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A RESULT ON OPERATORS ON $\mathscr{C}$ [0, 1]

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INTRODUCTION

Throughout the text, "subspace" means always "infinite dimensional subspace". For generalities about Banach spaces, we refer to [9]. $\mathscr{C}$ stands for $\mathscr{C}[0, 1]$.

Let (r_i) denote the Rademacker functions on [0, 1]. We recall that a Banach space $\mathscr{X}$ has cotype q $(2 \leqslant q < \infty)$ iff there exists a constant $\beta > 0$ such that

$$\int_0^1 \left\| \sum_{i=1}^m r_i(t)\, x_i \right\| \mathrm{d}t \geqslant \beta \left\{ \sum_{i=1}^m \|x_i\|^q \right\}^{1/q}$$

for all finite sequences $x_1, \ldots, x_m$ of elements of $\mathscr{X}$.

It is known that $\mathscr{X}$ has cotype q for some $q < \infty$ if and only if $\mathscr{X}$ does not contain $l^\infty(n)$ $(n = 1, 2, \ldots)$ uniformly, or in other words, c_0 is not finite dimensionally representable in $\mathscr{X}$ (see [10], for instance).

The following result is due to H. P. Rosenthal ([4] or [14]).

THEOREM 1. *Let $\mathscr{Y}$ be a Banach space and $T : \mathscr{C} \to \mathscr{Y}$ an operator such that $T^*\mathscr{Y}^*$ is not separable. Then there exists a subspace $\mathscr{X}$ of $\mathscr{C}$, $\mathscr{X}$ isometric to $\mathscr{C}$, such that $T|\mathscr{X}$ is an isomorphism.*

If $\mathscr{X}$ is a subspace of $\mathscr{C}$ and $\mathscr{H}$ a subset of $\mathscr{C}^*$, we say that $\mathscr{H}$ is norming for $\mathscr{X}$ provided

$$\sup_{\mu \in \mathscr{H}} \left| \int x \, \mathrm{d}\mu \right| \geqslant \|x\| \quad \text{for all } x \in \mathscr{X} .$$

(identifying $\mathscr{C}^*$ with the Radon measures μ on [0, 1]).

We will prove

THEOREM 2. *If $\mathscr{X}$ is a cotype subspace of $\mathscr{C}$ and $\mathscr{H}$ a w^*-compact subset of $\mathscr{C}^*$ which norms $\mathscr{X}$, then $\mathscr{H}$ is not separable.*

Taking $\mathscr{H} = \{T^*y^*; y^* \in \mathscr{Y}^*, \|y^*\| \leqslant M\}$ with M big enough, we obtain as immediate consequence of Theorem 1 and Theorem 2

COROLLARY 3. *If $\mathscr{Y}$ is a Banach space and $T:\mathscr{C}\to\mathscr{Y}$ an operator fixing a cotype subspace of $\mathscr{C}$, then T fixes a copy of $\mathscr{C}$.*

Applying [14], Corollary 1, we get also

COROLLARY 4. *Any complemented subspace of $\mathscr{C}$ which has a cotype subspace is isomorphic to $\mathscr{C}$.*

One may conjecture that Theorem 2 also holds under the weaker hypothesis that c_0 does not imbed in $\mathscr{X}$.

The rest of the paper is devoted to the proof of Theorem 2. Since $\mathscr{C}$ and $\mathscr{C}(\Delta)$, the continuous functions on the Cantor set, are isomorphic (see [11]), we may replace $\mathscr{C}$ by $\mathscr{C}(\Delta)$.

REDUCTION TO THE CASE OF POSITIVE MEASURES

Denote by $\mathscr{M}(\Delta)$ the space of Radon measures on $\Delta=\{0,1\}^{\mathbb{N}}$. If $\mu\in\mathscr{M}(\Delta)$, then $|\mu|$ is the variation of μ. It is clear that for a norm-separable w^*-compact subset $\mathscr{H}$ of $\mathscr{M}(\Delta)$, the w^*-closure of the set $\{|\mu|;\ \mu\in\mathscr{H}\}$ is not necessarily norm-separable. We will show how to restrict the variation of the measures in order to preserve separability.

We introduce first some notation.

Take $\mathscr{E}_0=\{\Delta\}$ and $\mathscr{E}_n=\{I_c;\ c\in\{0,1\}^n\}$ for $n\geqslant 1$, where $I_c=\{t\in\Delta;\ t_i=c_i$ if $i=1,\ldots,n\}$.Let $\mathscr{E}^n=\bigcup_{m\geqslant n}\mathscr{E}_m$ and denote $\mathscr{E}=\mathscr{E}^0$.

For $\mathscr{F}\subset\mathscr{E}$, $\cup\mathscr{F}$ denotes the union of the elements of $\mathscr{F}$. We remark that if $\mathscr{F}\subset\mathscr{E}$, it is possible to find some $\mathscr{G}\subset\mathscr{F}$ such that the members of $\mathscr{G}$ are mutually disjoint and $\cup\mathscr{F}=\cup\mathscr{G}$.

Let now $\mu\in\mathscr{M}(\Delta)$ and fix $n=0,1,2,\ldots$ and $\varepsilon>0$. Define

$$\mathscr{F}_{n,\varepsilon}(\mu)=\{I\in\mathscr{E}^n;\ |\mu(I)|<\varepsilon|\mu|(I)\}$$

and

$$G_{n,\varepsilon}(\mu)=\cup\ \mathscr{F}_{n,\varepsilon}(\mu),$$

which is an open subset of Δ.Finally, take $\mu[n,\varepsilon]$ the restriction $(\Delta\setminus G_{n,\varepsilon}(\mu))|\mu|$ of $|\mu|$ to $\Delta\setminus G_{n,\varepsilon}(\mu)$.

If S is a subset of Δ and f a function on Δ, $\mathrm{o}(f|S)$ means the oscillation of f on S.

LEMMA 1. 1. *$G_{n,\varepsilon}(\mu)$ decreases if n increases and ε decreases.*

2. *Let $f\in\mathscr{C}(\Delta)$, $\|f\|\leqslant 1$ be such that $\mathrm{o}(f|I)\leqslant\iota$ for all $I\in\mathscr{E}^n$.*

Then $\int_{G_{n,\varepsilon}(\mu)} f\,\mathrm{d}|\mu|\leqslant(\varepsilon+\iota)\|\mu\|$.

Proof. 1. Trivial.

2. Take a subfamily (I_i) of $\mathscr{F}_{n,\varepsilon}(\mu)$ whose members are mutually disjoint so that $G=G_{n,\varepsilon}(\mu)=\bigcup_i I_i$. Since $\mathrm{o}(f|I_i)\leqslant\iota$ for each i, there exists a function

$g = \sum_i \alpha_i \chi_{I_i}$ $(-1 \leqslant \alpha_i \leqslant 1)$ with $|f(t) - g(t)| \leqslant \iota$ if $t \in G$. We have

$$\left|\int_G f \, d\mu\right| \leqslant \left|\int_G g \, d\mu\right| + \iota\|\mu\| \leqslant \sum_i |\alpha_i| \, |\mu(I_i)| + \iota\|\mu\| \leqslant$$

$$\leqslant \varepsilon \sum_i |\alpha_i| \, |\mu|(I_i) + \iota\|\mu\| \leqslant (\varepsilon + \iota) \, \|\mu\| ,$$

as required.

If $\mathscr{H}$ is a subset of $\mathscr{M}(\Delta)$, n a positive integer and $\varepsilon > 0$, we define

$$\mathscr{H}_{n,\varepsilon} = \{\nu \in \mathscr{M}_+(\Delta) ; \ \nu \leqslant \mu[n, \varepsilon] \text{ for some } \mu \in \mathscr{H}\}.$$

LEMMA 2. *If $\mathscr{H}$ is w^*-compact, then the w^*-closure $\overline{\mathscr{H}}^*_{n,\varepsilon}$ of $\mathscr{H}_{n,\varepsilon}$ is contained in the set $\frac{1}{\varepsilon} \mathscr{H}_{n,\varepsilon}$.*

This result has the following immediate consequence

COROLLARY 3. *For a w^*-compact norm-separable set $\mathscr{H}$ in $\mathscr{M}(\Delta)$, the sets $\overline{\mathscr{H}}^*_{n,\varepsilon}$ are norm-separable.*

Proof of Lemma 2. We have to show that if (μ_k) is a sequence in $\mathscr{H}$ and ν a w^*-limit of the sequence $(\nu_k = \mu_k[n, \varepsilon])$, then $\nu \in \frac{1}{\varepsilon} \mathscr{H}_{n,\varepsilon}$.

For each k, the set $F_k = \Delta \setminus G_{n,\varepsilon}(\mu_k)$ is closed in Δ. Passing eventually to subsequences, we can find $\mu \in \mathscr{H}$, $\eta \in \mathscr{M}_+(\Delta)$ and a closed subset F of Δ, such that $\mu = \lim \mu_k(\omega^*)$, $\eta = \lim |\mu_k|(\omega^*)$ and $F = \lim F_k$ (for the Hausdorff-topology).

We show first that $|\mu(I)| \geqslant \varepsilon \, \eta(I)$ if $I \in \mathscr{S}^n$ and $I \cap F \neq \emptyset$. Indeed, there is some k_0 with $I \cap F_k \neq \emptyset$ for $k \geqslant k_0$. But then $I \notin \mathscr{F}_{n,\varepsilon}(\mu_k)$ or $|\mu_k(I)| \geqslant \varepsilon |\mu_k|(I)$. For $k \to \infty$, the result follows.

We show that $\nu \leqslant F \cdot \eta$. Fix a neighborhood O of F. There is some k_0 with $F_k \subset O$ for $k \geqslant k_0$. Then $\nu_k = F_k |\mu_k| \leqslant O \cdot |\mu_k|$. By passing to the limit, we get $\nu \leqslant \overline{O} \cdot \eta$. Thus also $\nu \leqslant F \cdot \eta$.

The next point is that $F \cdot \eta \leqslant \frac{1}{\varepsilon} F \cdot |\mu|$. To show this, we prove that if K is a closed subset of F and O a neighborhood of K, then $\eta(K) \leqslant \frac{1}{\varepsilon} |\mu|(O)$. Since $\mathscr{S}^n$ is a basis for the topology, there exists for each $t \in K$ some $I_t \in \mathscr{S}^n$ with $t \in I_t \subset O$ Let $\mathfrak{J}$ be a subfamily of $(I_t)_{t \in K}$ of mutually disjoint sets with same union. Then

$$\eta(K) \leqslant \sum_{I \in \mathfrak{J}} \eta(I) \leqslant \varepsilon^{-1} \sum_{I \in \mathfrak{J}} |\mu(I)| \leqslant \varepsilon^{-1} |\mu|(\bigcup \mathfrak{J}) \leqslant \varepsilon^{-1} |\mu|(O)$$

using the fact that $I \cap F \neq \emptyset$ if $I \in \mathfrak{J}$.

Consequently $\nu \leqslant \frac{1}{\varepsilon} F \cdot |\mu|$. It is also clear that $|\mu| \leqslant \eta$, since $|\mu(\varphi)| = \lim |\mu_k(\varphi)| \leqslant \leqslant \varliminf |\mu_k|(|\varphi|) = \eta(|\varphi|)$ for $\varphi \in \mathscr{C}(\Delta)$. Thus $|\mu(I)| \geqslant \varepsilon|\mu|(I)$ if $I \in \mathscr{S}^n$ and $I \cap F \neq \neq \emptyset$. It follows that $F \cap G_{n,\varepsilon}(\mu) = \emptyset$ and hence $\nu \leqslant \frac{1}{\varepsilon} \mu[n, \varepsilon]$. Thus ν belongs to $\frac{1}{\varepsilon} \mathscr{H}_{n,\varepsilon}$, completing the proof.

LEMMA 4. *If $\mathscr{X}$ is a subspace of $\mathscr{C}(\Delta)$ which does not contain c_0 isomorphically and if $\mathscr{H}$ is a bounded subset of $\mathscr{M}(\Delta)$ norming $\mathscr{X}$, then $\mathscr{X}$ will also be normed by a multiple of $\mathscr{H}_{n,\varepsilon}$, for some n and ε.*

Proof (ex absurdo). Suppose $\mathscr{H}$ uniformely bounded by $B < \infty$ and take $\varepsilon_i = 2^{-i}$ for each i. By induction, we construct an increasing sequence n_i of integers and a sequence x_i of vectors in $\mathscr{X}$, satisfying

1. $\|x_i\| = 1$
2. $\int |x_i| \mathrm{d}\nu < \varepsilon_i$, for all $\nu \in \mathscr{H}_{n_i, \varepsilon_i}$
3. $\mathrm{o}(x_i|I) < \varepsilon_{i+1}$, for all $I \in \mathscr{S}^{n_{i+1}}$.

Since x_i is a continuous function, there exists indeed some $n_{i+1} > n_i$ such that (3) is satisfied. Now, by hypothesis, there is some $x_{i+1} \in \mathscr{X}$, $\|x_{i+1}\| = 1$ so that $\left|\int x_{i+1} \mathrm{d}\nu\right| < \frac{1}{2} \varepsilon_{i+1}$ if $\nu \in \mathscr{H}_{n_{i+1}, \varepsilon_{i+1}}$. It follows from the definition of the $\mathscr{H}_{n,\varepsilon}$ that therefore the second condition is fulfilled.

We will show that $\sum_i \left|\int x_i \,\mathrm{d}\mu\right| \leqslant 1 + 2B$ for each $\mu \in \mathscr{H}$. Since $\mathscr{H}$ is norming, this implies that the sums of finitely many elements in the sequence (x_i) are uniformly bounded. But therefore (x_i) must have a subsequence equivalent to the c_0-basis, a contradiction.

Let thus $\mu \in \mathscr{H}$ be fixed and let $G_i = G_{n_i, \varepsilon_i}(\mu)$, which gives a decreasing sequence of open sets. For each i, we have

$$\left|\int x_i \mathrm{d}\mu\right| \leqslant \int_{\Delta \setminus G_i} |x_i| \,\mathrm{d}|\mu| + \int_{G_i \setminus G_{i+1}} |x_i| \,\mathrm{d}|\mu| + \left|\int_{G_{i+1}} x_i \mathrm{d}\mu\right| \cdot$$

It remains to estimate these integrals;

$$\int_{\Delta \setminus G_i} |x_i| \,\mathrm{d}|\mu| < \varepsilon_i,$$

since $(\Delta \setminus G_i)|\mu| = \mu[n_i, \varepsilon_i]$ is in $\mathscr{H}_{n_i, \varepsilon_i}$;

$$\int_{G_i \setminus G_{i+1}} |x_i| \,\mathrm{d}|\mu| \leqslant |\mu|(G_i \setminus G_{i+1}) \,.$$

Now, (3) allows us to apply Lemma 1, taking $n = n_{i+1}$ and $\varepsilon = \iota = \varepsilon_{i+1}$. Hence

$$\left|\int_{G_{i+1}} x_i \, d\mu\right| \leqslant (\varepsilon_{i+1} + \varepsilon_{i+1})\|\mu\| \leqslant \varepsilon_i B .$$

By summing, we get the bound

$$\sum_i \varepsilon_i + \sum_i |\mu|(G_i \setminus G_{i+1}) + B \sum_i \varepsilon_i \leqslant 1 + 2B.$$

This proves Lemma 4.

PROOF OF THEOREM 2.

Assume $\mathscr{X}$ a cotype subspace of $\mathscr{C}$ normed by a w^*-compact subset $\mathscr{H}$ of $\mathscr{C}^*$. If $\mathscr{X}^*$ is not separable, then $\mathscr{H}$ will obviously be non-separable. If now $\mathscr{X}^*$ is separable, then $\mathscr{X}$ contains a shrinking normalised basic sequence (e_r) (see [9]). Assume $\mathscr{H}$ separable. Since c_0 is not a subspace of $\mathscr{X}$, it follows from Lemma 2 and Lemma 4 that $\mathscr{X}$ is also normed by a w^*-compact norm-separablet et $\mathscr{P}$ of positive measures. Hence

$$\sup_{\mu \in \mathscr{P}} \int |x| \, d\mu \geqslant \|x\| \qquad \text{for all } x \in \mathscr{X} .$$

Take $f_r = |e_r|$ for each r.

LEMMA 5. 1. *If g is a positive linear combination of the f_r, then* $\sup_{\mu \in \mathscr{P}} \int g \, d\mu \geqslant \geqslant \|g\|$.

2. *Take q the cotype of $\mathscr{X}$. If $g_1, \ldots, g_m$ are positive linear combinations of the f_r, then*

$$\left\|\sum_{i=1}^{m} g_i\right\| \geqslant \beta \left(\sum_{i=1}^{m} \|g_i\|^q\right)^{\frac{1}{q}} .$$

3. *Any normalized bloc-subsequence of positive linear combinations of the f_r is weakly null.*

Proof. It is clear that if $g = \sum_r' a_r f_r$ and $t \in \Delta$, it is possible to find a sequence $\varepsilon = (\varepsilon_r)\,(\varepsilon_r = \pm 1)$ so that $g(t) = \sum_r \varepsilon_r a_r e_r(t)$. Particularly $\|g\| \leqslant \sup_\varepsilon \|\sum_r \varepsilon_r a_r e_r\|$ and we have equality if the a_r are positive.

1. Let $\varepsilon = (\varepsilon_r)$ be such that $\|g\| = \|x\|$, where $x = \sum_r \varepsilon_r a_r e_r$. We have

$$\sup_{\mu \in \mathscr{P}} \int g \, d\mu \geqslant \sup_{\mu \in \mathscr{P}} \int |x| \, d\mu \geqslant \|x\| = \|g\|.$$

2. Let $g_i = \sum_r{}' a_r^i f_r$ $(a_r^i \geqslant 0)$ and take $\varepsilon_i = (\varepsilon_r^i)$ such that $\|g_i\| = \|x_i\|$, where $x_i = \sum_i{}' \varepsilon_r^i a_r^i e_r$. For each $t \in [0, 1]$ we get

$$\left\| \sum_{i=1}^m r_i(t)\, x_i \right\| \leqslant \left\| \sum_{i=1}^m |x_i| \right\| \leqslant \left\| \sum_{i=1}^m g_i \right\|$$

implying that

$$\left\| \sum_{i=1}^m g_i \right\| \geqslant \int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\| \mathrm{d}t \geqslant$$

$$\geqslant \beta \left(\sum_{i=1}^m \|x_i\|^q \right)^{1/q} =$$

$$= \beta \left(\sum_{i=1}^m \|g_i\|^q \right)^{1/q} .$$

3. Let $g_i = \sum_r{}' a_r^i f_r$ $(a_r^i \geqslant 0)$ be a bloc-subsequence of (f_r) such that $\|g_i\| \leqslant 1$. We only have to show that $\lim_{i\to\infty} g_i(t) = 0$ for all $t \in \Delta$. Fixing $t \in \Delta$, we can find a sequence (ε_r) such that $f_r(t) = \varepsilon_r e_r(t)$ and hence $g_i(t) = x_i(t)$ where $x_i = \sum_r{}' a_r^i \varepsilon_r e_r$. Now, since (x_i) is a bloc-subsequence of (e_r) and (e_r) is shrinking, we have that $\lim_{i\to\infty} x_i(t) = 0$.

LEMMA 6. *Assume $g_1, \ldots, g_m$ positive linear combinations of the f_r. Let (Ω, μ) be a probability space, $\varphi_1, \ldots, \varphi_m$ in $L^\infty(\Omega, \mu)$ and $\delta > 0$ such that $0 \leqslant \varphi_i \leqslant 1$ and $\int \varphi_i \mathrm{d}\mu \leqslant \delta$ for all $i = 1, \ldots, m$. Then*

$$\int \left\| \sum_{i=1}^m \varphi_i(t)\, g_i \right\| \mathrm{d}\mu \leqslant K\, \delta^{1/q} \left\| \sum_{i=1}^m g_i \right\|$$

where we may take for $K = 2/\beta$.

Proof. Denote (Π, ν) the product measure space $(\Omega^{\mathrm{N}}, \otimes_{\mathrm{N}} \mu)$. Fix $i = 1, \ldots, m$. We introduce a sequence (φ_i^n) of functions on Π, taking

$$\varphi_i^1(u) = \varphi_i(t_1)$$

$$\varphi_i^{n+1}(u) = (1 - \varphi_i(t_1))(1 - \varphi_i(t_2)) \ldots (1 - \varphi_i(t_n))\varphi_i(t_{n+1})$$

where $u = (t_n)$ is the product variable. It is clear that $\varphi_i^n \geqslant 0$ and it easy to see that $\sum_n \varphi_i^n \leqslant 1$. Denote

$$\xi = \int \left\| \sum_{i=1}^m \varphi_i(t)\, g_i \right\| \mu(\mathrm{d}t) .$$

Then

$$\int \|\sum_i \varphi_i^1(u)\, g_i\| \quad \nu(\mathrm{d}u) = \xi$$

and

$$\int \|\sum_i \varphi_i^{n+1}(u)\, g_i\| \ \nu(\mathrm{d}u) =$$

$$= \int \left\| \sum_i \left[\prod_{k=1}^{n} (1 - \varphi_i(t_k)) \right] \varphi_i(t) g_i \right\| \mu(\mathrm{d}t_1) \ldots \mu(\mathrm{d}t_n)\, \mu(\mathrm{d}t) \geqslant$$

$$\geqslant \int \left\| \sum_i \left[\int \prod_{k=1}^{n} (1 - \varphi_i(t_k))\, \mu(\mathrm{d}t_1) \ldots \mu(\mathrm{d}t_n) \right] \varphi_i(t)\, g_i \right\| \mu(\mathrm{d}t) =$$

$$= \int \left\| \sum_i \left[1 - \int \varphi_i \,\mathrm{d}\mu \right]^n \varphi_i(t)\, g_i \right\| \mu(\mathrm{d}t) \geqslant (1 - \delta)^n \xi .$$

Hence

$$\int \|\sum_i \varphi_i^n(u)\, g_i\|^q \ \nu(\mathrm{d}u) \geqslant (1 - \delta)^{q(n-1)}\, \xi^q .$$

Applying then Lemma 5.2 for each $u \in \Pi$, we obtain

$$\|\sum_i g_i\|^q \geqslant \|\sum_i (\sum_n \varphi_i^n (u))\, g_i\,\|^q =$$

$$= \|\sum_n (\sum_i \varphi_i^n(u)\, g_i)\|^q \geqslant \beta^q \sum_n \|\sum_i \varphi_i^n(u)\, g_i\|^q .$$

Integration yields us

$$\left\| \sum_{i=1}^{m} g_i \right\|^q \geqslant \beta^q \xi^q \sum_n (1 - \delta)^{q(n-1)} = \frac{\beta^q \xi^q}{1 - (1 - \delta)^q}$$

and thus

$$\xi \leqslant \frac{[1 - (1 - \delta)^q]^{1/q}}{\beta} \|\sum_i g_i\| .$$

This is the required result, since $1 - (1 - \delta)^q \leqslant 2^q \delta$.

If (a_r) and (b_r) are sequences of real numbers, we write $(a_r) \leqslant (b_r)$ provided $a_r \leqslant b_r$ for each r. Denote $\underline{0}$ the 0-sequence and $\underline{1}$ the 1-sequence. For convenience, we introduce the following definition.

DEFINITION. Assume $\mathcal{Q}$ a set of positive measures, (a_r) a sequence of positive reals and $\rho, \iota > 0$. We say that $(\mathcal{Q}, (a_r), \rho, \iota)$ has property $(*)$ iff

1. $\sum_r a_r f_r$ is not convergent (which means that the partial sums are not bounded).

2. Let $0 \leqslant (b_r) \leqslant (a_r)$ such that $\sum_r b_r f_r$ is not convergent. Then there are positive sequences $(c_r) \leqslant (b_r)$ and (λ_r), finitely supported and satisfying the following conditions

i. $\|\sum_r c_r f_r\| \leqslant 1$

ii. $\sum_r \lambda_r \leqslant 1$

iii. $\sum_r \lambda_r \mu[c_r f_r \geqslant \iota\lambda_r] \geqslant \rho$ for some $\mu \in \mathcal{Q}$.

LEMMA 7. *There exists $\rho, \iota > 0$ such that $(\mathcal{P}, \underline{1}, \rho, \iota)$ has* $(*)$.

Proof. Assume $\mathcal{P}$ uniformely bounded by $B < \infty$. Take $\iota = \frac{1}{2}$, $M = 8^q K^q B^q$ and $\rho = \frac{1}{16\,BM}$, where K is as in Lemma 6. It follows from Lemma 5.2 that if $\sum_r b_r f_r$ does not converge there is a positive sequence $(c_r) \leqslant (b_r)$ so that $\|\sum_r c_r f_r\| = 1$. Let $\mu \in \mathcal{P}$ satisfy $\int (\sum_r c_r f_r)\, d\mu > \frac{1}{2}$ and take $\nu = \frac{\mu}{\|\mu\|}$. We obtain that

$$\sum_r c_r f_r = \sum_r c_r f_r \chi_{[M\nu(f_r) \geqslant f_r \geqslant \iota\nu(f_r)]} + \sum_r c_r f_r \chi_{[f_r < \iota\nu(f_r)]} + \sum_r c_r f_r \chi_{[f_r > M\nu(f_r)]} \leqslant$$

$$\leqslant M \sum_r c_r \nu(f_r)\, \chi_{[f_r > \iota\nu(f_r)]} + \iota \sum_r c_r\, \nu(f_r) + \sum_r c_r f_r\, \chi_{[f_r > M\nu(f_r)]}$$

and

$$\sum_r c_r f_r(t)\, \chi_{[f_r > M\nu(f_r)]}(t) \leqslant \|\sum_r c_r \varphi_r(t) f_r\|$$

where

$$\varphi_r = \chi_{[f_r > M\nu(f_r)]}\,.$$

Integration gives

$$(1 - \iota) \sum_r c_r \nu(f_r) \leqslant M \sum_r c_r\, \nu(f_r)\, \nu[f_r \geqslant \iota\nu(f_r)] + \int \|\sum_r c_r \varphi_r(t) f_r\|\, \nu(dt)\,.$$

If we take $\lambda_r = c_r\, \nu(f_r)$, then clearly $\lambda_r \geqslant 0$ and $\frac{1}{2B} \leqslant \sum_r \lambda_r \leqslant 1$.

On the other hand

$$\int \varphi_r \mathrm{d}\nu = \nu[f_r > M\nu(f_r)] \leqslant \frac{1}{M}$$

and Lemma 6 gives

$$\int \|\sum_r c_r \varphi_r(t) f_r\| \ \nu(\mathrm{d}t) \leqslant \frac{K}{M^{1/q}} \|\sum_r c_r f_r\| = \frac{K}{M^{1/q}}$$

Thus

$$M \sum_r \lambda_r \ \nu[c_r f_r \geqslant \iota\, \lambda_r] \geqslant \frac{1}{4B} - \frac{K}{M^{1/q}} = \frac{1}{8B}$$

and

$$\sum_r \lambda_r \mu[c_r f_r \geqslant \iota\, \lambda_r] \geqslant \frac{\|\mu\|}{8BM} > \frac{1}{16\, BM}$$

as required.

LEMMA 8. *Let (c_r) and (λ_r) be positive finitely supported sequences such that $\sum_r \lambda_r \leqslant 1$, let μ be a positive measure and $\iota, \varepsilon > 0$. Then*

$$\sum_r \lambda_r \mu[c_r f_r \geqslant \iota\, \lambda_r] \leqslant \mu[\sum_r c_r f_r \geqslant \iota\varepsilon] + \varepsilon\|\mu\|$$

holds.

Proof. Let $\Phi = \sum_r \lambda_r \chi_{[c_r f_r \geqslant \iota\lambda_r]}$. Obviously $\sum_r c_r f_r \geqslant \iota\, \Phi$. Since $\Phi \leqslant 1$, we get by integration

$$\sum_r \lambda_r \mu[c_r f_r \geqslant \iota\lambda_r] \leqslant \mu[\Phi \geqslant \varepsilon] + \varepsilon\|\mu\| \leqslant$$

$$\leqslant \mu[\sum_r c_r f_r \geqslant \iota\varepsilon] + \varepsilon\|\mu\|.$$

If $\mathscr{Q}$ is a set of positive measures, F a closed subset of Δ and $\delta > 0$, define $\mathscr{Q}_{F,\delta} = \{\mu \in \mathscr{Q}; \mu(F) \geqslant \delta\}$.

LEMMA 9. *Suppose $(\mathscr{Q}, (a_r), \rho, \iota)$ has $(*)$, $\mathscr{Q}$ bounded by $1 < B < \infty$ and let $0 < \varkappa < \frac{\rho}{4}$. Then there exist a sequence of positive reals $(b_r) \leqslant (a_r)$ and a function f in span $(f_r; a_r > 0)$, $\|f\| = 1$ such that $(\mathscr{Q}_{F,\delta}, (b_r), \rho - \varkappa, \iota')$ still has $(*)$, where $F = \left[f \geqslant \frac{\iota'\rho}{8B} \right]$, $\delta = \frac{\rho}{2}$ and $\iota' = \frac{\iota\varkappa}{3B}$.*

Proof (ex absurdo). Otherwise, we can construct by induction on i sequences (a_r^i), successive blocs E_i and finite sequences $(b_r)_{r\in E_i}$, so that the following conditions are satisfied

1. $0 \leqslant (a_r^i) \leqslant (a_r)$
2. $\sum_r a_r^i f_r$ diverges
3. $(a_r^{i+1}) \leqslant (a_r^i)$
4. $a_r > 0$ and $a_r^{i+1} = 0$ if $r \in E_i$
5. $0 \leqslant (b_r^i) \leqslant (a_r^i)$
6. $\|\sum_r b_r^i f_r\| = 1$
7. If $f \in \operatorname{span}\,(f_r\,;\, r \in \bigcup_{j=1}^{i-1} E_j)$, $\|f\| \leqslant 1$ and $\mu \in \mathcal{Q}$ such that $\mu\left[f \geqslant \frac{\iota'\rho}{4B}\right] \geqslant \delta$,

then $\sum_r \lambda_r \mu[c_r f_r \leqslant \iota'\lambda_r] < \rho - \varkappa$ for all positive finite sequences (c_r) and (λ_r) for which

$$(c_r) \leqslant (a_r^i), \qquad \|\sum_r c_r f_r\| \leqslant 1 \quad \text{and} \quad \sum_r \lambda_r \leqslant 1\,.$$

The construction is straightforward. In order to realize (7), we consider a finite $\frac{\iota'\rho}{8B}$ - net in the unit ball of $\operatorname{span}\left(f_r\,;\, r \in \bigcup_{j=1}^{i-1} E_j\right)$. The sequence (a_r^i) is then obtained by successive negations of (*) for the different members of this finite net. Consider now the sequence (b_r) defined by

$$b_r = \gamma b_r^i \qquad \text{if } r \in E_i$$

$$b_r = 0 \qquad \text{if } r \notin \bigcup_i E_i$$

where $\gamma = \frac{\varkappa\iota}{3B}$. Since $\sum_r b_r f_r$ does not converge and $(\mathcal{Q}, (a_r), \rho, \iota)$ has (*), there are positive finite sequences (c_r) and (λ_r) and $\mu \in \mathcal{Q}$ so that

$$(c_r) \leqslant (b_r), \qquad \|\sum_r c_r f_r\| \leqslant 1, \qquad \sum_r \lambda_r \leqslant 1$$

and

$$\sum_r \lambda_r \mu[c_r f_r \geqslant \iota\,\lambda_r] \geqslant \rho.$$

Let us first remark that for all i

$$\sum_{r\in E_i} \lambda_r \mu[c_r\, f_r \geqslant \iota\,\lambda_r] \leqslant \frac{\varkappa}{3}\cdot$$

Indeed, taking $\varepsilon > \frac{\gamma}{\iota}$ and applying Lemma 8, we get

$$\sum_{r \in E_i} \lambda_r \mu[c_r f_r \geqslant \iota\, \lambda_r] \leqslant \mu[\sum_{r \in E_i} c_r f_r \geqslant \iota\varepsilon] + \varepsilon\|\mu\| .$$

But since

$$\|\sum_{r \in E_i} c_r f_r\| \leqslant \|\sum_{r \in E_i} b_r f_r\| = \gamma \|\sum_{r \in E_i} b_r^i f_r\| = \gamma ,$$

we have

$$[\sum_{r \in E_i} c_r f_r \geqslant \iota\varepsilon] = \emptyset$$

and hence

$$\sum_{r \in E_i} \lambda_r \mu[c_r f_r \geqslant \iota\, \lambda_r] \leqslant \varepsilon B ,$$

as we claimed.

Therefore it is possible to find increasing integers $1 = i_0 < i_1 < \ldots < i_d$ such that

8. $\lambda_r = 0$ if $r \notin \bigcup_{i < i_d} E_i$

9. $\frac{\varkappa}{3} \leqslant \sum_{r \in F_e} \lambda_r \mu[c_r f_r \geqslant \iota\, \lambda_r] \leqslant \varkappa$

if for all $e = 1, \ldots, d$ we let $F_e = \bigcup_{i=i_{e-1}}^{i_e - 1} E_i$.

Particularly, we have $\sum_{r \in F_e} \lambda_r \geqslant \frac{\varkappa}{3B}$.

Introduce now the sequence (λ_r'), taking

$$\lambda_r' = \frac{\lambda_r}{\sum_{r \in F_e} \lambda_r} \qquad \text{if } r \in F_e \qquad \text{and}$$

$$\lambda_r' = 0 \qquad \text{if } r \notin \bigcup_{e=1}^{d} F_e .$$

For all $r \in E_i \subset F_e$ we have $c_r \leqslant b_r \leqslant b_r^i \leqslant a_r^i \leqslant a_r^{i_{e-1}}$, applying (3) and (5). We claim that the set

$$Z = \{e = 1, \ldots, d;\ \sum_{r \in F_e} \lambda_r' \mu[c_r f_r \geqslant \iota' \lambda_r'] \geqslant \rho - \varkappa\}$$

does not contain two distinct values $e < e'$. Assuming the inequality true for e and applying Lemma 8 with $\varepsilon = \frac{\rho}{4B}$ we find

$$\rho - \varkappa \leqslant \mu\left[\sum_{r \in F_e} c_r f_r \geqslant \frac{\iota' \rho}{4B}\right] + \frac{\rho\|\mu\|}{4B}$$

and hence $\mu\left[f \geqslant \frac{\imath'\rho}{4B}\right] \geqslant \frac{\rho}{2}$, if $f = \sum_{r \in F_e} c_r f_r$. Moreover $f \in \operatorname{span}\left(f_r; r \in \bigcup_{j=1}^{i_a-1} E_j\right)$ and $\|f\| \leqslant 1$. Since

$$(c_r)_{r \in F_{e'}} \leqslant (a_r^{i_{e'}-1}) \leqslant (a_r^{i_e}),$$
$$\|\sum_{r \in F_{e'}} c_r f_r\| \leqslant 1$$

and

$$\sum_{r \in F_{e'}} \lambda'_r = 1,$$

we deduce from (7) that

$$\sum_{r \in F_{e'}} \lambda'_r \mu[c_r f_r \geqslant \imath' \lambda'_r] < \rho - \varkappa,$$

which proves the claim. So finally we obtain

$$\sum_r \lambda_r \mu[c_r f_r \geqslant \imath\, \lambda_r] = \sum_e \sum_{r \in F_e} \lambda_r \mu[c_r f_r \geqslant \imath\, \lambda_r] \leqslant$$
$$\leqslant \varkappa + \sum_{e \notin Z} \Big(\sum_{r \in F_e} \lambda_r \Big) \sum_{r \in F_e} \lambda'_r \mu\left[c_r f_r \geqslant \frac{\imath\varkappa}{3B}\ ; \right] <$$
$$< \varkappa + (\rho - \varkappa) \sum_{e \notin Z} \Big(\sum_{r \in F_e} \lambda_r \Big),$$

a contradiction.

LEMMA 10. *Assume that $(\mathscr{Q}, (a_r), \rho, \imath)$ has $(*)$. Then there exist for each, $\varkappa > 0$ a sequence (φ_k) in $\mathscr{C}_+(\Delta)$, sequences $(b_r^k)_r$ of positive reals and a sequence $(\imath_k)$, such that*

1. $\|\varphi_k\| \leqslant 1$ *and* $\lim \varphi_k = 0$ *weakly*
2. $(\mathscr{Q}_k, (b_r^k), \rho - \varkappa, \imath_k)$ *has* $(*)$, *where* $\mathscr{Q}_k = \left\{\mu \in \mathscr{Q};\ \mu(\varphi_k) \geqslant \frac{\rho}{2}\right\}$.

Proof. Assume Q bounded by $1 < B < \infty$. Since $\sum_r a_r f_r$ does not converge, it is possible to find sequences $(a_r^k)_r$ such that

i. $0 \leqslant (a_r^k) \leqslant (a_r)$

ii. $\sum_r a_r^k f_r$ is not convergent

iii. The supports of the (a_r^k) are disjoint.

Fix k. Since $(\mathscr{Q}, (a_r^k), \rho, \imath)$ obviously satisfies $(*)$, Lemma 9 can be applied and yields us a positive sequence $(b_r^k) \leqslant (a_r^k)$ and a function g_k in span $(f_r\ ;\ a_r^k > 0)$, $\|g_k\| \leqslant 1$ such that $(\mathscr{Q}_{F_k,\delta}, (b_r^k), \rho - \varkappa, \imath_k)$ has $(*)$, where

$$F_k = \left[g_k \geqslant \frac{\imath_k \rho}{8B}\right], \quad \delta = \frac{\rho}{2} \quad \text{and} \quad \imath_k = \frac{\imath\varkappa}{3B}.$$

The g_k have disjoint supports and therefore we may assume (passing eventually to a subsequence) that (g_k) is a blocksubsequence of (f_r) and hence, by Lemma 5.3, is weakly null. Consider functions $\varphi_k \in \mathscr{C}_+(\Delta)$, $\|\varphi_k\| = 1$ such that

$$F_k \subset [\varphi_k = 1] \subset [\varphi_k \neq 0] \subset \left[g_k > \frac{\imath_k \rho}{9B} \right].$$

Since $\imath_k$ does not depend on k, it is clear that $\lim_{k\to\infty} \varphi_k = 0$ weakly. Finally

$$\mathscr{Q}_{F_{k,\delta}} \subset \mathscr{Q}_k, \quad \mu(\varphi_k) \geqslant \mu(F_k).$$

In order to establish non-separability, we need the notion of Szlenk-index. We explain this here briefly and refer the reader to [4], [6], [16] for a more complete treatment.

Fix $\varepsilon > 0$. For a given subset Q of $\mathscr{C}^*$, we define

$\delta(\mathscr{Q}) = \{\mu \in \mathscr{Q}$; there exist a sequence (φ_k) in the unit-ball of $\mathscr{C}(\Delta)$ and a sequence (μ_k) in $\mathscr{Q}$ such that $\lim \varphi_k = 0$ weakly, $\lim \mu_k = \mu$ weak* and $\mu_k(\varphi_k) \geqslant \varepsilon$ for each $k\}$.

This allows us to introduce a transfinite sequence $P_\alpha(\varepsilon, \mathscr{Q}) = \mathscr{Q}_\alpha$ of subsets of $\mathscr{C}^*$, taking

$$\mathscr{Q}_0 = \mathscr{Q}$$
$$\mathscr{Q}_{\alpha+1} = \delta(\mathscr{Q}_\alpha)$$
$$\mathscr{Q}_\gamma = \bigcap_{\alpha<\gamma} \mathscr{Q}_\alpha \quad \text{if } \gamma \text{ is a limit ordinal.}$$

It can be shown that $\mathscr{Q}$ is norm-separable iff $\mathscr{Q}_\alpha = \emptyset$ for some $\alpha < \omega_1$, assuming $\mathscr{Q}$ weak*-compact. The ordinal $\eta_\varepsilon(\mathscr{Q}) = \min\{\alpha < \omega_1; \mathscr{Q}_\alpha = \emptyset\}$ will then be the ε-Szlenk index of $\mathscr{Q}$.

LEMMA 11. *Assume $\mathscr{Q}$ ω^*-compact and $(\mathscr{Q}, (a_r), \rho, \imath)$ satisfying $(*)$. Then $\mathscr{P}_\alpha(\varepsilon, \mathscr{Q}) \neq \emptyset$ for all $\alpha < \omega_1$ and $0 < \varepsilon < \frac{\rho}{2}$.*

Proof. By induction on $\alpha < \omega_1$, assuming the statement true for all $\mathscr{Q}$ ω^*-compact, $(\mathscr{Q}, (a_r), \rho, \imath)$ satisfying $(*)$ and $0 < \varepsilon < \frac{\rho}{2}$. So let the property be true for all $\alpha < \beta < \omega_1$ and consider a sequence of ordinals $\alpha_k < \beta$ such that $\beta = \lim_{k\to\infty} (\alpha_k + 1)$. If $0 < \varepsilon < \frac{\rho}{2}$, some $\varkappa > 0$ can be find so that $0 < \varepsilon < \frac{\rho - \varkappa}{2} < \frac{\rho}{2}$. Assume (φ_k), (b_r^k) and $\imath_k$ as in Lemma 10. Fix k. Since $\mathscr{Q}_k = \left\{\mu \in \mathscr{Q}; \mu(\varphi_k) \geqslant \frac{\rho}{2}\right\}$ is still ω^*-compact, $\alpha_k < \beta$ and $0 < \varepsilon < \frac{\rho - \varkappa}{2}$, the induction hypothesis applies and we obtain some measure μ_k in $\mathscr{P}_{\alpha_k}(\varepsilon, \mathscr{Q}_k) \subset \mathscr{P}_{\alpha_k}(\varepsilon, \mathscr{Q})$. If now μ is a ω^*-clusterpoint of (μ_k), we have $\mu \in \mathscr{P}_\beta(\varepsilon, \mathscr{Q})$, which ends the proof.

Proof of Theorem 2. Consider again the w^*-compact norm-separable set $\mathscr{P}$ of positive measures norming $\mathscr{X}$. Lemma 7 asserts that $(\mathscr{P}, 1, \rho, \iota)$ has $(*)$ for some $\rho, \iota > 0$. But Lemma 11 will then contradict the separability of $\mathscr{P}$.

REMARK. Using the same technique as in the proof of Lemma 6, we obtain the following result:

PROPOSITION 3. *Let $\mathscr{X}$ be a Banach space and $\beta > 0$, $q < \infty$ such that*

$$\left(\int \left\| \sum_{i=1}^{m} r_i(t)\, x_i \right\|^q \mathrm{d}t\right)^{1/q} \geqslant \beta \left(\sum_{i=1}^{m} \|x_i\|^q\right)^{1/q},$$

for any finite sequence $x_1 \ldots, x_m$ of vectors in $\mathscr{X}$. Assume then $e_1, \ldots, e_n$ a 1-unconditional basic sequence in $\mathscr{X}$. If (Ω, μ) is a probability space, and $\varphi_1, \ldots, \varphi_n$ are members of $L^\infty(\Omega, \mu)$ such that $0 \leqslant \varphi_i \leqslant 1$, then

$$\left(\int \left\| \sum_{i=1}^{n} a_i \varphi_i(t)\, e_i \right\|^q \mathrm{d}t\right)^{1/q} \leqslant \frac{1}{\beta}\left(\max_i \int \varphi_i(t)\, \mathrm{d}t\right)^{1/q} \left\| \sum_{i=1}^{n} a_i e_i \right\|.$$

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