J. OPERATOR THEORY
6(1981), 25–37

NORM-CLOSE GROUP ACTIONS ON C^*-ALGEBRAS

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INTRODUCTION

This paper is in many ways an outgrowth of [5]. We deal with many of the same questions but work in the C^*-algebraic setting with groups more general than the real line. One aspect of the results of [5] was to determine conditions under which two representations of $\mathbf{R}$ as automorphisms of an operator algebra were cohomologous, more specifically exterior equivalent in Connes's sense. This, in fact, is the case [23, Lemma 4.1] if and only if the two representations appear as opposite "corners" of a representation on the 2×2 matrices over the algebra. We are, however, interested in determining analytical conditions under which this happens. As in [5], we see that one such condition is that the representations be norm close in a neighborhood of the identity.

In [5, Theorem 3.4], under the "closeness condition" just mentioned, it was shown that an obstruction in a certain twisted second cohomology group, with coefficients in the unitary group of the center of the (von Neumann) algebra, was zero. This gave the exterior equivalence of the two automorphism groups, and in turn a specific relation between the two generators. Since $H^2(\mathbf{R},T) = 0$, this result may not be surprising. But since for many commonly occurring groups G, e.g. $\mathbf{R}^n$ with $n \geqslant 2$, $H^2(G,T) \neq 0$, vanishing of the cohomological obstruction is not so automatic for general groups, and it is less clear whether positive results can be obtained. Thus, instead of appealing to general cross-section theorems or cohomology vanishing theorems, we directly choose certain unitary operators so that the relevant 2-cocycle is cohomologous to 0. We are able to do this for a finite, separable, simple C^*-algebra $\mathfrak{A}$ and G a connected, simply connected Lie group, one of whose representations on $\mathfrak{A}$ leaves a trace invariant. The results are better in the unital case and there we show, at least for separable C^*-algebras, that for $G = \mathbf{R}$ the results of [5] and of [6] are the same. We also examine the case where $H^2(G, T) = 0$ and the algebra has a non-trivial center on which G acts trivially. Here we restrict ourselves to algebras, all of whose derivations are inner [1,13]. To obtain that two center-fixing, norm-close automorphism groups are exterior equivalent, we need the analogue of a result of Moore [22, Theorem 1 and 5]; specifically, we show that $H^2(G,T) = 0$ implies $H^2\ (G,C(X,T)) = 0$ for G as above and X compact, metric.

In the final paragraph we characterize the connected component (in the topology of pointwise convergence) of the automorphism group of an AF algebra. As a corollary we obtain that all representations of a connected group by automorphisms of an AF-algebra leave any trace invariant. The results of this section of the paper stem, in part, from a collaboration with Ed Effros, and we would like to thank him for permission to include them here. Portions of Theorem 3.4 were independently obtained by Blackadar [4] by very similar techniques.

We also wish to thank Dick Kadison, for useful conversations, and Alain Connes, for pointing out to us the relevance of [8] to Remark 2.10.

This work was done while the second-named author was on leave at the Institute for Advanced Study and the University of Bielefeld; he is grateful for their gracious hospitality.

In this work G will generally denote a locally compact, second countable (separable) group. We assume that the reader is somewhat familiar with C.C. Moore's cohomology groups of G with coefficients in a Polish G-module (see [21]). If $\mathfrak{A}$ is a C^*-algebra, $\mathrm{Aut}(\mathfrak{A})$ denotes its $(*)$-automorphism group. This will generally be equipped with the topology of pointwise norm convergence, which has the advantage over the norm topology (for bounded operators on the Banach space $\mathfrak{A}$) of being Polish when $\mathfrak{A}$ is separable. For instance, when $\mathfrak{A} = C_\infty(X)$ is the abelian C^*-algebra of continuous functions on X vanishing at infinity, X a locally compact Hausdorff space, the Polish topology on $\mathrm{Aut}(\mathfrak{A})$ may be identified with the compact-open topology on the homeomorphism group of X.

§ 1.

In [5, Example 4.8], two one-parameter automorphism groups α and β of $\mathscr{B}(\mathscr{H})$ were given such that for a prescribed ε with $0 < \varepsilon < 2$, $\|\alpha_t - \beta_t\| = \varepsilon$ for all $t \in \mathbf{R} \setminus \{0\}$. As we shall see, this behaviour is impossible for group actions on unital separable C^*-algebras all of whose derivations are inner. First we need a

DEFINITION 1.1. Let G be a topological group. By a *representation of G as automorphisms of a C^*-algebra* $\mathfrak{A}$, we mean a homomorphism $\alpha: t \mapsto \alpha_t$ from G to $\mathrm{Aut}(\mathfrak{A})$, which is continuous with respect to the topology of pointwise convergence on $\mathrm{Aut}(\mathfrak{A})$.

Now we need to recall some facts about Polish groups (topological groups which admit a separable complete metric). For further details and references, see [21, § 2]. It is well-known that the closed graph theorem holds for such groups, that is, any homomorphism from one Polish group to another with a closed graph is continuous [18, Ch. 6, Problem R]. Not quite so well-known is a similar fact about crossed homomorphisms: however this reduces to the above by a standard trick (see [21; Theorem 3] for a similar but much harder result):

PROPOSITION 1.2 (Closed graph theorem for crossed homomorphisms). *Let G and H be Polish groups and suppose H is a G-module (that is, we are given a homomorphism $\beta: G \to \mathrm{Aut}(H)$ such that the map $(g,h) \mapsto \beta(g)h$ is continuous from $G \times H$ to H). Let $\gamma: G \to H$ be a crossed homomorphism, that is, a map such that*

$$\gamma(g_1 g_2) = \gamma(g_1)\beta(g_1)\,(\gamma(g_2)).$$

Then γ is continuous if and only if it has a closed graph.

Proof. Let S be the semi-direct product of G and H, that is, $S = G \times H$ as a topological space and is given the multiplication rule

$$(g_1, h_1)\,(g_2, h_2) = (g_1 g_2,\, h_1 \gamma(g_1)\,(h_2)).$$

Then S is a Polish group and $g \mapsto (g, \gamma(g))$ is a homomorphism into S. Now apply the ordinary closed graph theorem for homomorphisms $G \mapsto S$.

COROLLARY 1.3. *Let H be a Polish group and let K, L be Polish G-modules. Let $\gamma: H \to L$ be a continuous crossed homomorphism, let $\varphi: K \to L$ be a G-equivariant continuous injection, and suppose $\gamma(H) \subseteq \varphi(K)$. Then $\varphi^{-1} \circ \gamma: H \to K$ is continuous.*

Proof. By 1.2, it is enough to check that if $h_n \to h$ in H and $\varphi^{-1} \circ \gamma(h_n) \to k$ in K, then $\varphi^{-1} \circ \gamma(h) = k$. But by continuity of φ, $\gamma(h_n) \to \varphi(k)$, hence $\varphi(k) = \gamma(h)$ by continuity of γ and $k = \varphi^{-1}(\gamma(h))$.

The above corollary, although practically trivial, enables us to prove

PROPOSITION 1.4. *Let $\mathfrak{A}$ be a unital, separable C^*-algebra, all of whose derivations are inner. Let G be a second countable, locally compact group, and let $\alpha, \beta: G \to \mathrm{Aut}(\mathfrak{A})$ be two representations of G as automorphisms of $\mathfrak{A}$. Suppose that for all t in some neighborhood $\mathcal{N}$ of the identity e of G, $\|\alpha_t - \beta_t\| < 2$. Then $\|\alpha_t - \beta_t\| \to 0$ as $t \to e$ in G.*

Proof. Because $\mathfrak{A}$ is separable, $\mathrm{Aut}(\mathfrak{A})$, in its topology of pointwise convergence, and $\mathcal{U}(\mathfrak{A})$, the unitary group of $\mathfrak{A}$ in the norm topology, are Polish groups. Furthermore, if $\mathrm{Inn}(\mathfrak{A})$ denotes the inner automorphism group of $\mathfrak{A}$ and $\mathcal{U}(\mathcal{Z}(\mathfrak{A}))$ the unitary group of the center of $\mathfrak{A}$ (also the center of the unitary group), then the natural map $\varphi: \mathcal{U}(\mathfrak{A}) / \mathcal{U}(\mathcal{Z}(\mathfrak{A})) \to \mathrm{Inn}(\mathfrak{A}) \subseteq \mathrm{Aut}(\mathfrak{A})$ is a continuous injection onto $\mathrm{Inn}(\mathfrak{A})$.

For $t \in G$, let $\gamma_t = \alpha_t \circ \beta_{t^{-1}}$. Then if ι denotes the identity automorphism of $\mathfrak{A}$, we have $\|\gamma_t - \iota\| < 2$ for $t \in \mathcal{N}$. For such t, γ_t lies on a norm-continuous one-parameter subgroup of $\mathrm{Aut}(\mathfrak{A})$ by [16, Theorem 7], hence γ_t is exponentiated from a bounded derivation of $\mathfrak{A}$, which is inner by assumption. In other words, $\gamma_t \in \mathrm{Inn}(\mathfrak{A})$ for $t \in \mathcal{N}$. Now γ satisfies

$$\gamma_{ts} = \gamma_t \circ \gamma_s^{\beta_t},$$

where

$$\gamma_s^{\beta_t} = \beta_t \circ \gamma_s \circ \beta_{t^{-1}}.$$

It is easy to check that the map $(t, \dot{u}) \to \beta_t(\dot{u})$ is jointly continuous from $G \times (\mathscr{U}(\mathfrak{A}) / \mathscr{U}(\mathscr{Z}(\mathfrak{A})))$ to $\mathscr{U}(\mathfrak{A})/\mathscr{U}(\mathscr{Z}(\mathfrak{A}))$, and that φ is G-equivariant with respect to the conjugation action of G on $\mathrm{Aut}(\mathfrak{A})$ via β. Thus we can apply 1.3 with H the (open) subgroup of G generated by $\mathcal{N}$, with $L = \mathrm{Aut}(\mathfrak{A})$, and with $K = \mathscr{U}(\mathfrak{A}) / \mathscr{U}(\mathscr{Z}(\mathfrak{A}))$. (Note that because of the cocycle identity, $\gamma_t \in \mathrm{Inn}(\mathfrak{A})$ for $t \in \mathcal{N}$ implies $\gamma_t \in \mathrm{Inn}(\mathfrak{A})$ for all $t \in H$.) By definition of the quotient topology in K, γ_t is implemented by unitaries close to 1 for t near e and so $\gamma_t \to i$ is norm as $t \to e$. This completes the proof.

REMARKS 1.5. 1) As regards the last line of the above proof when $\mathfrak{A}$ has trivial center, a result of Wigner [29, p. 169] provides a local continuous cross-section from $\mathscr{U}(\mathfrak{A})/T$ to $\mathscr{U}(\mathfrak{A})$. For a generalization, see [6, Proposition 2.6].

2) The presence of a unit is absolutely necessary for the conclusion of 1.4, since when $\mathfrak{A} = \mathscr{K}(\mathscr{H})$ (the compact operators on an infinite-dimensional separable Hilbert space $\mathscr{H}$), a representation of $\mathbf{R}$ on $\mathfrak{A}$ is really just a one-parameter automorphism group of $\mathscr{B}(\mathscr{H})$. (Recall the example at the beginning of this section.) What goes wrong is that when $\mathfrak{A}$ has no unit, it is necessary to introduce the *multiplier algebra* $M(\mathfrak{A})$ of $\mathfrak{A}$, which may be defined as

$$\{x \in \mathfrak{A}^{**} : x\mathfrak{A} \subseteq \mathfrak{A},\ \ \mathfrak{A}x \subseteq \mathfrak{A}\}.$$

The *strict topology* on $M(\mathfrak{A})$ is given by the seminorms

$$x \mapsto \|xa\| + \|ax\|$$

for $a \in \mathfrak{A}$. The "inner" automorphisms of $\mathfrak{A}$ are now given by unitaries in $M(\mathfrak{A})$ and it is the strict topology, not the norm topology, which makes $\mathscr{U}(M(\mathfrak{A}))$ into a Polish group. (For all this, see [7, §§ 2—3] and [2, §§ 2—3].) When $\mathfrak{A} = \mathscr{K}(\mathscr{H})$, the strict topology and weak operator topology coincide on $\mathscr{U}(M(\mathfrak{A})) = \mathscr{U}(\mathscr{B}(\mathscr{H}))$.

§ 2.

In this section we show that the cohomological obstruction alluded to in the introduction often vanishes. For groups G with $H^2(G, T) = 0$, we obtain a result in the direction of [22, Theorem 5] which allows us to extend a result of Buchholz and Roberts [6, Proposition 5.3].

DEFINITION 2.1. Let G be a topological group and let $\alpha, \beta: G \to \mathrm{Aut}(\mathfrak{A})$ be two representations of G as automorphisms of a unital C^*-algebra $\mathfrak{A}$. We say α and β are *exterior equivalent* if there exists a continuous map $t \mapsto u_t$ from G to $\mathscr{U}(\mathfrak{A})$ such that

$$\alpha_t(x) = u_t\beta_t(x)u_t^* \qquad \text{for } t \in G, \quad x \in \mathfrak{A}$$

and

$$u_{ts} = u_t\beta_t(u_s) \qquad \text{for } t, s \in G,$$

or in short if α and β are "related by a unitary one-cocycle." Another condition equivalent to exterior equivalence is given in [23, Lemma 4.1] (note that commutativity of G is not used in the proof of that lemma).

THEOREM 2.2. *Let $\mathfrak{A}$ be a finite, unital, separable, simple C^*-algebra and let G be a connected, simply connected Lie group. Suppose that α and β are representations of G as automorphisms of $\mathfrak{A}$ and that β leaves a (normalized continuous) trace τ on $\mathfrak{A}$ invariant. If $\|\alpha_t - \beta_t\| < 2$ for t in some neighborhood of e in G, then α and β are exterior equivalent.*

Proof. By Proposition 1.4, $\|\alpha_t - \beta_t\| \to 0$ as $t \to e$. Choose a neighborhood $\mathscr{N}$ of e in G such that $\|\alpha_t - \beta_t\| < .33$ for $t \in \mathscr{N}$, and define $\gamma_t = \alpha_t \circ \beta_{t^{-1}}$. As in the proof of 1.4, γ_t is inner for all $t \in G$ because G is connected. For $t \in \mathscr{N}$, we see from [16, Lemma 5] and the fact that all derivations of $\mathfrak{A}$ are inner [27] that γ_t is of the form Adu for some unitary $u \in \mathscr{U}(\mathfrak{A})$ with spectrum in $\{z \in \mathbf{C}: \mathrm{Re}\, z \geqslant (4 - \|\gamma_t - \iota\|^2)^{1/2}/2\}$. Note that [16, Lemma 5] applies here by using a (automatically faithful) factor representation of $\mathfrak{A}$. Taking the principal branch of the logarithm enables us to write $u = \exp(ih)$ for some $h = h^* \in \mathfrak{A}$ with norm

$$\|h\| \leqslant \cos^{-1}((4 - \|\gamma_t - \iota\|^2)^{1/2}/2) = \sin^{-1}(\|\gamma_t - \iota\|/2).$$

By our choice of $\mathscr{N}$ this is less than $\sin^{-1}(.165) < 1/4 \log(2 - e^{-\pi})$. Furthermore by standard Borel cross-section theorems (e.g., [3, Theorem 3.4.1]), we may choose $u_t = \exp(ih_t)$ satisfying these conditions and depending in a Borel fashion on t. Now let $k_t = h_t - \tau(h_t)\cdot 1$. Then $\exp(ik_t)$ still depends measurably on t and implements γ_t, but $\tau(k_t) = 0$. For $t \notin \mathscr{N}$, choose $u_t \in \mathscr{U}(\mathfrak{A})$ implementing γ_t and depending measurably on t [ibid.]. A simple calculation shows that for $t, s \in G$, u_{ts} and $u_t\beta_t(u_s)$ both implement γ_{ts}, so that

$$z(t, s) = u_{ts}^* u_t\beta_t(u_s) \in T.$$

Note that z is a 2-cocycle representing the image of γ in $H^2(G, T)$ under the connecting map of the exact cohomology sequence

$$H^1(G, \mathscr{U}(\mathfrak{A})) \to H^1(G, \mathscr{U}(\mathfrak{A})/T) \to H^2(G, T).$$

(Since $\mathscr{U}(\mathfrak{A})$ is non-commutative, the H^1 spaces are only sets, not groups; nevertheless, we still have exactness in the sense that γ is liftable to a one-cocycle $G \to \mathscr{U}(\mathfrak{A})$ if and only if z is trivial.)

Since G is connected and simply connected, to show that z is trivial it suffices to show that z is locally trivial [25, Theorem 3.2]. Choose a neighborhood $\mathscr{M}$ of e in G with $\mathscr{M}^2 \subseteq \mathscr{N}$. We will show that z vanishes on $\mathscr{M} \times \mathscr{M}$. For $t, s \in \mathscr{M}$, we have

$$u_{ts} = \exp(ik_{ts}) \quad \text{and} \quad u_t\beta_t(u_s) = \exp(ik_t)\exp(i\beta_t(k_s)).$$

By construction, $\tau(k_t) = \tau(k_s) = 0$, and also $\tau(\beta_t(k_s)) = 0$ since τ was assumed β-invariant. Furthermore,

$$\|k_t\|, \|\beta_t(k_s)\| \leqslant 2 \sup_{t \in \mathscr{N}} \|h_t\| < 1/2 \log(2 - e^{-\pi}).$$

Therefore the Campbell-Baker-Hausdorff series for $\exp(ik_t)\exp(i\beta_t(k_s))$ converges absolutely (see [15, p. 112]) and

$$u_t\beta_t(u_s) = \exp(iK)$$

with

$$K = k_t + \beta_t(k_s) + (\text{commutator terms}), \quad \text{and} \quad \|K\| < \pi.$$

Since τ is a trace, it kills all commutators and $\tau(K) = 0$. But then $\exp(ik_{ts})$ and $\exp(iK)$ both implement γ_{ts} and satisfy $\tau(k_{ts}) = \tau(K)$ so that $K = k_{ts}$ and $u_{ts} = u_t\beta_t(u_s)$. This shows that z is locally trivial or, equivalently, that u satisfies the cocycle identity in a neighborhood of e. It follows that u can be modified by a coboundary so as to satisfy the cocycle identity everywhere. Then since u is Borel, it is automatically continuous [21, Theorem 3]. Since $\mathrm{Ad}u_t = \gamma_t = \alpha_t \circ \beta_{t^{-1}}$, we have $\alpha_t = (\mathrm{Ad}u_t) \circ \beta_t$, as required.

REMARK 2.3. We never used the fact that G is a Lie group, and could have assumed only that G is a connected, simply connected, locally compact group. However, by [19, Theorem 4.23], this is not much of a gain in generality, since every such group is a direct product of simply connected Lie groups.

As Remark 1.5 (2) indicates, the non-unital case is rather different. We can still give an analogue of Theorem 2.2, but unfortunately it only rarely applies, since not many non-unital simple C^*-algebras admit finite traces.

THEOREM 2.4. *Let $\mathfrak{A}$ be a finite, non-unital, separable, simple C^*-algebra and let G be a connected, simply connected Lie group. Suppose that β and α are representations of G as automorphisms of $\mathfrak{A}$, that β leaves a finite trace τ on $\mathfrak{A}$ invariant, and that $\|\alpha_t - \beta_t\| < .33$ for t in some neighborhood of e in G. Then α and*

β are exterior equivalent (via a one-cocycle $G \to \mathscr{U}(M(\mathfrak{A}))$ continuous in the strict topology of $M(\mathfrak{A})$).

Proof. This is the same as before except that we replace $\mathscr{U}(\mathfrak{A})$ with $\mathscr{U}(M(\mathfrak{A}))$ with the strict topology. $M(\mathfrak{A})$ is finite, has trivial center, and all derivations of $M(\mathfrak{A})$ are inner [28]. Note that we need to assume $\|\gamma_t - \iota\|$ is sufficiently small near e since it is no longer clear that $\|\gamma_t - \iota\| \to 0$ as $t \to 0$.

REMARKS 2.5. 1) It is clear that [5, Theorem 3.5] generalizes immediately to the situation of actions of amenable groups on von Neumann algebras. In the C^*-context it is harder to see what consequences can be drawn from the hypothesis that α_t and β_t are sufficiently close for all t, since we can no longer use an invariant mean argument.

2) Another application of the closed graph theorem (in the form of Corollary 1.3) shows that a representation of a Polish group by automorphisms of a separable unital C^*-algebra, if pointwise inner, must be norm-continuous. This raises the question (cf. [17, Theorem 4.3]) of whether or not the same holds for actions of $\mathbf{R}$ if we assume innernnes only at an uncountable number of points.

3) We are not sure to what extent all the hypotheses on $\mathfrak{A}$ in Theorem 2.2 and 2.4 are necessary for the conclusion. If β is trivial, the question just becomes whether a norm-continuous group action on a (unital, say) C^*-algebra must be implementable by a continuous homomorphism from the group into $\mathscr{U}(\mathfrak{A})$. Again it would appear there is an obstruction in $H^2(G, T)$, but Dixmier has shown [9] that for G as in 2.2 this obstruction vanishes. Dixmier's argument is ingenious, but unfortunately it uses the special nature of norm-continuous unitary representations of Lie groups and does not generalize to our situation.

Now recall that in [6, Proposition 5.3], Buchholz and Roberts showed that if α, β are representations of $\mathbf{R}$ as automorphisms of a simple C^*-algebra with unit, $\mathfrak{A}$, then α and β are exterior equivalent if $\|\alpha_t - \beta_t\| \to 0$ as $t \to e$. The simplicity of $\mathfrak{A}$ was used to insure that all derivations of $\mathfrak{A}$ are inner; however, the latter does not require that $\mathfrak{A}$ have trivial center [1,13]. We come now to consideration of the case where $\mathfrak{A}$ has non-trivial center but $H^2(G,T) = 0$ (as for $G = \mathbf{R}$). First we need an analogue of [22, Theorems 1 and 5].

THEOREM 2.6. *Let G be a connected, simply connected Lie group such that $H^2(G, T) = 0$, let X be a compact metric space, and let $C(X,T)$ be the group of continuous T-valued functions on X (viewed as a trivial G-module). Then $H^2(G, C(X,T)) = 0$.*

Proof. Let $\alpha: G \times G \to C(X,T)$ be any Borel 2-cocycle. Evaluation at points of X, when composed with α, gives a family of cocycles $G \times G \to T$ varying continuously in X. Moreover coboundaries go to coboundaries so we have a well-defined map $H^2(G, C(X,T)) \to C(X, H^2(G,T)) = 0$. Our problem is to show this is an injection. In other words, from the fact that $\alpha(x) \in \underline{B}^2(G,T)$ for each $x \in X$, we must show that α is a (global) coboundary. (The underlining in the notation $\underline{B}^2$ means, as in [21], that we identify cocycles agreeing almost everywhere; by [21, Theorem 5], this does not affect the cohomology groups.)

Now $\underline{B}^2(G,T)$ can be identified with $\underline{C}^1(G,T)/\underline{Z}^1(G,T)$, where $\underline{C}^1(G,T)$ is the group of Borel maps $G \to T$ with the topology of convergence in measure, with cochains agreeing a.e. identified, and where by [21, Theorem 3], $\underline{Z}^1(G, T)$ may be identified with $\mathrm{Hom}(G,T)$, the group of continuous homomorphisms $G \to T$. Thus α defines a continuous map

$$X \to \underline{C}^1(G,T)/\mathrm{Hom}(G, T);$$

if we can lift this map to a continuous map $\beta: X \to \underline{C}^1(G, T)$, then, considering β as an element of $\underline{C}^1(G, C(X, T))$, α will be the coboundary of β. Note that if $[G, G]$ is dense in G then $\mathrm{Hom}(G, T) = 0$ and the lifting problem is trivial. This case is already covered by [22, Proposition 4].

In general, the hypotheses on G imply that $\mathrm{Hom}(G, T)$ is a vector group V. So it will be enough to know the following:

THEOREM 2.7. *Let A be a Polish group with a closed vector subgroup V. Then any continuous map from a compact metric space X into A/V can be lifted to a continuous map $X \to A$.*

Proof. By results of Palais [24, § 4.1], $A \to A/V$ is a locally trivial principal V-bundle. (One could also prove this using the Whitney cross-section theorem for $\mathbf{R}$-actions — see the remark preceeding Theorem 1 in [20, p. 219].) Thus if $\varphi: X \to$ $\to A/V$ is continuous and X is compact, we may cover $\varphi(X)$ by finitely many open sets on which local liftings into A exist. Since V is contractible, elementary Čech theory shows that these local liftings can be patched together.

REMARKS 2.8. As noted in the proof of 2.6, one has $H^2(G, A) = 0$ for all Polish abelian groups A with trivial G-action (and in particular for $A = C(X, T)$ for some X) provided $H^2(G,T) = 0$ and $[G,G]$ is dense in G, whether or not G is a Lie group, by [22, Proposition 4]. However, the above theorem is nontrivial for G abelian, as indicated by the following example that shows that simple connectivity of G is necessary.

We claim that $H^2(T, C(S^1, T)) \neq 0$ even though $H^2(T,T) = 0$. (Here we write S^1 for T when we are not interested in the group structure.) Indeed, $C([0,1], T)$ contains a copy of $C(S^1, T)$ as the subgroup

$$\{f: [0, 1] \to T \mid f(0) = f(1)\},$$

with quotient group isomorphic to T. And clearly the topological group extension

$$1 \to C(S^1, T) \to C([0,1], T) \to T \to 1$$

is not split, since otherwise we would have $C([0,1],T) \cong C(S^1,T)\times T \cong C(S^1 \amalg \{\mathrm{pt}\}, T)$ and $[0,1] \cong S^1 \amalg \{\mathrm{pt}\}$. (Here $\amalg$ = disjoint union of topological spaces.) However, if we form the pull-back diagram

$$\begin{array}{ccccccc}
1 \to C(S^1,T) & \longrightarrow & E & \dashrightarrow & R & \to & 1\\
\| & & \vdots\downarrow & & \downarrow & & \\
1 \to C(S^1,T) & \to & C([0,1],T) & \to & T & \to & 1\\
 & & & & \downarrow & & \\
 & & & & 1 & &
\end{array}$$

that corresponds to inflating our 2-cohomology class to an element of $H^2(\mathbf{R}, C(S^1,T))$, then the extension

$$1 \to C(S^1,T) \to E \to \mathbf{R} \to 1$$

with

$$E = \{(f,t) \colon f \in C([0,1],T),\ t \in \mathbf{R},\ f(0) = \mathrm{e}^{-it} f(1)\}$$

is split; for a splitting map, let $g\colon [0,1] \to [0,1]$ be the identity map and send $t \in \mathbf{R}$ to $(\mathrm{e}^{itg}, t) \in E$.

Now we are ready for our application of cohomology theory to operator algebras.

THEOREM 2.9. *Let $\mathfrak{A}$ be a separable, unital C^*-algebra all of whose derivations are inner. Let G be a connected, simply connected Lie group with $H^2(G,T) = 0$ and let α and β be two representations of G as automorphisms of $\mathfrak{A}$, fixing the center of $\mathfrak{A}$. If $\|\alpha_t - \beta_t\| < 2$ for all t in some neighborhood of e in G, then α is exterior equivalent to β.*

Proof. By Proposition 1.4 and the preliminary steps of the proof of Theorem 2, it is enough to show that a certain obstruction in $H^2(G, \mathcal{U}(\mathcal{Z}(\mathfrak{A}))$ vanishes. This follows from Theorem 2.6.

REMARKS 2.10. A remaining question is what happens (say for $G = \mathbf{R}$) when α and β do not act trivially on $\mathcal{Z}(\mathfrak{A})$. When $\mathfrak{A}$ is a von Neumann algebra and α and β are pointwise σ-weakly continuous the cohomological obstruction to exterior equivalence lives in $H^2_\beta(G, U(X,T))$, where X is a standard measure space on which G acts via β and U has the meaning in [21]. It turns out rather remarkably that for $G = \mathbf{R}$, this group *always* vanishes, for any β. One can prove this by disintegrating X into ergodic parts for the G-action and then showing as in [22, Theorem 1] that the cohomology commutes with the operation of taking direct integrals. One is then left with four cases: a) trivial action, b) periodic action, c) free smooth action, and d) properly ergodic action. Cases (a), (b), and (c) are all handled by Moore's version of Shapiro's Lemma [21, Theorem 6]. Case (d) is considered in [8, Appendix, Proposition A.2]. (The proof given there actually works for any free action whether or not it is ergodic, and could be modified to work directly for the general case.) Note, in any event, that the fact that any non-trivial closed subgroup of **R** is free has been used in an essential way.

The case of actions of **R** on von Neumann algebras was actually handled in [6] and [5] without using the above deep facts about the Moore cohomology theory. Instead, vanishing theorems were proved for the portion of $H^2_\beta(G, U(X, T))$ representable by cocycles satisfying various continuity conditions, and these sufficed for the situation at hand.

We do not know if $H^2_\beta(\mathbf{R}, C(X, T))$ always vanishes for *continuous* flows on compact metric spaces X. In fact, the difficulty of the proof in the measurable case when decomposition into ergodic parts is possible suggests that this is highly unlikely. On the other hand, even without vanishing of the H^2 group, the conclusion of 2.9 may be correct whether or not the center of $\mathfrak{A}$ is fixed.

§ 3.

Theorem 2.2. raises the question of when there exists an invariant trace for a representation of a group by automorphisms of a unital C^*-algebra $\mathfrak{A}$. Existence of at least one trace on the algebra is of course necessary but by no means sufficient, as is clear from the example of the action of a non-compact semi-simple group G on the algebra of continuous functions on its Poisson boundary (or equivalently, on the uniformly continuous bounded harmonic functions on the symmetric space G/K) [14]. If the group is amenable and the algebra has at least one trace, an invariant trace will exist by the fixed-point property. If the algebra has a unique trace, as is true [30, Théorèmes 5.15 and 9.5] for the "irrational rotation algebra" of [10, § 7.3] or for the regular C^*-algebras of free groups [26, concluding remark], there is also no problem.

We wish to point out here that any action of a connected group on an AF-algebra preserves *all* traces. This will be an easy corollary of a characterization of the connected component of the identity in the automorphism group of an AF-algebra, but first we must recall some terminology from [11]. If $\mathfrak{A}$ is a C^*-algebra, we let $\mathfrak{A}^\sim$ denote the subalgebra of $M(\mathfrak{A})$ generated by $\mathfrak{A}$ and 1. An automorphism of $\mathfrak{A}$ is *approximately inner* if it is a point-norm limit of automorphisms of the form Adu, $u \in \mathscr{U}(M(\mathfrak{A}))$. It is *strongly approximately inner* if in addition we can choose the unitaries u in $\mathscr{U}(\mathfrak{A}^\sim)$. By [11, Proposition 2.1] the two definitions coincide for an automorphism α of $\mathfrak{A}$ if $\mathfrak{A}$ has an approximate identity consisting of α-fixed projections. Furthermore, any approximately inner automorphism (even in the weak sense) preserves all traces on $\mathfrak{A}$ [11, Lemma 2.9]. We will be concerned with AF (approximately finite dimensional) C^*-algebras, which are C^*-inductive limits of finite-dimensional, semi-simple algebras.

The following lemma is surely well known and could be deduced, for instance, from Bratteli's paper [Inductive limits of finite dimensional C^*-algebras, *Trans. Amer. Math. Soc.*, **171**(1972), 195—234].

LEMMA 3.1. *Let $\mathfrak{A}$ be an* AF-*algebra and let $\mathfrak{B}$ be a finite-dimensional sub-C^*-algebra of $\mathfrak{A}$. Then the relative commutant $\mathfrak{B}^c$ of $\mathfrak{B}$ in $\mathfrak{A}$ is an* AF-*algebra.*

Proof. We may as well assume $\mathfrak{A}$ has a unit and that $\mathfrak{B}$ contains the unit element of $\mathfrak{A}$. Choose finite-dimensional sub-C^*-algebras $\mathfrak{A}_n$ of $\mathfrak{A}$ with $\mathfrak{B} = \mathfrak{A}_1 \subseteq \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_3 \subseteq \ldots$ and with $\mathfrak{A} = \overline{\cup \mathfrak{A}_n}$. Since $\overline{\cup (\mathfrak{A}_n \cap \mathfrak{B}^c)}$ is clearly an AF-subalgebra of $\mathfrak{B}^c$, it is evidently enough to prove the inclusion $\mathfrak{B}^c \subseteq \overline{\cup (\mathfrak{A}_n \cap \mathfrak{B}^c)}$. Let $x \in \mathfrak{B}^c$ and choose $x_n \in \mathfrak{A}_n$ with $x_n \to x$. Let $y_n = \int_{\mathscr{U}(\mathfrak{B})} (ux_nu^*)\mathrm{d}u$, where $\mathrm{d}u$ denotes normalized Haar measure on the unitary group $\mathscr{U}(\mathfrak{B})$ of $\mathfrak{B}$ (which is compact!). Then since $\mathfrak{B} \subseteq \mathfrak{A}_n$, $y_n \in \mathfrak{A}_n$, and clearly $y_n \in \mathfrak{B}^c$. Also

$$\|y_n - x\| \leqslant \int_{\mathscr{U}(\mathfrak{B})} \|ux_nu^* - x\|\,\mathrm{d}u = \int_{\mathscr{U}(\mathfrak{B})} \|ux_n u^* - uxu^*\|\,\mathrm{d}u = \|x_n - x\| \to 0.$$

So $x \in \overline{\cup (\mathfrak{A}_n \cap \mathfrak{B}^c)}$.

LEMMA 3.2. *Let $\mathfrak{A}$ be a unital C^*-algebra. Then the unitary group $\mathscr{U}(\mathfrak{A})$ is locally arcwise connected in the norm topology.*

Proof. It is enough to show that for any ε satisfying $0 < \varepsilon < 2$ and any $u \in \mathscr{U}(\mathfrak{A})$ with $\|u - 1\| < \varepsilon$, there is a norm-continuous path $\{u_t\}$ in $\mathscr{U}(\mathfrak{A})$ with $u_0 = 1$, $u_1 = u$, and $\|u_t - 1\| < \varepsilon$ for all t. Using spectral theory, one can take $u_t = u^t$ (using the principal branch of $z \mapsto z^t$ defined in the complement of the negative real axis).

COROLLARY 3.3. *If $\mathfrak{A}$ is an* AF-*algebra with unit, the unitary group $\mathscr{U}(\mathfrak{A})$ is arcwise connected in the norm topology.*

Proof. By the lemma, it is enough to show that $\mathscr{U}(\mathfrak{A})$ is connected. But this is clear, since if $u \in \mathscr{U}(\mathfrak{A})$, one can choose a sequence $\{u_n\}$ of unitaries lying in finite-dimensional subalgebras of $\mathfrak{A}$ with $u_n \to u$ in norm, and each u_n lies in the connected component of the identity in $\mathscr{U}(\mathfrak{A})$.

THEOREM 3.4. *Let $\mathfrak{A}$ be an* AF-*algebra and let $\alpha \in$* Aut($\mathfrak{A}$). *Then the following are equivalent:*

a) *α is approximately inner;*
b) *α is strongly approximately inner;*
c) *α induces the identity automorphism of $\widetilde{K}_0(\mathfrak{A})$;*
d) *α lies in the connected component of the identity in* Aut($\mathfrak{A}$);
e) *α lies in the arcwise connected component of the identity in* Aut($\mathfrak{A}$).

Proof. The equivalence of (a), (b), and (c) was proved in [11, Theorem 3.8]. Also, it is clear that (e) $\Rightarrow$ (d). That (d) $\Rightarrow$ (c) is true for any C^*-algebra, since it was shown [11] that if $\widetilde{K}_0(\mathfrak{A})$ is given the discrete topology, then Aut($\mathfrak{A}$) operates continuously on $\widetilde{K}_0(\mathfrak{A})$ (and thus the connected component of the identity operates trivially). So it remains to prove that (b) $\Rightarrow$ (e).

Therefore assume (b), and let $\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \ldots$ be finite-dimensional sub-C^*-algebras of $\mathfrak{A}$ with $\mathfrak{A} = \overline{\bigcup \mathfrak{A}_n}$. We shall inductively construct norm-continuous arcs of unitaries $\{u_t^{(n)}\}$ in $\mathfrak{A}^\sim$ with $u_0^{(n)} = 1$, $(\mathrm{Ad}u_1^{(n)})|_{\mathfrak{A}_n} = \alpha|_{\mathfrak{A}_n}$, and $(\mathrm{Ad}u_t^{(n)})|_{\mathfrak{A}_{n-1}} = (\mathrm{Ad}u_t^{(n-1)})|_{\mathfrak{A}_{n-1}}$. This will prove (e) since then $\mathrm{Ad}u_t^{(n)}$ will converge (pointwise on $\mathfrak{A}$, as $n \to \infty$) to an automorphism α_t of $\mathfrak{A}$, and we will have $\alpha_0 = \mathrm{id}$, $\alpha_1 = \alpha$, and the map $t \mapsto \alpha_t$ will be norm-continuous on each $\mathfrak{A}_n$ hence point-norm continuous on all of $\mathfrak{A}$. To perform the construction, first choose as in [12, Lemma 4.1] the unitaries $u_1^{(n)}$ so that $(\mathrm{Ad}u_1^{(n)})|_{\mathfrak{A}_n} = \alpha|_{\mathfrak{A}_n}$. We will choose our arcs with these unitaries as endpoints. To construct $\{u_t^{(1)}\}$, we need only apply Corollary 3.3. Hence assume $\{u_t^{(n-1)}\}$ is constructed, and let us construct $\{u_t^{(n)}\}$. By assumption, $\mathrm{Ad}u_1^{(n)}$ and $\mathrm{Ad}u_1^{(n-1)}$ coincide on $\mathfrak{A}_{n-1}$, so $u_1^{(n-1)*}u_1^{(n)} \in \mathfrak{A}_{n-1}^c$. By Lemma 1, $\mathfrak{A}_{n-1}^c$ is an AF-algebra, so by the Corollary 3.3, we may choose an arc $\{w_t^{(n)}\}$ of unitaries in $(\mathfrak{A}_{n-1}^c)^\sim$ such that $w_0^{(n)} = 1$ and $w_1^{(n)} = u_1^{(n-1)*}u_1^{(n)}$. Now let $u_t^{(n)} = u_t^{(n-1)}w_t^{(n)}$. Then $\mathrm{Ad}u_t^{(n)}$ and $\mathrm{Ad}u_t^{(n-1)}$ coincide on $\mathfrak{A}_{n-1}$, $u_0^{(n)} = 1$, and $u_1^{(n)}$ is as desired. This completes the inductive step.

First author: Partially supported by National Science Foundation Grant MCS 77-04921.

Second author: On leave January-July 1979, at the Institute for Advanced Study, Princeton, New Jersey 08540 and at Fakultät für Mathematik, Universität Bielefeld, 4800 Bielefeld 1, Fed. Rep. of Germany. Supported in part by National Science Foundation Grants MCS 77–18723A0 to the Institute for Advanced Study and MCS 77–02831.

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Received June 13, 1980