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# EXTENSIONS OF NORMAL OPERATORS AND HYPERINVARIANT SUBSPACES

H. W. KIM and C. M. PEARCY

## 1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. If $N$ is a nonscalar normal operator in $\mathscr{L}(\mathscr{H})$, then the spectral theorem guarantees that $N$ has a generous supply of nontrivial hyperinvariant subspaces. (Recall that a subspace $\mathfrak{M}$ of $\mathscr{H}$ is called a *nontrivial hyperinvariant subspace* for an operator $T$ in $\mathscr{L}(\mathscr{H})$ if $(0) \neq \mathfrak{M} \neq \mathscr{H}$ and $T'\mathfrak{M} \subset \mathfrak{M}$ for every $T'$ in $\mathscr{L}(\mathscr{H})$ that commutes with $T$.) For this reason classes of operators that are intimately associated with normal operators have long been favorite objects of study with a view to establishing the existence of nontrivial invariant and hyperinvariant subspaces. In this connection the classes of *n-normal* operators and *subnormal* operators come quickly to mind. (Recall that an $n$-normal operator may be defined as an $n \times n$ operator matrix whose entries are mutually commuting normal operators, and a subnormal operator is the restriction of a normal operator to an invariant subspace.) In [5] it was shown that every nonscalar $n$-normal operator has nontrivial hyperinvariant subspaces (cf. also [9, p. 76] and [11]). The corresponding problem for subnormal operators remains unsolved (cf. [8]), but in the pioneering paper [1], Scott Brown recently proved that every subnormal operator in $\mathscr{L}(\mathscr{H})$ does have nontrivial *invariant* subspaces.

The purpose of this note is to make a beginning on the hyperinvariant subspace problem for another class of operators closely related to the normal operators — namely, the class of *extensions of normal operators*. (Recall that an operator $T$ is an extension of a normal operator if $T$ restricted to some nontrivial invariant subspace of $T$ is normal.) In what follows, we call such operators *extnormal* operators. Thus, by definition, extnormal operators have nontrivial invariant subspaces, and it is only the question of the existence of nontrivial *hyperinvariant* subspaces that is of interest here. The authors have worked on this problem for a considerable period, and have found it surprisingly difficult going. We have chosen to publish our modest results because the techniques would seem to have some independent interest, and because thereby others may be motivated to make additional progress on this problem.

## 2. THE RESULTS

Let $T$ be a nonscalar extnormal operator in $\mathcal{L}(\mathcal{H})$, and let $\mathfrak{M}$ be a nontrivial invariant subspace for $T$ such that $T|\mathfrak{M}$ is a normal operator. If $\mathcal{H} \ominus \mathfrak{M}$ is finite dimensional, then $T^*|(\mathcal{H} \ominus \mathfrak{M})$ must have point spectrum. In this case, $T^*$ itself has an eigenvalue, the eigenspace corresponding to this eigenvalue is a nontrivial hyperinvariant subspace for $T^*$, and its orthocomplement is a nontrivial hyperinvariant subspace for $T$. Hence we may assume that $\mathcal{H} \ominus \mathfrak{M}$ is infinite dimensional. Arguing similarly, one sees that if the normal operator $T|\mathfrak{M}$ has point spectrum, then again $T$ has a nontrivial hyperinvariant subspace. Thus we may assume that $T$ has empty point spectrum and, in particular, that $\mathfrak{M}$ is infinite dimensional. Hence, without loss of generality, we may suppose that our nonscalar extnormal operator $T$ acts on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$ and has a matrix of the form

$$T = \begin{pmatrix} N & C \\ 0 & B \end{pmatrix} \tag{1}$$

where $N$ is a normal operator in $\mathcal{L}(\mathcal{H})$ with empty point spectrum, and $B, C \in \mathcal{L}(\mathcal{H})$. Once again there are some cases in which the result is trivial, and we dispose of these first. We write $\sigma(A)$, $\sigma_p(A)$, and $\sigma_e(A)$ for the spectrum, point spectrum, and essential (i.e., Calkin) spectrum, respectively, of an operator $A$ in $\mathcal{L}(\mathcal{H})$, and we denote the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ by $\mathbf{K}$.

PROPOSITION 2.1. *If $\sigma_p(B^*) \neq \emptyset$, or $C = 0$, or the range of the mapping $\Delta_{N,B}: \mathbf{K} \to \mathbf{K}$ defined by $\Delta_{N,B}(X) = NX - XB$ contains an operator of rank one, then the extnormal operator $T$ in (1) has a nontrivial hyperinvariant subspace. In particular, if $\sigma(N) \cap \sigma(B) = \emptyset$, this is the case.*

*Proof.* If $\sigma_p(B^*) \neq \emptyset$, then $T^*$ has point spectrum, and the result follows as above. If $C = 0$, then the result follows from the fact that $N$ has nontrivial hyperinvariant subspace and [4, Theorem 1.4]. If the range of the mapping $\Delta_{N,B}$ contains an operator of rank one (which happens, in particular, if $\sigma(N) \cap \sigma(B) = \emptyset$, cf. [12]), then the argument goes exactly as in [7, Theorem 4].

It turns out that most of our results are valid for operators in a somewhat larger class than the class of extnormal operators, and we prefer to expose these results in this more general setting. For this purpose, we need the following definition.

DEFINITION. An operator $A$ in $\mathcal{L}(\mathcal{H})$ is said to have property (K) if for every $\lambda$ in $\sigma(A)$ and for every $\varepsilon > 0$, there exists a unit vector $x_{\lambda,\varepsilon}$ in $\mathcal{H}$ such that

$$\limsup_n \|(A - \lambda)^n x_{\lambda,\varepsilon}\|^{1/n} < \varepsilon.$$

It is clear that every normal operator in $\mathscr{L}(\mathscr{H})$ has property (K), as does every decomposable operator (cf. [2]). Moreover, it is not hard to see that if $S$ is a subnormal operator such that $\sigma_p(S^*) = \emptyset$, then $S^*$ has property (K). It can also be shown, using the results and techniques of [3], that every $n$-normal operator has property (K). Finally, we note for future use that if $A$ is any operator with property (K), and $\sigma_p(A) = \emptyset$, then $\sigma_e(A) = \sigma(A)$.

We shall thus treat operators in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ of the form

$$T = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{2}$$

where $A$ has property (K). Of course, to avoid the trivial cases that have already been dealt with, we may assume that $A$ and $B^*$ have empty point spectra, that $\sigma(A) \cap \sigma(B) \neq \emptyset$, and that $C \neq 0$. We shall need the following elementary proposition, which is an improvement of Theorem 1 and Corollaries 1 and 2 of [6], and does not depend on the Lomonosov technique. This result was pointed out to us by Ciprian Foiaş.

PROPOSITION 2.2. (C. Foiaş). *Suppose $T \in \mathscr{L}(\mathscr{H})$ and there exist nonzero vectors $x$ and $y$ in $\mathscr{H}$ such that $y$ is the weak limit of a sequence of vectors $\{T^{*n}y_n\}_{n=1}^{\infty}$ where $\|T^n x\| \, \|y_n\| \to 0$. Then $T$ has a nontrivial hyperinvariant subspace.*

*Proof.* The following calculation shows that the hyperinvariant linear manifold $\{T'x : T' \in \mathscr{L}(\mathscr{H}) \text{ and } T'T = TT'\}$ is orthogonal to $y$ and thus not dense in $\mathscr{H}$:

$$|(T'x, y)| = |\lim_n (T'x, T^{*n}y_n)| = \lim_n |(T'T^n x, y_n)| \leqslant$$

$$\leqslant \lim_n \|T'\| \, \|T^n x\| \, \|y_n\| = 0.$$

Our first results on operators of the form (2) where $A$ has property (K) concern spectral hypotheses that yield hyperinvariant subspaces.

PROPOSITION 2.3. *Every operator $T$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ of the form (2) where $A$ has property (K) and $\sigma(A) \setminus \sigma(B) \neq \emptyset$ has a nontrivial hyperinvariant subspace.*

*Proof.* By translating $T$ if necessary we may assume that $0 \in \sigma(A) \setminus \sigma(B)$. Thus $B$ is invertible and we can choose $\varepsilon > 0$ such that $\|B^{-1}\| < 1/2\varepsilon$. Since (every translate of) $A$ has property (K), there exists a unit vector $x$ in $\mathscr{H}$ such that for $n$ sufficiently large, $\|A^n x\| < \varepsilon^n$. If $y$ is any unit vector in $\mathscr{H}$, and we set

$$\tilde{x} = x \oplus 0; \ \tilde{y} = 0 \oplus y; \ \tilde{y}_n = 0 \oplus (B^*)^{-n}y, \quad n = 1, 2, \ldots,$$

then $(T^*)^n \tilde{y}_n = \tilde{y}$ for all $n$, and

$$\|T^n\tilde{x}\| \, \|\tilde{y}_n\| = \|A^n x\| \, \|(B^*)^{-n}y\| < \varepsilon^n(1/2\varepsilon)^n = (1/2)^n$$

for $n$ sufficiently large, so the result follows from Proposition 2.2.

PROPOSITION 2.4. *Every operator $T$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ of the form* (2) *where $A$ has property* (K) *and either $\partial\sigma(A) \setminus \sigma_e(B) \neq \emptyset$ or $(\sigma(B) \setminus \sigma_e(B)) \setminus \sigma(A) \neq \emptyset$ has a nontrivial hyperinvariant subspace.*

*Proof.* We first assume that $\partial\sigma(A) \setminus \sigma_e(B) \neq \emptyset$. By translating if necessary, we may suppose that $0 \in \partial\sigma(A) \setminus \sigma_e(B)$. If $0 \notin \sigma(B)$, the argument goes as in Proposition 2.3, so we may assume that $0 \in \sigma(B)$. If 0 is an isolated point of $\sigma(B) \setminus \sigma_e(B)$, then $0 \in \sigma_p(B^*)$, so $T^*$ has point spectrum, and since $T$ cannot be scalar, the result follows. If 0 lies in a hole of $\sigma_e(B)$ and the Fredholm index associated with the hole is either negative or zero, then again $B^*$ and $T^*$ have point spectrum, so we may assume that the hole has positive associated Fredholm index (cf. [10, Chapter 1]). Since $0 \in \partial\sigma(A)$ there exists a nearby point $\lambda$ (in the same hole) such that $A-\lambda$ is invertible and the Fredholm index of $B-\lambda$ remains positive. If $y \neq 0$ belongs to the kernel of $B-\lambda$ and we set $x = -(A-\lambda)^{-1}Cy$, then calculation shows that $(T-\lambda)(x \oplus y) = 0$, so $\sigma_p(T) \neq \emptyset$, and the result follows in the case under consideration. The above arguments apply to the case $(\sigma(B) \setminus \sigma_e(B)) \setminus \sigma(A) \neq \emptyset$ equally well, so the proposition is proved.

To summarize, these propositions show that when looking for nontrivial hyperinvariant subspaces for operators $T$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ of the form (2), where $A$ has property (K), we may suppose that the following spectral relationships hold:

$$(*) \qquad \begin{cases} \sigma_p(A^*) = \sigma_p(B^*) = \emptyset, \ \sigma(A) = \sigma_e(A) \subset \sigma(B), \\ \partial\sigma(A) \subset \sigma_e(B), \ \sigma(B) \setminus \sigma_e(B) \subset \sigma(A). \end{cases}$$

This seems to be as far as we can go with purely spectral information. The next proposition assumes that we know something more about the operator $B$.

PROPOSITION 2.5. *Suppose $T$ is a nonscalar operator in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ of the form* (2), *where $A$ has property* (K). *If $B$ does not have property* (K) *and $\sigma(B) = \sigma(A)$, then $T$ has a nontrivial hyperinvariant subspace. On the other hand, if $A$ is normal and $B^*$ does have property* (K), *then the same conclusion holds.*

*Proof.* Suppose first that $B$ does not have property (K) and that $\sigma(B) = \sigma(A)$. Let $\lambda$ in $\sigma(B)$ and $\varepsilon > 0$ be such that there exists no nonzero vector $y$ in $\mathscr{H}$ for which $\limsup_n \|(B-\lambda)^n y\|^{1/n} < \varepsilon$. Since $A$ does have property (K), there exists a nonzero vector $x = x_{\lambda,\varepsilon}$ such that $\limsup_n \|(A-\lambda)^n x\|^{1/n} < \varepsilon$. If $S = (S_{ij})_{i,j=1}^2$ is an arbitrary operator in the commutant of $T$, then calculation shows that $S_{21}(A-\lambda)^n x = (B-\lambda)^n S_{21} x$ for every positive integer $n$, and hence that

$$\limsup_n \|(B-\lambda)^n S_{21} x\|^{1/n} \leqslant \limsup_n \|S_{21}\|^{1/n} \limsup_n \|(A-\lambda)^n x\|^{1/n} < \varepsilon,$$

which implies that $S_{21}x = 0$. It follows easily that the commutant of $T$ is not a transitive algebra, and hence that $T$ has a nontrivial hyperinvariant subspace.

We turn now to the case that $A$ is normal and $B^*$ has property (K). By translating if necessary we may assume that $0 \in \sigma(B^*)$, and clearly $T^*$ is unitarily equivalent to the operator matrix

$$\begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix}.$$

Since we may suppose that $A$ is not a scalar (Prop. 2.1), $A^*$ has reducing spectral subspaces on which it is invertible. One now proceeds, exactly as was done in Proposition 2.3, to construct nonzero vectors $\tilde{x}$, $\tilde{y}$, and $\{\tilde{y}_n\}_{n=1}^{\infty}$ such that $T^n \tilde{y}_n = \tilde{y}$ for all $n$ and $\|T^{*n}\tilde{x}\| \, \|\tilde{y}_n\| \to 0$. Thus the result follows from Proposition 2.2 and the fact that $T$ has a nontrivial hyperinvariant subspace whenever $T^*$ does.

We now ask "what are some examples of extnormal operators which do not have hyperinvariant supspaces by virtue of Propositions 2.1 — 2.5?" To get an idea of what sort of extnormal operators are left to be dealt with, we set forth an example of such an extnormal operator that satisfies condition (*). This example is perhaps as simple as any that could be given.

EXAMPLE 2.6. Let $N$ be a normal operator in $\mathscr{L}(\mathscr{H})$ such that $\sigma(N) = \{\zeta : |\zeta| \leqslant 1\}$ and such that $\sigma_p(N) = \emptyset$. Furthermore, let $U$ be an unweighted unilateral shift operator in $\mathscr{L}(\mathscr{H})$ of multiplicity one, and let $C$ be any nonzero operator in $\mathscr{L}(\mathscr{H})$. Then the operator $T$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ given by

$$T = \begin{pmatrix} N & C \\ 0 & U^* \end{pmatrix} \tag{3}$$

is clearly extnormal, and some elementary checking shows that $T$ satisfies condition(*).

The present authors are unable to show that every such extnormal operator $T$ has a nontrivial hyperinvariant subspace, but we can deal with a couple of special cases.

PROPOSITION 2.7. *Let the extnormal operator $T$ be as in* (3), *where $N$, $U$, and $C$ are as described in Example* 2.6 *and the normal operator $N$ has the additional property that when it is decomposed into a direct sum of pieces of uniform multiplicity, there is a piece $N_0$ of uniform multiplicity greater than one such that $\sigma(N_0)$ is not contained in the unit circle. Then $T$ has a nontrivial hyperinvariant subspace.*

*Proof.* We may and do assume, without loss of generality, that $N$ is unitarily equivalent to a normal operator $N_1 \oplus N_2$ where the invertible normal operator $N_1$ has uniform multiplicity greater than one and satisfies $\sigma(N_1) \subset \{\zeta : 0 \neq |\zeta| < 1\}$. Thus $T$ is unitarily equivalent to an operator matrix of the form

$$T_1 = \begin{pmatrix} N_1 & 0 & C_1 \\ 0 & N_2 & C_2 \\ 0 & 0 & U^* \end{pmatrix}$$

acting on $\mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$. If $C_1 = 0$, then $T_1$ has a normal direct summand and thus a nontrivial hyperinvariant subspace by [4, Theorem 1.4]. Thus we may suppose that $C_1 \neq 0$. An easy computation shows that $T_1$ is similar to the operator matrix

$$T_2 = \begin{pmatrix} N_1 & 0 & XU^* - N_1X + C_1 \\ 0 & N_2 & C_2 \\ 0 & 0 & U^* \end{pmatrix}$$

where $X$ is completely arbitrary, and it clearly suffices to show that $T_2$ has a nontrivial hyperinvariant subspace. Write $X = YU$, and observe that

$$XU^* - N_1X + C_1 = N_1\{N_1^{-1}Y - YU\} + C_1 - YP$$

where $P$ is the projection on the (one-dimensional) kernel of $U^*$. Since $\sigma(N_1^{-1}) \cap \sigma(U) = \emptyset$ and $C_1 \neq 0$, by a theorem of Rosenblum [12] there exists a unique nonzero operator $Y$ in $\mathscr{L}(\mathscr{H})$ such that $N_1^{-1}Y - YU = -N_1^{-1}C_1$. Hence $XU^* - N_1X + C_1 = -YP = -Ye_0 \otimes e_0$ where $e_0$ is a unit vector in the range of $P$, and we have

$$T_2 = \begin{pmatrix} N_1 & 0 & -Ye_0 \otimes e_0 \\ 0 & N_2 & C_2 \\ 0 & 0 & U^* \end{pmatrix}. \tag{4}$$

If $Ye_0 = 0$, then $T_2$ has a normal direct summand and the argument is complete. Thus we may suppose that $Ye_0 \neq 0$. Since $N_1$ has uniform multiplicity greater than one and thus has no *-cyclic vectors, there exists a nonzero vector $f$ in $\mathscr{H}$ such that $f$ is orthogonal to the cyclic linear manifold

$$\{p(N_1, N_1^*)Ye_0 : p(x, y) \text{ a polynomial}\}.$$

In other words, $Ye_0$ is orthogonal to the subspace

$$\mathfrak{M} = \{p(N_1^*, N_1)f : p(x, y) \text{ a polynomial}\}^-.$$

Using this fact it is easy to see that $\mathfrak{M} \oplus 0 \oplus 0$ is a reducing subspace for $T_2$ and that $T_2|(\mathfrak{M} \oplus 0 \oplus 0) = N_1|\mathfrak{M}$. Hence $T_2$ has a normal direct summand, and the argument is complete.

PROPOSITION 2.8. *Let the extnormal operator $T$ be as in* (3), *where $N$, $U$, and $C$ are as described in Example* 2.6 *and the normal operator $N$ has the additional property that it can be written as $N = D + K$ where $D$ is a diagonal operator and $K$ belongs to the trace class. Then $T$ has a nontrivial hyperinvariant subspace.*

*Proof.* The proof begins like that of Proposition 2.7, with the only condition imposed upon the normal direct summand $N_1$ of $N$ being that $\sigma(N_1) \subset \{\zeta : |\zeta| < 1\}$, and continues until the operator $T_2$ in (4) similar to $T$ is obtained. Of course it suffices to show that $T_2$ has a nontrivial hyperinvariant subspace, and, as before, we may suppose that $f = Ye_0 \neq 0$. Let $S = (S_{ij})_{i,j=1}^3$ be any operator in $\mathscr{L}(\mathscr{H} \oplus \oplus \mathscr{H} \oplus \mathscr{H})$ that commutes with $T_2$. An easy calculation shows that the matrix entry $S_{11}$ satisfies the equation $N_1S_{11} - S_{11}N_1 = f \otimes S_{31}^*e_0$. Thus if $S_1$ is the operator on $\mathscr{H}$ that is unitarily equivalent to $S_{11} \oplus 0$ on $\mathscr{H} \oplus \mathscr{H}$ via the same unitary equivalence that sends $N$ to $N_1 \oplus N_2$, we have

$$(f \otimes S_{31}^*e_0) \oplus 0 = [N_1 \oplus N_2, S_{11} \oplus 0] \simeq [N, S_1] = \\ = [D + K, S_1] = [D, S_1] + [K, S_1].$$

Since $(f \otimes S_{31}^*e_0) \oplus 0$ is an operator of rank one and $K$ belongs to the trace class, it follows that the commutator $[D, S_1]$ belongs to the trace class. Since $D$ is a diagonal operator and the trace of $[D, S_1]$ can be calculated relative to the basis with respect to which the matrix for $D$ is diagonal, an easy computation shows that the trace of $[D, S_1]$ is zero. Since the trace of the commutator $[K, S_1]$ is obviously zero, it follows that the trace of $(f \otimes S_{31}^*e_0) \oplus 0$ is zero. But this trace can also be calculated to be $(f, S_{31}^*e_0)$, so we conclude that for any $S = (S_{ij})_{i,j=1}^3$ that commutes with $T_2$, we have $(S_{31}f, e_0) = 0$. This shows that the commutant of $T_2$ is not transitive, and completes the argument.

Propositions 2.7 and 2.8 treat special cases of the class of examples of extnormal operators introduced in Example 2.6, but they certainly do not cover all of the operators in that class. In particular, it is well-known that if $N$ is taken to be $M_\lambda$ — the operator of multiplication by $\lambda$ on the Hilbert space of square integrable functions on the unit disc under planar Lebesgue measure — then the operator $T$ in (3) does not satisfy the hypotheses of either Proposition 2.7 or Proposition 2.8. Thus further progress in the theory of hyperinvariant subspaces for extnormal operators would seem to depend upon being able to handle the operators of Example 2.6.

We turn now to consideration of one other interesting class of extnormal operators. By definition, an operator $T$ in $\mathscr{L}(\mathscr{H})$ is said to be an $\aleph_0$-*normal* operator if $T$ is unitarily equivalent to an operator matrix $(N_{ij})_{i,j=1}^{\infty}$ acting on $\mathscr{H}_\infty = \mathscr{H} \oplus \oplus \mathscr{H} \oplus \ldots$ with the property that the $N_{ij}$, $1 \leqslant i, j < \infty$, are mutually commuting normal operators generating a (commutative) $C^*$-algebra whose spectrum is not finite. It is not known whether every $\aleph_0$-normal operator has nontrivial hyperinvariant subspaces. In fact, it was asked in [7, Problem 8] whether every $\aleph_0$-normal operator $(N_{ij})_{i,j=1}^{\infty}$ in $\mathscr{L}(\mathscr{H}_\infty)$ with the additional property that $N_{ij} = 0$ whenever $i > j$ has a nontrivial hyperinvariant subspace, and this question remains unresolved even though all of these operators are clearly extnormal operators. Here is a partial result.

PROPOSITION 2.9. *Let* $T = (N_{ij})_{i,j=1}^{\infty}$ *be an* $\aleph_0$*-normal operator in* $\mathcal{L}(\mathcal{H}_\infty)$ *such that* $N_{ij} = 0$ *whenever* $i > j$ *and such that* $\sum_{i,j} \|N_{ij}\|^2 < +\infty$. *Then* $T$ *has a nontrivial hyperinvariant subspace.*

*Proof.* If the normal operator $N_{11}$ is zero, then $(N_{ij})$ has nonempty point spectrum, and the result follows. Thus we may suppose that there exists a number $\lambda \neq 0$ in $\sigma(N_{11})$. We observe that it follows from the hypothesis on the $\|N_{ij}\|$ that there exists a positive integer $k$ with the property that if we set $\mathcal{K}_1$ equal to the direct sum of the first $k$ copies of $\mathcal{H}$ in $\mathcal{H}_\infty$, then relative to the decomposition $\mathcal{H}_\infty = \mathcal{K}_1 \oplus \mathcal{K}_2$, the operator $(N_{ij})$ can be written as

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

where $\|B\| < |\lambda|$. Clearly the operator $A$ acting on $\mathcal{K}_1$ is $n$-normal, and, as mentioned earlier, $n$-normal operators have property (K). Since $\lambda \in \sigma(A) \setminus \sigma(B)$, the result follows from Proposition 2.3.

There are thus two specific classes of extnormal operators on which to test any proposed theory of hyperinvariant subspaces—the class set forth in Example 2.6 and the class of "upper triangular" $\aleph_0$-normal operators.

We close this note with a short discussion of operators with property (K). It is easy to see that if $T$ is an operator in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ of the form (2), where $A$ has property (K) and $\sigma(B) \subset \sigma(A)$, then $T$ also has property (K). This naturally leads one to ask whether every nonscalar operator with property (K) may be shown to have nontrivial hyperinvariant subspaces. One result in this direction is as follows.

PROPOSITION 2.10. *If* $T \in \mathcal{L}(\mathcal{H})$, $\lambda$ *is any scalar, and* $\varepsilon$ *is any positive number, then the set*

$$\mathfrak{M} = \{x \in \mathcal{H} : \limsup_n \| (T - \lambda)^n x \|^{1/n} < \varepsilon\} \tag{5}$$

*is a linear manifold in* $\mathcal{H}$ *that is hyperinvariant for* $T$.

*Proof.* One shows by direct calculation that $\mathfrak{M}$ is closed under addition and scalar multiplication. Another such calculation shows that if $x \in \mathfrak{M}$ and $T'$ commutes with $T$, then also $T'x \in \mathfrak{M}$.

It follows from this result that if $T$ is any operator in $\mathcal{L}(\mathcal{H})$ with property (K), $\lambda$ is taken to be any point in $\sigma(T)$, and $\varepsilon$ is taken to by any positive number, then the linear manifold $\mathfrak{M}$ in (5) is different from (0). Thus, in order to prove that such a $T$ has a nontrivial hyperinvariant subspace, it would suffice to show that $\mathfrak{M}^- \neq \mathcal{H}$. If $T$ is a normal operator and $\varepsilon$ is chosen properly, this can be done. However, if $T$ is an extnormal operator, — say of the form (1) — the present authors have been unable to establish that any such $\mathfrak{M}^-$ is a proper subspace.

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*H. W. KIM*
*Mathematics Department*
*Bucknell University*
*Lewisburg, PA 17837*
*U.S.A.*

*C. M. PEARCY*
*Mathematics Department*
*University of Michigan*
*Ann Arbor, MI 48104*
*U.S.A.*

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