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# ON THE SPECTRAL BOUND OF THE GENERATOR OF SEMIGROUPS OF POSITIVE OPERATORS

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## INTRODUCTION

During the last two decades $C_0$-semigroups of positive linear operators became more and more important in applications (e.g. transport theory) as well as in their own right (see e.g. [2, 3, 4, 8, 9, 10, 12, 13, 14, 18, 21, 24]).

In particular for the study of the limit behaviour of such a semigroup $(T_t)_{t\geqslant 0}$ good knowledge is needed about the spectrum $\sigma(A)$ of its infinitesimal generator $A$ as well as about its type $\omega_0 = \inf\{t^{-1}\log \|T_t\|:\ t > 0\}$.

The main aim of the present paper is to prove that the spectral bound

$$\mathrm{s}(A) = \sup\{\mathrm{Re}\, z: z \in \sigma(A)\}$$

is always contained in $\sigma(A)$ (provided that the order of the underlying Banach space is not pathological). Moreover, if $\mathrm{s}(A)$ is a pole of the resolvent of $A$, we show that it is of maximal order on the line $\mathrm{s}(A) + \mathrm{i}\mathbf{R}$.

The first mentioned assertion was claimed already in [8]. The proof given there, however, is incomplete since there it is assumed tacitly that $\mathrm{s}(A)$ equals $\omega_0$. This, however, is not true in general for positivity preserving semigroups as we will show by an example in § 4. More surprisingly this example, furnished by the semigroup $(T_t)$ of translations on a suitable Banach lattice of functions on $\mathbf{R}_+$, turns out to be a rather spectacular counterexample to a spectral mapping theorem of the kind "$\exp(t\sigma(A)) = \sigma(\exp(tA)),\ t > 0$". In fact $\sigma(A) = \emptyset$, and $\sigma(\exp(tA)) = \{z \in \mathbf{C}:\ |z| \leqslant 1\}$ holds in our case. (Note that in the nonpositive case examples of similar kind are already well-established, see [7, 23]).

Our proof of the announced result is based on a Pringsheim-Landau theorem for the Laplace transform of functions on $\mathbf{R}_+$ taking their values in a weakly normal cone of a sequentially complete locally convex vector space. This theorem is proved in § 2, whereas § 1 contains some information on the abscissa of convergence for Laplace transforms of operator-valued functions; this result enables us to refine the assertion about $\mathrm{s}(A)$ mentioned above.

Among other consequences of our main result in § 3 we obtain that the spectrum $\sigma(A)$ of the generator $A$ of a group (not merely a semigroup) of positive operators is never empty (which is not true for arbitrary $C_0$-groups, see [7], sect. 23.16).

We conclude the paper with some open problems.

## 1. ON THE ABSCISSA OF CONVERGENCE FOR LAPLACE TRANSFORMS

In the following let $G$, $H$ denote Banach spaces over $\mathbf{C}$. Let $f$ be a function from $\mathbf{R}_+ = \{u \in \mathbf{R}\colon u \geqslant 0\}$ into the space $L(G, H)$ of all bounded linear operators from $G$ into $H$. Assume that for all $x \in G$ the function $t \to f(t)x$ is locally Bochner integrable (i.e. Bochner integrable with respect to the Lebesgue measure over all compact subintervals of $\mathbf{R}_+$).

We want to define the Laplace transform

$$\mathscr{L}(f)(z) = \int_0^\infty \mathrm{e}^{-zt} f(t)\,\mathrm{d}t$$

in a suitable sense. To this end we denote by s-lim the limit in the strong operator topology and by u-lim the limit with respect to the norm. Then we can define two abscissas of convergence, namely

$$\sigma_{\mathrm{s}}(f) = \inf\left\{v \in \mathbf{R}\colon \operatorname*{s-lim}_{t\to\infty}\int_0^t \mathrm{e}^{-vs} f(s)\,\mathrm{d}s \quad \text{exists}\right\}.$$

$$\sigma_{\mathrm{u}}(f) = \inf\left\{v \in \mathbf{R}\colon \operatorname*{u-lim}_{t\to\infty}\int_0^t \mathrm{e}^{-vs} f(s)\,\mathrm{d}s \quad \text{exists}\right\}.$$

Obviously $\sigma_{\mathrm{s}}(f) \leqslant \sigma_{\mathrm{u}}(f)$. For the purpose of later applications we consider the set $W(f)$ of all $v \in \mathbf{R}$ such that for every $(x, y') \in G \times H'$ the function $t \to \mathrm{e}^{-vt}\langle f(t)\,x, y'\rangle$ is Lebesgue-integrable over $\mathbf{R}_+$.

Then we call $\sigma_{\mathrm{w,a}}(f) := \inf W(f)$ the *abscissa of weak absolute convergence.*

1.1. PROPOSITION. *If* $u \in W(f)$ *then* $\operatorname*{u-lim}_{t\to\infty}\int_0^t \mathrm{e}^{-vs} f(s)\,\mathrm{d}s$ *exists for all* $v > u$. *As a consequence, we have* $\sigma_{\mathrm{u}}(f) \leqslant \sigma_{\mathrm{w,a}}(f)$.

(Note that the integral $\int_0^t$ may only exist in the strong topology.)

*Proof.* Let $u \in W(f)$. Then $B(x, y')(t) := \mathrm{e}^{-ut}\langle f(t)\,x, y'\rangle$ defines a bilinear map $B\colon G \times H' \to L^1(\mathbf{R}_+)$. For fixed $y' \in H'$, it is easy to see that the partial map $B(\cdot, y')\colon G \to L^1(\mathbf{R}_+)$ has closed graph, hence is continuous. Similarly the partial map $B(x, \cdot)\colon H' \to L^1(\mathbf{R}_+)$ is continuous for all $x \in G$. This shows that $B$ is separately continuous; and therefore is continuous (cf. [15], 5.1, Corollary 1), i.e.

$$\int_0^\infty \mathrm{e}^{-ut}|\langle f(t)\,x, y'\rangle|\,\mathrm{d}t \leqslant \|B\|\,\|x\|\,\|y'\| \quad (x \in G,\ y' \in H').$$

Now let $v > u$. If $0 \leqslant s \leqslant t < \infty$, then

$$\left|\left\langle\left(\int_s^t e^{-vr} f(r)\,\mathrm{d}r\right)x, y'\right\rangle\right| \leqslant \int_s^t e^{(u-v)r} e^{-ur}|\langle f(r)\,x, y'\rangle|\,\mathrm{d}r \leqslant$$

$$\leqslant e^{(u-v)s}\,\|B\|\,\|x\|\,\|y'\| \qquad (x \in G,\ y' \in H')$$

implies

$$\left\|\int_s^t e^{-vr} f(r)\,\mathrm{d}r\right\| \leqslant e^{(u-v)s}\,\|B\|.$$

This shows that $\text{u-}\lim_{t\to\infty}\int_0^t e^{-vr} f(r)\,\mathrm{d}r$ exists.

## 2. ON THE LAPLACE TRANSFORM OF VECTOR-VALUED FUNCTIONS WITH VALUES IN A CONE

In this section let $E$ denote a sequentially complete locally convex linear space over $\mathbf{C}$, and let $C$ be a weakly normal, closed cone in $E$ (cf. [15], V, 3). Weak normality of $C$ is equivalent to $E_0' = D - D$, where $E_0$ is the underlying real space of $E$, and $D = \{x' \in E_0' \colon \langle x, x'\rangle \geqslant 0 \text{ for all } x \in C\}$ is the (relative) polar of $(-C)$ with respect to $\langle E_0, E_0'\rangle$ (cf. [15], V, 3.3, Corollary 3).

The following main theorem of this section is a generalization of the Pringsheim-Landau theorem on the Laplace transform of nonnegative functions (see [5], Satz 1 on p. 153, and [19], Theorem 5b on p. 58). For the corresponding theorem on power series see [15], App., 2.1.

2.1. THEOREM. *Let $f$ be a continuous mapping from $\mathbf{R}_+$ into the cone $C$. Let $\sigma$ denote the abscissa of convergence of the Laplace transform* $\mathscr{L}(f)(z) := = \lim_{t\to\infty}\int_0^t e^{-zs} f(s)\,\mathrm{d}s$, *i.e.* $\sigma := \inf\{u \in \mathbf{R} \colon \mathscr{L}(f)(z)$ *exists for all $z$ with* $\mathrm{Re}z > u\}$. *Then the following assertions are true:*

a) *The analytic function represented by this integral formula is singular at $\sigma$ whenever $\sigma \neq \pm\infty$.*

b) *$\mathscr{L}(f)(u)$ lies in $C$ for all $u > \sigma$.*

c) *Assume that $\sigma$ is a pole of order $m$ of the analytic function represented by $\mathscr{L}(f)$ on the half plane $\{z \colon \mathrm{Re}z > \sigma\}$. If $z = \sigma + \mathrm{i}v$ ($v \in \mathbf{R}$) is another pole then its order is $\leqslant m$.*

*Proof.* First we note the equality

$$\sigma = \inf\left\{\mathrm{Re}(z) \colon z \in \mathbf{C},\ \left\{\int_0^t e^{-zs} f(s)\,\mathrm{d}s \colon t \geqslant 0\right\} \text{ is bounded in E}\right\}.$$

The inequality "$\geqslant$" follows from the definition. For the converse, assume that $\left\{\int_0^t e^{-zs} f(s)\,ds: t \geqslant 0\right\}$ is bounded for some $z \in \mathbf{C}$. If $w \in \mathbf{C}$, $\mathrm{Re}(w) > \mathrm{Re}(z)$, then the identity

$$\int_0^t e^{-ws} f(s)\,ds = e^{-(w-z)t} \int_0^t e^{-zs} f(s)\,ds + (w-z) \int_0^t e^{-(w-z)s} \left(\int_0^s e^{-zr} f(r)\,dr\right) ds$$

(cf. [7], Section 6.2) together with the sequential completeness of $E$ implies that $\mathscr{L}(f)(w)$ exists.

Obviously $\mathscr{L}(f)$ is analytic on $H(\sigma) := \{z: \mathrm{Re}\, z > \sigma\}$.

a) Assume that $\mathscr{L}(f)$ is regular at $\sigma \neq \pm\infty$. This means that there exists an analytic $E$-valued extension $h$ of $\mathscr{L}(f)$ and an $\varepsilon > 0$ such that $h$ is regular in $\{z: |z-\sigma| < \varepsilon\}$. By the classical Pringsheim-Landau theorem ([5], Satz 1 on p. 153) this implies that for $x' \in D$ (see the paragraph preceding the theorem) the Laplace transform $\mathscr{L}(f_{x'})$ of the continuous positive function $f_{x'}: t \to \langle f(t), x'\rangle$ is regular in $H(\sigma-\varepsilon)$, and that

$$\left\langle \int_0^t e^{-us} f(s)\,ds, x' \right\rangle = \int_0^t e^{-us} \langle f(s), x'\rangle\,ds \to \langle h(u), x'\rangle \quad (t \to \infty)$$

for all $u > \sigma - \varepsilon$. Now $E_0' = D - D$ implies $\int_0^t e^{-us} f(s)\,ds \to h(u)$ $(t \to \infty)$ with respect to $\sigma(E, E')$, in particular $\left\{\int_0^t e^{-us} f(s)\,ds: t \geqslant 0\right\}$ is bounded in $E$ for all $u > \sigma - \varepsilon$. Now the identity established at the beginning of this proof would imply $\sigma < \sigma - \varepsilon$, which is contradictory.

b) This follows since $C$ is closed.

c) Suppose that $\sigma$ is a pole of order $m$. Without loss of generality we assume $\sigma = 0$. Then for all $p > m$ we obtain $\lim_{u \to 0} |u|^p \mathscr{L}(f)(u) = 0$.

Let $z = iv$ be another pole. Then for $w = u + iv$, $u > 0$, and $x' \in D$ we get from $\langle f(r), x'\rangle \geqslant 0$

$$|z - w|^p |\langle \mathscr{L}(f)(w), x'\rangle| = u^p \left|\int_0^\infty e^{-wr} \langle f(r), x'\rangle\,dr\right| \leqslant$$

$$\leqslant u^p \int_0^\infty e^{-ur} \langle f(r), x'\rangle\,dr = u^p \langle \mathscr{L}(f)(u), x'\rangle.$$

Since $z$ is a pole by assumption, its order does not exceed $m$.

## 3. APPLICATIONS TO $C_0$-SEMIGROUPS OF POSITIVE OPERATORS

3.1. PRELIMINARIES. In the following let $E$ be a real Banach space ordered by a closed, normal cone $E_+$ satisfying $E_+ - E_+ = E$. Denote by $E_{\mathbf{C}}$ the complexification of $E$, i.e. $E_C = E + iE$, equipped with an appropriate norm inducing the product topology and such that $E_C$ becomes a complex Banach space (e.g. $\|x + iy\| = \sup_{0 \leq \theta < 2\pi} \|(\cos\theta)x + (\sin\theta)y\|$). Then $E_C$ is called an *ordered Banach space over* **C**.

A linear operator $T$ on $E_C$ is called *positive* ($T \geqslant 0$) if $T(E_+) \subset E_+$. Such an operator is necessarily bounded (apply [15], V, 5.6 together with 5.5). Let $C$ be the cone of all positive operators in the space $L(E_C)$ of all bounded linear operators on $E_C$. Then $C$ is closed and normal with respect to the stro ng operator topology (use [15], V, 5.2). We set $S \geqslant T$ whenever $S - T \in C$.

Before we apply Section 2 to the present situation we give some examples.

3.2. EXAMPLES. a) The positive cone of a real Banach lattice is clearly closed and normal. A complex Banach lattice is defined as the complexification of a real Banach lattice; in particular the classical Banach lattices of functions fit into our framework (see [16], II.11 as well as [11]).

b) Every complexification of a real order unit space (see [1]).

c) Every $C^*$-algebra $A$. The real space $A_0$ consists of the self-adjoint elements, $A_+ = \{x \in A: x \text{ is self-adjoint and positive}\}$.

The main theorem of this section now reads as follows:

3.3. THEOREM. *Let $\mathscr{T} = (T_t)_{t \geqslant 0}$ denote a $C_0$-semigroup of positive operators on the ordered Banach space $E_C$ over* **C**. *Let $A$ be the infinitesimal generator of $\mathscr{T}$ and denote by $\sigma(A)$ its spectrum and by $\mathrm{s}(A) = \sup\{\mathrm{Re}\, z: z \in \sigma(A)\}$ its spectral bound. Then the following assertions are true:*

a) *If $\sigma(A)$ is nonempty then $\mathrm{s}(A) \in \sigma(A)$.*

b) *For $u > \mathrm{s}(A)$ the resolvent $(u - A)^{-1} =: \mathrm{R}(A)(u)$ is positive. Moreover for $\mathrm{Re}\, z > \mathrm{s}(A)$ the net $\left(\int_0^t e^{-zs} T_s \, ds\right)_{t \geqslant 0}$ converges to $\mathrm{R}(A)(z)$ with respect to the operator norm (for $t \to \infty$).*

c) *Let $\mathrm{s}(A)$ be a pole of order $m$ of the resolvent of $A$. If $z = \mathrm{s}(A) + iv$ ($v \in \mathbf{R}$) is another pole then its order is $\leqslant m$.*

*Proof.* I) As was pointed out above, $C = \{T \in L(E_C): T \geqslant 0\}$ is a closed normal cone in $F = L(E_C)$, equipped with the strong operator topology, for which $F$ is sequentially complete. $f: \mathbf{R}_+ \to C$, given by $f(s) = T_s$, is continuous. As is well-known the resolvent of $A$ is a holomorphic function from $\rho(A) := \mathbf{C} \setminus \sigma(A)$ to $F$, which agrees with $\mathscr{L}(f)(z)$ on $\{z: \mathrm{Re}\, z > \omega_0\}$, where $\mathscr{L}(f)$ is the Laplace transform of $f$ and $\omega_0$ is the type of $\mathscr{T}$.

Thus by the uniqueness of holomorphic functions, $\mathrm{R}(A)$ agrees with $\mathscr{L}(f)$ on $\{z: \mathrm{Re}\, z > \sigma\}$ where $\sigma$ is as in 2.1. The identity shown at the beginning of the proof of 2.1 implies $\sigma = \sigma_s(f)$ of § 1. Now by 2.1 we obtain a), c), and the first part of b).

II) The second part of b) will follow from 1.1 if we show $\sigma_s(f) \geq \sigma_{w,a}(f)$. If $s > \sigma_s(f)$ and if $x \in E_+$, $x' \in E'_+ = \{x' \in E' : x'(E_+) \subset \mathbf{R}_+\}$, then $\lim_{t\to\infty} \int_0^t e^{-sr} \langle T_r x, x' \rangle \, dr$ exists by I) and the integrand is nonnegative, hence in $L^1(\mathbf{R}_+)$. Since $E_+$ generates $E_\mathbf{C}$, and $E'_+$ generates $(E_\mathbf{C})'$ we obtain $s \geq \sigma_{w,a}(f)$.

3.4. Corollary. *Let $(T_t)_{t\in\mathbf{R}}$ be a $C_0$-group of positive operators on an ordered Banach space $E_\mathbf{C} \neq \{0\}$. Then the spectrum of its infinitesimal generator $A$ is non-empty; more precisely $\sigma(A) \cap \mathbf{R} \neq \emptyset$.*

*Proof.* The operator $(-A)$ is the generator of the semigroup $(S_t)_{t\geq 0}$ where $S_t = T_{-t}$. Both semigroups consist of positive operators.

Assume that the spectrum $\sigma(A) = \emptyset$. Then $\sigma(-A) = \emptyset$, hence $A^{-1} \geq 0$ and $(-A)^{-1} \geq 0$ by 3.3b. This implies $A^{-1}x = 0$ for all $x \in E_+$, since the normal cone $E_+$ satisfies $E_+ \cap (-E_+) = \{0\}$. But $E_\mathbf{C}$ is the linear hull of $E_+$, hence $A^{-1} = 0$, i.e. $\dim(E_\mathbf{C}) = 0$.

3.5. Corollary. *Assume that the resolvent $\mathrm{R}(A)$ of the infinitesimal generator $A$ of a positive $C_0$-semigroup $(T_t)_{t\geq 0}$ is compact and that $\mathrm{s}(A)$ is a pole of order 1 of $\mathrm{R}(A)$. Then all singularities of $\mathrm{R}(A)$ on the line $\mathrm{s}(A) + \mathrm{i}\mathbf{R}$ are poles of order 1.*

We now want to give another perhaps more important application of the above theorem. To this end we have to introduce some more notions.

3.6. Definition. Let $E_\mathbf{C}$ be an ordered Banach space over $\mathbf{C}$. A closed cone $K \subset E_+$ satisfying $(K - E_+) \cap E_+ \subset K$ is called *a solid subcone*.

Remarks. a) Some authors prefer "hereditary" instead of "solid".

b) If we use the order $x \leq y$ whenever $y - x \in E_+$, then the closed cone $K$ is solid iff $x \in K$ and $0 \leq y \leq x$ always implies $y \in K$.

3.7. Definition. Let $\mathscr{S}$ be a set of positive operators on the ordered Banach space $E$ and $K$ a solid subcone of $E_+$.

a) $K$ is called *$\mathscr{S}$-invariant* if for every $S \in \mathscr{S}$ $S(K) \subseteq K$.

b) $\mathscr{S}$ is called *irreducible* if there is no $\mathscr{S}$-invariant solid subcone $K \neq \{0\}, E_+$.

The definition of irreducibility agrees with the usual ones in case of a Banach lattice ([16], p. 186) or in case of $C^*$-algebras ([6]). The irreducibility of $C_0$-semigroups can be characterized as follows:

3.8. Proposition. *Let $\mathscr{T} = (T_t)_{t\geq 0}$ be a $C_0$-semigroup of positive operators on the ordered Banach space $E_\mathbf{C} \neq \{0\}$, $A$ its generator. The following two assertions are equivalent:;*

a) *$\mathscr{T}$ is irreducible;*

b) *There exists $u > \omega_0$ (the type of $\mathscr{T}$) such that $(u - A)^{-1}$ is irreducible. If $E_\mathbf{C}$ is an order unit space or a Banach lattice, then* a) *and* b) *are equivalent to*

c) *For every $x > 0$ and every $x' > 0$ there exists $t \geq 0$ satisfying $\langle T_t x, x' \rangle > 0$.*

*Sketch of the proof.* a) $\Rightarrow$ c) If $x' > 0$, then $\{x \in E_+ : \sup_{t \geqslant 0}\langle T_t x, x'\rangle = 0\}$ is a solid $\mathscr{T}$-invariant subcone of $E_+$.

c) $\Rightarrow$ b) If $E_{\mathbf{C}}$ is an order unit space or a Banach lattice, and if $K$ is a proper solid subcone, then there exists an $x_0' > 0$ such that $x_0'(K) = \{0\}$. Now suppose $0 \neq K \neq E_+$ and $(u - A)^{-1}(K) \subseteq K$, then for $0 < x \in K$ one has

$$\int_0^\infty e^{-ut}\langle T_t x, x_0'\rangle \, dt = \langle (u - A)^{-1}x, x_0'\rangle = 0,$$

hence $\langle T_t x, x_0'\rangle = 0$ for all $t \geqslant 0$.

b) $\Rightarrow$ a) The formula

$$(u - A)^{-1} = \int_0^\infty e^{-ut} T_t \, dt \quad (u > \omega_0)$$

implies that each $\mathscr{T}$-invariant subcone is $(u - A)^{-1}$-invariant.

The application of 3.3 we have in mind is the following:

3.9. PROPOSITION. *Let $\mathscr{T} = (T_t)_{t \geqslant 0}$ be an irreducible $C_0$-semigroup of positive operators on the ordered Banach space $E_{\mathbf{C}} \neq \{0\}$, let $A$ be its infinitesimal generator and assume that the spectral bound* $s(A)$ *is a pole of the resolvent* $R(A)$. *Then:*

a) *All poles of* $R(A)$ *on the line* $s(A) + i\mathbf{R}$ *are of order one.*

b) *If $E_{\mathbf{C}}$ is an order unit space or a Banach lattice, the geometric multiplicity of* $s(A)$ *is one.*

*Proof.* Assume without loss of generality that $s(A) = 0$.

a) In view of 3.3 all we have to show is that the order $n$ of $s(A) = 0$ is one. Let $Q := \lim_{u \to 0} u^n R(A)(u)$ be the leading coefficient, then $Q$ is positive and $Q \neq 0$. Since $T_t Q = Q T_t$ for all $t \geqslant 0$, the solid subcone $K := \{x \in E_+ : Q(x) = 0\}$ is $\mathscr{T}$-invariant and the irreducibility implies $K = \{0\}$. If $n > 1$, then $Q^2 = 0$ (see [22], p. 228), that is $Q(E_+) \subset K = \{0\}$, hence $Q = 0$, a contradiction.

b) If we can show that the residuum $P := \lim_{u \to 0} uR(A)(u)$ is irreducible, the assertion follows from [15], App. 3.2.

First we note that $P$ is a positive projection, $PT_t = T_t P = P$ for all $t \geqslant 0$ and $Px > 0$ if $x > 0$ (cf. proof of a)). Now suppose that $K$ is a solid $P$-invariant subcone of $E_+$, $K \neq \{0\}$. If $0 < x_0 \in K$, then $x_1 := Px_0 > 0$ and $T_t x_1 = x_1$ for all $t \geqslant 0$. Therefore the closure of the solid subcone

$$K_1 := \{x \in E_+ : \text{ there exists } n \in \mathbf{N} \text{ such that } 0 \leqslant x \leqslant nx_1\}$$

is $\mathscr{T}$-invariant. Since $\{0\} \neq \overline{K_1} \subseteq K$, the irreducibility of $\mathscr{T}$ implies $\overline{K_1} = K = E_+$.

We want to point out that Theorem 3.3 and Proposition 3.9 should be seen in relation with the results that follow from the Kreĭn-Rutman theorem (cf. [15], Appendix). In order to make this relation more precise let $A$ be as above, and let $B$ be a positive operator. Letting correspond the line $\{z \in \mathbf{C}: \operatorname{Re} z = \mathrm{s}(A)\}$ and the spectral circle $\{z \in \mathbf{C}: |z| = r(B)\}$, one should compare Theorem 3.3 with [15], App. 2.4, and Proposition 3.9 with [15], App. 3.2.

We want to finish this section with an example clarifying the role of the condition that a semigroup is positivity preserving. Let $(T_t)_{t \geq 0}$ be a $C_0$-semigroup on the ordered Banach space $E_{\mathbf{C}}$. Denote as usual its generator by $A$ and the resolvent of $A$ by $\mathrm{R}(A)$. Then $T_t \geqslant 0$ for all $t \geqslant 0$ if and only if $\mathrm{R}(A)(u) \geqslant 0$ for all $u > \mathrm{s}(A)$. On the other hand, if there exists $u_0 > \mathrm{s}(A)$ such that $\mathrm{R}(A)(u_0) \geqslant 0$, then the expansion $\mathrm{R}(A)(u) = \sum_0^\infty (u_0 - u)^n (\mathrm{R}(A)(u_0))^{n+1}$ yields $\mathrm{R}(A)(u) \geqslant \mathrm{R}(A)(u_0) \geqslant 0$ for all $u$ satisfying $u_0 \geqslant u > \mathrm{s}(A)$. But this condition alone does not imply $\mathrm{s}(A) \in \sigma(A)$ as the following example shows.

3.10. Example. Set $E = \mathbf{R}^3$, $E_+ = \{(x, y, z): x, y, z \geqslant 0\}$, and $B = \begin{pmatrix} 3 & 0 & 3 \\ 4 & 2 & 0 \\ 0 & 4 & 2 \end{pmatrix} \geqslant 0$. Then the spectrum $\sigma(B)$ equals $\{6, 20^{-1}(1 \pm \mathrm{i}\sqrt{39})\}$. Hence for $A = -B^{-1}$ we obtain $\sigma(A) = \left\{-\frac{1}{6}, -\frac{1}{20}(1 \pm \mathrm{i}\sqrt{39})\right\}$, in particular $\mathrm{s}(A) = -\frac{1}{20} \notin \sigma(A)$. Thus $e^{tA}$ cannot be positive for all $t \geqslant 0$ (use 3.3). But $\mathrm{R}(A)(u) \geqslant 0$ for $-1/6 < u \leqslant 0 =: u_0$.

## 4. EXAMPLES OF SEMIGROUPS OF POSITIVE OPERATORS WHOSE GENERATOR HAS EMPTY SPECTRUM

In this section we construct a $C_0$-semigroup of positive operators of type $\omega_0 = 0$ on a reflexive Banach lattice, such that the infinitesimal generator has empty spectrum. To this end we recall some notions and facts.

4.1. Preliminaries. Let $\{0\} \neq E$ denote a complex Banach lattice (see [16], II.11). A linear operator $T$ on $E$ is called a lattice homomorphism if $|Tx| = T|x|$ holds for all $x \in E$. Such an operator is obviously positive. If $T$ is an invertible linear operator on $E$ such that $T$ and $T^{-1}$ are positive, then $T$ is a lattice isomorphism. Thus a group of positive operators (with the identity as its unit) is always a group of lattice isomorphisms.

The generator of a $C_0$-group of lattice isomorphisms has always nonempty spectrum by 3.4. *The case which may be considered closest to such a group is a $C_0$-semigroup of lattice homomorphisms.* We shall study such a semigroup in the next paragraph.

4.2. EXAMPLE. We construct a $C_0$-semigroup $\mathcal{T} = (T_t)_{t \geqslant 0}$ of lattice homomorphisms on a reflexive Banach lattice $E$ with the following properties:

a) $\|T_t\| = 1$ for all $t \geqslant 0$, in particular $\omega_0 = 0$.

b) $s(A) = -\infty$ where $A$ denotes the infinitesimal generator of $\mathcal{T}$.

c) The spectrum $\sigma(T_t)$ of $T_t$ ($t > 0$) is the whole unit disk $\{z: |z| \leqslant 1\}$.

The idea behind the example is the following: the semigroup will consist of the translations on the intersection of $L^q$ and weighted $L^p$. The choice of $L^q$ will guarantee a). The eigenfunctions of the generator to be expected are exponentials. The choice of the weight in $L^p$ prevents these expected eigenfunctions from being in the space.

We give the construction in a series of particular steps.

STEP 1. *Construction of E.* Let $1 \leqslant p < q < \infty$ be arbitrary real numbers. For a Lebesgue-measurable complex valued function $f$ on $\mathbf{R}_+$ define

$$m(f) := \left( \int_0^\infty e^{px^2} |f(x)|^p \, dx \right)^{1/p}$$

and

$$n(f) := \|f\|_q = \left( \int_0^\infty |f(x)|^q dx \right)^{1/q}.$$

Then we set $E_1 := \{f: \mathbf{R}_+ \to \mathbf{C}: f \text{ measurable}, m(f) < \infty\}$, $E_2 = L^q(\mathbf{R}_+)$ (with respect to Lebesgue-measure), and $E := E_1 \cap E_2$. For $f \in E$ we define $\|f\| = m(f) + n(f)$. (As usual we identify $f$ with its equivalence class mod null functions.)

As is easily seen $(E, \|\cdot\|)$ is a Banach lattice which is reflexive whenever $1 < p$. ($E$ is a closed subspace of $E_1 \times E_2$.)

STEP 2. *Construction of the semigroup.* If $f$ is in $E_i$ ($i = 1, 2$) and $t \geqslant 0$ then by $(T_t f)(x) := f(t + x)$ (a.e.) there is defined a $C_0$-semigroup $\mathcal{T}$ of contractive lattice homomorphisms on each of $E_i$. Its restriction to $E$ which we also denote by $\mathcal{T}$ is the $C_0$-semigroup we want to consider.

Its generator $A$ is given by $Af = f'$ (the derivative in the distributional sense) on a suitable domain in each of the spaces $E$, $E_1$, $E_2$. We claim:

(i) $m(T_t f) \leqslant e^{-t^2} m(f)$;

(ii) $n(T_t f) \leqslant n(f)$;

(iii) $\|T_t\| = 1$ when $T_t$ is considered as an operator on $E$.

*Proof.* (i) and (ii) are easy to prove and both together imply $\|T_t\| \leqslant 1$ (on $E$). Now let $\varepsilon > 0$ be arbitrary and denote by $f$ the indicator function of the interval $[t, t + \varepsilon^q]$. Then $n(f) = n(T_t f) = \varepsilon$, $m(f) \leqslant \varepsilon^{q/p} \exp(t + \varepsilon^q)^2$, and $m(T_t f) \geqslant \varepsilon^{q/p}$. Thus we obtain

$$1 \geqslant \|T_t\| \geqslant \|f\|^{-1} \|T_t f\| \geqslant (\varepsilon^{q/p} + \varepsilon)(\varepsilon^{q/p} \exp(t + \varepsilon^q)^2 + \varepsilon)^{-1}.$$

Since $p < q$ the last expression tends to 1 as $\varepsilon$ goes to 0. This proves (iii).

STEP 3. *The spectrum $\sigma(A)$ of the generator $A$ of $\mathcal{T}$ (considered on $E$) is empty.*

*Proof.* By Proposition 1.1 it is sufficient to show $\sigma_{w,a}(h) = -\infty$ for the function $h(t) = T_t$. Since $E$ is a closed subspace of $E_1 \times E_2$, under the imbedding $E \ni f \to (f, f) \in E_1 \times E_2$, we obtain $E'$ as a quotient of $E_1' \times E_2'$. Let $f \in E$, and let $u \in \mathbf{R}$. For $g \in E_1'$ we obtain $\int_0^\infty e^{-ut}|\langle T_t f, g\rangle| \, dt < \infty$ from $m(T_t f) \leqslant e^{-t^2} m(f)$.

For $g \in E_2' = L^{q'}(\mathbf{R}_+)$ $\left(\frac{1}{q} + \frac{1}{q'} = 1\right)$ we start estimating

$$\int_0^\infty e^{-ut} |\langle T_t f, g\rangle| \, dt \leqslant \int_0^\infty e^{-ut} \int_0^\infty |f(t+x) g(x)| \, dx dt.$$

In order to estimate this expression further we may assume $u < 0$, and we set $w = -u$. Then

$$\int_0^\infty \int_0^\infty e^{wt} |f(t+x) \; g(x)| \, dt dx = \int_0^\infty \int_x^\infty e^{w(v-x)} \, | f(v) \, g(x) | dv dx \leqslant$$

$$\leqslant \int_0^\infty e^{-wx} |g(x)| \, dx \cdot \int_0^\infty e^{wv} |f(v)| dv.$$

We estimate each factor separately by Hölder's inequality.

$$\int_0^\infty e^{-wx} |g(x)| \, dx \leqslant (wq)^{-1/q} \|g\|_q < \infty,$$

and

$$\int_0^\infty e^{wv} \, |f(v)| \, dv \leqslant \left( \int_x^\infty e^{((w-v)vp')} \, dv \right)^{1/p'} \cdot m(f) < \infty.$$

(The necessary modification in case $p = 1$ is obvious.) This shows

$$\int_0^\infty e^{-ut} \, |\langle T_t f, x'\rangle| \, dt < \infty$$

for all $x' \in E'$. Since $u \in \mathbf{R}$ was arbitrary we obtain $\sigma_{w,a}(h) = -\infty$.

STEP 4. *For every $t > 0$ the spectrum $\sigma(T_t)$ of $T_t$ (considered on $E$) equals $\{z : |z| \leqslant 1\}$.*

*Proof.* For fixed $t > 0$ denote by $U_0$, $U_1$, $U_2$ the operator $T_t$ acting on $E$, $E_1$, $E_2$, respectively. The spectral radius $r(U_0)$ of $U_0$ equals 1 because of (iii) in step 2, and $r(U_0) \in \sigma(U_0)$ by [8], Th. 4. Also $r(U_1) = 0$ by (i) in step 2.

I) The Neumann-series and the resolvent equation together show

$$(s - U_j)^{-1} =: \mathrm{R}(U_j)(s) \geqslant \mathrm{R}(U_j)(t) \geqslant 0 \quad \text{for } r(U_j) < s \leqslant t.$$

In addition since $r(U_1) = 0$ we obtain $R(U_0)(s) = R(U_1)(s)|_E$ (the restriction to $E$) whenever $s \neq 0$ is in the resolvent set of $U_0$.

Suppose that there is an $s \notin \sigma(U_0)$ satisfying $0 < s < 1$. Then

$$R(U_0)(s) = R(U_1)(s)|_E \geqslant R(U_1)(t)|_E = R(U_0)(t) \quad \text{for all} \quad t > 1,$$

contradicting $1 \in \sigma(U_0)$. Thus $[0, 1] \subset \sigma(U_0)$.

II) Let now $v \in \mathbf{R}$ be arbitrary and denote by $M$ the operator on $E$ which is defined by $(Mf)(x) = e^{ivx}f(x)$. Obviously $M$ is well-defined, continuous, and $M^{-1}U_0M = e^{ivt}U_0$. Thus $\sigma(U_0)$ is invariant under multiplication by any $z$ with $|z| = 1$. Now the assertion of step 4 follows.

4.3. A MODIFICATION OF EXAMPLE 4.2. As was mentioned in the introduction for any $C_0$-semigroup of positive operators on $C(K)$ the type of the semigroup is equal to the spectral bound of its generator. So it may be of interest, that we can modify our example in such a manner that the underlying space $E$ consists of continuous functions but the semigroup itself has nevertheless all the properties a)-c) of 4.2.

We define $m(f)$ and $E_1$ as in 4.2 and we take $E_2 = \{f\colon \mathbf{R}_+ \to \mathbf{C}\colon f \text{ continuous}, f(\infty) = 0\}$, and we set $n(f) = \|f\|_\infty$. Then we use $E = E_1 \cap E_2$, $\|f\| = m(f) + n(f)$ and we take again the semigroup of translations on $E$. It is easy to modify the particular steps in 4.2 to get the desired result in the same way.

4.4. FINAL REMARKS AND OPEN PROBLEMS. From now on we consider only $C_0$-semigroups $\mathscr{T}$ of positive operators. Their generator will be denoted by $A$ as before, its spectral bound by $s(A)$, and the type of $\mathscr{T}$ by $\omega_0$.

a) It is known (cf. [3]) that $s(A) = \omega_0$ holds whenever the underlying space is an AL-space or an AM-space with unit. Does this assertion hold in $L^p$-spaces, too?

b) Our semigroup in 4.2 is far from being irreducible (see 3.7). Does $s(A) = \omega_0$ hold for irreducible semigroups?

c) We know already (3.4) that $s(A) \neq -\infty$ if $\mathscr{T}$ is a group. Does $s(A) = \omega_0$ hold in this case? Here we remark that there are semigroups with the property $-\infty < s(A) < \omega_0$, e.g. if one chooses the function $e^{px}$ instead of $e^{px^2}$ in 4.2, step 1, one obtains another space $E$ where the semigroup of translations has this property.

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