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A VARIATION OF LOMONOSOV'S THEOREM. II

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Let $\mathscr{X}$ be an infinite dimensional complex Banach space and let $L(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. A closed subspace M of $\mathscr{X}$ is said to be invariant for an operator T in $L(\mathscr{X})$ if $Tx \in M$ for all $x \in M$; M is hyperinvariant for T if M is invariant for every operator that commutes with T.

The following fact has been proved in several places ([1], [3], [5]):

THEOREM A. *Suppose that T and K are operators in $L(\mathscr{X})$ such that T is nonscalar and K is compact and nonzero. If there is a complex number λ such that $KT = \lambda TK$, then T has a nontrivial hyperinvariant subspace.*

In [5], we obtained a partial generalization of Theorem A:

THEOREM B. *Suppose that T and K are operators in $L(\mathscr{X})$ such that T is nonscalar and K is compact and nonzero. Let f be a function analytic on an open neighborhood $\mathscr{U}$ of $\sigma(K)$, the spectrum of K, such that either $KT = Tf(K)$ or $TK = f(K)T$. Then T has a nontrivial hyperinvariant subspace under any one of the following conditions:*

(i) $|f'(0)| < 1$,

(ii) $|f'(0)| > 1$ *and K is quasinilpotent, or*

(iii) $|f'(0)| > 1$ *and K has trivial kernel.*

The proof of Theorem B given in [5] depended on Lomonosov's result [6] concerning transitive algebras, and involved considerable manipulation of n-fold compositions of the function f. If in addition to the stated hypotheses, one assumes that $f(\mathscr{U}) \subset \mathscr{U}$, then one can apply the notion of Schroeder functions, which is of interest in stability theory, to simplify the proof of Theorem B, while at the same time allowing a slight relaxation of the other hypotheses. In fact, in this case Theorem B does not extend Theorem A as much as it appears to: if T and T^* have no eigenvalues, if T satisfies the hypotheses of Theorem B, and if $f(\mathscr{U}) \subset \mathscr{U}$ then, unless $f'(0) = 0$, T also satisfies those of Theorem A (with a different compact operator involved). The purpose of this note is to prove this rather surprising fact.

We require some facts from [5]. The first lemma is slightly different from Lemma 2 of [5], but the proof is the same, with the obvious modifications. We follow the practice of the earlier paper by writing $\sigma_p(T)$ for the point spectrum of T, and $f^{(n)}$ for the n-fold composition of f with itself; that is, $f^{(0)}(\zeta)=\zeta$ and $f^{(n)}(\zeta)=f(f^{(n-1)}(\zeta))$ for every positive integer n.

LEMMA 1. *Suppose K and T are operators in $L(\mathcal{X})$ such that T is nonscalar and K is nonzero and compact, and suppose f is a function defined and analytic on an open neighborhood $\mathcal{U}$ of $\sigma(K)$ such that $KT=Tf(K)$. Suppose in addition that ψ is a function analytic on a neighborhood of $\sigma(K)\cup\sigma(f(K))$. Then*

$$\psi(K)\,T=T\psi(f(K)).$$

Furthermore, if $f(0)=0$ and if $\sigma_p(T)=\sigma_p(T^*)=\emptyset$, then f maps $\sigma(K)$ onto itself in such a way that for every positive integer k, the function $f^{(k)}$ has no fixed point in $\sigma(K)$ except 0.

The next lemma also appears in [5].

LEMMA 2. *Suppose K, T, f, and $\mathcal{U}$ are as in Lemma 1, including the hypotheses that $\sigma_p(T)=\sigma_p(T^*)=\emptyset$. Then the compact operator K is quasinilpotent provided either $|f'(0)|<1$ or $|f'(0)|>1$ and K has a trivial kernel.*

We now introduce the notion of Schroeder functions. Suppose that f is a complex function defined and analytic on an open neighborhood $\mathcal{U}$ containing the origin, and suppose that $f(0)=0$. A Schroeder function for f is a function ψ, univalent on an open subset $\mathcal{V}$ of $\mathcal{U}$, such that $0\in\mathcal{V}$, $\psi'(0)=1$, and for some complex λ,

$$\psi(f(z))=\lambda\psi(z)$$

for all z in $\mathcal{V}$.

Some remarks are in order. First, if a Schroeder function ψ exists for f, and if $\lambda\neq 1$, then $\psi(0)=\psi(f(0))=\lambda\psi(0)$, and it follows that $\psi(0)=0$. Second, we have

$$\psi'(f(0))f'(0)=\lambda\psi'(0)$$

and since $\psi'(0)=1$, λ is necessarily equal to $f'(0)$. (For an expository account of Schroeder functions, we refer the reader to [2, p. 197].)

Most surprising, perhaps, is the fact that the existence of a Schroeder function depends primarily on $f'(0)$. Let $J=\{\alpha\in[0,1]$: there exist positive numbers ε,μ such that $|n\alpha-m|>\varepsilon n^{-\mu}$ for all integers m,n with $n>0\}$. If α is a number in J, then for all m, n, $|\alpha-m/n|>\varepsilon n^{-1-\mu}$, so one might think of J as the collection of numbers in the unit interval that cannot be "well approximated" by rational numbers. By considering the complement of J in $[0,1]$, we can see that the Lebesgue measure of J is 1 [7, p. 191]. Let

$$\Omega=\{z\in\mathbf{C}\colon\ z=\exp(2\pi i\alpha)\ \text{for some}\ \alpha\in J\}.$$

The following theorem is due to C. L. Siegel [7, p. 191—198]:

THEOREM 3. *Suppose $f: \mathcal{U} \to \mathcal{U}$ is analytic on a subset $\mathcal{U}$ of the complex plane. Let $f(0) = 0$ and $f'(0) = \lambda$. Then there exists a Schroeder function ψ for f, univalent on a neighborhood $\mathcal{V}$ of the origin, under either of the conditions*

(i) $|\lambda| \neq 0, 1$

(ii) $\lambda \in \Omega$.

It follows that, in a rather strong sense, "almost every" function $f: \mathcal{U} \to \mathcal{U}$ such that $f(0) = 0$ has a Schroeder function. We now use Theorem 3 to extend Lemma 2.

LEMMA 4. *Suppose K, T, f, and $\mathcal{U}$ are as in Lemma 1, including the hypotheses that $f(0) = 0$ and $\sigma_p(T) = \sigma_p(T^*) = \emptyset$, and suppose that $f(\mathcal{U}) \subset \mathcal{U}$. Then K is quasinilpotent under any one of the following conditions:*

(i) $|f'(0)| < 1$;

(ii) $|f'(0)| > 1$ *and K has trivial kernel;*

(iii) $f'(0) \in \Omega$.

Proof. Cases (i) and (ii) are covered by Lemma 2, so we assume that $f'(0) = \lambda \in \Omega$. Suppose that $\sigma(K) \neq \{0\}$ and let μ_0 be a nonzero complex number in $\sigma(K)$. By Lemma 1, $f(\sigma(K)) = \sigma(K)$ so there is an element μ_1 of $\sigma(K)$ such that $f(\mu_1) = \mu_0$. By continuing in this fashion we construct a sequence $\{\mu_n\}_{n=0}^{\infty}$ such that $f(\mu_{n+1}) = \mu_n$ for all n. Since $f(0) = 0$ and $\mu_0 \neq 0$ it follows that for each n, $\mu_n \neq 0$. Moreover, since $f^{(k)}(\mu_n) = \mu_{n-k}$ for $k \leqslant n$, and since, by Lemma 1, $f^{(k)}$ has no nonzero fixed point in $\sigma(K)$, we have $\mu_n \neq \mu_m$ whenever $n \neq m$. Hence the sequence $\{\mu_n\}$ converges to 0.

Let ψ be the Schroeder function for f with domain $\mathcal{V} \subset \mathcal{U}$ guaranteed by Theorem 3. Since $\mu_n \to 0$, we also have $\psi(\mu_n) \to 0$. On the other hand, since $\mathcal{V}$ is an open neighborhood of 0, there exists an integer N such that $\mu_n \in \mathcal{V}$ for all $n \geqslant N$. For any such n,

$$\psi(\mu_n) = \psi(f(\mu_{n+1})) = \lambda\psi(\mu_{n+1})$$

and thus

$$\psi(\mu_N) = \lambda\psi(\mu_{N+1}) = \lambda^2\psi(\mu_{N+2}) = \dots$$

and since $|\lambda| = 1$, $|\psi(\mu_n)| = |\psi(\mu_N)|$ for all $n \geqslant N$. However, the univalence of ψ and the fact that $\psi(0) = 0$ ensure that $\psi(\mu_n) \neq 0$, and thus that $\{\psi(\mu_n)\}$ cannot converge to 0.

A contradiction has been obtained and it follows that $\sigma(K) = \{0\}$.

We are now ready to show that Theorem B reduces in most cases to Theorem A.

THEOREM 5. *Let K, T, f, and $\mathcal{U}$ be as in Lemma 4, including the hypotheses that $f(0) = 0$ and $\sigma_p(T) = \sigma_p(T^*) = \emptyset$. Suppose that one of the following conditions holds:*

(i) $0 < |f'(0)| < 1$;

(ii) $|f'(0)| > 1$ *and K has trivial kernel;*

(iii) $|f'(0)| > 1$ *and K is quasinilpotent;*

(iv) $f'(0) \in \Omega$.

Then there exist a nonzero compact operator K and complex number μ such that $K_1T = \mu TK_1$. If K is not nilpotent then we can choose $\mu = f'(0)$.

Proof. By the preceding lemma we can assume that $\sigma(K) = \{0\}$ in any of the cases (i)-(iv). Let ψ be the Schroeder function for f, with domain $\mathcal{V}$. Let Γ be a curve in $\mathcal{V}$ surrounding the origin. By the Riesz functional calculus,

$$\psi(f(K)) = \frac{1}{2\pi i}\int_\Gamma \psi(f(z))\,(z-K)^{-1}dz =$$

$$= \frac{1}{2\pi i}\int_\Gamma \lambda\psi(z)\,(z-K)^{-1}dz = \lambda\psi(K).$$

From Lemma 1 we know that $\psi(K)\,T = T\psi(f(K)) = \lambda T\psi(K)$. Since $\psi(0) = 0$, $\psi(K)$ is compact, and if $\psi(K)$ is nonzero we can choose $K_1 = \psi(K)$ and $\mu = \lambda = f'(0)$.

Suppose that $\psi(K) = 0$. By a result of Halmos [4, p. 860], there is a polynomial p such that $p(K) = 0$, and since K has no eigenvalues we see that K is actually nilpotent, say of index m. By Lemma 1, $K^{m-1}T = T(f(K))^{m-1}$. Since $f(0) = 0$, we can write $f(z) = zg(z)$, so

$$(f(K))^{m-1} = K^{m-1}(g(K))^{m-1} = K^{m-1}(\alpha_0 + \alpha_1 K + \dots) = \alpha_0 K^{m-1}$$

because $K^m = 0$. Hence

$$K^{m-1}T = \alpha_0 TK^{m-1},$$

and in this instance we choose $K_1 = K^{m-1}$ and $\mu = \alpha_0$.

COROLLARY. *If K, T, f, and $\mathcal{U}$ are as above without the conditions $f(0) = 0$ and $\sigma_p(T) = \sigma_p(T^*) = \emptyset$, and if any one of conditions* (i)-(iv) *hold, or if $f'(0) = 0$, then T has a nontrivial hyperinvariant subspace.*

Proof. If T (or T^*) has an eigenvalue then the corresponding eigenspace (or its orthogonal complement) is hyperinvariant for T. If $f(0) = \alpha \neq 0$, write $f(z) = \alpha + zg(z)$. The equation $KT = Tf(K)$ becomes $KT = \alpha T + TKg(K)$, and thus T is compact and has a hyperinvariant subspace by Lomonosov's theorem [6]. Thus we can suppose that $f(0) = 0$ and that $\sigma_p(T) = \sigma_p(T^*) = \emptyset$. If $f'(0) = 0$, Theorem B provides the conclusion, and the other cases follow from Theorem 5 and Theorem A.

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