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ON THE SPECTRUM OF HYPONORMAL OR SEMI-HYPONORMAL OPERATORS

DAOXING XIA

§ 1

Let $\mathscr{H}$ be a complex separable Hilbert space, $\mathscr{L}(\mathscr{H})$ be the algebra of all linear bounded operators in $\mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *semi-hyponormal* [10], [11], if

$$(T^*T)^{1/2} - (TT^*)^{1/2} \geqslant 0;$$

and T is called *hyponormal*, if

$$T^*T - TT^* \geqslant 0.$$

If T is semi-hyponormal, then there is an isometric operator U such that $T = U(T^*T)^{1/2}$ [10]. Let $U^{[n]} = U^n$ and $U^{[-n]} = U^{*n}$ for $n = 1, 2, 3, \ldots$. By results in [10], the polar symbols

$$T^{\pm} = \operatorname*{st-lim}_{n \to \mp\infty} U^{[n]} T U^{[-n]}$$

exist. The operator T^+ is normal and the operator T^- is subnormal. However if U is unitary then T^- is also normal.

If $T = X + \mathrm{i}Y$ is hyponormal, X and Y are self-adjoint, then the symbols [2], [13],

$$T_{\pm} = \operatorname*{st-lim}_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}Xt} T \mathrm{e}^{-\mathrm{i}Xt}$$

exist and are normal.

We construct the operators

$$T_k = kT_+ + (1-k)\,T_-, \quad T^{(k)} = kT^+ + (1-k)\,T^-, \quad 0 \leqslant k \leqslant 1.$$

It is easy to verify that these operators are normal when the operator U in the polar decomposition $T=U(T^*T)^{1/2}$ is unitary in the semi-hyponormal case.

In a previous paper [11], the author proved that if T is in a special subclass of semi-hyponormal operators, then

$$\sigma(T)=\bigcup_{0\leqslant k\leqslant 1}\sigma(T^{(k)}). \tag{1}$$

The aim of the present paper is to prove that (1) is true for all semi-hyponormal operators and

$$\sigma(T)=\bigcup_{0\leqslant k\leqslant 1}\sigma(T_k), \tag{2}$$

if T is hyponormal.

§ 2

We shall consider the singular integral model of a hyponormal operator.

LEMMA 1. [8], [9], [6]. *If $T=X+iY$ is completely non-normal hyponormal operator, X and Y are self-adjoint, $\mathscr{B}$ is the σ-algebra of all Borel sets in $\sigma(X)$, m is the Lebesgue measure on $(\sigma(X),\mathscr{B})$ and $\Omega=(\sigma(X),\mathscr{B},m)$, then there are an auxiliary complex separable Hilbert space $\mathscr{D}$, a strongly measurable projection-valued function $Q(\cdot)$ with $Q(x)\in\mathscr{L}(\mathscr{D})$, a uniformly bounded strongly measurable $\mathscr{L}(\mathscr{D})$-valued function $\alpha(\cdot)$, $\beta(\cdot)$ on $\Omega=(\sigma(X),\mathscr{B},m)$ satisfying*

$$\alpha Q=Q\alpha=\alpha,\qquad \beta Q=Q\beta=\beta,\qquad \alpha=\alpha^*,\qquad \beta=\beta^*,$$

a unitary operator $W:\mathscr{H}\mapsto\tilde{\mathscr{H}}$, where $\tilde{\mathscr{H}}$ is the Hilbert space of all strongly measurable, square integrable $\mathscr{D}$-valued functions f satisfying $Qf=f$, and an operator $\tilde{T}$ in $\tilde{\mathscr{H}}$,

$$(\tilde{T}f)(x)=(x+i\beta(x))f(x)+i\alpha(x)P(\alpha f),\qquad \text{for } f\in\tilde{\mathscr{H}}, \tag{3}$$

where

$$P(g)=\text{st-}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\sigma(X)}\frac{g(s)\,ds}{x-(s+i\varepsilon)},$$

such that $T=W\tilde{T}W^{-1}$.

In this case $(W^{-1}T_kWf)(x)=T_k(x)f(x)$, where

$$T_k(x)=\beta(x)+k\alpha(x)^2.$$

Without loss of generality, in the following we shall assume that $Q(\cdot)=I$, since we can use $\tilde{T}\oplus 0|_{\mathscr{H}_1}$ instead of $\tilde{T}$, where $\mathscr{H}_1$ is the Hilbert space of all strongly measurable and square integrable $\mathscr{D}$-valued functions f satisfying $(I-Q(\cdot))f(\cdot)=f(\cdot)$.

§ 3

We shall consider the hyponormal case first.

THEOREM 1. *If T is hyponormal, then* (2) *is true.*

Proof. (i) Without loss of generality, we assume that T is completely non-normal. By Lemma 1, we also assume that $\mathscr{H}=\tilde{\mathscr{H}}$, T is the singular integral operator (3) and $Q(\cdot)\equiv I$. Let $E=\bigcup_{0\leqslant k\leqslant 1}\sigma(T_k)$ and $M=\operatorname{ess\,sup}\|\alpha(x)\|$. If $\operatorname{dist}(z_0,E)=0$, then there are a sequence of numbers $k_n, 0\leqslant k_n\leqslant 1$, $k_n\to k_0$, and a sequence of unit vectors $\{f_n\}\subset\mathscr{H}$ such that

$$\|(T_{k_n}-z_0I)f_n\|\to 0.$$

Since $\operatorname{ess\,sup}_x\|T_k(x)-T_{k'}(x)\|\leqslant M^2|k-k'|$, we have $\|(T_{k_0}-z_0I)f_n\|\to 0$. Thus $z_0\in\sigma(T_{k_0})\subset E$. Therefore, E is closed.

(ii) Let $z_0=x_0+iy_0\notin E$. We have to prove that there is a number δ,

$$0<\delta<K\eta^2/(3M^2(1+2K)) \tag{4}$$

such that

$$\|\alpha(x)\,(\beta(x)+\alpha(x)^2/2-y_0I)^{-1}\alpha(x)\|\leqslant(K+1/2)^{-1} \tag{5}$$

for almost all $x\in[x_0-\delta,\,x_0+\delta]$, where $\eta=\operatorname{dist}(z_0,E)$, $K=\eta/(6M^2)$.

For any $f\in\mathscr{H}$, $\|f\|=1$ and $0\leqslant k\leqslant 1$, we have

$$\int_{\sigma(X)}\|(T_k(x)^*-\bar{z}_0)f(x)\|^2\,dx=\|(T_k^*-\bar{z}_0)f\|^2\geqslant\eta^2.$$

Thus

$$\operatorname*{ess\,sup}_x\|(T_k(x)^*-\bar{z}_0)\,h\|\geqslant\eta\|h\|$$

for any $h\in\mathscr{D}$ and $0\leqslant k\leqslant 1$. Hence, there is a number δ, satisfying (4), and a null set F_δ such that

$$\|(\beta(x)+k(\alpha(x)+sI)^2-y_0I)\,h\|\geqslant\frac{2}{3}\,\eta\,\|h\| \tag{6}$$

for all

$$x\in[x_0-\delta,x_0+\delta]-F_\delta,\qquad -K<k<1+K,\qquad 0\leqslant s\leqslant\delta.$$

In this case, we have $(\beta(x)+k(\alpha(x)+sI)^2-y_0I)^{-1}\in\mathscr{L}(\mathscr{D})$. Hence the spectrum of the self-adjoint operator

$$(\alpha(x)+sI)^{-1}(\beta(x)+(\alpha(x)+sI)^2/2-y_0I)\,(\alpha(x)+sI)^{-1}$$

is contained in $(-\infty, -K-1/2] \cup [K+1/2, \infty)$ for $0 < s < \delta$ and $x \in [x_0 - \delta, x_0 + \delta] - F_\delta$. Hence

(7) $$\|(\alpha(x) + sI)(\beta(x) + (\alpha(x) + sI)^2/2 - y_0 I)^{-1}(\alpha(x) + sI)\| \leqslant (K + 1/2)^{-1}$$

for $0 < s < \delta$, $x \in [x_0 - \delta, x_0 + \delta] - F_\delta$. Put $s \to 0$, in (7), we obtain (5).

(iii) Now we have to prove that $z_0 \in \rho(T)$. We suppose on the contrary, $z_0 \in \sigma(T)$. Let $\Delta = [x_0 - \delta, x_0 + \delta]$, $\mathscr{H}_\Delta = \{f \mid f \in \mathscr{H}, f(x) = 0 \text{ for } x \in \Delta\}$, P_Δ be the projection $\mathscr{H} \to \mathscr{H}_\Delta$ and $T_\Delta = P_\Delta T|_{\mathscr{H}_\Delta}$. It is well known that

$$\sigma(T_\Delta) \supset \{\lambda \mid \operatorname{Re}(\lambda) \in (x_0 - \delta,\ x_0 + \delta),\ \lambda \in \sigma(T)\}.$$

Hence $z_0 \in \sigma(T_\Delta)$. Since $\sigma(T_\Delta) = \{\bar{\lambda} \mid \lambda \in \sigma_a(T_\Delta^*)\}$ [10], where $\sigma_a(A)$ is the approximate point spectrum of A, there is a sequence of unit vectors $\{f_n\} \subset \mathscr{H}_\Delta$ such that

$$c_n = \|(T_\Delta^* - \bar{z}_0 I) f_n\| \to 0.$$

Since $0 \leqslant P \leqslant I$, we have

(8) $$\left\| P(f) - \frac{1}{2} f \right\| \leqslant \frac{1}{2} \|f\|.$$

On the other hand

$$\alpha f_n + \alpha(T_{1/2}(x) - y_0 I)^{-1} \alpha \left(P_\Delta P(\alpha f_n) - \frac{1}{2} \alpha f_n \right) =$$

$$= i\alpha(T_{1/2}(x) - y_0 I)^{-1}[(T_\Delta^* - \bar{z}_0 I) f_n - (x - x_0) f_n],$$

by means of (5)—(8), we have

$$\|\alpha f_n\|(1 - (1 + 2K)^{-1}) \leqslant 3M(c_n + \delta)/(2\eta).$$

Thus

(9) $$\begin{aligned} \|(T_\Delta^* - \bar{z}_0) f_n\| &\geqslant \|(T_0(x)^* - \bar{z}_0 I) f_n\| - \|\alpha P(\alpha f_n)\| \geqslant \\ &\geqslant \eta - 3M^2(c_n + \delta)(1 + 2K)/(4K\eta). \end{aligned}$$

Put $\eta \to \infty$ in (9). We obtain $\delta \geqslant K\eta^2/[3M^2(1 + 2K)]$. This contradicts (4). Thus $z_0 \in \rho(T)$, i.e. $\sigma(T) \subset E$.

(iv) Let $z_0 \in \sigma(T_k)$. We have to prove $z_0 \in \sigma(T)$. We suppose on the contrary that $z_0 \in \rho(T)$, then there is a positive b such that

(10) $$\{x + iy \mid |x - x_0| = b,\ |y - y_0| \leqslant b\} \subset \rho(T).$$

Let

$$L(a) = \operatorname*{ess\,inf}_{x} \|(T_k(x) - z_0 I)\, a\|, \quad \text{for} \quad a \in \mathscr{D}.$$

Since

$$\|(T_k - z_0 I) f\| \geqslant \inf_{\|a\|=1} L(a)\, \|f\|, \quad \text{for } f \in \mathscr{H},$$

and $z_0 \in \sigma(T_k)$, we have $\inf_{\|a\|=1} L(a) = 0$, i.e. there is a sequence of unit $\{a_n\} \subset \mathscr{D}$ such that $L(a_n) \to 0$. Let $\{\eta_n\}$ be a sequence of positive numbers such that $\eta_n \to 0$ and $L(a_n) < \eta_n \leqslant b$. There is a sequence of measurable sets $\{E_n\}$ in the real line such that $m(E_n) > 0$ and

$$\sup_{x \in E_n} \sqrt{\|x - x_0\|^2 + \|Y(x)\, a_n\|^2} \leqslant \eta_n,$$

where $Y(x) = \beta(x) + k\alpha(x)^2 - y_0 I$. Evidently $E_n \subset [x_0 - \eta_n, x_0 + \eta_n]$ and

$$\sup_{x \in E_n} \|Y(ax) a_n\| \leqslant \eta_n. \tag{11}$$

Since $\alpha(x)a_n$ and $\alpha(x)^2 a_n$ are strongly measurable vector-valued functions, $\|\alpha(x)a_n\| \leqslant M$ and $\|\alpha(x)^2 a_n\| \leqslant M^2$, there is a measurable set $F_n \subset E_n$, $m(F_n) > 0$ and the vectors e_n, $v_n \in \mathscr{D}$ with $\|e_n\| \leqslant M$, $\|v_n\| \leqslant M^2$ such that

$$\sup_{x \in F_n} \|\alpha(x) a_n - e_n\| \leqslant \eta_n/(1+M) \qquad \sup_{x \in F_n} \|\alpha(x)^2 a_n - v_n\| \leqslant \eta_n. \tag{12}$$

We may suppose that $\lim_{n\to\infty} \|e_n\| = a$, $\lim_{n\to\infty} \|v_n\| = a'$ exist. It is obvious that

$$a^2 \leqslant a' \quad \text{and} \quad a' \leqslant Ma. \tag{13}$$

Since $F_n \subset [x_0 - \eta_n,\ x_0 + \eta_n]$ and $m(F_n) > 0$ there is an interval

$$\Delta_n = [x_n - q_n,\ x_n + q_n] \subset [x_0 - \eta_n,\ x_n + \eta_n]$$

such that

$$m(\Delta_n - F_n) < m(\Delta_n)\eta_n^2. \tag{14}$$

From (13) and (14), it is easily to verify that

$$\frac{1}{m(\Delta_n)} \int_{\Delta_n} \|\alpha(x) a_n - e_n\|^2 \,\mathrm{d}x \leqslant (1 + 4M^2)\eta_n^2 \tag{15}$$

and

$$\frac{1}{m(\Delta_n)} \int_{\Delta_n} \|\alpha(x)^2 a_n - v_n\|^2 \,\mathrm{d}x \leqslant (1 + 4M^4)\, \eta_n^2. \tag{16}$$

We now construct an operator T_n in $\mathscr{H}_{\Delta_n}$

$$(T_n f)(x) = \left(\frac{x - x_n}{q_n} + \mathrm{i}\,\frac{\beta(x) - y_0}{b}\right) f(x) + \mathrm{i}\,\frac{\alpha(x)}{b} P(\alpha f).$$

By a spectral mapping theorem [4], [13]

$$\sigma(T_n) \subset \left\{\left(\frac{x - x_n}{q_n} + \mathrm{i}\,\frac{y - y_0}{b}\right) \middle|\, x + \mathrm{i}y \in \sigma(T),\ x \in \Delta_n\right\} \cup$$

$$\cup \left\{(\pm 1 + \mathrm{i}y) \,\middle|\, -\infty < y < \infty\right\},$$

Thus

$$\{x + \mathrm{i}y \mid |x - x_0| < 1,\ |y - y_0| < 1\} \subset \rho(T_n).$$

From (17), it is obvious

$$\|T_n^* f\| \geqslant \operatorname{dist}(0, \sigma(T_n))\|f\| \geqslant \|f\|. \tag{18}$$

Let $\gamma=[-1,1]$ $\mathscr{B}_\gamma$ be the σ-algebra of all Borel sets in γ, $\Omega_1 = (\gamma, \mathscr{B}_\gamma, m)$, $\mathscr{F}$ be the family of all functions in $L^2(\Omega_1)$ satisfying

$$\operatorname{ess\,sup} |h(t)| < \infty, \quad \operatorname{ess\,sup} |P(h)| < \infty.$$

Evidently, $\mathscr{F}$ is dense in $L^2(\Omega_1)$.

If $f_n(x) = a_n h((x - x_n)/q_n) q_n^{-1/2}$, where $h \in \mathscr{F}$ and $\|h\| = 1$, then $\|f_n\| = 1$. From (11), (15) and (16), we obtain

$$\lim_{n\to\infty} \|T_n^* f_n\|^2 = \begin{cases} \|th\|^2 & a = a' = 0 \\ \left\| \dfrac{a^2 t}{a'} h - \dfrac{\mathrm{i}a'}{b}(P(h) - kh)\right\|^2 + \left(1 - \dfrac{a^4}{a'^2}\right)\|th\|, & a' > 0. \end{cases} \tag{19}$$

If $a' > 0$, the spectrum of the operator

$$T' : h \mapsto \frac{a^2 t + \mathrm{i}a'^2 k/b}{a'} h - \frac{\mathrm{i}a'}{b} P(h)$$

in $L^2(\Omega_1)$ is $\left\{ \dfrac{a^2 t + \mathrm{i}a'^2 k/b}{a'} - \dfrac{\mathrm{i}a' y}{b} \,\middle|\, t \in [-1, 1],\ y \in [0,1]\right\}$ which contains 0, then

we can choose a sequence $\{h_n\} \subset \mathscr{F}$ such that $\|h_n\| = 1$ and

$$\lim_{n\to\infty} \|T' h_n\| = 0.$$

From (18) and (19), we have $1 \leqslant 1 - a^4/a'^2$; this contradicts to (13).

If $a' = 0$, then $a = 0$ by (13). In this case, (18) and (19) implies

$$\|th\| \geqslant \|h\|, \qquad \text{for } h \in \mathcal{F}.$$

But it is impossible. Hence $E \subset \sigma(T)$ and (2) is proved.

§ 4

Let us consider the class H_1 of all hyponormal operators $X + iY$ with non-negative immaginary parts Y and the class S_1 of all semi-hyponormal operator $T = U(T^*T)^{1/2}$ with unitary U satisfying $1 \notin \sigma(U)$.

The mapping

$$L : X + iY \mapsto (X + iI)(X - iI)^{-1} Y$$

is bijective from H_1 to S_1. The mapping $x + iy \mapsto (x + i)(x - i)^{-1} y$ from the upper half-plan to the complex plane is also denoted by L.

Lemma 2. [3], [12] *Let R be a set in the complex plane, $T(t)$ be an operator-valued function of $t \in [0,1]$ which is continuous with respect to the operator norm, $\{\tau_t,\ t \in [0,1]\}$ be a family of topological mappings from R to it-self such that $\tau_t(z)$ is a continuous function of $t \in [0,1]$ for every $z \in R$. If τ_0 is the identity mapping and*

$$\sigma_a(T(t)) \cap R = \tau_t(\sigma_a(T(0)) \cap R) \qquad \text{for } t \in [0, 1], \tag{20}$$

then

$$\sigma(T(t)) \cap R = \tau_t(\sigma(T(0)) \cap R) \qquad \text{for } t \in [0,1]. \tag{21}$$

Theorem 2. *If $T \in H_1$, then*

$$L(\sigma(T)) = \sigma(L(T)). \tag{22}$$

Proof. Let $R = \{z \mid \operatorname{Im}(z) > 0\}$, $\varphi_t(x) = (1 - itx)/(tx - i)$, $\psi_t(y) = (1 - t^2)/2 + ty$,

$$\tau_t(x + iy) = \begin{cases} (\varphi_t(x)\psi_t(y) - i/2)/t, & 0 < t \leqslant 1, \\ x + iy, & t = 0, \end{cases}$$

and

$$T(t) = \begin{cases} (\varphi_t(X)\psi_t(Y) - i/2)/t, & 0 < t \leqslant 1, \\ X + iY, & t = 0, \end{cases}$$

where $X + iY = T$, X and Y are self-adjoint. In this case

$$\tau_1(x + iy) = -iL(x + iy) - i/2, \quad T(1) = -iL(X + iY) - i/2.$$

It is easy to verify that τ_t and $T(t)$ satisfy all the assumptions of Lemma 2 except, (20). Now we have to verify (20).

It is obvious that

$$\|T(t)f - \tau_t(x_0 + iy_0)f\|^2 = \|(\psi_t(Y) - \psi_t(y_0)I)f/t\|^2 +$$

$$+ 2\psi_t(y_0)\mathrm{Re}\{((\psi_t(Y) - \varphi_t(x_0)\psi_t(Y)\varphi_t(X)^*)f, f)\}/t^2.$$

Since $\mathrm{Re}(((1 - \varphi_t(x_0)\varphi_t(X)^*)f, f)) \geqslant 0$ and $\mathrm{Re}(((Y - \varphi_t(x_0)Y\varphi_t(X)^*) f, f)) \geqslant 0$ we have

$$\|T(t)f - \tau_t(x_0 + iy_0)f\|^2 \geqslant \|(Y - y_0I)f\|^2. \tag{23}$$

If $x_0 + iy_0 \in R$ and $\tau_t(x_0 + iy_0) \in \sigma_a(T(t))$, then there is a sequence of unit vectors $\{f_n\}$ such that $\|T(t)f_n - \tau_t(x_0 + iy_0)f_n\| \to 0$. From (23), we see that

$$\lim_{n\to\infty} \|(Y - y_0)f_n\| = 0.$$

Then $\|(\psi_t(Y) - \psi_t(y_0))f_n\| \to 0$ and $\|(\varphi_t(X) - \varphi_t(x_0))f_n\| \to 0$. Hence $\|(X - x_0)f_n\| \to 0$. Thus (20) holds. From (21) we have

$$\sigma(L(T)) \cap R = L(\sigma(T) \cap R). \tag{24}$$

On the other hand, it is well-known [7] that if a real $x_0 \in \sigma(T)$ then $0 \in \sigma(Y)$ and $0 \in \sigma(L(T))$. Similarly if $0 \in \sigma(L(T))$ then $0 \in \sigma(Y)$ and there is a real x_0 such that $x_0 \in \sigma(T)$. Let R_1 be the real line, then

$$\sigma(L(T)) \cap R_1 = L(\sigma(T) \cap R_1). \tag{25}$$

(24) and (25) imply (22).

LEMMA 3. [13]. *If* $T = X + iY \in H_1$, *then*

$$L(T_\pm) = (L(T))^\pm \tag{26}$$

§ 5

THEOREM 3. *If T is semi-hyponormal then (1) is true.*

Proof. First we assume that $T = U(T^*T)^{1/2}$ where U is unitary. From Theorem 1, Theorem 2 and Lemma 3, it is easy to prove that (1) is true for $T \in S_1$. Now we consider the general case, $T \notin S_1$.

Let γ be any open arc in the unit circle $C_1 = \{z \mid |z| = 1\}$ satisfying $\gamma \neq C_1$. For simplicity, we suppose that $1 \notin \gamma$. Let

$$U = \int_{C_1} \lambda E(\mathrm{d}\lambda)$$

be the spectral decomposition of the unitary operator U, $\mathscr{H}_\gamma = E(\gamma)\mathscr{H}$, $T(\gamma) = E(\gamma)T \mid \mathscr{H}_\gamma$ and

$$\mathscr{D}_\gamma = \{z \mid z \neq 0, \quad z/|z| \in \gamma\}.$$

By [10], we have

$$\sigma(T) \cap \mathscr{D}_\gamma = \sigma(T_\gamma) \cap \mathscr{D}_\gamma. \tag{27}$$

It is easily to verify that

$$(T_\gamma)^{\pm} = E(\gamma)T^{\pm}|_{\mathscr{H}_\gamma} \tag{28}$$

and

$$\sigma(T_\gamma^{(k)}) = \sigma(T^{(k)}) \cap \mathscr{D}_\gamma. \tag{29}$$

Since $T_\gamma \in S_1$, we have

$$\sigma(T_\gamma) = \bigcup_{0 \leqslant k \leqslant 1} \sigma(T_\gamma^{(k)}). \tag{30}$$

From (1), (27), (29) and (30) we have

$$\sigma(T) \cap \mathscr{D}_\gamma = \bigcup_{0 \leqslant k \leqslant 1} (\sigma(T^{(k)} \cap \mathscr{D}_\gamma).$$

Since γ is arbitrary, we have

$$\sigma(T) - \{0\} = \bigcup_{0 \leqslant k \leqslant 1} (\sigma(T^{(k)}) - \{0\}).$$

But it is obvious that $\sigma(T) \cap \{0\} = (\bigcup_{0 \leqslant k \leqslant 1} \sigma(T^{(k)})) \cap \{0\}$, thus (1) is true when U it unitary.

If the operator U in the polar decompositon $T = U(T^*T)^{1/2}$ is not unitary, by a technique used in [10], we extend T to be an operator $\widetilde{T}$ in a larger space such that the corresponding operator U becomes unitary and then $\widetilde{T}$ satisfy (1). From this, we can easily verify that T aslo satisfies (1).

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DAOXING XIA
Research Institute of Mathematics,
Fudan University,
Shanghai,
China.

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