded triangular operators, then it is easy to verify that the set of triangular operators is a band in and a sub-algebra of $\mathscr{L}_b$. We now look closer at maximal chains. In case L does not contain any atoms, then we have for each $P_{\tau_0} \in \Pi_0$ that

$$P_{\tau_0} = \sup (P_\tau \in \Pi_0 : P_\tau < P_{\tau_0}) = \inf(P_\tau \in \Pi : P_\tau > P_{\tau_0})$$

in $\mathscr{B}(L)$. In case L does contain atoms, then it can happen that e.g.

$$P_{\tau_0}^- = \sup(P_\tau \in \Pi_0 : P_\tau < P_{\tau_0}) < P_{\tau_0},$$

but this happens only if $P_{\tau_0} - P_{\tau_0}^-$ is an atomic projection, i.e., only if $\dim(P_{\tau_0} - P_{\tau_0}^-)(L) = 1$.

LEMMA 3.1. *Let T be a positive linear operator from L into L. If T is triangular and $T \wedge I = 0$, then $T(P_{\tau_0}(L)) \subset P_{\tau_0}^-(L)$ for all $\tau_0 \in \{\tau\}$.*

Proof. Let $L_0 = P_{\tau_0}(L)$, $L_1 = P_{\tau_0}^-(L)$ and $(P_{\tau_0} - P_{\tau_0}^-)(L) = \{\lambda e\}$ for $e \geqslant 0$. Then $L_0 = L_1 \oplus \{\lambda e\}$. Observe then that $T(L_1) \subset L_1$ and $T(L_0) \subset L_0$, so it suffices to show that $Te \in L_1$. Assume that $Te \notin L_1$. Then $Te = f + \lambda e$ for some $\lambda > 0$ and some $f \in L_1^+$. It is then obvious that

$$0 < \lambda(P_{\tau_0} - P_{\tau_0}^-) \leqslant T$$

on L, which contradicts the assumption that $T \wedge I = 0$.

THEOREM 3.2. *Let L be a Dedekind complete Banach lattice and $T: L \to L$ a positive triangular, order continuous compact operator with $T \wedge I = 0$. Then T is quasi-nilpotent.*

Proof. Define $f : \{\tau\} \to \mathbf{R}^+$ by $f(\tau) = r(TP_\tau) = r(P_\tau TP_\tau)$. We note that f is a monotone mapping and that $f(\tau) \in \sigma(T)$ for each τ, since $f(\tau)$ equals the spectral radius of the restriction of T to $P_\tau(L)$. Assume now that $r(T) > 0$. The range of f is at most countable, say $\{r_n\}$ with $r_n \downarrow 0$. Then $r_1 = r(T)$. Let $P_{\tau_0} = \sup (P_\tau : f(\tau) = r_2)$. From Theorem 2.4 it follows that $f(\tau_0) = r_2 < r(T)$. We consider now two cases:

(1) $P_{\tau_0} = P_{\tau_1}^-$ for some $\tau_1 \in \{\tau\}$ such that $P_{\tau_1}^- < P_{\tau_1}$,

(2) $P_{\tau_0} = \inf(P_\tau : P_\tau > P_{\tau_0})$.

In case 1) there exists $0 \neq u \in L^+$ such that $Tu = r(T)u$ and $P_{\tau_1}u = u$, since $f(\tau_1) = r(T)$. By the above Lemma $TP_{\tau_1}(L) \subset P_{\tau_0}(L)$, so $u \in P_{\tau_0}(L)$. This however implies that $f(\tau_0) = r(T)$, contradicting the choice of τ_0. It remains therefore to derive a contradiction in case 2). In this case there exists $0 \neq u_\tau \in P_\tau(L^+)$ such that $Tu_\tau = r(T)u_\tau$ for all τ with $P_\tau > P_{\tau_0}$. We claim now that this implies that there exists τ' with $P_{\tau'} > P_{\tau_0}$ such that $Tu_{\tau'} = r(T)u_{\tau'}$ and $0 \neq u_{\tau'} \in P_{\tau_0}(L)$, which gives then the required contradiction $f(\tau_0) = r(T)$. Assume that this claim does not hold. Let $0 < u_1 \in L$ be such that $Tu_1 = r(T)u_1$. Then $P_{\tau_0}u_1 < u_1$, so there exists $P_{\tau_1} > P_{\tau_0}$

such that $P_{\tau_1}u_1 < u_1$, since otherwise $P_{\tau_0}u_1 = \inf(P_\tau u_1 : P_\tau > P_{\tau_0}) = u_1$. Let now $0 < u_2 \in P_{\tau_1}(L)$ such that $Tu_2 = r(T)u_2$. Then again $P_{\tau_0}u_2 < u_2$, so we can find P_{τ_2} with $P_{\tau_0} < P_{\tau_2} < P_{\tau_1}$ such that $P_{\tau_2}u_2 < u_2$. Repeating this argument we find $0 < u_n \in$ $\in P_{\tau_{n-1}}(L)$ such that $Tu_n = r(T)u_n$, $P_{\tau_0} < P_{\tau_{n-1}} < P_{\tau_{n-2}} < \ldots < P_{\tau_1}$ and $P_{\tau_n}u_n < u_n$. It is clear from this construction that $\{u_1, \ldots, u_n\}$ is linearly independent for all n, which contradicts the finite dimensionality of the eigenspace corresponding to $r(T)$. The proof is therefore complete.

REMARKS. 1) In case L does not contain any atoms, then we can omit the condition that $T \wedge I = 0$ in the above theorem.

2) The above theorem was already proved by T. Ando [1] for reflexive Banach lattices without atoms.

3) It is still an open problem, whether every positive quasi-nilpotent compact operator is triangular with respect to some maximal chain Π_0. It is known that there exist positive quasi-nilpotent operators without non-trivial invariant bands ([19]), therefore the compactness assumption is essential. However in case the operator is a positive integral operator on a Banach function space, then the compactness condition can be omitted and every positive quasi-nilpotent integral operator is triangular with respect to some maximal chain. This is a consequence of the Perron-Frobenius-Jentzsch theorem for integral operators (see [10]). We now recall some definitions and notations. For $T \in \mathscr{L}_b(L, L)$ we denote by $\sigma_0(T)$ the order spectrum of T, i.e., the spectrum of T as an element of the Banach algebra $\mathscr{L}_b(L, L)$. By $r_0(T)$ we denote the spectral radius of T as an element of $\mathscr{L}_b(L, L)$. The order spectrum has been studied by H. H. Schaefer in [20]. In general we have $\sigma(T) \subset \sigma_0(T)$ and if $T \geqslant 0$, then $r(T) = r_0(T)$.

THEOREM 3.3. *Let L be a Dedekind complete Banach lattice and let $T : L \to L$ be a positive linear operator such that $T = R + S$ with $0 \leqslant R$ compact and quasi-nilpotent and with $0 \leqslant S \in$ Orth(L). Then $\sigma(T) = \sigma_0(T) = \sigma(S)$.*

Proof. First we prove $\sigma_0(T) \subset \sigma(S)$. Let $\lambda \in \sigma(S)$. Then, by Theorem 1.4,

$$|S - \lambda I)^{-1}| \leqslant \|(S - \lambda I)^{-1}\| I.$$

From

$$T - \lambda I = R + S - \lambda I = (R(S - \lambda I)^{-1} + I)(S - \lambda I)$$

we conclude that it is sufficient to show that $(R(S - \lambda I)^{-1} + I)$ is invertible in $\mathscr{L}_b(L, L)$. From the inequalities

$$|R(S - \lambda I)^{-1}| \leqslant R|(S - \lambda I)^{-1}| \leqslant \|(S - \lambda I)^{-1}\| R$$

it follows that

$$r_0(R(S - \lambda I)^{-1}) \leqslant r_0(|R(S - \lambda I)^{-1}|) \leqslant \|(S - \lambda I)^{-1}\|\, r_0(R) = 0\,,$$

so $I + R(S - \lambda I)^{-1}$ is invertible in $\mathscr{L}_b(L, L)$ and it follows that $\lambda \notin \sigma_0(T)$. Hence $\sigma_0(T) \subset \sigma(S)$. Let now $\lambda \notin \sigma(T)$. Then $T - \lambda I = R + S - \lambda I$ implies that

$$I - (T - \lambda I)^{-1}R = (T - \lambda I)^{-1}(S - \lambda I),$$

so it suffices to show that $I - (T - \lambda I)^{-1}R$ is invertible. Since $\sigma(S) \subset [0, \infty)$ we can find $\lambda_n \notin \sigma(S)$, e.g. $\lambda_n \in \mathbf{C} \setminus \mathbf{R}$, such that $\lambda_n \to \lambda$ as $n \to \infty$. Denote $S_n = (S - \lambda_n I)^{-1} \in \in \mathrm{Orth}(L)$. Then it follows from part 1 of the proof that

$$(T - \lambda_n I)^{-1} = S_n(RS_n + I)^{-1}.$$

For each n we have

$$|S_n(RS_n + I)^{-1}R| \leqslant |S_n|\, |(RS_n + I)^{-1}|R \leqslant$$

$$\leqslant \|S_n\|\, (I + R|S_n| + \ldots + (R|S_n|)^k + \ldots)R \leqslant$$

$$\leqslant \|S_n\|\, (I + R\, \|S_n\| + \ldots + \|S_n\|^k R^k + \ldots)R =$$

$$= \|S_n\|R + \ldots + \|S_n\|^k R^k + \ldots .$$

It follows now from the spectral mapping theorem that the operator $\|S_n\|R + \ldots \ldots + \|S_n\|^k R^k + \ldots$ is quasi-nilpotent, so also $S_n(RS_n + I)^{-1}R$ is quasi-nilpotent. From

$$S_n(RS_n + I)^{-1} \to (T - \lambda I)^{-1}$$

we conclude that

$$S_n(RS_n + I)^{-1}R \to (T - \lambda I)^{-1}R.$$

Since $(T - \lambda I)^{-1}R$ is compact, it follows (see [15]) that also $(T - \lambda I)^{-1}R$ is quasi-nilpotent, so the operator $I + (T - \lambda I)^{-1}R$ is invertible and the proof of the theorem is complete.

REMARK. In case L does not contain any atoms, we can remove the positivity condition: Let $T = R + S$ with R compact and $\sigma(|R|) = \{0\}$ and with $S \in \mathrm{Orth}(L)$. Then $\sigma(T) = \sigma_0(T) = \sigma(S)$.

Proof. The inclusion $\sigma_0(T) \subset \sigma(S)$ is proved in the same way as in the above theorem. To prove $\sigma(S) \subset \sigma(T)$ we use now Theorem 1.11 to get $\sigma(S) = \sigma_e(S) = = \sigma_e(T) \subset \sigma(T)$.

It is not clear whether one can remove the positivity condition in the above theorem, in case L does contain atoms. We present as an application of Theorem 3.3 some examples.

EXAMPLE 1: Let $L = L_2[0, 1]$ and $Tf(x) = xf(x) + \int_x^1 f(y)\mathrm{d}y$. Then by the above theorem $\sigma(T) = \sigma_0(T) =$ ess range of $x = [0, 1]$. See also Sarason [18] for a discussion of this operator.

EXAMPLE 2. Let $L = L_p[0, 1]$, $1 < p < \infty$ and $T : L \to L$ given by

$$Tf(x) = g(x)f(x) + \int_x^1 T(x, y)f(y)\mathrm{d}y \quad \text{a.e.}$$

for some fixed $g \in L^\infty[0, 1]$. If $f \mapsto \int_x^1 T(x, y)f(y)\mathrm{d}y$ defines a compact operator, e.g. if $T(x, y)$ is a Hille-Tamarkin kernel, then by the remark made after Theorem 3.3 we have that $\sigma(T) =$ ess range of $g(x)$. We note that integral equations corresponding to the operator T are sometimes called Volterra equations of the third kind.

We conclude this section with some bibliographical remarks.

1. There exist a more general theory of triangular operators on Banach and Hilbert spaces (see [4] and [17]). There an operator is called triangular if it has a maximal chain of closed linear subspaces. Our theorems do not follow from these more general theories, since a maximal chain of bands is not always a maximal chain of closed linear subspaces (in ℓ^∞ one can find counterexamples). Another difference is that in these other theories one associates with compact triangular operators diagonal values, which are then proved to be equal to the eigenvalues of the operators. Here we associated with a triangular operator a diagonal operator and proved, under certain conditions, that the spectrum of the triangular operator equals the spectrum of its diagonal component.

2. In Section 1 of this paper we studied the projection $\mathfrak{D}$ from $\mathscr{L}_b^+$ onto $\mathrm{Orth}^+(L)$, where L is a Dedekind complete Riesz space. It is easy to see that for $0 \leqslant T, S \in \mathscr{L}_b(L, L)$ we have that $\mathfrak{D}(T)\mathfrak{D}(S) \leqslant \mathfrak{D}(TS)$, but in general it is not true that $\mathfrak{D}(TS) = \mathfrak{D}(T)\mathfrak{D}(S)$. From the results in Section 1 we get equality in two special cases. In the first place equality holds if T or S is a positive orthomorphism. This follows immediately from Theorem 1.1. Secondly, equality holds wherever L is a Dedekind complete Banach lattice without atoms and T or S is compact. This follows from Corollary 1.7. The problem of the multiplicativity of diagonal maps, like $\mathfrak{D}$, has been investigated in the more general context of partially ordered linear algebras by R. DeMarr and T. Y. Dai in a series of papers ([5], [6] and [7]). It seems now a natural conjecture that the map $\mathfrak{D}$ is multiplicative on the algebra of order bounded triangular operators, but the author has not been able to prove this in the general case, but only for special cases.

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