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ERGODIC ACTIONS OF COMPACT ABELIAN GROUPS

DORTE OLESEN, GERT K. PEDERSEN and MASAMICHI TAKESAKI

1. INTRODUCTION

A few years ago Størmer raised the question whether a von Neumann algebra $\mathscr{M}$ is finite, if it admits an ergodic representation $\alpha: G \to \mathrm{Aut}(\mathscr{M})$ for some compact group G. He showed in [28, 3.4] that the answer is yes if G is abelian. Well-aimed, but unsuccessful attempts at a positive solution of the general problem (cf. [10] and [1]) made it clear that the question is hard, and the answer may, in fact, be negative.

By contrast, the abelian case is relatively straightforward and, as we shall show, it is possible to give a complete description of the finite von Neumann algebras $\mathscr{M}$ appearing in the ergodic representations $\alpha: G \to \mathrm{Aut}(\mathscr{M})$ of some fixed compact abelian group.

Given such a von Neumann algebra $\mathscr{M}$ one first observes that it has a complete set $\{u(p) \mid p \in \hat{G}\}$ of unitary eigenoperators for the action α and that the eigenspaces are all one-dimensional and span $\mathscr{M}$. This means that $p \to u(p)$ is a projective representation of $\hat{G}$, and we are led to consider the 2-cocyle $m(p, q) = u(p)u(q)u(p+q)^*$ with values in the circle group $\mathbf{T}$. Thus our algebra $\mathscr{M}$ is a crossed product of $\hat{G}$ and $\mathbf{C}$ over the 2-cocycle m, as described by Zeller-Meier in [34]. The next observation is that the group $\mathscr{G}_\alpha$ of all unitary eigenoperators is an extension of $\mathbf{T}$ by $\hat{G}$ (with the map $p \to u(p)$ as a cross-section for the quotient map of $\mathscr{G}_\alpha$ on $\hat{G}$). Classifying group extensions is a well-known exercise in homological algebra, the invariants being the elements in the second cohomology group $H^2(\hat{G}, \mathbf{T})$. Since $\hat{G}$ is discrete, and $\mathbf{T}$ is a direct summand in any abelian locally compact group in which it is a closed subgroup, one may describe $H^2(\hat{G}, \mathbf{T})$ as the set $X^2(\hat{G}, \mathbf{T})$ of symplectic bicharacters $\chi: \hat{G} \times \hat{G} \to \mathbf{T}$. Alternatively, $X^2(\hat{G}, \mathbf{T})$ is the set of homomorphisms $\chi: \hat{G} \to G$ for which $\langle \chi(p), p \rangle = 1$ for every p in $\hat{G}$. Thus the pair $(\mathscr{M}, \alpha)$ is completely determined (up to conjugacy) by the corresponding symplectic bicharacter χ_α on $\hat{G} \times \hat{G}$.

In [9] it was shown that corresponding to each factor there is a unique ergodic flow, the flow of weights, and that the flow of weights on a tensor product is constructed by simple arithmetics from the flows of weights on the components. In this paper we show that the exact same arithmetics gives rise to a binary operation in the category of ergodic actions of G, and that the operation turns out to give a commutative group structure on the set $[G]$ of conjugacy classes in the category. Furthermore, we topologize the subcategory of ergodic actions of G on von Neumann algebras on a fixed Hilbert space (take e.g. $L^2(\hat{G})$), using pointwise convergence on the set of unit balls with the strong topology. Giving $[G]$ the quotient topology, we find that it is compatible with the group structure, so that $[G]$ becomes a compact abelian group isomorphic to $X^2(\hat{G}, \mathbf{T})$, the compact group of symplectic bicharacters on $\hat{G} \times \hat{G}$ with pointwise product and convergence. The unit in $[G]$ correspond to the translation action of G on $L^\infty(G)$, and the inverse of a class $[\alpha]$, corresponding to an action α on $\mathscr{M}$, is given by taking α acting on the opposite algebra of $\mathscr{M}$. We should like to point out here, that the topology mentioned above, on the set of all von Neumann algebras on a fixed Hilbert space, is quite badly behaved. For instance it is not T_1 and the isomorphism classes are not closed. However, on the subset of von Neumann algebras which admit an ergodic action of G the topology is Hausdorff and each conjugacy class is closed.

Given an ergodic representation $\alpha : G \to \mathrm{Aut}(\mathscr{M})$, let $\chi_\alpha : \hat{G} \to G$ be the associated element in $X^2(\hat{G}, \mathbf{T})$. If $\chi_\alpha(\hat{G}) = \{0\}$ then $\mathscr{M} = L^\infty(G)$ with α acting as translation. The other extreme occurs when χ_α is injective (equivalently, has dense range), and corresponds to the case where $\mathscr{M}$ is a factor. We show that any representation is induced from an ergodic representation $\beta: G_{\mathscr{Z}} \to \mathrm{Aut}(\mathscr{N})$, where $\mathscr{N}$ is a factor and $G_{\mathscr{Z}}$ is the kernel of α restricted to the center of $\mathscr{M}$. Therefore, assuming that G is second countable,

$$\mathscr{M} = L^\infty(G/G_{\mathscr{Z}}) \otimes \mathscr{N};$$

and since $\mathscr{M}$ is injective, $\mathscr{N}$ is either $\mathbf{M}_n$ (if $\mathscr{M}$ is of type I) or the hyperfinite II_1-factor.

A similar theory of ergodic representations of G on C^*-algebras is possible, and runs parallel with the von Neumann theory. Indeed, there is a bijective correspondence between ergodic representations of G on C^*-algebras and on von Neumann algebras. However, a simple description of the C^*-algebras that occur in the representations is no longer possible; though they are all induced from subrepresentations $\beta: G_{\mathscr{Z}} \to \mathrm{Aut}(\mathscr{B})$, where $\mathscr{B}$ is a simple, nuclear C^*-algebra with unit and unique trace.

Our theory can be applied to certain ergodic representations of non-compact abelian groups. Such almost periodic representations $\alpha: G \to \mathrm{Aut}(\mathscr{M})$ can be extended to $\tilde{\alpha}: \tilde{G} \to \mathrm{Aut}(\mathscr{M})$, where $\tilde{G}$ is a compactification of G (arising as the dual group of the pure point spectrum for α). In this way we obtain a non-commutative generalization of von Neumann's classification of ergodic transformations with pure point spectrum.

The paper uses the combined knowledge of its authors, but is written in order to match their combined ignorance. Therefore most of the arguments are self-contained, even though they are not all original. In particular, the arguments in Section 3 can presumably be found in any standard text on homological algebra, and some results overlap with [1] and [3]. The paper is neatly complemented by the recent work of V. F. R. Jones ([16], [17]), classifying all actions of a finite (non-abelian) group on the hyperfinite II_1-factor.

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2. G-SYSTEMS

2.1. Let G be a fixed compact abelian group with normalized Haar measure. A pair $(\mathcal{M}, \alpha)$ consisting of a von Neumann algebra $\mathcal{M}$ and a faithful, ergodic, continuous representation $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ will be called a *G-system*. The continuity requirement on α is that each function $s \to \alpha_s(x)$, $x \in \mathcal{M}$, is σ-weakly continuous from G to $\mathcal{M}$. Thanks to the Haar measure on G this is equivalent to the demand that each function $s \to \varphi(\alpha_s(\cdot))$, $\varphi \in \mathcal{M}_*$, is norm continuous from G to $\mathcal{M}_*$, although the two conditions correspond to different topologies on $\mathrm{Aut}(\mathcal{M})$.

2.2. We say that two G-systems $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ are *conjugate* (and write $(\mathcal{M}, \alpha) \sim (\mathcal{N}, \beta)$) if there is an isomorphism $\Phi : \mathcal{M} \to \mathcal{N}$ such that

$$\Phi \circ \alpha_s \circ \Phi^{-1} = \beta_s, \quad s \in G.$$

The conjugacy class of $(\mathcal{M}, \alpha)$ is denoted by $[\alpha]$ and the set of conjugacy classes of G-systems is denoted by $[G]$.

Define a product of two G-systems $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ as follows: Take the action $\alpha \otimes \beta$ of $G \times G$ on $\mathcal{M} \otimes \mathcal{N}$ and note that $(\mathcal{M} \otimes \mathcal{N}, \alpha \otimes \beta)$ is a $G \times G$-system. Let $\mathcal{P}$ denote the fixed-point subalgebra of $\mathcal{M} \otimes \mathcal{N}$ under the action $\alpha_s \otimes \beta_{-s}$, $s \in G$, and consider the action $\gamma_t = \alpha_t \otimes \iota$, $t \in G$, on $\mathcal{P}$. It is elementary to check that $(\mathcal{P}, \gamma)$ is a G-system. We call it the *product* and denote it by $(\mathcal{M}, \alpha) \times (\mathcal{N}, \beta)$.

In general no G-system is a unit for the product (which is clearly associative and commutative) so that we only obtain a semi-group structure on the set of G--systems. However, if $(\mathcal{M}_i, \alpha^i) \sim (\mathcal{N}_i, \beta^i)$ via a conjugacy $\Phi_i : \mathcal{M}_i \to \mathcal{N}_i$ for $i = 1, 2$, then $\Phi_1 \otimes \Phi_2$ is an isomorphism of $\mathscr{M}_1 \otimes \mathscr{M}_2$ on $\mathscr{N}_1 \otimes \mathscr{N}_2$ intertwining $\alpha^1 \otimes \alpha^2$ with $\beta^1 \otimes \beta^2$. Consequently

$$(\mathscr{M}_1, \alpha^1) \times (\mathscr{M}_2, \alpha^2) \sim (\mathscr{N}_1, \beta^1) \times (\mathscr{N}_2, \beta^2).$$

Thus the definition $[\alpha^1] . [\alpha^2] = [\alpha^1 \times \alpha^2]$ gives a well-defined product in $[G]$. As we shall see later (4.5) this product gives a group structure on $[G]$.

2.3. Given a G-system $(\mathscr{M}, \alpha)$ take p in $\hat{G}$ and for each x in $\mathscr{M}$ put

$$\hat{x}(p) = \int \alpha_s(x) \langle s, p \rangle^{-1} ds.$$

Then $\alpha_s(\hat{x}(p)) = \langle s, p \rangle \hat{x}(p)$ for all s in G. Consequently, if $x, y \in \mathscr{M}$ then $\hat{x}(p)\hat{y}(p)^*$ is a fixedpoint for G, hence a scalar multiple of 1. Thus if $\hat{x}(p) \neq 0$ then $u(p) = \|\hat{x}(p)\|^{-1}\hat{x}(p)$ is unitary and for each y in $\mathscr{M}$, $\hat{y}(p)$ is a scalar multiple of $u(p)$.

The existence of unitary eigenoperators for α shows immediately that the Arveson spectrum, $\mathrm{Sp}(\alpha)$ is a subgroup of $\hat{G}$ (see [24, 8.1.6]). But

$$\mathrm{Sp}(\alpha)^{\perp} = \{s \in G \mid \alpha_s = \iota\},$$

and since α is assumed to be faithful, $\mathrm{Sp}(\alpha)^{\perp} = \{0\}$, i.e. $\mathrm{Sp}(\alpha) = \hat{G}$. We can therefore choose a complete set $\{u(p) \mid p \in \hat{G}\}$ of unitary eigenoperators for α such that $\mathscr{M}$ is the σ-weakly closed linear span of the $u(p)$'s.

2.4. Proposition (cf. [2, III, 3.3]). *If $(\mathscr{M}, \alpha)$ is a G-system then α_G is a maximally abelian subgroup of* $\mathrm{Aut}(\mathscr{M})$.

Proof. Let σ be an automorphism of $\mathscr{M}$ commuting with α_G. Then $\sigma(\hat{x}(p)) = \sigma(x)^\wedge(p)$ for all x and p. Choosing $u(p)$, $p \in \hat{G}$, as in 2.3 it follows that $\sigma(u(p)) \in \mathbf{C}\, u(p)$. Since σ is an automorphism this implies that the map $p \to \sigma(u(p))u(p)^*$ is a character on $\hat{G}$. Thus there is an s in G with

$$\sigma(u(p)) = \langle s, p \rangle\, u(p) = \alpha_s(u(p))$$

for all p. Since the $u(p)$'s generate $\mathscr{M}$ it follows that $\sigma = \alpha_s$, whence α_G is maximally abelian in $\mathrm{Aut}(\mathscr{M})$.

2.5. Proposition (see [28], [15], [10], [1]). *If $(\mathscr{M}, \alpha)$ is a G-system then the unique G-invariant state τ on $\mathscr{M}$ is a faithful, normal trace.*

Proof. Identifying $\mathbf{C}$ with $\mathbf{C}1$ we define

$$\tau(x) = \int \alpha_s(x) dx = \hat{x}(0), \quad x \in \mathscr{M}.$$

Then τ is a faithful normal G-invariant state on $\mathscr{M}$, and clearly the only one. To show that τ is a trace, choose unitary eigenoperators $\{u(p) \mid p \in \hat{G}\}$ as in 2.3. Since the $u(p)$'s generate $\mathscr{M}$ linearly it suffices to show that

$$\tau(u(p)u(q)) = \tau(u(q)u(p))$$

for all p, q in $\hat{G}$. But if $p + q \neq 0$ then

$$\tau(u(p)u(q)) = \int \langle s, p + q \rangle \, u(p)u(q) \mathrm{d}s = 0 \ ;$$

whereas if $p + q = 0$ we have

$$u(p)u(q) = \lambda 1 = u(q)u(p) \, ,$$

from which the result follows.

2.6. If $(\mathscr{M}, \alpha)$ is a G-system with trace τ we use the GNS construction to obtain a Hilbert space $\mathfrak{H}_\tau$ with a unit vector ξ_τ and a unitary representation λ^τ of G on $\mathfrak{H}_\tau$ such that we may regard $\mathscr{M}$ as a von Neumann algebra on $\mathfrak{H}_\tau$ with

$$\alpha_s(x) = \lambda^\tau(s) x \lambda^\tau(-s) \quad \text{and} \quad \tau(x) = (x\xi_\tau \mid \xi_\tau)$$

for all x in $\mathscr{M}$. We shall call this the *standard representation* of $(\mathscr{M}, \alpha)$.

2.7. Let $(\mathscr{M}, \alpha)$ be a G-system and denote by $\mathscr{G}_\alpha$ the set of unitaries u in $\mathscr{M}$ such that $u^*\alpha_s(u) \in \mathbf{C}$ for all s in G. It is easy to verify that $\mathscr{G}_\alpha$ is a group, and clearly every unitary eigenoperator belongs to $\mathscr{G}_\alpha$, so that $\mathscr{G}_\alpha'' = \mathscr{M}$ by 2.3. On the other hand, each u in $\mathscr{G}_\alpha$ is an eigenoperator corresponding to the eigenvalue $\pi(u)$ in $\hat{G}$ determined by

$$s \to u^*\alpha_s(u) = \langle s, \pi(u) \rangle, \ s \in G \, .$$

The map $\pi : \mathscr{G}_\alpha \to \hat{G}$ is a homomorphism and the faithfulness of α implies that π is surjective. Since α is ergodic the kernel of π is the circle group $\mathbf{T}$, so that we have a short exact sequence

$$\{1\} \to \mathbf{T} \to \mathscr{G}_\alpha \to \hat{G} \to \{0\}.$$

In other words, $\mathscr{G}_\alpha$ is an extension of $\mathbf{T}$ by $\hat{G}$, and this extension is an invariant for $(\mathscr{M}, \alpha)$. This observation motivates our little excursion into algebra in the next section.

Note that if $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ are conjugate via Φ and if $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are the associated extensions then $\Phi(\mathcal{G}_\alpha) = \mathcal{G}_\beta$. Since of course Φ is the identity map on $\mathbf{T}$ it follows that $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$ are equivalent extensions in the natural sense.

2.8. We topologize the set of von Neumann algebras on a Hilbert space $\mathfrak{H}$, using elementwise convergence in the strong topology on the unit balls. Thus if $\mathcal{M} \subset \mathbf{B}(\mathfrak{H})$ a subbasis for the neighbourhood system of $\mathcal{M}$ is given by sets of the form

$$\mathcal{U}(\xi, x) = \{\mathcal{N} \subset \mathbf{B}(\mathfrak{H}) \mid \exists\, y \in \mathcal{N} : \|y\| \leq 1,\ \|(x - y)\xi\| < 1\},$$

where $\xi \in \mathfrak{H}$ and $x \in \mathcal{M}$ with $\|x\| \leq 1$. It follows that a net $\{\mathcal{N}_i\}$ of von Neumann algebras converges to $\mathcal{M}$ if for each x in $\mathcal{M}$ there is a net $\{x_i\}$ converging strongly to x with $\|x_i\| \leq \|x\|$ and x_i in $\mathcal{N}_i$ for all i. The topology is not T_1 since every set $\mathcal{U}(\xi, x)$ contains all von Neumann algebras larger than $\mathcal{M}$. It is, however, T_0 so that points can be separated.

Let now $(u, \mathfrak{H})$ be a faithful continuous unitary representation of G and denote by $G(u, \mathfrak{H})$ the set of G-systems $(\mathcal{M}, \alpha)$ for which $\mathcal{M} \subset \mathbf{B}(\mathfrak{H})$ and $\alpha = \operatorname{Ad} u$. Consider $G(u, \mathfrak{H})$ with the topology described above. Thus if a net $\{(\mathcal{M}_i, \alpha^i)\}$ converges to $(\mathcal{M}, \alpha)$ there is for each x in $\mathcal{M}$ a net $\{x_i\}$ converging strongly to x with $\|x_i\| \leq \|x\|$ and x_i in $\mathcal{M}_i$ for all i. However, for each ξ in $\mathfrak{H}$ the set $\{u(s)\xi \mid s \in G\}$ is compact. We may therefore assume that $\alpha^i_s(x_i) \to \alpha_s(x)$ strongly, uniformly on G. But then by integration $\{\hat{x}_i(p)\}$ converges strongly to $\hat{x}(p)$ for each p in $\hat{G}$ and $\|\hat{x}_i(p)\| \leq \|x\|$.

2.9. Lemma. *A net $\{(\mathcal{M}_i, \alpha^i)\}$ in $G(u, \mathfrak{H})$ converges to an element $(\mathcal{M}, \alpha)$ if and only if for each v in $\mathcal{G}_\alpha$ there is a net $\{v_i\}$ of unitary eigenoperators for u converging strongly to v, such that $v_i \in \mathcal{G}_{\alpha^i}$ and $\pi(v_i) = \pi(v)$ in $\hat{G}$ for all i.*

Proof. If $(\mathcal{M}_i, \alpha^i) \to (\mathcal{M}, \alpha)$ in $G(u, \mathfrak{H})$ and $v \in \mathcal{G}_\alpha$ with $\pi(v) = p$ in $\hat{G}$ there is by 2.8 a net $\{x_i\}$, where $x_i \in \mathcal{M}_i$ and $\|\hat{x}_i(p)\| \leq 1$ for all i, such that $\{\hat{x}_i(p)\}$ converges strongly to $\hat{v}(p) = v$. Since the norm is strongly lower semi-continuous it follows that $\|\hat{x}_i(p)\| \to 1$. Set $v_i = \|\hat{x}_i(p)\|^{-1}\hat{x}_i(p)$ and note that $v_i \in \mathcal{G}_{\alpha^i}$ with $\pi(v_i) = p$, and that $v_i \to v$ strongly.

Conversely, assume that the net satisfies the conditions in the lemma and take x in $\mathcal{M}$ with $\|x\| \leq 1$. Put

$$y = (1 + (1 - xx^*)^{\frac{1}{2}})^{-1}x$$

so that

$$x = 2(1 + yy^*)^{-1}y = 2y(1 + y^*y)^{-1}.$$

The linear span $\operatorname{Lin}(\mathcal{G}_\alpha)$ of $\mathcal{G}_\alpha$ is a $*$-algebra which is strongly-$*$dense in $\mathcal{M}$ (cf. [31, 2.2]). Moreover, since the involution is strongly continuous on the unitary group of $\mathbf{B}(\mathfrak{H})$, each element in $\mathrm{Lin}(\mathscr{G}_\alpha)$ can be approximated in the strong-* topology by elements from $\mathrm{Lin}(\mathscr{G}_{\alpha^i})$. It follows that there is a net $\{y_i\}$ converging strongly-* to y with y_i in $\mathscr{M}_i$ for each i. Set

$$x_i = 2(1 + y_i y_i^*)^{-1} y_i$$

and observe that $\|x_i\| \leqslant 1$. Moreover, by Kaplansky's argument from the density theorem ([31, 2.4.8])

$$\frac{1}{2}(x_i - x) = (1 + y_i y_i^*)^{-1} y_i - y(1 + y^* y)^{-1} =$$

$$= (1 + y_i y_i^*)^{-1}(y_i - y)(1 + y^* y)^{-1} + (1 + y_i y_i^*)^{-1} y_i (y_i^* - y^*) y (1 + y^* y)^{-1},$$

so that $x_i \to x$ strongly. Since x was arbitrary, $(\mathscr{M}_i, \alpha^i) \to (\mathscr{M}, \alpha)$ in $G(u, \mathfrak{H})$.

2.10. Proposition. *The topological space $G(u, \mathfrak{H})$ defined in 2.8 is Hausdorff.*

Proof. Suppose that a net $\{(\mathscr{M}_i, \alpha^i)\}$ converges to both $(\mathscr{M}, \alpha)$ and $(\mathscr{N}, \beta)$ in $G(u, \mathfrak{H})$. For each v in $\mathscr{G}_\alpha$ choose w in $\mathscr{G}_\beta$ such that $\pi(v) = \pi(w)$ in $\hat{G}$. By 2.9 there are nets $\{v_i\}$ and $\{w_i\}$ converging strongly to v and w, respectively, with v_i and w_i in $\mathscr{G}_{\alpha^i}$ and $\pi(v_i) = \pi(v)$, $\pi(w_i) = \pi(w)$ for all i. But then $\pi(v_i) = \pi(w_i)$, whence $v_i = \lambda_i w_i$ for some scalar λ_i with $|\lambda_i| = 1$. It follows that $\lambda_i \to \lambda$ and that $v = \lambda w$. Since v is arbitrary, $\mathscr{G}_\alpha \subset \mathscr{G}_\beta$, whence $\mathscr{G}_\alpha = \mathscr{G}_\beta$ by symmetry. Thus $\mathscr{M} = \mathscr{N}$ and since $\alpha = \mathrm{Ad}\, u = \beta$ we see that $(\mathscr{M}, \alpha) = (\mathscr{N}, \beta)$ in $G(u, \mathfrak{H})$ which is therefore a Hausdorff space.

2.11. Analogously to the von Neumann algebra case define a *C^*-G-system* as a pair $(\mathscr{A}, \alpha)$, where $\mathscr{A}$ is a C^*-algebra with unit and $\alpha : G \to \mathrm{Aut}(\mathscr{A})$ is a faithful, ergodic, continuous representation of G. The continuity requirement here is that each function $s \to \alpha_s(x)$, $x \in \mathscr{A}$, is norm continuous.

Given a G-system $(\mathscr{M}, \alpha)$ let $\mathscr{M}^c$ denote the set of elements x in $\mathscr{M}$ for which the function $s \to \alpha_s(x)$ is norm continuous. Then $\mathscr{M}^c$ is a weakly dense G-invariant C^*-subalgebra of $\mathscr{M}$ generated by elements of the form $x = \int \alpha_s(y) f(s) \mathrm{d}s$, $f \in L^1(G)$, $y \in \mathscr{M}$ (cf. [24, 7.5.1]). It follows that $(\mathscr{M}^c, \alpha \mid \mathscr{M}^c)$ is a C^*-G-system. As we shall see (6.1) any C^*-G-system has this form. By direct verification we see that for any C^*-G-system $(\mathscr{A}, \alpha)$ there is a complete set $\{u(p) \mid p \in \hat{G}\}$ of unitary eigenoperators for α in $\mathscr{A}$ such that $\mathscr{A}$ is the norm closed linear span of the $u(p)$'s. Thus 2.4, 2.5 and the group extension $\mathscr{G}_\alpha$ in 2.7 have direct analogues for C^*-G-systems.

3. GROUP EXTENSIONS

3.1. Let Γ be a discrete abelian group and $\mathbf{T}$ the circle group. An extension of $\mathbf{T}$ by Γ is a topological group $\mathscr{G}$, necessarily locally compact but not necessarily abelian, for which we have a short exact sequence.

$$\{1\} \to \mathbf{T} \xrightarrow{\iota} \mathscr{G} \xrightarrow{\pi} \Gamma \to \{0\}.$$

Let $p \to u(p)$ be a right inverse for π (a cross-section). Then the function $m(p, q) = u(p)u(q)u(p+q)^{-1}$ on $\Gamma \times \Gamma$ takes values in $\iota(\mathbf{T})$ (identified with $\mathbf{T}$) and satisfies the 2-cocycle equation

$$(*) \qquad m(q, r)m(p, q+r) = m(p+q, r)m(p, q).$$

Conversely, given a 2-cocycle m we obtain an extension $\mathscr{G}$ by defining a product on $\mathbf{T} \times \Gamma$ by

$$(**) \qquad (s, p)(t, q) = (m(p, q)st, p+q).$$

Let $Z^2(\Gamma, \mathbf{T})$ denote the compact abelian group (under pointwise multiplication) of all 2-cocycles from $\Gamma \times \Gamma$ to $\mathbf{T}$ and denote by $B^2(\Gamma, \mathbf{T})$ the closed subgroup of 2-coboundaries in $Z^2(\Gamma, \mathbf{T})$ i.e. elements of the form $m(p, q) = f(p)f(q)f(p+q)^{-1}$ for some function $f: \Gamma \to \mathbf{T}$. Thus the second cohomology group

$$H^2(\Gamma, \mathbf{T}) = Z^2(\Gamma, \mathbf{T})/B^2(\Gamma, \mathbf{T})$$

is a compact abelian group. It is well-known (and easy to prove) that two extensions of $\mathbf{T}$ by Γ are equivalent in the natural sense if and only if $m_1m_2^{-1} \in B^2(\Gamma, \mathbf{T})$ for some (hence any) 2-cocycles m_1, m_2 determining the extensions. Hence $H^2(\Gamma, \mathbf{T})$ is a complete invariant for the (classes of) extensions of $\mathbf{T}$ by Γ.

Due to the special choice of $\mathbf{T}$ as the subgroup in the extensions, a more tractable description of $H^2(\Gamma, \mathbf{T})$ can be given.

3.2. Proposition. *Given m in $Z^2(\Gamma, \mathbf{T})$ define m^* by $m^*(p, q) = m(q, p)^{-1}$ and let $X^2(\Gamma, \mathbf{T})$ denote the subgroup of $Z^2(\Gamma, \mathbf{T})$ consisting of the symplectic bicharacters on $\Gamma \times \Gamma$. Then the endomorphism $m \to mm^*$ of $Z^2(\Gamma, \mathbf{T})$ has $B^2(\Gamma, \mathbf{T})$ as its kernel and $X^2(\Gamma, \mathbf{T})$ as its range. In particular $H^2(\Gamma, \mathbf{T})$ is isomorphic to $X^2(\Gamma, \mathbf{T})$.*

Proof. Since $m \to m^*$ is an automorphism of $Z^2(\Gamma, \mathbf{T})$ and $m^* = m^{-1}$ for each m in $B^2(\Gamma, \mathbf{T})$ it follows that $m \to mm^*$ is an endomorphism with $B^2(\Gamma, \mathbf{T})$ in its kernel. However, if $mm^* = 1$ then m is symmetric, so that the corresponding extension $\mathscr{G}$ is abelian (cf. $(**)$ in 3.1). The identity map of $\mathbf{T}$ is a character on the closed subgroup $\iota(\mathbf{T})$ of $\mathscr{G}$ and can therefore be extended to a character on $\mathscr{G}$. Thus ι has a left inverse which means that the sequence splits, i.e. $\mathscr{G} = \mathbf{T} \oplus \Gamma$. Consequently $m \in B^2(\Gamma, \mathbf{T})$ so that $B^2(\Gamma, \mathbf{T})$ is equal to the kernel of the map $m \to mm^*$.

Let X_0 denote the closed subgroup of $Z^2(\Gamma, \mathbf{T})$ consisting of elements mm^*. From the preceding we know that

$$X_0 = Z^2(\Gamma, \mathbf{T})/B^2(\Gamma, \mathbf{T}) = H^2(\Gamma, \mathbf{T}).$$

Take $\chi = mm^*$ in X_0. Clearly $\chi(p, q) = \chi(q, p)^{-1}$, i.e. χ is a symplectic form. To show that χ is a bicharacter let $\mathscr{G}$ be the extension and $p \to u(p)$ the cross-section of Γ in

$\mathscr{G}$ given by m, cf. 3.1. Thus $m(p, q) = u(p)u(q)u(p+q)^{-1}$, whence $\chi(p, q) = u(p)u(q)$ $u(p)^{-1}u(q)^{-1}$. Consequently

$$\chi(p+q, r) = u(p+q)u(r)u(p+q)^{-1}u(r)^{-1} =$$
$$= m(p, q)^{-1}u(p)u(q)u(r)u(p+q)^{-1}u(r)^{-1} =$$
$$= m(p, q)^{-1}u(p)\chi(q, r)u(r)u(q)u(p+q)^{-1}u(r)^{-1} =$$
$$= m(p, q)^{-1}\chi(p, r)\chi(q, r)u(r)u(p)u(q)u(p+q)^{-1}u(r)^{-1} =$$
$$= m(p, q)^{-1}\chi(p, r)\chi(q, r)u(r)m(p, q)u(r)^{-1} = \chi(p, r)\chi(q, r)\,.$$

Thus $\chi(\cdot, q)$ is a character on Γ and since χ is symplectic it is a bicharacter. It follows that $X_0 \subset X^2(\Gamma, \mathbf{T})$.

To prove the reverse inclusion take χ in $X^2(\Gamma, \mathbf{T})$. Let (Σ, m) be a pair such that Σ is a subgroup of Γ and $m \in Z^2(\Sigma, \mathbb{T})$ with $mm^* = \chi|\Sigma \times \Sigma$. If $a \in \Gamma \setminus \Sigma$ we shall extend m to $\Sigma_1 \times \Sigma_1$, where $\Sigma_1 = \mathbb{Z}a + \Sigma$, such that $mm^* = \chi|\Sigma_1 \times \Sigma_1$. To do so note first that $\mathbb{Z}a \cap \Sigma = \mathbb{Z}b$ for some b. Then let $\mathscr{G}$ be the extension of $\mathbb{T}$ by Σ given by m and consider the short exact sequence

$$\{1\} \to \mathbb{T} \xrightarrow{\iota} \mathscr{G} \xrightarrow{\pi} \Sigma \to \{0\}.$$

Define a group structure on $\mathscr{G} \times \mathbb{Z}a$ by

$$(u, p)(v, q) = (\chi(p, \pi(v))\, uv, p+q)\,.$$

With $\tilde{\iota} = \iota \times \iota$ and $\tilde{\pi} = \pi \times \iota$ we then obtain a short exact sequence

$$\{1\} \to \mathbf{T} \xrightarrow{\tilde{\iota}} \mathscr{G} \times \mathbf{Z}a \xrightarrow{\tilde{\pi}} \Sigma \oplus \mathbf{Z}a \to \{0\}.$$

Choose an element c in $\mathscr{G}$ with $\pi(c) = b$. If $nb = 0$ then $c^n \in \operatorname{Ker} \pi = \mathbf{T}$. Replacing c by $\theta^{-1}c$, where $\theta \in \mathbf{T}$ such that $\theta^n = c^n$, we may assume that b and c have the same order. Now consider the subgroup of $\mathscr{G} \times \mathbf{Z}a$ given by

$$\mathscr{C} = \{(c^n, -nb) \mid n \in \mathbf{Z}\}.$$

Since $\chi = mm^*$ it follows that if $u, v \in \mathscr{G}$ then $uvu^{-1} = \chi(\pi(u), \pi(v))v$. From this we see that for any pair $(u, p), (v, q)$ in $\mathscr{G} \times \mathbf{Z}a$,

$$(u, p)\,(v, q)\,(u, p)^{-1} =$$
$$= (\chi(p, \pi(v))uv, p+q)\,(\chi(p, \pi(u))u^{-1}, -p) =$$
$$= (\chi(p, \pi(v))\chi(p, \pi(u))\chi(p+q, \pi(u^{-1}))uvu^{-1}, q) =$$
$$= (\chi(p, \pi(v))\chi(\pi(u), q)\chi(\pi(u), \pi(v))v, q) =$$
$$= (\chi(p, q+\pi(v))\chi(\pi(u), q+\pi(v))v, q) =$$
$$= (\chi(p+\pi(u), q+\pi(v))v, q)$$

(note that $\chi(p, q) = 0$, since $p, q \in \mathbf{Z}a$). Replacing (v, q) with $(c^n, -nb)$ it is immediate that $\mathscr{C}$ belongs to the center of $\mathscr{G} \times \mathbf{Z}a$; in particular $\mathscr{C}$ is a normal subgroup. Clearly $\tilde{\pi}(\mathscr{C})$ (viz. $\{(nb, -nb) \mid n \in \mathbf{Z}\}$) is the kernel of the natural homomorphism of $\mathscr{G} \oplus \mathbf{Z}a$ onto $\Sigma_1 (= \Sigma + \mathbf{Z}a)$, and since c and b have the same order, $\tilde{\iota}(\mathbf{T}) \cap \mathscr{C} = \{1\}$ so that we have an exact sequence

$$\{1\} \to T \xrightarrow{\iota} \mathscr{G} \times \mathbf{Z}a/\mathscr{C} \xrightarrow{\rho} \Sigma_1 \to \{0\} .$$

Choose m_0 in $\mathbf{Z}^2(\Sigma_1, \mathbf{T})$ associated with this extension. The computation above can be rewritten as

$$(u, p)\,(v, q)\,(u, p)^{-1} = (\chi(\rho(u, p), \rho(v, q))v, q) .$$

Thus for any pair d, e in $\mathscr{G} \times \mathbf{Z}a/\mathscr{C}$ we have

$$ded^{-1} = \chi(\rho(d), \rho(e))e .$$

It follows that $m_0 m_0^* = \chi \mid \Sigma_1 \times \Sigma_1$. Since therefore m and $m_0 \mid \Sigma \times \Sigma$ have the same image in $H^2(\Sigma, \mathbf{T})$ there is a function $f : \Sigma \to \mathbf{T}$ such that

$$m(p, q) = f(p)f(q)f(p + q)^{-1}m_0(p, q)$$

for all p, q in Σ. Let $f_1 : \Sigma_1 \to \mathbf{T}$ be any extension of f and define

$$m_1(p, q) = f_1(p)f_1(q)f_1(p + q)^{-1}m_0(p, q)$$

for all p, q in Σ_1. Then the pair (Σ_1, m_1) extends (Σ, m) and $m_1 m_1^* = \chi \mid \Sigma_1 \times \Sigma_1$ since $m_1 m_0^{-1} \in B^2(\Sigma_1, \mathbf{T})$. Applying Zorn's lemma to the set of pairs (Σ, m) in their natural ordering we find a pair (Γ, m_∞), i.e. an element m_∞ in $Z^2(\Gamma, \mathbf{T})$, such that $m_\infty m_\infty^* = \chi$. This establishes the equality $X_0 = X^2(\Gamma, \mathbf{T})$ and completes the proof.

4. CLASSIFICATION OF G-SYSTEMS

4.1. LEMMA. *Let $(\lambda, L^2(\hat{G}))$ denote the regular representation of G on $L^2(\hat{G})$ given by*

$$(\lambda(s)\xi)\,(p) = \langle s, p\rangle\, \xi(p), \;\; \xi \in L^2(\hat{G}) .$$

For each m in $\mathbf{Z}^2(\hat{G}, \mathbf{T})$ there is a canonical G-system $(\mathscr{M}_m, \alpha^m)$ in $G(\lambda, L^2(\hat{G}))$ (cf. 2.8) *and a cross-section $p \to v(p)$ from $\hat{G}$ into $\mathscr{G}_{\alpha^m}$ such that the extension $\mathscr{G}_{\alpha^m}$ of $\mathbf{T}$ by $\hat{G}$*

described in 2.7 is given by m. Moreover, the map $m \to (\mathcal{M}_m, \alpha^m)$ *is continuous from* $Z^2(\hat{G}, \mathbf{T})$ *into* $G(\lambda, L^2(\hat{G}))$.

Proof Let $\{\eta_p | p \in \hat{G}\}$ be the usual basis for $L^2(\hat{G})(=l^2(\hat{G}))$ and define unitary operators $v(p)$, $p \in \hat{G}$, by

$$(*) \qquad v(p)\eta_q = m(p, q)\eta_{p+q}, \quad q \in \hat{G}.$$

Elementary computations using the cocycle equation (*) in 3.1 show that $v(p)v(q)v(p+q)^* = m(p, q)$.

Since $\lambda(s)v(p)\lambda(-s) = \langle s, p\rangle\, v(p)$ for all s and p it follows that $\alpha^m = \mathrm{Ad}\,\lambda$ is a faithful representation of G as automorphisms on the von Neumann algebra $\mathcal{M}_m$ generated (linearly) by the $v(p)$'s. If x is a fixed-point for G in $\mathcal{M}_m$ there is a net $\{x_i\}$ of linear combinations of the $v(p)$'s converging strongly to x. But then $\hat{x}_i(0) \to \hat{x}(0)$ $(=x)$ strongly (cf. 2.8), and since $\hat{x}_i(0) \in \mathbf{C}1$ it follows that $x \in \mathbf{C}1$; so that $(\mathcal{M}_m, \alpha^m)$ is a G-system. Clearly $p \to v(p)$ is the cross-section from $\hat{G}$ to $\mathcal{G}_{\alpha^m}$ that defines m, so that $\mathcal{G}_{\alpha^m}$ is the extension of $\mathbf{T}$ by $\hat{G}$ given by m.

If $m_i \to m$ in $Z^2(\hat{G}, \mathbf{T})$ then clearly $v^i(p) \to v(p)$ strongly on $L^2(\hat{G})$ for each p (cf. (*)), whence $(\mathcal{M}_{m_i}, \alpha^{m_i}) \to (\mathcal{M}_m, \alpha^m)$ by 2.9.

4.2. Lemma. *Given a G-system* $(\mathcal{M}, \alpha)$ *let* χ_α *denote the symplectic bicharacter associated with the extension* $\mathcal{G}_\alpha$ *of* $\mathbf{T}$ *by* $\hat{G}$ *(identifying* $H^2(\hat{G}, \mathbf{T})$ *with* $X^2(\hat{G}, \mathbf{T})$ *by 3.2). For all* $(\mathcal{M}, \alpha)$ *in* $G(\lambda, L^2(\hat{G}))$ *the map* $(\mathcal{M}, \alpha) \to \chi_\alpha$ *is continuous.*

Proof. Suppose that $(\mathcal{M}_i, \alpha^i) \to (\mathcal{M}, \alpha)$ in $G(\lambda, L^2(\hat{G}))$. Take any cross-section $p \to u(p)$ of $\hat{G}$ in $\mathcal{G}_\alpha$ and note that $\chi_\alpha(p, q) = u(p)u(q)u(p)^*u(q)^*$. For fixed p and q in $\hat{G}$ there are by 2.9 nets $\{u_i(p)\}$ and $\{u_i(q)\}$ converging strongly to $u(p)$ and $u(q)$, respectively, with $u_i(p)$ and $u_i(q)$ in $\mathcal{M}_i$ and $\pi(u_i(p)) = p$, $\pi(u_i(q)) = q$ for all i. Consequently

$$\chi_i(p, q) = u_i(p)u_i(q)u_i(p)^*u_i(q)^* \to \chi(p, q);$$

whence $\chi_i \to \chi$ pointwise on $\hat{G} \times \hat{G}$.

4.3. Lemma. *Every G-system* $(\mathcal{M}, \alpha)$ *is conjugate to an element in* $G(\lambda, L^2(\hat{G}))$.

Proof. Given $(\mathcal{M}, \alpha)$ choose a cross-section $p \to u(p)$ from $\hat{G}$ into $\mathcal{G}_\alpha$ as described in 2.3 and 2.4, and let m be the corresponding element in $Z^2(\hat{G}, \mathbf{T})$. Taking $(\mathcal{M}, \alpha)$ in the standard representation (cf. 2.6) we note that the system $\{u(p)\xi_\tau | p \in \hat{G}\}$ is an orthonormal basis for $\mathfrak{H}_\tau$. Define an isometry $w : \mathfrak{H}_\tau \to L^2(\hat{G})$ by $wu(p)\xi_\tau = \eta_p$

for all p. Now let $(\mathcal{M}_m, \alpha^m)$ denote the G-system in $G(\lambda, L^2(\hat{G}))$ defined in 4.1 together with the cross-section $p \to v(p)$. A straightforward calculation gives

$$wu(p)w^* = v(p) \text{ and } w\lambda^\gamma(s)w^* = \lambda(s)$$

for all p and s. Consequently, with $\Phi = \mathrm{Ad}\, w$ we get $(\mathcal{M}, \alpha) \sim (\mathcal{M}_m, \alpha^m)$.

4.4. Proposition. *If $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ are G-systems then $[\alpha] = [\beta]$ if and only if $\chi_\alpha = \chi_\beta$.*

Proof. If $(\mathcal{M}, \alpha) \sim (\mathcal{N}, \beta)$ then $\mathcal{G}_\alpha \sim \mathcal{G}_\beta$ by 2.7 whence $\chi_\alpha = \chi_\beta$ by 3.2.

Conversely, if $\chi_\alpha = \chi_\beta$ and if $p \to u(p)$ and $p \to v(p)$ are cross-sections for $\hat{G}$ in $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, respectively, then the associated 2-cocycles are homologous by 3.2. There is therefore a function $f : \hat{G} \to \mathbb{T}$ such that

$$m(p, q) = u(p)u(q)u(p+q)^* = f(p)f(q)f(p+q)^{-1}v(p)v(q)v(p+q)^*.$$

Replacing $v(p)$ by $f(p)v(p)$ we see from the proof of 4.3 that $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ are both conjugate to the G-system $(\mathcal{M}_m, \alpha^m)$ in $G(\lambda, L^2(\hat{G}))$.

4.5. Theorem. *Let $[G]$ denote the set of conjugacy classes of G-systems equipped with the product defined in 2.2 and the quotient topology obtained from $G(\lambda, L^2(\hat{G}))$ under the equivalence map. Then $[G]$ is a compact abelian group and the map $(\mathcal{M}, \alpha) \to \chi_\alpha$ defined in 4.2 gives an isomorphism of $[G]$ onto $X^2(\hat{G}, \mathbf{T})$ $(= H^2(\hat{G}, \mathbf{T}))$.*

Proof. Consider the diagram

$$\begin{array}{ccc} Z^2(\hat{G},\mathbf{T}) & \xrightarrow[\pi]{} & X^2(\hat{G},\mathbf{T}) \\ {\scriptstyle f}\downarrow & \nearrow{\scriptstyle g} & \downarrow{\scriptstyle h} \\ G(\lambda, L^2(\hat{G})) & \xrightarrow[\rho]{} & [G] \end{array}.$$

Here π and ρ are the quotient maps (and ρ is surjective by 4.3) and f, g and h are the maps defined in 4.1, 4.2 and 4.4, respectively. It is immediate from the definitions that the diagram is commutative, and we know from 4.4 that h is a bijection.

If $\chi_i \to \chi$ in $X^2(\hat{G}, \mathbf{T})$ we can find $m_i \to m$ in $Z^2(\hat{G}, \mathbf{T})$ with $\pi(m_i) = \chi_i$ and $\pi(m) = \chi$, because π is an open map. Since both f and ρ are continuous it follows that $h(\chi_i) \to h(\chi)$. Thus h is a continuous bijection and therefore a homeomorphism, since $X^2(G, \mathbf{T})$ is compact.

Take G-systems $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ and choose cross-sections $p \to u(p)$ and $p \to v(p)$ for $\hat{G}$ in $\mathcal{G}_\alpha$ and $\mathcal{G}_\beta$, respectively. Then the function $p \to u(p) \otimes v(p)$ is a cross-section for $\hat{G}$ in $\mathcal{G}_{\alpha\times\beta}$ (cf. 2.2). It follows that

$$\chi_{\alpha\times\beta}(p, q) = (u(p) \otimes v(p))(u(q) \otimes v(q))(u(p) \otimes v(p))^*(u(q) \otimes v(q))^* =$$
$$= \chi_\alpha(p, q)\, \chi_\beta(p, q).$$

Since the product respects conjugacy by 2.2 we see that

$$h^{-1}([\alpha][\beta]) = h^{-1}([\alpha \times \beta]) = h^{-1}([\alpha])h^{-1}([\beta]) .$$

Consequently $[G]$ is also algebraically isomorphic to $X^2(\hat{G}, \mathbf{T})$ and thus a compact abelian group.

4.6. Corollary. *If $(\mathcal{M}, \alpha)$ is a G-system let $\mathcal{M}^\sim$ denote the opposite algebra (i.e. $\tilde{x}\tilde{y} = (yx)^\sim$) and define $\tilde{\alpha}_s(\tilde{x}) = (\alpha_s(x))^\sim$. Then $(\mathcal{M}^\sim, \tilde{\alpha})$ is a G-system and $[\tilde{\alpha}] = [\alpha]^{-1}$ in $[G]$. In particular, $[\alpha] = 1$ if and only if $\mathcal{M}$ is abelian.*

Proof. Since $\chi_{\tilde{\alpha}} = \chi_\alpha^{-1}$ we see from 4.5 that $[\tilde{\alpha}] = [\alpha]^{-1}$. If $\mathcal{M}$ is abelian then $\mathcal{M} = \mathcal{M}^\sim$ whence $[\alpha] = 1$. Conversely, if $\chi_\alpha = 1$ then $\mathcal{G}_\alpha$ is abelian because $u(p)u(q)$ $u(p)^*u(q)^* = 1$ for all p, q in G, so that $\mathcal{M}$ is abelian.

4.7. Proposition (cf. [20, III. 5]). *Let $(\mathcal{M}, \alpha)$ be a G-system and β a perturbed action of α : i.e. $\beta_s = (\mathrm{Ad}\, w(s)) \circ \alpha_s$, $s \in G$, for some 1-cocycle w in $\mathcal{M}$ with respect to α. Then $(\mathcal{M}, \beta)$ is a G-system conjugate to $(\mathcal{M}, \alpha)$.*

Proof. Let $\mathcal{N} = \mathcal{M} \otimes \mathbf{M}_2$ and define the action $\bar{\alpha}$ of G on $\mathcal{N}$ by

$$(*) \qquad \bar{\alpha}_s \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} \alpha_s(x_{11}) & \alpha_s(x_{12})\, w(s)^* \\ w(s)\, \alpha_s(x_{21}) & \beta_s(x_{22}) \end{pmatrix}$$

(cf. [24, 8.11.3]). Since $\bar{\alpha}$ is exterior equivalent with $\alpha \otimes \iota$ (via the 1-cocycle $\overline{w}(s) = w(s) \otimes e_{11} + 1 \otimes e_{22}$) we know that the three crossed products

$$G \underset{\bar{\alpha}}{\times} \mathcal{N}, \quad G \underset{\alpha\otimes\iota}{\times} \mathcal{N}, \; (G \underset{\alpha}{\times} \mathcal{M}) \otimes \mathbf{M}_2$$

are isomorphic. As $\Gamma(\alpha) = \hat{G}$ and α is ergodic on the center of $\mathcal{M}$ it follows from [9, III. 3.4] that $G \underset{\alpha}{\times} \mathcal{M}$ is a factor, whence $G \underset{\bar{\alpha}}{\times} \mathcal{N}$ is a factor. Let p be the function $p(s) = 1$ on G, identified with a projection in $G \underset{\bar{\alpha}}{\times} \mathcal{N}$ and check that

$$p(G \underset{\bar{\alpha}}{\times} \mathcal{N})\, p = \mathcal{N}^G p \sim \mathcal{N}^G$$

(or use [9, III. 2.15]). We conclude that $\mathcal{N}^G$ is a factor. There is a unique normalized trace on $\mathcal{N}^G$ given by

$$\tau_0(x) = \frac{1}{2}(\tau(x_{11}) + \tau(x_{22})), \;\; x = (x_{ij}) \in \mathcal{N}^G.$$

Since e_{11} and e_{22} belong to $\mathcal{N}^G$ with $\tau_0(e_{11}) = \tau_0(e_{22}) (= 1/2)$ it follows that $e_{11} \sim e_{22}$ in $\mathcal{N}^G$. This implies the existence of a partial isometry $v = u \otimes e_{21}$ in $\mathcal{N}^G$ such that $u^*u = uu^* = 1$ in $\mathcal{M}$. Thus $u = w(s)\,\alpha_s(u)$; i.e. $w(s) = u\alpha_s(u^*)$ so that w is a coboundary. From the formula

$$\beta_s(x) = u\alpha_s(u^*xu)\,u^*, \;\; s \in G\,, \;\; x \in \mathcal{M}\,,$$

it follows immediately that β is faithful and ergodic, so that $(\mathcal{M}, \beta)$ is a G-system. Taking $\Phi = \mathrm{Ad}u$ we have $\beta_s = \Phi \circ \alpha_s \circ \Phi^{-1}$, whence $(\mathcal{M}, \alpha) \sim (\mathcal{M}, \beta)$.

4.8. Remark. The stability result in 4.7 (any cocycle cobounds) remains valid for an action α of a compact group G provided that $\mathcal{M}$ is finite and $\Gamma(\alpha) = \hat{G}$. A proof can be obtained by disintegrating $\mathcal{M}$ into components $\mathcal{M}(\lambda)$ on which G acts centrally ergodic. Then $G \underset{\alpha}{\times} \mathcal{M}(\lambda)$ is a factor and using the proof above we obtain a unitary $u(\lambda)$ in $\mathcal{M}(\lambda)$ such that $u(\lambda) = w(s)(\lambda)\,\alpha_s(u(\lambda))$. Taking $u = \int u(\lambda) \mathrm{d}\lambda$ the result follows. The stability result even extends to non-abelian compact groups (and finite algebras) provided one replaces the spectrum condition $\Gamma(\alpha) = \hat{G}$ by a suitable condition which guarantees non-degeneracy of the action — for example the factor property of $G \underset{\alpha}{\times} \mathcal{M}$.

4.9. Remark. If $(\mathcal{M}, \alpha)$ is a G-system and σ is an automorphism of G then, defining $\alpha \circ \sigma$ by $(\alpha \circ \sigma)_s = \alpha_{\sigma(s)}$, we obtain a G-system $(\mathcal{M}, \alpha \circ \sigma)$. If χ_α is the symplectic bicharacter associated with $[\alpha]$ then $(p, q) \to \chi_\alpha(\hat{\sigma}(p), \hat{\sigma}(q))$ is associated with $[\alpha \circ \sigma]$. Motivated by this observation we say that two G-systems $(\mathcal{M}, \alpha)$ and $(\mathcal{N}, \beta)$ are *weakly conjugate* if there is an isomorphism $\Phi : \mathcal{M} \to \mathcal{N}$ and an automorphism σ of G such that $\Phi \circ \alpha \circ \sigma = \beta \circ \Phi$.

At first glance, weak conjugacy may seem a more natural concept than mere conjugacy. After all, changing α by an automorphism of G has no effect on $\mathcal{M}$ and is just a "change of scale" on G. Indeed, the simple G-systems for a finite group are usually not all conjugate, but they are weakly conjugate by 5.9. This fact is a special case of the so-called "uniqueness of the Weyl relations", and the reader will recall that the relations are in fact only unique up to automorphisms of the group.

The objections to the use of weak conjugacy are, first, that the group structure of conjugacy classes is lost. Second, and more important, the classification space is no longer smooth. A few examples will illustrate the problem.

4.10. EXAMPLE. Let $G = \mathbf{T}^n$ so that $\hat{G} = \mathbf{Z}^n$. Any bicharacter on $\mathbf{Z}^n \times \mathbf{Z}^n$ is given by

$$\chi(p, q) = \prod a_{ij}^{p_i q_j}, \quad p, q \in \mathbf{Z}^n,$$

for some $n \times n$ matrix $A = (a_{ij})$ with entries in $\mathbf{T}$. Moreover, χ is symplectic if $a_{ji} = a_{ij}^{-1}$. Thus χ is determined by the upper triangular part of A and, product of χ's corresponding to elementwise product of A's, we conclude that

$$X^2(\mathbf{Z}^n, \mathbf{T}) = \mathbf{T}^{n(n-1)/2}.$$

An automorphism of $\mathbf{Z}^n$ is given by

$$\sigma(p)_i = \sum b_{ij} p_j, \quad p \in \mathbf{Z}^n,$$

for some invertible $n \times n$ matrix $B = (b_{ij})$ with entries in $\mathbf{Z}$; composition of automorphisms corresponding to multiplication of matrices.

Given χ and σ corresponding to matrices A and B as above let $\tilde{\chi}$ denote the transformed bicharacter and $\tilde{A}$ its corresponding matrix. Thus

$$\tilde{\chi}(p, q) = \chi(\sigma(p), \sigma(q)) = \prod a_{ij}^{\sigma(p)_i \sigma(q)_j} = \\ = \prod a_{ij}^{b_{ik} p_k b_{jl} p_l} = \prod \tilde{a}_{kl}^{p_k p_l},$$

where

$$\tilde{a}_{kl} = \prod a_{ij}^{b_{ik} b_{jl}}.$$

If $n = 1$ there are no symplectic bicharacters. If $n = 2$ we have $X^2(\mathbf{Z}^2, \mathbf{T}) = \mathbf{T}$. This case is extremely interesting (see 6.8 and 6.9) but does not help us to understand weak conjugacy. Indeed, since $s \to s$ and $s \to s^{-1}$ are the only automorphisms of $\mathbf{T}$ it follows that the set of weak conjugacy classes of $\mathbf{T}^2$-systems is isomorphic with the unit interval (identifying s and s^{-1} in $\mathbf{T}$). Now consider $\mathbf{T}^3$ where $X^2(\mathbf{Z}^3, \mathbf{T}) = \mathbf{T}^3$. An explicit isomorphism from the 3×3 matrices $A = (a_{ij})$, corresponding to elements in $X^2(\mathbf{Z}^3, \mathbf{T})$, onto $\mathbf{T}^3$ is given by

$$\varphi(A)_m = a_{ij}, \text{ where } i < j \text{ and } m = i + j - 2.$$

Take χ, σ, $\tilde{\chi}$ and A, B, $\tilde{A}$ as above (but for $n = 3$). Then if $k < l$ and $r = k + l - 2$ we have

$$\varphi(A)_r = \tilde{a}_{kl} = \prod_{i<j} a_{ij}^{b_{ik} b_{jl} - b_{jk} b_{il}} = \prod_m \varphi(A)_m^{c_{mr}},$$

putting $c_{mr} = b_{ik} b_{jl} - b_{jk} b_{il}$. The last expression is the typical form of an automorphism $\tilde{\sigma}$ of $\mathbf{T}^3$ (compare with the automorphisms of $\mathbf{Z}^3$). We may therefore write

$\tilde{A} = \varphi^{-1} \circ \tilde{\sigma} \circ \varphi(A)$. Consequently the set of weak conjugacy classes of $\mathbf{T}^3$-systems is isomorphic to the set of automorphism classes of $\mathbf{T}^3$. This space is not smooth. Indeed Aut($\mathbf{T}^3$) is a countable group (isomorphic with $GL(3, \mathbf{Z})$), but contains ergodic elements. Thus there are dense orbits in $\mathbf{T}^3$ under the action of Aut($\mathbf{T}^3$), yet the action is not transitive.

5. STRUCTURE OF G-SYSTEMS

5.1. If $(\mathcal{M}, \alpha)$ is a G-system let χ_α (or just χ) denote the associated symplectic bicharacter (cf. 4.5). There is then a unique homomorphism $\tilde{\chi} : \hat{G} \to G$ such that $\langle \tilde{\chi}(p), p \rangle = 1$ for all p in $\hat{G}$. The bijective correspondence $\chi \to \tilde{\chi}$ is given by

$$\langle \tilde{\chi}(p), q \rangle = \chi(p, q), \quad p, q \in \hat{G} .$$

For notational simplicity we shall drop the tilde and identify $\tilde{\chi}$ and χ.

Throughout this section we denote by $\mathcal{Z}$ the center of the von Neumann algebra $\mathcal{M}$. If $\mathcal{Z} = \mathcal{M}$ we say that the G-system is *abelian* (cf. 4.6), and if $\mathcal{Z} = \mathbf{C}1$, i.e. $\mathcal{M}$ is a factor, we say that the system is *simple*.

5.2. PROPOSITION. *If $(\mathcal{M}, \alpha)$ is a G-system and $\mathcal{N}$ is a (globally) G-invariant von Neumann subalgebra of $\mathcal{M}$ let $G_{\mathcal{N}} = \ker(\alpha|\mathcal{N})$. Then $(\mathcal{N}, \dot{\alpha})$ is a $G/G_{\mathcal{N}}$-system ($\dot{\alpha}$ denoting the quotient map) and $\chi_{\dot{\alpha}}$ is the restriction of χ_α to $G_{\mathcal{N}}^{\perp} \times G_{\mathcal{N}}^{\perp}$ in $\hat{G} \times \hat{G}$. The map $\mathcal{N} \to G_{\mathcal{N}}$ is a bijective order reversing correspondence between G-invariant von Neumann subalgebras of $\mathcal{M}$ and closed subgroups of G.*

Proof. It is clear from the definition of $G_{\mathcal{N}}$ that $(\mathcal{N}, \dot{\alpha})$ is a $G/G_{\mathcal{N}}$-system. Choose a cross-section $p \to u(p)$ of $\hat{G}$ into $\mathcal{G}_\alpha$ and note that

$$G_{\mathcal{N}}^{\perp} \supset \{p \in \hat{G} \mid u(p) \in \mathcal{N}\} .$$

On the other hand, since $(\mathcal{N}, \dot{\alpha})$ is a $G/G_{\mathcal{N}}$-system, $\mathcal{N}$ is spanned by unitary eigenoperators $v(p), p \in (G/G_{\mathcal{N}})^\wedge = G_{\mathcal{N}}^{\perp}$, and since the eigenspaces in $\mathcal{M}$ are one-dimensional we must have

$$G_{\mathcal{N}}^{\perp} = \{p \in \hat{G} \mid u(p) \in \mathcal{N}\} .$$

Thus for p, q in $G_{\mathcal{N}}^{\perp}$ we have

$$\chi_{\dot{\alpha}}(p, q) = u(p)\, u(q)\, u(p)^*\, u(q)^* = \chi_\alpha(p, q) .$$

Conversely, if H is a closed subgroup of G let $\mathcal{M}^H$ be the von Neumann algebra generated by $\{u(p) \mid p \in H^{\perp}\}$. Then $\mathcal{M}^H$ is G-invariant and is a G/H-system. Also

$\mathcal{M}^H$ is precisely the fixed-point algebra for H. Evidently the maps $\mathcal{N} \to G_{\mathcal{N}}$ and $H \to \mathcal{M}^H$ are the inverse of each other.

5.3. Proposition. *Let $(\mathcal{M}, \alpha)$ be a G-system and $\mathcal{N}$ a G-invariant von Neumann subalgebra of $\mathcal{M}$ corresponding to the closed subgroup $G_{\mathcal{N}}$ of G* (cf. 5.2). *Then $\mathcal{N}$ is maximal abelian in $\mathcal{M}$ if and only if $\mathcal{N} \cap \mathcal{G}_\alpha$ is maximal abelian in $\mathcal{G}_\alpha$ and again if and only if $G_{\mathcal{N}}$ is the closure of $\chi_\alpha(G_{\mathcal{N}}^\perp)$ in G.*

Proof. If $\mathcal{N}$ is maximal abelian in $\mathcal{M}$ then clearly $\mathcal{G}_0 = \mathcal{N} \cap \mathcal{G}_\alpha$ is maximal abelian in $\mathcal{G}_\alpha$. The converse follows from the fact that if $x \in \mathcal{N}' \cap \mathcal{M}$ then $\hat{x}(p) \in \mathcal{N}'$ for each p in $\hat{G}$ by the G-invariance of $\mathcal{N}$. Thus the relative commutant of $\mathcal{N}$ is spanned by the commutant of $\mathcal{G}_0$ in $\mathcal{G}_\alpha$.

Suppose now that $\mathcal{G}_0$ is maximal abelian in $\mathcal{G}_\alpha$ and take $u(p)$, $u(q)$ in $\mathcal{G}_\alpha$ with images p, q in $\hat{G}$. Since $u(p)$ and $u(q)$ commutes if and only if $\langle \chi_\alpha(p), q\rangle = 1$ it follows immediately that

$$G_{\mathcal{N}}^\perp = \chi_\alpha(G_{\mathcal{N}}^\perp)^\perp \quad \text{i.e} \quad G_{\mathcal{N}} = \chi_\alpha(G_{\mathcal{N}}^\perp)^- .$$

Conversely, if $G_{\mathcal{N}}$ is the closure of $\chi_\alpha(G_{\mathcal{N}}^\perp)$ then $u(q) \in \mathcal{G}_0'$ if and only if $q \in G_{\mathcal{N}}^\perp$, and since $\mathcal{G}_0$ is the inverse image of $G_{\mathcal{N}}^\perp$ in $\mathcal{G}_\alpha$ by 5.2 this shows that $\mathcal{G}_0$ is maximal abelian.

5.4. Proposition. *Let $\mathcal{Z}$ denote the center of $\mathcal{M}$. Then $(\mathcal{Z}, \dot{\alpha})$ is a $G/G_{\mathcal{Z}}$-system conjugate to*

$$(L^\infty(G/G_{\mathcal{Z}}), \operatorname{Ad} \lambda) ,$$

λ denoting translation on $L^\infty(G/G_{\mathcal{Z}})$.

Proof. With $\mathcal{G}_0 = \mathcal{G}_\alpha \cap \mathcal{Z}$ the short exact sequence

$$\{1\} \to \mathbf{T} \to \mathcal{G}_0 \to (G/G_{\mathcal{Z}}) \hat{\to} \{0\}$$

splits, since $\mathcal{G}_0$ is abelian. Thus we may assume that $p \to u(p)$, $p \in G_{\mathcal{Z}}^\perp$, is a faithful unitary representation. Since furthermore $\dot{\alpha}_s(u(p)) = \langle s, p\rangle\, u(p)$ for every s in $G/G_{\mathcal{Z}}$ it follows that $\mathcal{Z} = \{u(p)\}''$ is isomorphic with $L^\infty(G/G_{\mathcal{Z}})$ and that the natural isomorphism carries the action of $\dot{\alpha}$ into translation of functions on $G/G_{\mathcal{Z}}$.

5.5. Corollary. *All abelian G-systems are conjugate to* $(L^\infty(G), \operatorname{Ad} \lambda)$.

5.6. Lemma. *Given a G-system $(\mathcal{M}, \alpha)$ choose a cross-section $p \to u(p)$ of $\hat{G}$ in $\mathcal{G}_\alpha$. Then the map $\check{\alpha} : p \to \operatorname{Ad} u(p)$ is a representation of $\hat{G}$ in $\operatorname{Aut}(\mathcal{M})$, independent of the choice of the $u(p)$'s. Moreover $\check{\alpha} = \alpha \circ \chi$ and (denoting by $\operatorname{Sp}(\check{\alpha})$ and $\Gamma(\check{\alpha})$ the Arveson and Connes spectra of $\check{\alpha}$)*

$$\Gamma(\check{\alpha}) = \operatorname{Sp}(\check{\alpha}) = G_{\mathcal{Z}} = \chi(\hat{G})^- .$$

Proof. Since $p \to u(p)$ is a projective representation of $\hat{G}$ it follows that $p \to \mathrm{Ad}\, u(p)$ is a representation of $\hat{G}$ in $\mathrm{Aut}(\mathcal{M})$ (even in $\mathrm{Aut}(\mathbf{B}(\mathfrak{H}))$). Changing the $u(p)$'s by scalar multiples from $\mathbf{T}$ makes no difference in $\mathrm{Ad}\, u(p)$, so the representation $\check{\alpha}$ is independent of the choice of the $u(p)$'s.

For all p, q in $\hat{G}$ we have

$$\check{\alpha}_p(u(q)) = u(p)\, u(q)\, u(p)^* = \langle \chi(p), q\rangle\, u(q) = \alpha_{\chi(p)}(u(q)) .$$

It follows that $\check{\alpha}_p = \alpha_{\chi(p)}$, whence $\check{\alpha} = \alpha \circ \chi$. By [7, 3.4.3] this implies (since $\mathrm{Sp}(\alpha) = \hat{G}$) that

$$\mathrm{Sp}(\check{\alpha}) = \chi(\mathrm{Sp}(\alpha))^- = \chi(\hat{G})^- .$$

Clearly $\check{\alpha}_p = \iota$ if and only if $u(p) \in \mathscr{Z}$, which by 5.2 means that $p \in G_{\mathscr{Z}}^{\perp}$. On the other hand, if $\check{\alpha}_p \neq \iota$ then $u(p) \notin \mathscr{Z}$. Consequently, $\mathrm{Sp}(\check{\alpha}) = G_{\mathscr{Z}}$. The set of fixed-points under $\check{\alpha}$ is $\mathscr{Z}$ and for each non-zero central projection z we have $\check{\alpha}_p(zu(q)) = \langle \chi(p), q\rangle\, zu(q)$. Therefore

$$\mathrm{Sp}(\check{\alpha} \mid z\mathcal{M}) = \mathrm{Sp}(\check{\alpha})$$

for all z. By the very definition of the Connes spectrum [7, p. 174] we obtain

$$\Gamma(\check{\alpha}) = \mathrm{Sp}(\check{\alpha}) = G_{\mathscr{Z}} = \chi(\hat{G})^-.$$

5.7. Proposition. *Let $(\mathcal{M}, \alpha)$ be a G-system and χ the associated homomorphism of $\hat{G}$ into G. Then*

$$\chi(\hat{G}) = \{s \in G \mid \alpha_s = \mathrm{Ad}\, w,\ w \text{ unitary in } \mathcal{M}\} .$$

Proof. From 5.6 it is immediate that every α_s where $s = \chi(p)$ is inner. Indeed $\alpha_s = \mathrm{Ad}\, u(p)$.

Conversely, if $\alpha_s = \mathrm{Ad}\, w$ for some unitary w in $\mathcal{M}$ then $\alpha_s = \mathrm{Ad}\, \alpha_t(w)$ for every t since G is abelian. But then with $p \to u(p)$ the usual cross-section,

$$\hat{w}(p)u(q) = \int \alpha_t(w)u(q)\,\overline{\langle t, p\rangle}\, \mathrm{d}t = \alpha_s(u(q))\, \hat{w}(p) =$$

$$= \langle s, q\rangle\, u(q)\, \hat{w}(p) = \langle s, q\rangle\, \chi(q, p)\, \hat{w}(p)\, u(q) ,$$

since $\hat{w}(p) \in \mathbf{C}\, u(p)$. Consequently,

$$\hat{w}(p) = \langle s, q\rangle\, \chi(-p, q)\, \hat{w}(p) = \langle s - \chi(p), q\rangle\, \hat{w}(p)$$

for all p and q. Choosing p such that $\hat{w}(p) \neq 0$ it follows that $s = \chi(p)$, as desired.

5.8. THEOREM. *Let $(\mathscr{M}, \alpha)$ be a G-system and χ the associated homomorphism of $\hat{G}$ into G. The following conditions are equivalent:*

(i) *χ is injective.*

(ii) *$\chi(\hat{G})$ is dense in G.*

(iii) *$(\mathscr{M}, \alpha)$ is a simple system* (i.e. *$\mathscr{M}$ is a factor).*

Proof. (i) $\Leftrightarrow$ (ii). Since α is faithful we see from 5.6 that

$$\ker(\chi) = \ker(\check{\alpha}) = \mathrm{Sp}(\check{\alpha})^{\perp} = \chi(\hat{G})^{\perp}\,.$$

(i) $\Leftrightarrow$ (iii). From the definition of $\check{\alpha}$ we have $\ker(\check{\alpha}) = \{p \mid u(p) \in \mathscr{Z}\}$. Thus $\mathscr{Z} = \mathbf{C}1$ if and only if $\check{\alpha}$ is faithful, i.e. χ is injective.

5.9. THEOREM. *Let $(\mathscr{M}, \alpha)$ be a simple G-system for some finite group G. Then $G = \hat{\Gamma} \times \Gamma$ and $\mathscr{M} = \mathbf{M}_n$, where n^2 is the order of G. Moreover, $(\mathscr{M}, \alpha)$ is conjugate to the Weyl system* ($\mathbf{M}_n$, Ad $\lambda \times$ Ad $\hat{\lambda}$), *where λ and $\hat{\lambda}$ are the regular representations of Γ and $\hat{\Gamma}$ on $l^2(\Gamma)$ as translation and multiplication operators, respectively.*

Proof. Since G is a finite abelian group, any cyclic subgroup $G_1 = \mathbf{Z}s_1$, whose order k is maximal among all orders of cyclic subgroups of G, is a direct summand. Indeed, if p_1 is a character on G such that $\langle s_1, p_1\rangle = \exp(2\pi i/k)$, then $\langle s, p_1\rangle$ is a power of $\exp(2\pi i/k)$ for all s in G, since otherwise we could find a cyclic subgroup with higher order than k. Consequently, $G_0 = \ker p_1$ is a subgroup of G with $G_0 \oplus G_1 = G$.

By 5.8 we know that $\chi : \hat{G} \to G$ is an isomorphism. Choose s_1 and p_1 as above and put $t_1 = \chi(p_1)$. If $s \in \mathbf{Z}s_1 \cap \mathbf{Z}t_1$, say $s = rs_1$ then since $\langle \chi(p_1), p_1\rangle = 1$ we have

$$1 = \langle s, p_1\rangle = \exp(2\pi i r/k)\,,$$

whence $r = 0$ (modulo k), i.e. $s = 0$. Thus if we write $G = G_0 \oplus G_1$ then $\mathbf{Z}t_1 \subset G_0$ and, being of maximal order k, it is a direct summand of G_0; whence $G = \mathbf{Z}s_1 \oplus \oplus\, \mathbf{Z}t_1 \oplus G'$. Put $q_1 = \chi^{-1}(s_1)$ so that $\hat{G} = \mathbf{Z}p_1 \oplus \mathbf{Z}q_1 \oplus \chi^{-1}(G')$. Since $\hat{\chi} = -\chi$ (χ being symplectic) we have

$$\langle t_1, q_1\rangle = \langle \chi(p_1), \chi^{-1}(s_1)\rangle = \langle s_1, p_1\rangle^{-1} = \exp(2\pi i/k)^{-1}\,,$$

from which we conclude that $\chi^{-1}(G') = (G')^{\wedge}$. Consequently the whole argument can be repeated with G' in place of G. After a finite number of steps we arrive at decompositions

$$G = \bigoplus_m (\mathbf{Z}s_m \oplus \mathbf{Z}t_m)\,; \quad \hat{G} = \bigoplus_m (\mathbf{Z}p_m \oplus \mathbf{Z}q_m)\,,$$

satisfying the relations

$$\chi(p_m) = t_m\,,\ \chi(q_m) = s_m\,,$$

$$\langle s_m, p_m\rangle = \langle t_m, q_m\rangle^{-1} = \exp(2\pi i/k(m))\,,$$

for all m, where $k(m)$ is the common order of s_m, t_m, p_m and q_m.

Both $\Gamma = \oplus\, \mathbf{Z}p_m$ and $\Gamma' = \oplus\, \mathbf{Z}q_m$ correspond to maximal abelian subalgebras of $\mathcal{M}$ by 5.3. We can therefore choose a cross-section $p \to u(p)$ of $\hat{G}$ onto $\mathcal{G}_\alpha$ such that $u|\Gamma$ and $u|\Gamma'$ are unitary representations and $u(p+p') = u(p)u(p')$ if $p \in \Gamma$, $p' \in \Gamma'$. Define the isomorphism $\sigma : \Gamma' \to \hat{\Gamma}$ by $\langle p, \sigma(p')\rangle = \chi(p, p')$ and put $\lambda(p) = u(p), p \in \Gamma$; $\hat{\lambda}(\hat{p}) = u(\sigma^{-1}(\hat{p}))$, $\hat{p} \in \hat{\Gamma}$. Then

$$\lambda(p)\,\hat{\lambda}(\hat{p})\,\lambda(p)^*\,\hat{\lambda}(\hat{p})^* = \chi(p, \sigma^{-1}(\hat{p})) = \langle p,\ \hat{p}\rangle\,.$$

Consequently λ and $\hat{\lambda}$ satisfy the Weyl commutation relations and are therefore (up to unitary equivalence) equal to the regular representations of Γ and $\hat{\Gamma}$ on $l^2(\Gamma)$. In particular $\mathcal{M} = \mathbf{M}_n$ where n is the order of Γ.

Identifying $\hat{G} = \Gamma \oplus \Gamma'$ with $\Gamma \times \hat{\Gamma}$ via the isomorphism $\varphi : p + p' \to (-p, \sigma(p'))$ we compute the dual isomorphism $\hat{\varphi} : \hat{\Gamma} \times \Gamma \to G(=\chi(\hat{G}))$ as $\hat{\varphi}(\hat{p}, p) = \chi(p + \sigma^{-1}(\hat{p}))$. Indeed,

$$\langle \chi(p + \sigma^{-1}(\hat{p})),\ q+q'\rangle = \chi(p, q')\,\chi(q, \sigma^{-1}(\hat{p}))^{-1} =$$

$$= \langle p, \sigma(q')\,\rangle\,\langle q, \hat{p}\,\rangle^{-1} = \langle (p, \hat{p}), (-q, \sigma(q'))\rangle =$$

$$= \langle (p, \hat{p}), \varphi(q + q')\rangle\,.$$

Thus if $s = \hat{\varphi}(\hat{p}, p)$ then by 5.6

$$\alpha_s = \check{\alpha}_{\chi^{-1}(s)} = \mathrm{Ad}\, u(\chi^{-1}(s)) = \mathrm{Ad}\, u(p + \sigma^{-1}(\hat{p})) =$$

$$= \mathrm{Ad}\, u(p)u(\sigma^{-1}(\hat{p})) = \mathrm{Ad}\, \lambda(p)\,\hat{\lambda}(\hat{p}).$$

Consequently $\alpha \circ \hat{\varphi} = \mathrm{Ad}\,\lambda \times \hat{\lambda}$, as desired.

5.10. Suppose that $(\mathcal{N}, \beta)$ is an H-system for some closed subgroup H of G. Define $\mathcal{M}$ as the von Neumann subalgebra of $L^\infty(G) \otimes \mathcal{N}(= L^\infty(G, \mathcal{N}))$ consisting of functions $x : G \to \mathcal{N}$ such that

$$x(s - t) = \beta_t(x(s)),\ s \in G,\ t \in H\,.$$

With $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ defined by translation, i.e.

$$\alpha_s(x)(r) = x(r - s), \quad s, r \in G,$$

it is readily verified that $(\mathcal{M}, \alpha)$ is a G-system. We say that $(\mathcal{M}, \alpha)$ is *induced* from $\mathcal{N}, \beta)$, see [30, 10.2].

If $(\mathcal{B}, \beta)$ is a C^*-H-system (cf. 2.11) we define the *induced* C^*-G-system $(\mathcal{A}, \alpha)$ in the analogous manner, replacing the σ-weak by the norm topology.

5.11. Lemma. *Given a G-system $(\mathcal{M}, \alpha)$ take $\mathcal{M}^c$ as in 2.11. There is then a C^*-$G_{\mathcal{Z}}$- system $(\mathcal{B}, \beta)$, where $\mathcal{B}$ is simple, such that $(\mathcal{M}^c, \alpha)$ is induced from $(\mathcal{B}, \beta)$.*

Proof. The set of tracial states on $\mathcal{M}^c$ is a non-empty simplex (since it contains τ) and has therefore an extreme point τ_0. Let $(\pi_0, \mathfrak{H}_0, \xi_0)$ denote the cyclic representation of $\mathcal{M}^c$ associated with τ_0 via the GNS construction and put $\mathcal{B} = \pi_0(\mathcal{M}^c)$. Since τ_0 is extremal, π_0 is a factor representation, in particular the center of $\mathcal{B}$ is $\mathbb{C}1$.

If $t \in \chi_\alpha(\hat{G})$ then $\alpha_t = \mathrm{Ad}\, u(p)$ for some eigenoperator $u(p)$ by 5.7. Since $u(p) \in \mathcal{M}^c$ if follows that, with $\tau_s = \tau_0 \circ \alpha_s$, we have $\tau_t = \tau_0$ for all t in $\chi_\alpha(\hat{G})$. As the map $s \to \tau_s$ is weak* continuous this implies that $\tau_t = \tau_0$ for every t in $G_{\mathcal{Z}}$ $(= \chi_\alpha(\hat{G})^-$ by 5.6). A similar argument shows that if $\mathcal{I}$ is a norm closed ideal in $\mathcal{M}^c$ then $\alpha_t(\mathcal{I}) = \mathcal{I}$ for all t in $G_{\mathcal{Z}}$.

If $\mathcal{I}$ is strictly larger than $\ker \pi_0$, choose a positive element x in $\mathcal{I} \setminus \ker \pi_0$. Then

$$(*) \qquad z = \int_{G_{\mathcal{Z}}} \alpha_t(x)\, \mathrm{d}t \in \mathcal{I} \setminus \ker \pi_0 .$$

However, from 5.2 we see that $\mathcal{Z}$ coincides with the $G_{\mathcal{Z}}$-fixed points (the integral above is the center-valued trace on $\mathcal{M}$). Thus $z \in \mathcal{Z}^c$. Since the center of $\mathcal{B}$ is trivial it follows that $\pi_0(z)$ is a non-zero multiple of 1, whence $\pi_0(\mathcal{I}) = \mathcal{B}$, i.e. $\mathcal{I} = \mathcal{M}^c$. Thus $\ker \pi_0$ is a maximal ideal so that $\mathcal{B}$ is simple.

Define a continuous unitary representation w of $G_{\mathcal{Z}}$ on $\mathfrak{H}_0$ by

$$w(t)\, \pi_0(x)\, \xi_0 = \pi_0(\alpha_t(x))\, \xi_0, \quad t \in G_{\mathcal{Z}}, x \in \mathcal{M}^c .$$

This is possible only because $\tau_t = \tau_0$ for all t in $G_{\mathcal{Z}}$. Set $\beta = \mathrm{Ad}\, w$ and note that

$$(**) \qquad \beta_t(\pi_0(x)) = w(t)\, \pi_0(x)\, w(t)^* = \pi_0(\alpha_t(x))$$

for all x in $\mathcal{M}^c$. Since the center of $\mathcal{B}$ is trivial we see from $(*)$ that $(\mathcal{B}, \beta)$ is a C^*-$G_{\mathcal{Z}}$- -system.

Define a map $\Phi : \mathcal{M}^c \to C(G, \mathcal{B})$ by

$$\Phi(x)(s) = \pi_0(\alpha_{-s}(x)), \quad x \in \mathcal{M}^c, \quad s \in G .$$

Clearly Φ is a morphism and since $x \in \ker \Phi$ implies $\alpha_s(x) \in \ker \pi_0$ for all s, whence $\tau(x)1 \in \ker \pi_0$; we see that $\tau(x)=0$, and thus Φ is faithful. Let $(\mathrm{Ind}(\mathscr{B}, \beta), \tilde{\alpha})$ denote the C^*-G-system induced from $(\mathscr{B}, \beta)$ (cf. 5.10). Thus the elements in $\mathrm{Ind}(\mathscr{B}, \beta)$ are the functions y in $C(G, \mathscr{B})$ such that

$$y(s-t) = \beta_t(y(s)), s \in G, t \in G_{\mathscr{Z}} .$$

Since for each x in $\mathscr{M}^c$ and r, s in G

$$\Phi(\alpha_s(x))(r) = \pi_0(\alpha_{s-r}(x)) = \Phi(x)(r-s)$$

we see from (**) that Φ is a covariant injection of $\mathscr{M}^c$ into $\mathrm{Ind}(\mathscr{B}, \beta)$.

It follows from 5.4 that

$$(\mathscr{Z}^c, \dot{\alpha}) \sim (C(G/G_{\mathscr{Z}}), \mathrm{Ad}\,\lambda).$$

Moreover, since $(\pi_0, \mathfrak{H}_0)$ is a factor representation there is an $\dot{r}_0$ in $G/G_{\mathscr{Z}}$ such that

$$\Phi(z)(s) = \pi_0(\alpha_{-s}(z)) = z(\dot{r}_0 - \dot{s})$$

for all z in $\mathscr{Z}^c$ and s in G.

Given y in $\mathrm{Ind}(\mathscr{B}, \beta)$ and $\varepsilon > 0$ there is for each point s_i in G an element x_i in $\mathscr{M}^c$ and a neighbourhood Ω_i of s_i such that

$$\|y(s) - \Phi(x_i)(s)\| < \varepsilon, \ s \in \Omega_i .$$

Since G is compact we can choose $s_1, \ldots, s_n$ such that $\cup \Omega_i = G$. Let $\{f_i \mid 1 \leqslant i \leqslant n\}$ be a partion of the unit on $G/G_{\mathscr{Z}}$ relative to the covering $\cup (\Omega_i/G_{\mathscr{Z}})$. Identifying $\mathscr{Z}^c$ with $C(G/G_{\mathscr{Z}})$ as above we find elements $\{z_i \mid 1 \leqslant i \leqslant n\}$ such that $\Phi(z_i) = f_i$ for all i. Put $x = \sum z_i x_i$ and take s in G. If $s \in \Omega_i + G_{\mathscr{Z}}$, say $s = r - t$, then

$$\|y(s) - \Phi(x_i)(s)\| = \|w(t)(y(r) - \Phi(x_i)(r))w(t)^*\| < \varepsilon .$$

If $s \notin \Omega_i + G_{\mathscr{Z}}$ then $\Phi(z_i)(s) = 0$. Consequently

$$\|y(s) - \Phi(x)(s)\| = \| \sum \Phi(z_i)(s)(y(s) - \Phi(x_i)(s))\| \leqslant$$

$$\leqslant \sum f_i(\dot{s})\|y(s) - \Phi(x_i)(s)\| < \varepsilon .$$

It follows that $\Phi(\mathscr{M}^c) = \mathrm{Ind}(\mathscr{B}, \beta)$ so that

$$(\mathscr{M}^c, \alpha) \sim (\mathrm{Ind}(\mathscr{B}, \beta), \tilde{\alpha}) ,$$

as desired.

5.12. Theorem. *Each G-system $(\mathscr{M}, \alpha)$ is induced from a simple $G_{\mathscr{Z}}$-system $(\mathscr{N}, \beta)$.*

Proof. Choose $(\pi_0, \mathfrak{H}_0, \xi_0)$ as in the proof of 5.11 and put $\mathcal{N} = \pi_0(\mathcal{M}^c)''$. Then $\mathcal{N}$ is a finite factor (with ξ_0 as trace vector) and with $\beta_t = \operatorname{Ad} w_t$, $t \in G_{\mathcal{X}}$, we obtain a simple $G_{\mathcal{X}}$-system $(\mathcal{N}, \beta)$.

The morphism $\Phi : \mathcal{M}^c \to C(G, \mathcal{B})$ defined in 5.11 can be considered as a representation of $\mathcal{M}^c$ on $L^2(G, \mathfrak{H}_0)$, where

$$(\Phi(x)\, \xi)(s) = \Phi(x)(s)\xi(s), \quad s \in G .$$

If $\xi_\tau = 1 \otimes \xi_0$ we have

$$(\Phi(x)\, \xi_\tau \mid \xi_\tau) = \int (\Phi(x)(s)\, \xi_0 | \xi_0)\, \mathrm{d}s = \int \tau_0(\alpha_{-s}(x))\, \mathrm{d}s = \tau(x) ,$$

since clearly $\int \tau_s\, \mathrm{d}s = \tau$. Let $\mathfrak{H}$ be the closure of $\Phi(\mathcal{M}^c)\xi_\tau$ in $L^2(G, \mathfrak{H}_0)$. Then ξ_τ is a cyclic vector for the representation $(\Phi, \mathfrak{H})$ which is therefore unitarily equivalent with the standard representation $(\pi_\tau, \mathfrak{H}_\tau)$ of $\mathcal{M}^c$, cf. 2.6. Thus it extends by normality to a faithful representation of $\mathcal{M}$ onto the weak closure of $\Phi(\mathcal{M}^c)$ in $\mathbf{B}(\mathfrak{H})$.

The weak closure of $\Phi(\mathcal{M}^c)$ in $\mathbf{B}(L^2(G, \mathfrak{H}_0))$ is evidently $\operatorname{Ind}(\mathcal{N}, \beta)$. Moreover, if $y \in \operatorname{Ind}(\mathcal{N}, \beta)$, regarded as a function from G to $\mathcal{N}$, there is a unique z in $L^\infty(G/G_{\mathcal{X}})$ such that

$$z(\dot{s}) = \int_{G_{\mathcal{X}}} \beta_t(y(s))\, \mathrm{d}t .$$

Consequently,

$$(y\xi_\tau \mid \xi_\tau) = \int_G (y(s)\, \xi_0 | \xi_0)\, \mathrm{d}s = \int_{G/G_{\mathcal{X}}} \int_{G_{\mathcal{X}}} (y(s-t)\xi_0|\xi_0)\mathrm{d}t\, \mathrm{d}\dot{s} = \int_{G/G_{\mathcal{X}}} z(\dot{s})\, \mathrm{d}\dot{s} .$$

It follows that ξ_τ is a separating vector for $\operatorname{Ind}(\mathcal{N}, \beta)$ so that the reduction from $L^2(G, \mathfrak{H}_0)$ to $\mathfrak{H}$ is an isomorphism. Thus Φ is an isomorphism of $\mathcal{M}$ into $\operatorname{Ind}(\mathcal{N}, \beta)$, intertwining the actions α and $\tilde{\alpha}$, and the proof is complete.

5.13. Corollary. *If $G \to G/G_{\mathcal{X}}$ admits a Borel cross-section, in particular if G is second countable, then*

$$\mathcal{M} = L^\infty(G/G_{\mathcal{X}}) \otimes \mathcal{N} .$$

Proof. By assumption there is a Borel set R in G, isomorphic with $G/G_{\mathcal{X}}$, such that each point s in G has a unique representation $s = r - t$, $r \in R$, $t \in G_{\mathcal{X}}$. For each y in $L^\infty(R, \mathcal{N})$ define an extension $\tilde{y}$ in $L^\infty(G, \mathcal{N})$ by $\tilde{y}(s) = \beta_t(y(r))$. Clearly $\tilde{y} \in \operatorname{Ind}(\mathcal{N}, \beta)$. The restriction map $x \to x|R$ from $\operatorname{Ind}(\mathcal{N}, \beta)$ to $L^\infty(R, \mathcal{N})$ is the inverse for the map $y \to \tilde{y}$ and we conclude from 5.12 that

$$\mathcal{M} \cong \operatorname{Ind}(\mathcal{N}, \beta) \cong L^\infty(R, \mathcal{N}) = L^\infty(G/G_{\mathcal{X}}) \otimes \mathcal{N} .$$

5.14. THEOREM. *The following conditions on a G-system* $(\mathcal{M}, \alpha)$ *are equivalent:*

(i) $\mathcal{M} = L^\infty(G/G_\mathcal{Z}) \otimes \mathbf{M}_n$.

(ii) $\mathcal{M}$ *is of type* I.

(iii) $\chi_\alpha(\hat{G})$ *is closed in* G.

(iv) $G_\mathcal{Z}$ *has finite order.*

In this case $G_\mathcal{Z} = \Gamma \times \Gamma$ *where* Γ *has order* n.

Proof. (i) $\Rightarrow$ (ii) is trivial and so is (iii) $\Rightarrow$ (iv) (given 5.6), since any discrete subgroup of a compact group is finite.

(ii) $\Rightarrow$ (iii). An automorphism of $\mathcal{M}$ is inner if and only if it leaves $\mathcal{Z}$ pointwise fixed (see e.g. [24, 8.9.2]). Combining 5.6 and 5.7 it follows that

$$\chi_\alpha(\hat{G}) = G_\mathcal{Z} = \chi_\alpha(\hat{G})^- .$$

(iv) $\Rightarrow$ (i). Since $G_\mathcal{Z}(= \chi_\alpha(\hat{G}))$ is finite, each point in G has a neighbourhood Ω such that $\pi|\Omega$ is injective (π denoting the quotient map onto $G/G_\mathcal{Z}$). Choose a finite covering of $G/G_\mathcal{Z}$ by sets $\pi(\Omega_1), \ldots, \pi(\Omega_m)$ and put $R = \cup \Omega_j'$ where Ω_j' is defined inductively by

$$\Omega_1' = \Omega_1, \; \Omega_m' = \Omega_m \setminus \bigcup_{j<m} (\Omega_j + G_\mathcal{Z}) .$$

Then $\pi(R) = G/G_\mathcal{Z}$ and R is a cross-section for π. Thus 5.13 applies to show that $\mathcal{M} = L^\infty(G/G_\mathcal{Z}) \otimes \mathcal{N}$, where $\mathcal{N}$ is a factor. Moreover, from 5.12 we know that there is an action β for which $(\mathcal{N}, \beta)$ is a $G_\mathcal{Z}$-system. Since $G_\mathcal{Z}$ is finite we have $G_\mathcal{Z} = \Gamma \times \Gamma$ and $\mathcal{N} = \mathbf{M}_n$, where n is the order of Γ, by 5.9.

5.15. THEOREM. *If G is second countable the following conditions on a G-system* $(\mathcal{M}, \alpha)$ *are equivalent:*

(i) $\mathcal{M} = L^\infty(G/G_\mathcal{Z}) \otimes \mathcal{R}$, *where* $\mathcal{R}$ *is the unique hyperfinite* II_1*-factor.*

(ii) $\mathcal{M}$ *is not of type* I.

(iii) $G_\mathcal{Z}$ *is infinite.*

Proof. Clearly (i) $\Rightarrow$ (ii), and since (ii) $\Leftrightarrow$ (iii) by 5.14 we only have to prove (ii) $\Rightarrow$ (i).

Since G is second countable we have

$$\mathcal{M} = L^\infty(G/G_\mathcal{Z}) \otimes \mathcal{N}$$

for some factor $\mathcal{N}$ by 5.13, and by assumption $\mathcal{N}$ is not of type I, hence is of type II_1 since $\mathcal{M}$ is finite by 2.5. Moreover, by 5.12 $(\mathcal{N}, \beta)$ is a $G_\mathcal{Z}$-system for some action β, so it suffices to show that $\mathcal{M} = \mathcal{R}$ whenever $(\mathcal{M}, \alpha)$ is a simple G-system not of type I.

Consider $(\mathcal{M}, \alpha)$ in the standard representation on $\mathfrak{H}_\tau$ (2.6) and choose a cross-section $p \to u(p)$ of $\hat{G}$ in $\mathcal{G}_\alpha$. Then as in 5.6 the map $\check{\alpha} : p \to \mathrm{Ad}\, u(p)$ is a representation of $\hat{G}$ as automorphisms of $\mathbf{B}(\mathfrak{H}_\tau)$. Since $\{u(p) \mid p \in G\}'' = \mathcal{M}$ it follows that the fixed-point algebra of $\hat{G}$ under $\check{\alpha}$ is precisely $\mathcal{M}'$. Choose an invariant mean μ on $\hat{G}$ (which is possible since abelian groups are amenable), and define $\pi' : \mathbf{B}(\mathfrak{H}_\tau) \to \mathcal{M}'$ by

$$(\pi'(x)\xi \mid \eta) = \mu\{s \to (\check{\alpha}_s(x)\xi \mid \eta)\}, \quad \xi, \eta \in \mathfrak{H}_\tau.$$

Then π' is a projection of $\mathbf{B}(\mathfrak{H}_\tau)$ onto $\mathcal{M}'$ so that $\mathcal{M}'$ is injective. Consequently $\mathcal{M}$ is an injective factor of type II_1 and by Connes fundamental result [8], $\mathcal{M} = \mathcal{R}$.

6. C^*-ALGEBRAIC SYSTEMS

6.1. PROPOSITION. *The map $(\mathcal{M}, \alpha) \to (\mathcal{M}^c, \alpha)$ defined in 2.11 gives a bijective correspondence between the sets of (von Neumann algebraic) G-systems and C^*-G-systems.*

Proof. Given a C^*-G-system $(\mathcal{A}, \alpha)$ let τ be the unique G-invariant state of $\mathcal{A}$ $\left(\text{defined by } \tau(x) = \int \alpha_s(x) ds\right)$ and use τ to construct a covariant representation $(\pi_\tau, \lambda^\tau, \mathfrak{H}_\tau)$ of the C^*-dynamical system $(\mathcal{A}, G, \alpha)$. Since τ is faithful, so is π_τ, and we may identify $\mathcal{A}$ with $\pi_\tau(\mathcal{A})$. Let $\mathcal{M}$ be the weak closure of $\mathcal{A}$ in $\mathbf{B}(\mathfrak{H}_\tau)$ and consider the extension of α to $\mathcal{M}$ given by $\alpha_s = \mathrm{Ad}\, \lambda^\tau(s)$. Since G is compact it is immediate that $(\mathcal{M}, \alpha)$ is a G-system. Furthermore we see that the system is in its standard representation (cf. 2.6).

If $(\mathcal{M}, \alpha)$ is a G-system then clearly $\mathcal{M}$ is the weak closure of $\mathcal{M}^c$ in the standard representation of $\mathcal{M}$ on $\mathfrak{H}_\tau$. Conversely, if $(\mathcal{A}, \alpha)$ is a C^*-G-system represented on $\mathfrak{H}_\tau$ and if x is an element in the weak closure $\mathcal{M}$ of $\mathcal{A}$ such that the function $s \to \lambda^\tau(s) x \lambda^\tau(-s)$ is norm continuous, then x is the norm limit of the net $\{\hat{x}(\Omega)\}$, where Ω is a finite subset of $\hat{G}$ and $\hat{x}(\Omega) = \sum \hat{x}(p)$, $p \in \Omega$. But as in 2.3 we can choose a complete set of unitary eigenoperators $\{u(p) \mid p \in \hat{G}\}$ for α in $\mathcal{A}$, whence $x \in \mathcal{A}$. Thus $\mathcal{M} = \mathcal{A}$, so that the correspondence $\mathcal{A} \to \mathcal{M}$ is bijective.

6.2. LEMMA (cf. [11], [14]). *If $(\mathcal{A}, \alpha)$ is a C^*-G-system then $\mathcal{A}$ is nuclear.*

Proof. Consider $\mathcal{A}$ in its universal representation on a Hilbert space $\mathfrak{H}_u$ and choose a cross-section $p \to u(p)$ of $\hat{G}$ in $\mathcal{G}_\alpha$. Then $\check{\alpha} : p \to \mathrm{Ad}\, u(p)$ is a representation of $\hat{G}$ in Aut $\mathbf{B}(\mathfrak{H}_u)$ and as in the proof of 5.15 we obtain a projection of $\mathbf{B}(\mathfrak{H}_u)$ on $\mathcal{A}'$ using the amenability of $\hat{G}$. Thus $\mathcal{A}'$ is injective, whence $\mathcal{A}''$ is injective. But $\mathcal{A}'' = \mathcal{A}^{**}$ so that $\mathcal{A}$ is nuclear; see [5], [6], [33].

6.3. THEOREM. *Every C^*-G-system $(\mathscr{A}, \alpha)$ is induced from a simple $G/G_{\mathscr{X}}$-system $(\mathscr{B}, \beta)$. If $G_{\mathscr{X}}$ is finite then $\mathscr{B} = \mathbf{M}_n$($n^2$ being the order of $G_{\mathscr{X}}$) and $\mathscr{A}$ is homogeneous of degree n with spectrum $G/G_{\mathscr{X}}$. If $G_{\mathscr{X}}$ is infinite $\mathscr{B}$ is a simple, nuclear C^*-algebra not of type I with a unique tracial state.*

Proof. Identifying $\mathscr{A}$ with $\mathscr{M}^c$ for some G-system $(\mathscr{M}, \alpha)$, the first statement is immediate from 5.11. If $G_{\mathscr{X}}$ is finite then $\mathscr{B} = \mathbf{M}_n$ by 5.9 and we may realize $\mathscr{A}$ as the C^*-subalgebra of functions x in $C(G, \mathbf{M}_n)$ such that

$$x(s - t) = \beta_t(x(s)), \quad s \in G, \ t \in G_{\mathscr{X}},$$

cf. 5.11. Consequenty, $\hat{\mathscr{A}}$ is homeomorphic with $G/G_{\mathscr{X}}$ and $\mathscr{A}$ is homogeneous of degree n, cf. Section 2 of [32].

If $G_{\mathscr{X}}$ is infinite we know from 5.11 and 6.2 that $\mathscr{B}$ is a simple nuclear C^*-algebra with a faithful, ergodic action β of $G_{\mathscr{X}}$. Since $1 \in \mathscr{B}$ and $\mathscr{B}$ is infinite dimensiona we see that $\mathscr{B}$ is not of type I. The uniqueness of the trace on $\mathscr{B}$ (as well as the simplicity) is [11, 32]. The argument goes as follows: Let τ_0 be a tracial state of $\mathscr{B}$ and put $\tau_t = \tau_0 \circ \beta_t$. Since β_t is inner for each t in the dense subgroup $\chi_\beta(\hat{G}_{\mathscr{X}})$ of $G_{\mathscr{X}}$ (5.7 and 5.8), and the function $t \to \tau_t$ is weak* continuous, it follows that $\tau_t(x) = \tau_0(x)$ for every x in $\mathscr{B}$. Consequently τ_0 is $G_{\mathscr{X}}$-invariant and thus unique.

6.4. LEMMA. (cf. [25, p. 281] and [4, 4.4]). *If $(\mathscr{A}, \mathbf{R}, \gamma)$ is a C^*-dynamical system with an invariant, finite trace τ, then there is a complex homomorphism Δ on the group $\mathscr{V}$ of invertible elements in $\mathscr{A}$ (or $\mathscr{A} \oplus \mathbf{C}1$ if $1 \notin \mathscr{A}$) such that Δ is constant on the connected components of $\mathscr{V}$.*

Proof. (Oral communication by G. A. Elliott). Let δ denote the infinitesimal generator for γ, so that δ is a derivation (probably unbounded) on $\mathscr{A}$ with dense domain $\mathscr{D}(\delta)$. Set

$$\Delta(u) = \tau(\delta(u)u^{-1}), \qquad u \in \mathscr{V} \cap \mathscr{D}(\delta).$$

It follows from the derivation property of δ that if u, v belong to $\mathscr{V} \cap \mathscr{D}(\delta)$ then $\Delta(uv) = \Delta(u) + \Delta(v)$.

If $u = \exp(x)$ for some x in $\mathscr{D}(\delta)$ then since τ is an invariant trace

$$\Delta(u) = \tau(\delta(\exp(x)) \exp(-x)) =$$

$$= \tau(\delta(x)\exp(x)\exp(-x)) = \tau(\delta(x)) = 0.$$

Since $\|u - v\| < \|v^{-1}\|^{-1}$ implies $u = \exp(x)v$ for some x in $\mathscr{D}(\delta)$ it follows that Δ is locally constant on $\mathscr{V} \cap \mathscr{D}(\delta)$, in particular Δ is continuous. Extending Δ by continuity we obtain a locally constant complex homomorphism on $\mathscr{V}$.

6.5. THEOREM. *Assume that G is second countable and let $(\mathscr{A}, \alpha)$ be a C^*-G-system. Thea $\mathscr{A}$ is approximately finite dimensional if and only if G is totally disconnected.*

Proof. If G is totally disconnected then every element in $\hat{G}$ has finite order. Otherwise $\hat{G}$ would have a non-zero countable, torsion free quotient which could be embedded as a subgroup of $\mathbf{R}$. The dual map would give a non-zero, continuous homomorphism of $\mathbf{R}$ into G, hence a path-connected subset of G; a contradiction. Thus $\hat{G}$ is the union of an increasing sequence $\{H_n\}$ of finite subgroups. Let $\{u(p) \mid p \in \hat{G}\}$ be a complete set of unitary eigenoperators for α in $\mathscr{A}$ and let $\mathscr{A}_n$ be the C^*-algebra generated by $\{u(p) \mid p \in H_n\}$. Then each $\mathscr{A}_n$ is finite dimensional whence $\mathscr{A}$, being the closure of $\cup \mathscr{A}_n$, is approximately finite dimensional.

Conversely, if G is not totally disconnected there is a non-zero, torsion free quotient of $\hat{G}$ and thus, as we saw above, a non-zero, continuous homomorphism $\beta : \mathbf{R} \to G$. Consider the C^*-dynamical system $(\mathscr{A}, \mathbf{R}, \gamma)$, where $\gamma = \alpha \circ \beta$, and let Δ be the homomorphism constructed in 6.4. Since β is non-zero there is a p in $\hat{G}$ such that

$$\gamma_t(u(p)) = \langle \beta(t), p \rangle u(p) = \exp(\mathrm{i}st)u(p)$$

for all real t, with $s \neq 0$. Consequently,

$$\Delta(u(p)) = \tau(\delta(u(p))u(p)^{-1}) = \mathrm{i}s \neq 0\,,$$

and we conclude from 6.4 that the group of invertible elements in $\mathscr{A}$ is not connected. Now in a finite dimensional C^*-algebra the invertible elements form a connected group (because every element has a logaritm), and the same is therefore true in an approximately finite dimensional C^*-algebra. Consequently $\mathscr{A}$ is not approximately finite dimensional.

6.6. Remark. The argument in 6.5 also gives the following generalization of [4, 4.4]: An approximately finite dimensional C^*-algebra $\mathscr{A}$ can not be the cocycle crossed product of a C^*-algebra $\mathscr{B}$ with unit and a discrete abelian group H with elements of infinite order.

Indeed, suppose that $\beta : H \to \mathrm{Aut}(\mathscr{B})$ is an automorphic representation of H and $m : H \times H \to C_u(\hat{\mathscr{B}})$ is a 2-cocycle on H with values in the group of central unitary elements of $\mathscr{B}$ (identified with $C_u(\hat{\mathscr{B}})$ — the circle-valued functions on $\hat{\mathscr{B}}$) such that

$$\mathscr{A} = (H, m) \underset{\beta}{\times} \mathscr{B}$$

(cf. Definition 2.24 of [34]). Denote by $\alpha : \hat{H} \to \mathrm{Aut}(\mathscr{A})$ the dual action of β. Since $\mathscr{B}$ has a unit and H is discrete, $\mathscr{A}$ has a unit. Being approximately finite dimensional it therefore has a finite trace τ, and since $\hat{H}$ is compact we may assume that τ is α-invariant. Now use 6.4 to construct a complex homomorphism on the invertible elements in $\mathscr{A}$ and, as in 6.5, use the unitary eigenoperators $u(h), h \in H$, arising from the embedding of H into $\mathscr{A}$ to show that the homomorphism is non-zero, in contradiction with $\mathscr{A}$ being approximately finite dimensional.

6.7. REMARK. If $(\mathscr{A}, \alpha)$ is a C^*-G-system then the crossed product $G \times_\alpha \mathscr{A}$ is isomorphic with $\mathscr{C}(L^2(G))$ by [18, 2], and if $\hat{\alpha}$ denotes the dual action of $\hat{G}$ on $G \times_\alpha \mathscr{A}$ then by Takai's duality result [24, 7.9.3], we have

$$\mathscr{A} \otimes \mathscr{C}(L^2(G)) = \hat{G} \times_{\hat{\alpha}} G \times_\alpha \mathscr{A} = \hat{G} \times_{\hat{\alpha}} \mathscr{C}(L^2(G)) .$$

Thus $\mathscr{A}$ is stably isomorphic to the crossed product of the compact operators by a discrete group; from which we may again deduce the nuclearity of $\mathscr{A}$.

The primitive ideal space of $\mathscr{A}$ being Hausdorff $(= G/G_{\mathscr{Z}})$, it follows from [22, 3.2] (or by passing to the von Neumann systems) that the Connes spectrum for $\hat{\alpha}$ is $G_{\mathscr{Z}}$.

Choose a unitary eigenoperator $u(p)$ in $\mathscr{A}$ corresponding to p in $\hat{G}$ and define a unitary $v(p)$ in the multiplier algebra of $G \times_\alpha \mathscr{A}$ by $v(p) = u(p)^* \otimes \delta_0$. Then for each x in $C(G, \mathscr{A})$ (considered as a dense subalgebra of $G \times_\alpha \mathscr{A}$) we have

$$(v(p)xv(p)^*)(s) = \iint v(p)(r)\alpha_r(x(t-r))\alpha_t(v(p)^*(s-t))\mathrm{d}r\,\mathrm{d}t =$$

$$= u(p)^*x(s)\alpha_s(u(p)) = \langle s, p\rangle\, u(p)^*x(s)u(p) .$$

Thus if $p \in G_{\mathscr{Z}}^{\perp}$ (i.e. $u(p) \in \mathscr{Z}^c$) we have

$$v(p)xv(p)^*(s) = \hat{\alpha}_p(x)\,(s) ,$$

and the map $p \to v(p)$ is a unitary representation of $G_{\mathscr{Z}}^{\perp}$ into the fixed-points of $\hat{\alpha}$ in $M(G \times_\alpha \mathscr{A})$.

6.8. EXAMPLE. If $G = \Gamma \times \Gamma$ then for any homomorphism $\varphi : \hat{\Gamma} \to \Gamma$ we obtain a symplectic bicharacter $\chi : \hat{G} \to G$ by

$$\chi(p, q) = (-\hat{\varphi}(q), \varphi(p)) , \quad p, q \in \Gamma ,$$

where $\hat{\varphi} : \hat{\Gamma} \to \Gamma$ is the dual map of φ. Since $\hat{\varphi}$ is injective if φ has dense range we see that χ is injective if φ is injective with dense range. The C^*-G-system $(\mathscr{A}, \alpha)$ corresponding to χ can be described in terms of crossed products as follows:

Consider the action β of $C(\Gamma)$ given by

$$\beta_p(x)\,(s) = x(s - \varphi(p)) , \quad x \in C(\Gamma),\ p \in \hat{\Gamma} .$$

The translation action Adλ of Γ on $C(\Gamma)$ commutes with β, hence lifts to an action γ of Γ on the crossed product $\mathscr{A} = \hat{\Gamma} \times_{\beta} C(\Gamma)$. On $\mathscr{A}$ we also have the dual action $\hat{\beta}$ of Γ and it is easy to verify that $\hat{\beta}$ and γ commute and that $\alpha = \hat{\beta} \times \gamma$ is a faithful ergodic action of $G = \Gamma \times \Gamma$ on $\mathscr{A}$. Realizing $\mathscr{A}$ as functions on $\hat{\Gamma}$ with values in $C(\Gamma)$ we define elements

$$u(p, q) = \delta_p \otimes (-q)\,, \ p, q \in \hat{\Gamma}\,.$$

Then for s, t in Γ we compute

$$\alpha_{s,t}(u(p, q)) = \hat{\beta}_s(\delta_p) \otimes \gamma_t(-q) =$$

$$= \langle s, p\rangle\, \delta_p \otimes \langle t, q\rangle\, (-q) = \langle (s, t), (p, q)\rangle\, u(p, q)\,.$$

Thus the function $(p, q) \to u(p, q)$ is a cross-section for $\hat{G}\ (= \hat{\Gamma} \times \hat{\Gamma})$ into $\mathscr{G}_\alpha$. Having verified the equation

$$u(p, q)u(p', q') = \langle \varphi(p), q'\rangle\, u(p + p', q + q')$$

it is immediate to see that the bicharacter associated with $(\mathscr{A}, \alpha)$ is

$$\chi(p, q) = (-\hat{\varphi}(q), \varphi(p))\,, \ (p, q) \in \hat{\Gamma} \times \hat{\Gamma}\,,$$

as claimed.

6.9 EXAMPLE. Taking $G = \mathbf{T}^2$ (i.e. $\Gamma = \mathbf{T}$) in 6.8, the symplectic bicharacters described above are the only ones. Each homomorphism $\varphi : \mathbf{Z} \to \mathbf{T}$ is given by $\varphi(n) = \theta^n$ for some θ in $\mathbf{T}$, and there are thus an uncountable infinity of simple $\mathbf{T}^2$-systems, parametrized by the irrational numbers (modulo 2π) in $\mathbf{R}$.

The simple C^*-algebras arising in this way have been studied by Rieffel [26], [27] who shows that they all contain projections and that there is an uncountable infinity of non-isomorphic examples, in sharp contrast to the von Neumann algebra case.

6.10. EXAMPLE. Every (UHF) Glimm algebra can appear in a simple C^*-G-product for some group G. It suffices to realize that if $\{\mathscr{A}_n, \alpha^n\}$ is a set of (simple) C^*-G_n-systems, then with

$$\mathscr{A} = \otimes \mathscr{A}_n\,, \quad G = \textstyle\prod' G_n, \quad \alpha = \otimes \alpha^n$$

we exhibit $(\mathscr{A}, \alpha)$ as a (simple) G-system. Taking $\mathscr{A}_n = \mathbf{M}_n$, let $G_n = \hat{\mathbf{Z}}_n \times \hat{\mathbf{Z}}_n$, where $\mathbf{Z}_n$ is the cyclic group of order n, and use 5.9 to find α^n making $(\mathbf{M}_n, \alpha^n)$ a G_n-system. Since the Glimm algebras are infinite tensor products of matrix algebras the claim is established. Note that the group G is totally disconnected in accordance with 6.5.

7. ALMOST PERIODIC G-SYSTEMS

7.1. From now on G is only assumed to be a *locally* compact abelian group. We say that a continuous unitary representation $(u, \mathfrak{H})$ of G has *pure point spectrum* f $\mathfrak{H}$ has an orthonormal basis $\{\xi_p | p \in S\}$, where $S \subset \hat{G}$, consisting of eigenvectors, i.e. $u(s)\xi_p = \langle s, p \rangle \xi_p$ for all p in S.

7.2. PROPOSITION. *Let* $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ *be a faithful, ergodic, continuous representation of G as automorphisms of the von Neumann algebra* $\mathcal{M}$. *The following conditions are equivalent:*

(i) $\mathcal{M}$ *is generated by the eigenoperators for* α *and has a (necessarily unique) normal G-invariant state.*

(ii) $\mathcal{M}$ *has a normal G-invariant state* τ *and if* $(\pi_\tau, \lambda^\tau, \mathfrak{H}_\tau)$ *denotes the associated covariant representation of* $(\mathcal{M}, G, \alpha)$ *then* $(\lambda^\tau, \mathfrak{H}_\tau)$ *has pure point spectrum.*

(iii) *There is a compactification* $\iota : G \to \tilde{G}$ *of G and a* $\tilde{G}$*-system* $(\mathcal{M}, \tilde{\alpha})$ *such that* $\tilde{\alpha}_{\iota(s)} = \alpha_s$ *for every s in G.*

(iv) *The commutant* α'_G *of* α_G *in* $\mathrm{Aut}(\mathcal{M})$ *is compact (in any of the two topologies on* $\mathrm{Aut}(\mathcal{M})$ *mentioned in* 2.1).

When these conditions are satisfied then the pure point spectrum $\mathrm{Sp}_d(\alpha)$ *of* $(\lambda^\tau, \mathfrak{H}_\tau)$ *isa dense (discrete) subgroup of* $\hat{G}$ *with* $\tilde{G}$ *as its dual group, and* $\alpha'_G = \alpha_{\tilde{G}}$. *Furthermore* τ *is a trace.*

Proof. The uniqueness of τ is a consequence of the ergodicity of α, see e.g. 24, 7.12.4].

(i) $\Rightarrow$ (ii). Let $\{\xi_p \mid p \in S\}$ be a maximal family of unit eigenvectors for λ^τ corresponding to distinct p's, and note that they form an orthogonal set. If η annihilates all ξ_p then for every eigenoperator x in $\mathcal{M}$ we have $(\eta | \pi_\tau(x)\xi_\tau) = 0$, since $\pi_\tau(x)\xi_\tau$ is an eigenvector. As ξ_τ is cyclic and $\mathcal{M}$ is generated (and, in fact, generated linearly) by its eigenoperators it follows that $\eta = 0$.

(ii) $\Rightarrow$ (iii). Let $\mathrm{Sp}_d(\alpha)$ denote the subgroup of $\hat{G}$ generated by the pure point spectrum and give it the discrete topology. Since α, hence also λ^τ, is faithful, $\mathrm{Sp}_d(\alpha)$ is dense in $\hat{G}$. Thus if $\tilde{G}$ denotes the compact dual of $\mathrm{Sp}_d(\alpha)$ we have a continuous injection $\iota : G \to \tilde{G}$ with dense range, i.e. G is a compactification of G. Define $\tilde{\lambda} : \tilde{G} \to \mathbf{B}(\mathfrak{H}_\tau)$ by $\tilde{\lambda}(\tilde{s})\xi_p = \langle \tilde{s}, p \rangle \xi_p$ for all p in the pure point spectrum and note that since the ξ_p 's form an orthonormal basis for $\mathfrak{H}_\tau$, $\tilde{\lambda}$ is a unitary representation of $\tilde{G}$ extending λ^τ. Since $\iota(G)$ is dense in $\tilde{G}$ and $\tilde{\lambda}$ is strongly continuous we see that

$$\tilde{\lambda}(\tilde{s})\pi_\tau(\mathcal{M})\tilde{\lambda}(-\tilde{s}) = \pi_\tau(\mathcal{M}), \quad \tilde{s} \in \tilde{G} .$$

Thus with $\tilde{\alpha} = \mathrm{Ad}\,\tilde{\lambda}$ we have a $\tilde{G}$-system $(\mathcal{M}, \tilde{\alpha})$ such that $\tilde{\alpha}_{\iota(s)} = \alpha_s$ for all s in G.

(iii) $\Rightarrow$ (i) is obvious from 2.3, since an eigenoperator for $\tilde{\alpha}$ is also an eigenoperator for α. At the same time we see that τ is a trace (2.5) and that the eigenvectors for λ^τ in $\mathfrak{H}_\tau$ are parametrized by the dual group of $\tilde{G}$, so that the pure point spectrum is a subgroup of $\hat{G}$ (i.e. equals $\mathrm{Sp_d}(\alpha)$).

(iii) $\Rightarrow$ (iv). Since $\tilde{G}$ and G are abelian $\tilde{\alpha}_{\tilde{G}} \subset \alpha'_G \subset \tilde{\alpha}'_{\tilde{G}}$. However, by 2.4 $\tilde{\alpha}_{\tilde{G}}$ is maximal abelian in $\mathrm{Aut}(\mathscr{M})$, whence $\alpha'_G = \tilde{\alpha}_{\tilde{G}}$ which is compact.

(iv) $\Rightarrow$ (iii). Let $\tilde{G}$ denote the closure of α_G in α'_G. Then $\tilde{G}$ is a compact abelian group containing G as a dense subgroup, and if $\tilde{\alpha}$ denotes the identity map on $\tilde{G}$ we have a $\tilde{G}$-system $(\mathscr{M}, \tilde{\alpha})$ extending $(\mathscr{M}, \alpha)$.

7.3. We say that a pair $(\mathscr{M}, \alpha)$ satisfying the conditions in 7.2 is an *almos periodic G-system*. We are indebted to T. Hamachi for drawing our attention to condition (iv) in 7.2.

Given an almost periodic G-system $(\mathscr{M}, \alpha)$ we associate to it the group $\mathrm{Sp_d}(\alpha)$ and the symplectic bicharacter χ_α in $X^2(\mathrm{Sp_d}(\alpha), \mathbf{T})$, corresponding to the $\tilde{G}$-system $(\mathscr{M}, \tilde{\alpha})$. Furthermore, we define conjugacy for almost periodic G-systems exactly as for G-systems (2.2).

7.4. THEOREM. *Let $(\mathscr{M}, \alpha)$ and $(\mathscr{N}, \beta)$ be almost periodic G-systems and let* $\mathrm{Sp_d}(\alpha)$, $\mathrm{Sp_d}(\beta)$ *and* χ_α, χ_β *be their associated pure point spectra and symplectic bicharacters. Then $(\mathscr{M}, \alpha)$ and $(\mathscr{N}, \beta)$ are conjugate if and only if*

$$\mathrm{Sp_d}(\alpha) = \mathrm{Sp_d}(\beta) \text{ and } \chi_\alpha = \chi_\beta.$$

Proof. The necessity is trivial. Assume therefore that the conditions are satisfied. By 4.5 the $\tilde{G}$-systems $(\mathscr{M}, \tilde{\alpha})$ and $(\mathscr{N}, \tilde{\beta})$ are conjugate. There is therefore an i somorphism $\Phi : \mathscr{M} \to \mathscr{N}$ such that

$$\Phi \circ \alpha_s = \Phi \circ \tilde{\alpha}_{\iota(s)} = \tilde{\beta}_{\iota(s)} \circ \Phi = \beta_s \circ \Phi$$

for all s in G, whence $(\mathscr{M}, \alpha) \sim (\mathscr{N}, \beta)$.

7.5. The result above is the generalization to non-commutative algebras of the classification theorem by von Neumann on ergodic flows with pure point spectrum. See [21, Sats 5], [13, p. 46] or [28, 2.6]. Clearly, when $\mathscr{M}$ is commutative the bicharacter χ_α is zero (4.6), so that $\mathrm{Sp_d}(\alpha)$ alone is a complete invariant.

7.6. Theorem 7.4 can be used to classify all ergodic almost periodic automorphisms, up to conjugacy, of the hyperfinite II_1 factor $\mathscr{R}$. The invariants are:

(i) A dense subgroup S of $\mathbf{T}$.

(ii) An injective symplectic bicharacter in $X^2(S, \mathbf{T})$.

In the converse direction the result can be used to give examples of such automorphisms. If namely G is an infinite compact monothetic group (i.e. G contains $\mathbf{Z}$ as a dense subgroup or, equivalently, $\hat{G}$ is a dense subgroup of $\mathbf{T}$) and if we can find a simple G-system, then by 5.15 it has the form $(\mathscr{R}, \alpha)$. Now let $\varphi : \mathbf{Z} \to G$ be an injection and let $\sigma = \alpha_{\varphi(1)}$. Then σ is an ergodic almost periodic automorphism of $\mathscr{R}$. The groups considered in 6.9 and 6.10 offer ample opportunities for using this trick.

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DORTE OLESEN and GERT K. PEDERSEN
Matematisk Institut,
Universitetsparken 5,
DK-2100 Copenhagen Ø,
Denmark

MASAMICHI TAKESAKI
Department of Mathematics,
University of California
Los Angeles, California 90024
U.S.A.

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