J. OPERATOR THEORY
6(1981), 351–361

THE PRODUCT OF SPECTRAL MEASURES

W. RICKER

1. INTRODUCTION

The example of S. Kakutani [3] shows that the construction of the tensor product of two commuting spectral measures is not always possible. Accordingly, if S and T are commuting scalar operators on a space X (see [2]), then $S+T$ and ST may not be of scalar type.

It was pointed out by C. Foiaş (for a discussion see [1]) that the tensor product of two commuting spectral measures always exists if they are interpreted as spectral distributions. Then the product is, of course, only a spectral distribution and not necessarily a spectral measure. Accordingly, the sum and the product of two commuting scalar operators are generalized scalar operators. As such, they admit a functional calculus for smooth functions only.

An alternative solution is possible if the operators S and T have extensions acting on a suitable larger space containing X. The technique of going to a larger space is often used in mathematical physics.

For example, the (unbounded) operator of differentiation in $L^2(\mathbf{R})$ does not admit any eigenfunctions. However, $L^2(\mathbf{R})$ can be considered as part of a larger space which accommodates the complete set of eigenfunctions, $x \to \exp(i\lambda x)$, of the differentiation operator.

Or, let D be a self-adjoint, second order, non-singular differential operator with C^∞ coefficients on $[a, b]$, and let $Du = \psi$ be an associated Sturm-Liouville problem with appropriate boundary conditions. Let $u = m(\psi)$ be its solution for $\psi \in C[a, b]$. Then $m : C[a, b] \to C^1[a, b]$ is a Radon measure which does not have a density in the space $C^1[a, b]$. However, using the density of m with values in $C[a, b] \supseteq C^1[a, b]$, the Green's function for the problem can be constructed (see [9]).

In this note we apply this time-honoured technique to construct the tensor product of two commuting spectral measures P and Q. Let P have domain $\mathscr{M}$ and Q have domain $\mathscr{N}$. We shall seek a space Y, containing X as a dense subspace, such that for each $E \in \mathscr{M}$ and $F \in \mathscr{N}$, the operators $P(E)$ and $Q(F)$ have unique continuous extensions $P_Y(E) : Y \to Y$ and $Q_Y(F) : Y \to Y$ respectively, and the so obtained spectral measures P_Y and Q_Y have a tensor product. If S and T are scalar operators with resolutions of the identity P and Q respectively, they have continuous extensions S_Y and T_Y on to the whole of Y, then the operators $S_Y + T_Y$ and $S_Y T_Y$ are again of scalar type. Accordingly, the operators $S_Y + T_Y$ and $S_Y T_Y$ admit a rich functional calculus.

2. PRODUCT OF SPECTRAL MEASURES

Suppose that P and Q are given commuting spectral measures in X. In this section, the notion of an *admissible space* for P and Q is introduced. It is a space including X, in which the tensor product of P and Q can be constructed.

Let $\mathcal{M}$ and $\mathcal{N}$ be σ-algebras of subsets of the sets Ω and Λ respectively. Let $\mathcal{M} \otimes \mathcal{N}$ denote the algebra generated by the rectangles $\mathcal{M} \times \mathcal{N} = \{E \times F : E \in \mathcal{M}, F \in \mathcal{N}\}$ and $\mathcal{M} \underset{\sigma}{\otimes} \mathcal{N}$ the σ-algebra generated by $\mathcal{M} \otimes \mathcal{N}$. Each $G \in \mathcal{M} \otimes \mathcal{N}$ is of the form

$$G = \bigcup_{i=1}^{n} (E_i \times F_i), \tag{1}$$

where the sets $E_i \times F_i \in \mathcal{M} \times \mathcal{N}$, $i = 1, \ldots, n$, are pairwise disjoint.

Let $\text{sim}(\mathcal{M} \otimes \mathcal{N})$ denote the algebra of all simple functions based on $\mathcal{M} \otimes \mathcal{N}$. Assume that $\text{sim}(\mathcal{M} \otimes \mathcal{N})$ separates the points of $\Omega \times \Lambda$. The closure of $\text{sim}(\mathcal{M} \otimes \mathcal{N})$ with respect to the uniform norm is denoted by $\overline{\text{sim}}(\mathcal{M} \otimes \mathcal{N})$. There exists a unique compact Hausdorff space $(\Omega \times \Lambda)^\wedge$, containing $\Omega \times \Lambda$ as a dense subset, such that each member of $\overline{\text{sim}}(\mathcal{M} \otimes \mathcal{N})$ has a unique continuous extension to $(\Omega \times \Lambda)^\wedge$.

If $G \in \mathcal{M} \otimes \mathcal{N}$, then $\hat{G}$ denotes the closure of G in $(\Omega \times \Lambda)^\wedge$. Let $(\mathcal{M} \otimes \mathcal{N})^\wedge = \{\hat{G} : G \in \mathcal{M} \otimes \mathcal{N}\}$ and let $\hat{\mathcal{B}}$ be the σ-algebra generated by $(\mathcal{M} \otimes \mathcal{N})^\wedge$. If μ is an additive scalar-valued function of finite variation on $\mathcal{M} \otimes \mathcal{N}$, then the set function $\hat{\mu}$, defined by $\hat{\mu}(\hat{G}) = \mu(G)$ for each $\hat{G} \in (\mathcal{M} \otimes \mathcal{N})^\wedge$, is σ-additive on $(\mathcal{M} \otimes \mathcal{N})^\wedge$.

Let X be a locally convex Hausdorff space, X' its continuous dual, X^* its algebraic dual and $L(X)$ the space of all continuous linear operators on X. The space $L(X)$ will always have the topology of pointwise convergence. If we wish to consider X equipped with a locally convex Hausdorff topology τ, other than its original topology, we will denote it by (X, τ). The identity operator is denoted by I. The adjoint of an operator $T \in L(X)$ is denoted by T'.

Let Ω be a set and $\mathcal{M}$ a σ-algebra of subsets of Ω. A map $P : \mathcal{M} \to L(X)$ is called a *spectral measure* if,

(i) $P(\emptyset) = 0$ and $P(\Omega) = I$;

(ii) $P(E \cap F) = P(E)P(F)$ for each $E, F \in \mathcal{M}$,

and

(iii) $E \mapsto P(E)(x)$, $E \in \mathcal{M}$ is σ-additive for each $x \in X$.

A spectral measure P is said to be *equicontinuous* if $\{P(E) : E \in \mathcal{M}\}$ is an equicontinuous part of $L(X)$.

Let $P : \mathcal{M} \to L(X)$ and $Q : \mathcal{N} \to L(X)$ be commuting spectral measures. Define the set function $R : \mathcal{M} \otimes \mathcal{N} \to L(X)$ by setting

$$R(G) = \sum_{i=1}^{n} P(E_i)Q(F_i),$$

whenever $G \in \mathcal{M} \otimes \mathcal{N}$ has the representation (1). This definition is, of course, independent of the representation. Furthermore, R is finitely additive, $R(\Omega \times \Lambda) = I$ and $R(G \cap H) = R(G)R(H)$ for all $G, H \in \mathcal{M} \otimes \mathcal{N}$.

A locally convex Hausdorff space Y is said to be *admissible for* P and Q if the following conditions are satisfied:

(i) Y is barrelled, X is continuously included in Y and X is dense in Y.

(ii) For each $E \in \mathcal{M}$ and $F \in \mathcal{N}$, $P(E)$ has a unique extension $P_Y(E) \in L(Y)$ and $Q(F)$ has a unique extension $Q_Y(F) \in L(Y)$.

(iii) There exists a σ-additive measure $\mathcal{H} : \hat{\mathcal{B}} \to L(Y)$ such that

$$\mathcal{H}(\hat{G}) = R_Y(G) = \sum_{i=1}^{n} P_Y(E_i)Q_Y(F_i),$$

for every $G \in \mathcal{M} \otimes \mathcal{N}$ given in the form (1).

It follows, from (i), that Y' can be identified with a subspace of X' and that Y' separates the points of X.

PROPOSITION 2.1. *The map* $\mathcal{H} : \hat{\mathcal{B}} \to L(Y)$ *is a spectral measure.*

Proof. It is to be shown that if $\hat{G}_1, \hat{G}_2 \in \hat{\mathcal{B}}$ then,

$$\mathcal{H}(\hat{G}_1 \cap \hat{G}_2) = \mathcal{H}(\hat{G}_1)\mathcal{H}(\hat{G}_2). \tag{2}$$

The multiplicativity of R_Y shows that (2) holds whenever $\hat{G}_1, \hat{G}_2 \in (\mathcal{M} \otimes \mathcal{N})^\wedge$.

Let $\hat{G}_1 \in (\mathcal{M} \otimes \mathcal{N})^\wedge$. Denote by M_1 the system of all sets $\hat{G}_2 \in \hat{\mathcal{B}}$ such that (2) is valid. Clearly $(\mathcal{M} \otimes \mathcal{N})^\wedge \subseteq M_1$. Due to the σ-additivity of $\mathcal{H}$ the collection M_1 is a monotone system. Consequently $\hat{\mathcal{B}} \subseteq M_1$. Now let $\hat{G}_2$ be an arbitrary element of $\hat{\mathcal{B}}$. Denote by M_2 the system of all sets $\hat{G}_1 \in \hat{\mathcal{B}}$ such that (2) is valid. Since $(\mathcal{M} \otimes \mathcal{N})^\wedge \subseteq M_2$ and M_2 is a monotone system it follows that $\hat{\mathcal{B}} \subseteq M_2$.

PROPOSITION 2.2. *Let* $P : \mathcal{M} \to L(X)$ *and* $Q : \mathcal{N} \to L(X)$ *be commuting spectral measures. Let* Y *be an admissible space for* P *and* Q *and let* $\tilde{Y}$ *be the completion of* Y. *Then* $\tilde{Y}$ *is an admissible space for* P *and* Q.

Proof. It is clear that $\tilde{Y}$ is barrelled, X is continuously included in $\tilde{Y}$ and X is dense in $\tilde{Y}$. Furthermore, for each $E \in \mathcal{M}$, the operator $P(E)$ has a unique extension $P_{\tilde{Y}}(E) \in L(\tilde{Y})$, and for each $F \in \mathcal{N}$, $Q(F)$ has a unique extension $Q_{\tilde{Y}}(F) \in L(\tilde{Y})$.

By hypothesis there exists a spectral measure $\mathscr{H}: \hat{\mathscr{B}} \to L(Y)$ such that $\mathscr{H}(\hat{G}) = R_Y(G)$ for each $G \in \mathscr{M} \otimes \mathscr{N}$. Since $\mathscr{H}$ is σ-additive and the space Y is barrelled, $\{\mathscr{H}(B) : B \in \hat{\mathscr{B}}\}$ is an equicontinuous part of $L(Y)$. Each $\mathscr{H}(B) \in L(Y)$ has a unique extension $\tilde{\mathscr{H}}(B) \in L(\tilde{Y})$, and the family $\{\tilde{\mathscr{H}}(B) : B \in \hat{\mathscr{B}}\}$ is an equicontinuous part of $L(\tilde{Y})$.

Let $z \in \tilde{Y}$ and let B_n be sets in $\hat{\mathscr{B}}$, such that $B_n \supseteq B_{n+1}$, for every $n = 1, 2, \ldots$, and the intersection of the sets is empty. Let U be a neighbourhood of zero in $\tilde{Y}$. Choose a neighbourhood W of zero such that $W + W \subseteq U$. By equicontinuity, there is a neighbourhood V of zero such that $\tilde{\mathscr{H}}(B_n)(V) \subseteq W$ for all n. Since Y is dense in $\tilde{Y}$, there exists $y \in Y$ with $z - y \in V$. As $\mathscr{H}(B_n)(y) \to 0$ in Y, there is a positive integer N such that $\mathscr{H}(B_n)(y) \in W$, whenever $n \geqslant N$. Hence, the identity

$$\tilde{\mathscr{H}}(B_n)(z) = \tilde{\mathscr{H}}(B_n)(z - y) + \mathscr{H}(B_n)(y) \in W + W \subseteq U,$$

valid for every $n \geqslant N$, shows that $\tilde{\mathscr{H}}$ is σ-additive.

A family $\mathscr{C}$ of subsets of a set Ω, is said to be *compact* if it has the (countable) finite intersection property. That is, if the sets $C_n \in \mathscr{C}$, $n = 1, 2, \ldots$, have empty intersection, there exists a finite number of them having empty intersection.

Let $\mathscr{A}$ be an algebra of subsets of Ω. An additive, scalar-valued function μ defined on $\mathscr{A}$ is said to be *$\mathscr{C}$-regular* if, for every $A \in \mathscr{A}$ and $\varepsilon > 0$, there exist sets $B \in \mathscr{A}$ and $C \in \mathscr{C}$ such that $B \subseteq C \subseteq A$ and $|\mu(E)| < \varepsilon$ for every $E \in \mathscr{A}$ such that $E \subseteq A \setminus B$. An additive map $\mathscr{S} : \mathscr{A} \to L(X)$ is said to be *$\mathscr{C}$-regular* if, for every $x \in X$ and $x' \in X'$, the additive function

$$E \mapsto \langle \mathscr{S}(E)(x), x' \rangle, \quad E \in \mathscr{A},$$

is $\mathscr{C}$-regular. An additive, scalar-valued function μ on $\mathscr{A}$ or an additive map $\mathscr{S} : \mathscr{A} \to L(X)$ is called *regular*, if it is $\mathscr{C}$-regular for some compact family $\mathscr{C}$ of subsets of Ω.

LEMMA 2.3. *Let μ be a bounded regular additive scalar-valued function defined on an algebra $\mathscr{A}$. Then μ is σ-additive on $\mathscr{A}$.*

Proof. Since the variation of μ is again bounded and regular the result follows from 4(i) of [6].

LEMMA 2.4. *Let $P : \mathscr{M} \to L(X)$ be a spectral measure, $\mathscr{C}$-regular with respect to some compact family $\mathscr{C}$. For every $A \in \mathscr{M}$, $\varepsilon > 0$, $x \in X$ and every equicontinuous subset $W \subseteq X'$, there exist sets $B \in \mathscr{M}$ and $C \in \mathscr{C}$ such that $B \subseteq C \subseteq A$ and*

$$\sup\{|\langle P(E)(x), x' \rangle| : x' \in W\} < \varepsilon$$

for all $E \in \mathscr{M}$ such that $E \subseteq A \setminus B$.

Proof. The statement follows from Rybakov's theorem ([5], p. 121) and the fact that the gauge of the polar of W is a continuous semi-norm on X.

PROPOSITION 2.5. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. Let Y be an admissible space for P and Q. If the operator valued measures $P_Y : \mathscr{M} \to L(Y)$ and $Q_Y : \mathscr{N} \to L(Y)$ are regular, then there exists a unique spectral measure $\mathscr{K} : \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N} \to L(Y)$, such that $\mathscr{K}(E \times F) = P_Y(E)Q_Y(F)$ for every $E \times F \in \mathscr{M} \times \mathscr{N}$.*

Proof. By hypothesis there exists a spectral measure $\mathscr{H} : \hat{\mathscr{B}} \to L(Y)$ such that $\mathscr{H}(\hat{G}) = R_Y(G)$ for each $G \in \mathscr{M} \otimes \mathscr{N}$. Since Y is barrelled, $\mathscr{H}$ is equicontinuous and it follows that $\{P_Y(E) : E \in \mathscr{M}\}$ and $\{Q_Y(F) : F \in \mathscr{N}\}$ are equicontinuous parts of $L(Y)$. Lemma 2.3 and the Orlicz-Pettis lemma imply that P_Y and Q_Y are spectral measures. In particular, $R_Y : \mathscr{M} \otimes \mathscr{N} \to L(Y)$ is finitely additive and multiplicative.

Using Lemma 2.4, it can be shown as in Proposition 2.4 of [8], that the map

$$G \mapsto \langle R_Y(G)(y), y' \rangle, \quad G \in \mathscr{M} \otimes \mathscr{N},$$

is regular for every $y \in Y$ and $y' \in Y'$. Lemma 2.3 then shows that R_Y is σ-additive on $\mathscr{M} \otimes \mathscr{N}$. Since, for fixed $y \in Y$, the set $\{R_Y(G)(y) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is contained in the relatively weakly compact set $\{\mathscr{H}(B)(y) : B \in \hat{\mathscr{B}}\}$, by the Theorem of Extension in [4], there exists a unique σ-additive measure $\mathscr{K}(\cdot)(y) : \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N} \to Y$, such that $\mathscr{K}(G)(y) = R_Y(G)(y)$ for every $G \in \mathscr{M} \otimes \mathscr{N}$.

If $M = \{A \in \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N} : \mathscr{K}(A) \in L(Y)\}$, then M contains $\mathscr{M} \otimes \mathscr{N}$. It follows from the Banach-Steinhaus theorem that M is a monotone class. Hence, $M = \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N}$. The multiplicativity of $\mathscr{K}$ can be proved as in Proposition 2.1.

The condition that the measures $P_Y : \mathscr{M} \to L(Y)$ and $Q_Y : \mathscr{N} \to L(Y)$ are regular is weaker than the requirement that the original measures $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be regular.

LEMMA 2.6. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be regular, commuting spectral measures. Let Y be an admissible space for P and Q. Then $P_Y : \mathscr{M} \to L(Y)$ and $Q_Y : \mathscr{N} \to L(Y)$ are regular spectral measures.*

Proof. Let $\mathscr{C}$ be a compact family such that P is $\mathscr{C}$-regular. Since Y is admissible, Y' is a part of X' and $\{P_Y(E) : E \in \mathscr{M}\}$ is an equicontinuous part of $L(Y)$.

Let $y \in Y$, $y' \in Y'$, $\varepsilon > 0$ and $A \in \mathscr{M}$. Since

$$V = \{z \in Y : |\langle z, y' \rangle| < \varepsilon/2\}$$

is a neighbourhood of zero in Y, there is a neighbourhood U of zero such that $P_Y(E)(U) \subseteq V$ for each $E \in \mathscr{M}$. Choose $x \in X$ such that $y - x \in U$, then for every

$E \in \mathscr{M}$,

$$|\langle P_Y(E)(y - x), y' \rangle| < \varepsilon/2.$$

As P is regular, there exist sets $B \in \mathscr{M}$ and $C \in \mathscr{C}$ such that $B \subseteq C \subseteq A$ and

$$|\langle P(E)(x), y' \rangle| < \varepsilon/2,$$

for each $E \in \mathscr{M}$ with $E \subseteq A \setminus B$. The regularity of P_Y follows.

3. A CONSTRUCTION

In this section, a way of constructing admissible spaces for P and Q by weakening the topology of X is discussed.

Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. A locally convex Hausdorff topology τ on X, weaker than the original topology on X, is said to be *admissible for P and Q*, if the completion of (X, τ) is barrelled and if $\{R(G) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is an equicontinuous part of $L((X, \tau))$.

Let τ be an admissible topology for P and Q. Denote by Z the completion of (X, τ). For $G \in \mathscr{M} \otimes \mathscr{N}$, consider $R(G)$ as a continuous linear operator from (X, τ) into Z. The operator $R(G)$ has a unique extension $R_Z(G) \in L(Z)$. It follows that $\{R_Z(G) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is an equicontinuous part of $L(Z)$. In particular, for each $E \in \mathscr{M}$ and $F \in \mathscr{N}$, there are unique extensions $P_Z(E) \in L(Z)$ and $Q_Z(F) \in L(Z)$ of $P(E)$ and $Q(F)$ respectively.

LEMMA 3.1. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. Let τ be an admissible topology for P and Q. Then the maps $P_Z : \mathscr{M} \to L(Z)$ and $Q_Z : \mathscr{N} \to L(Z)$ are spectral measures.*

Proof. The multiplicativity of P_Z and Q_Z follows from the multiplicativity of P and Q. The σ-additivity of P_Z and Q_Z can be proved in the same way that the σ-additivity of $\widetilde{\mathscr{H}}$ was proved in Proposition 2.2.

Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures and Y an admissible space for P and Q. Let τ be the topology that X inherits from Y. Then τ is an admissible topology for P and Q. Furthermore, the completion of (X, τ) is precisely $\widetilde{Y}$, the completion of Y (cf. Proposition 2.2). Hence, every admissible space for P and Q is a dense, barrelled subspace of the completion of (X, τ), for some admissible topology τ for P and Q.

A locally convex space X is said to be *weakly Σ-complete* [9] if every sequence $\{x_n\}_{n=1}^{\infty}$ of its elements, such that $\{\langle x_n, x' \rangle\}_{n=1}^{\infty}$ is absolutely summable for each $x' \in X'$, is itself summable with the sum belonging to X. In [4], such a space is said to have the *B-P* property. Weakly sequentially complete spaces, in particular reflexive spaces, are weakly Σ-complete.

According to a theorem of Ju. B. Tumarkin [10], generalizing the well known result of C. Bessaga and A. A. Pełczyński, a space X is weakly Σ-complete if and only if it does not contain an isomorphic copy of the space c_0.

THEOREM 3.2. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. Let τ be an admissible topology for P and Q, and Z the completion of (X, τ). If Z is weakly Σ-complete, then Z is an admissible space for P and Q.*

Proof. It was noted that for each $E \in \mathscr{M}$ and $F \in \mathscr{N}$, the operators $P(E)$ and $Q(F)$ have unique extensions $P_Z(E) \in L(Z)$ and $Q_Z(F) \in L(Z)$ respectively.

For each $z \in Z$ and $z' \in Z'$, the map

$$\hat{G} \mapsto \langle R_Z(G)(z), z' \rangle, \quad \hat{G} \in (\mathscr{M} \otimes \mathscr{N})^\wedge,$$

is bounded and finitely additive, hence, is σ-additive. That is, the map $m : (\mathscr{M} \otimes \mathscr{N})^\wedge \to \to L(Z)$ defined by $m(\hat{G}) = R_Z(G)(z)$, is weakly σ-additive for each $z \in Z$. Furthermore, as m is bounded and its range is contained in the weakly Σ-complete space Z, by the Theorem of Extension in [4], there exists a σ-additive map $\mathscr{H}(\cdot)(z) : \hat{\mathscr{B}} \to Z$ such that $\mathscr{H}(\hat{G})(z) = R_Z(G)(z)$ for each $G \in \mathscr{M} \otimes \mathscr{N}$.

That $\mathscr{H}$ is a spectral measure can be shown as in the proof of Proposition 2.5.

COROLLARY 3.3. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be regular, commuting spectral measures. Let τ be an admissible topology for P and Q, and Z the completion of (X, τ). If Z is weakly Σ-complete, then there exists a unique spectral measure $\mathscr{K} : \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N} \to L(Z)$, such that $\mathscr{K}(G) = R_Z(G)$ for each $G \in \mathscr{M} \otimes \mathscr{N}$.*

Proof. The statement follows immediately from Proposition 2.5, Lemma 2.6 and Theorem 3.2.

In the above corollary, it sufficies to assume that $P : \mathscr{M} \to L((X, \tau))$ and $Q : \mathscr{N} \to L((X, \tau))$ are regular.

Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. Let Z' be a subspace of X' which separates the points of X such that $R(G)'(Z') \subseteq Z'$, for each $G \in \mathscr{M} \otimes \mathscr{N}$. Let $\tau = \sigma(X, Z')$ be the weakest topology on X making all the elements of Z' continuous. The maps $P : \mathscr{M} \to L((X, \tau))$ and $Q : \mathscr{N} \to L((X, \tau))$ are spectral measures. In this case, the completion Z of (X, τ) is $(Z')^*$ equipped with the topology $\sigma(Z, Z')$. The space Z is barrelled and weakly sequentially complete. For each $G \in \mathscr{M} \otimes \mathscr{N}$ the operator $R(G)$ has a unique extension $R_Z(G) \in L(Z)$. Since $R(G)'$ maps Z' into Z', the operator $R_Z(G)'$ can be interpreted as the operator dual to the restriction of $R(G)'$ to Z', with respect to the duality of Z and Z'.

If, in addition, $\{R(G) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is an equicontinuous part of $L((X, \tau))$, then P_Z and Q_Z are σ-additive.

PROPOSITION 3.4. *Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. Let Z' be a subspace of X' such that Z' separates the points of X, $R(G)'(Z') \subseteq Z'$ for each $G \in \mathscr{M} \otimes \mathscr{N}$, and $\{R(G) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is an equicontinuous part of*

$L((X, \sigma(X, Z')))$. Then $\sigma(X, Z')$ is an admissible topology for P and Q and, consequently, the completion of $(X, \sigma(X, Z'))$ is an admissible space for P and Q.

Proof. The statement is a particular case of Theorem 3.2.

4. EXAMPLES

Given two commuting spectral measures P and Q, it is relatively easy to find subspaces Z' of X' which separate the points of X, such that $R(G)'(Z') \subseteq Z'$, for every $G \in \mathcal{M} \otimes \mathcal{N}$. However, it is usually more difficult to show that $\{R(G) : G \in \mathcal{M} \otimes \mathcal{N}\}$ is an equicontinuous part of $L((X, \sigma(X, Z')))$, because the equicontinuity of $R : \mathcal{M} \otimes \mathcal{N} \to L((X, \sigma(X, Z')))$ does not follow from the equicontinuity of $R : \mathcal{M} \otimes \mathcal{N} \to L(X)$.

EXAMPLE 4.1. Let Ω be the set of positive integers and $\mathcal{M}$ the σ-algebra of all subsets of Ω. Let $X = \ell^1(\Omega)$. For any $E \in \mathcal{M}$, define $P(E)$ to be the operator of pointwise multiplication by the characteristic function of E. Then $P : \mathcal{M} \to L(X)$ is an equicontinuous spectral measure.

Let $Z' = c_0 \subseteq X'$. Then for each $E \in \mathcal{M}$, $P(E)' : X' \to X'$ is again multiplication by the characteristic function of E. Hence, $P(E)'(Z') \subseteq Z'$ for each $E \in \mathcal{M}$, and it follows that $\{P(E) : E \in \mathcal{M}\}$ is a subset of $L((X, \sigma(X, Z')))$. However, $P : \mathcal{M} \to L((X, \sigma(X, Z')))$ is not equicontinuous.

LEMMA 4.2. *Let X be a locally convex Hausdorff space, $\{T_i : i \in \mathcal{I}\}$ a subset of $L(X)$ and W' a subset of X'. Assume that W' separates the points of X, that $T_i'(W') \subseteq W'$ for each $i \in \mathcal{I}$, and that for each $w' \in W'$ there is a constant $\alpha > 0$, such that*

$$\sup\{|\langle T_i(x), w'\rangle| : i \in \mathcal{I}\} \leqslant \alpha|\langle x, w'\rangle|, \tag{3}$$

for each $x \in X$. If Z' is the linear span of W', then $\{T_i : i \in \mathcal{I}\}$ is an equicontinuous part of $L((X, \sigma(X, Z')))$.

Proof. A subset $M' \subseteq Z'$ is equicontinuous for $(X, \sigma(X, Z'))$ if and only if there exist finitely many points $z_1', \ldots, z_k' \in Z'$, such that M' is contained in the convex hull of $\{z_j' : 1 \leqslant j \leqslant k\}$. Hence, the family of operators $\{T_i : i \in \mathcal{I}\}$ is an equicontinuous part of $L((X, \sigma(X, Z')))$ if and only if for each $z' \in Z'$ there exists a constant $\gamma > 0$ and points $z_1', \ldots, z_k' \in Z'$, such that

$$\sup\{|\langle T_i(x), z'\rangle| : i \in \mathcal{I}\} \leqslant \gamma \max\{|\langle x, z_j'\rangle| : 1 \leqslant j \leqslant k\},$$

for each $x \in X$. This inequality follows from (3).

EXAMPLE 4.3. (S. Kakutani [3]). Let $\Omega = \Lambda$ be the Cantor set in [0,1]. Consider the linear manifold $C(\Omega) \otimes C(\Lambda)$ of $C(\Omega \times \Lambda)$ consisting of all finite sums of the form

$$x(s, t) = \sum_{i=1}^{n} f_i(s)g_i(t), \quad (s, t) \in \Omega \times \Lambda, \tag{4}$$

where $f_i \in C(\Omega)$, $g_i \in C(\Lambda)$ for $i = 1, \ldots, n$. Define the norm of an element $x \in C(\Omega) \otimes C(\Lambda)$ by

$$\|x\| = \inf \sum_{i=1}^{n} \|f_i\|_\infty \|g_i\|_\infty, \tag{5}$$

where the infimum is taken over all representations of x in the form (4), and $\|\cdot\|_\infty$ denotes the supremum norm. Let X be the completion of $C(\Omega) \otimes C(\Lambda)$ with respect to the norm (5).

Let $\mathscr{M} = \mathscr{N}$ be the σ-algebra of all closed and open subsets of the Cantor set. Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be defined by letting

$$(P(E)(x))(s, t) = \chi_E(s)x(s, t), \quad (s, t) \in \Omega \times \Lambda,$$

for every $E \in \mathscr{M}$ and $x \in X$, and

$$(Q(F)(x))(s, t) = \chi_F(t)x(s, t), \quad (s, t) \in \Omega \times \Lambda,$$

for every $F \in \mathscr{N}$ and $x \in X$. As shown by S. Kakutani, P and Q are equicontinuous, commuting spectral measures such that the product $R : \mathscr{M} \otimes \mathscr{N} \to L(X)$ is not uniformly bounded, and so the extension of R to $\mathscr{M} \underset{\sigma}{\otimes} N$, with values in $L(X)$, is impossible.

For $w \in \Omega \times \Lambda$ define $w' \in X'$ by $w'(x) = x(w)$, for every $x \in X$. Let $W' = \{w' : w \in \Omega \times \Lambda\}$ and Z' be the linear span of W'. Then $R(G)'(W') \subseteq W'$ for each $G \in \mathscr{M} \otimes \mathscr{N}$ and it follows from Lemma 4.2 that $\{R(G) : G \in \mathscr{M} \otimes \mathscr{N}\}$ is an equicontinuous part of $L((X, \sigma(X, Z')))$. Furthermore, the spectral measures $P : \mathscr{M} \to L((X, \sigma(X, Z')))$ and $Q : \mathscr{N} \to L((X, \sigma(X, Z')))$ are regular with respect to the family of all compact subsets of the Cantor set. Hence, if Y denotes the completion of $(X, \sigma(X, Z'))$, Corollary 3.3 and Proposition 3.4 imply that there exists a spectral measure $\mathscr{K} : \mathscr{M} \underset{\sigma}{\otimes} \mathscr{N} \to L(Y)$, such that $\mathscr{K}(E \times F) = P_Y(E)Q_Y(F)$ for every $E \times F \in \mathscr{M} \times \mathscr{N}$.

Let X be a locally convex Hausdorff space. Let $\mathscr{I}$ be an interval of ordinal numbers. A family, $\{e_i : i \in \mathscr{I}\}$, of points in X is said to be a *basis for X* if the following conditions are satisfied:

(i) For every $x \in X$ there are unique scalars α_i, $i \in \mathscr{I}$, such that

$$x = \sum_{i \in \mathscr{I}} \alpha_i e_i = \lim_j \sum_{i=0}^{j} \alpha_i e_i.$$

(ii) The associated coefficient functionals e_i', $i \in \mathscr{I}$, given by $e_i'(x) = \alpha_i$, whenever x has the expansion (i), belong to X'.

For example, if X is a Banach space then any Schauder basis for X is a basis for X.

LEMMA 4.4. *Let $\{e_i : i \in \mathscr{I}\}$ be a basis for the locally convex Hausdorff space X, $W' = \{e_i' : i \in \mathscr{I}\}$ be the associated coefficient functionals, and Z' be the linear span of W'. Let $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ be commuting spectral measures. If $R(G)'(W') \subseteq W'$ for each $G \in \mathscr{M} \otimes \mathscr{N}$, and if for each $w' \in W'$ there is a positive constant α such that*

$$\sup\{|\langle R(G)(x), w'\rangle| : G \in \mathscr{M} \otimes \mathscr{N}\} \leqslant \alpha |\langle x, w'\rangle|,$$

for all $x \in X$, then the completion of $(X, \sigma(X, Z'))$ is an admissible space for P and Q.

Proof follows from Lemma 4.2 and Theorem 3.2.

EXAMPLE 4.5. (C. A. McCarthy [7]). Let $S_n = T_n = \{1, 2, \ldots, 2^n\}$ for $n = 1, 2, \ldots$, and X_n denote the projective tensor product of the spaces $C(S_n)$ and $C(T_n)$, $n = 1, 2, \ldots$, (cf. Example 4.3). Then X_n is a Banach space of dimension 4^n whose elements can be interpreted as matrices x with entries $x(s, t)$, $(s, t) \in$ $\in S_n \times T_n$.

Let Γ be the disjoint union of the sets $S_n \times T_n$, $n = 1, 2, \ldots$. Let X denote the space of all functions x on Γ such that, if x_n is the restriction of x to $S_n \times T_n$ (considered as an element of X_n) for every $n = 1, 2, \ldots$, then

$$\|x\| = \left(\sum_{n=1}^{\infty} \|x_n\|^2\right)^{1/2} < \infty.$$

If we identify X_n' with X_n, $n = 1, 2, \ldots$, then X' can be identified with X as a vector space.

Let Ω denote the disjoint union of the sets S_n, $n = 1, 2, \ldots$, and Λ denote the disjoint union of the sets T_n, $n = 1, 2, \ldots$. Let $\mathscr{M}$ be the σ-algebra of all subsets of Ω and $\mathscr{N}$ the σ-algebra of all subsets of Λ. Define equicontinuous, commuting spectral measures $P : \mathscr{M} \to L(X)$ and $Q : \mathscr{N} \to L(X)$ by letting

$$(P(E)(x))(s, t) = \chi_E(s)x(s, t), \quad (s, t) \in \Gamma,$$

for every $E \in \mathscr{M}$ and $x \in X$, and

$$(Q(F)(x))(s, t) = \chi_F(t)x(s, t), \quad (s, t) \in \Gamma,$$

for every $F \in \mathscr{N}$ and $x \in X$. It was shown by C. A. McCarthy that the product $R : \mathscr{M} \otimes \mathscr{N} \to L(X)$ is not uniformly bounded.

For each $n = 1, 2, \ldots$, let B_n denote the subset of X given by

$$\{\chi_{\{(s,t)\}} : (s, t) \in S_n \times T_n\}.$$

Define a collection $\{e_m : m = 1, 2, \ldots\} \subseteq X$ inductively as follows:

(i) For $n = 1$, let $\{e_1, e_2, e_3, e_4\}$ be an enumeration of B_1.

(ii) For $n > 1$, let $\{e_{r+1}, \ldots, e_{r+4^n}\}$ be an enumeration of B_n, where

$$r = \sum_{i=1}^{n-1} 4^i.$$

The collection $\{e_m : m = 1, 2, \ldots\}$ is a Schauder basis for X. Let e'_m denote the element e_m, $m = 1, 2, \ldots$, considered as a member of X'. Then $W' = \{e'_m : m = 1, 2, \ldots\}$ is the collection of associated coefficient functionals. If Z' denotes the linear span of W', then the assumptions of Lemma 4.4 hold. Hence, the completion of $(X, \sigma(X, Z'))$ is an admissible space for P and Q.

Acknowledgement. The author thanks Professor I. Kluvánek for the theme of this note and valuable discussions.

REFERENCES

1. Colojoară, I. ; Foiaş, C., *Theory of generalized spectral operators*, Gordon and Breach Science Publishers, 1968.
2. Dunford, N. ; Schwartz, J. T., *Linear operators*. III, Interscience Publishers, New York, 1971.
3. Kakutani, S., An example concerning uniform boundedness of spectral measures, *Pacific J. Math*, **4**(1954), 363—372.
4. Kluvánek, I., The extension and closure of vector measure, *Vector and operator valued measures and applications*, Academic Press Inc., 1973, 175—189.
5. Kluvánek, I. ; Knowles, G., *Vector measures and control systems*, North Holland Publ. Co., 1976.
6. Marczewski, E., On compact measures, *Fund. Math.*, **40**(1953), 113—124.
7. McCarthy, C. A., Commuting Boolean algebras of projections, *Pacific J. Math.*, **11**(1961), 295—307.
8. Oberai, K. K., Sum and product of commuting spectral operators, *Pacific J. Math.*, **25**(1968), 129—146.
9. Thomas, E., The Lebesgue-Nikodym theorem for vector valued Radon measures, *Mem. Amer. Math. Soc.*, **139**(1974).
10. Tumarkin, Ju. B., On locally convex spaces with basis, *Dokl. Akad. Nauk SSSR*, **195**, 1278—1281 ; *Soviet Math. Dokl.*, **11**(1970), 1672—1675.

W. RICKER
School of Mathematical Sciences,
The Flinders University of South Australia,
Bedford Park, 5042,
South Australia.

Received October 9, 1980.