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FINITELY CONNECTED DOMAINS G, REPRESENTATIONS OF $H^{\infty}(G)$, AND INVARIANT SUBSPACES

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Recall that if $\mathscr{A}$ is a subalgebra of $\mathscr{L}(\mathscr{H})$ and $\mathscr{M}$ is a subspace of $\mathscr{H}$ such that $(0) \neq \mathscr{M} \neq \mathscr{H}$ and $B\mathscr{M} \subset \mathscr{M}$ for every B in $\mathscr{A}$, then $\mathscr{M}$ is a *nontrivial invariant subspace for* $\mathscr{A}$. If $\mathscr{A}$ is the algebra of all polynomials in a fixed operator A, then $\mathscr{M}$ is a *nontrivial invariant subspace for* A, and if $\mathscr{A}$ is the commutant of A, then $\mathscr{M}$ is a *nontrivial hyperinvariant subspace for* A.

The question whether every operator in $\mathscr{L}(\mathscr{H})$ has a (nontrivial) invariant subspace is, of course, one of the most important unsolved problems in operator theory. As a result there is considerable interest in whether all operators of a given sort in $\mathscr{L}(\mathscr{H})$ have invariant subspaces.

A big breakthrough in this area was made about three years ago by Scott Brown, who showed in [12] that every subnormal operator in $\mathscr{L}(\mathscr{H})$ has invariant subspaces and simultaneously originated a technique for constructing invariant subspaces that could be applied to a much wider class of operators. Brown's pioneering work was rapidly followed by a sequence of papers exploiting this breakthrough (see the attached bibliography). The present paper is another in that sequence, although herein we take a somewhat different point of view. In any case, this paper should be regarded as a sequel to [13]. In that paper it was shown that if A is a contraction in $\mathscr{L}(\mathscr{H})$ with the property that the intersection of the spectrum $\sigma(A$ of A and the open unit disc D in $\mathbf{C}$ is a dominating subset of D (see § 2 for definitions), then A has invariant subspaces. One of the purposes of this paper is to generalize that theorem. We replace D by an arbitrary bounded domain G in $\mathbf{C}$ and we study operators A for which G^{-} is an M-spectral set for A and $\sigma(A) \cap G$ is a dominating subset of G (see §§ 2 and 4 for definitions). Eventually we show

(Theorem 5.2) that if G is a finitely connected Jordan domain, then a certain algebra of rational functions of A has an invariant subspace, thereby generalizing considerably the main theorem of [13] mentioned above.

Anyone familiar with [12] or [13] (or any of the other papers exploiting the Scott Brown technique listed in the bibliography) knows that the two main components of this technique are 1) the construction of a suitable representation Φ of some $H^\infty(G)$ into $\mathscr{L}(\mathscr{H})$ and 2) the demonstration that the image algebra $\Phi(H^\infty(G))$ has an invariant subspace. Thus, after dealing with some preliminaries in §2, we turn our attention immediately in §3 to the question of when representations of $H^\infty(G)$ (for G an arbitrary bounded domain in $\mathbf{C}$) have nontrivial invariant subspaces, and in Theorem 3.2 we give sufficient conditions on Φ for this to happen. In §4 we show that in the important special case in which G is finitely connected, one of the continuity hypotheses on Φ in Theorem 3.2 may be omitted, thus yielding a better result (Theorem 4.1). In §5 we apply Theorem 4.1 to obtain the invariant subspace theorem for single operators mentioned above (Theorem 5.2). Then in §6 we apply Theorem 5.2 to conclude that many invertible bilateral weighted shift operators have hyperinvariant subspaces (Theorem 6.2). Section 7, which is independent of the preceding sections, is devoted to the construction of an $H^\infty(G)$ functional calculus with certain properties that is needed in §§ 4 and 5. Finally, § 8 consists of some remarks concerning the existence of norm-discontinuous representations of $H^\infty(G)$ and the non-uniqueness of norm-continuous representations. These remarks also shed some light on the interplay between components 1) and 2) of the Scott Brown technique that is an essential feature of the present paper.

2. PRELIMINARIES

In this section we set forth some preliminaries that we shall need later. As far as possible, the notation and terminology used herein will agree with that of [13]. Throughout the paper G will denote a (nonempty) bounded domain in $\mathbf{C}$, and $H^\infty(G)$ will denote, as usual, the Banach algebra of all bounded holomorphic functions on G in the supremum norm:

$$\|f\|_\infty = \sup_{\lambda \in G} |f(\lambda)|, \quad f \in H^\infty(G).$$

A subset Γ of G is said to be a *dominating subset of G* (or a *dominating set for* $H^\infty(G)$) if

$$\|f\|_\infty = \sup_{\lambda \in \Gamma} |f(\lambda)|, \quad f \in H^\infty(G).$$

In case G is the open disc D, the dominating sets were characterized in [11, Theorem 3] as follows: a subset Γ of D is a dominating set in D if and only if almost every boundary point ζ of D is a non-tangential limit point of Γ (that is, ζ is the limit of a sequence in Γ that lies inside some proper angular opening with vertex at ζ).

The space $H^\infty(G)$ may be regarded as a subspace of $L^\infty(G)$ (where the measure in question is Lebesgue area measure on G). The space $L^\infty(G)$ has a weak* topology since it is the dual space of $L^1(G)$, and it is known that $H^\infty(G)$ is a weak* closed subspace of $L^\infty(G)$. (See, for example, the proof of Theorem 4.5 in [25].) Furthermore a sequence $\{f_n\}$ in $H^\infty(G)$ is weak* convergent to zero if and only if it is bounded and converges pointwise to zero on G; cf. [25, 3.11 and 4.6]. Of course the unit ball in $H^\infty(G)$ is compact and metrizable in the weak* topology (since $L^1(G)$ is separable).

The following elementary proposition is an analog of [13, Proposition 3.1].

PROPOSITION 2.1. *If $\lambda_0 \in G$ and E_{λ_0} is the bounded linear functional on $H^\infty(G)$ obtained by setting $E_{\lambda_0}(f) = f(\lambda_0)$, $f \in H^\infty(G)$, then E_{λ_0} is weak* continuous. Moreover, every h in $H^\infty(G)$ can be decomposed as*

$$h(\zeta) = h(\lambda_0) + (\zeta - \lambda_0)\, g(\zeta), \quad \zeta \in G,$$

where $g \in H^\infty(G)$ and satisfies $\|g\|_\infty \leqslant 2\|h\|_\infty/\mathrm{dist}(\lambda_0, \partial G)$.

Recall next that a set S in a complex vector space X is said to be *balanced* if $\alpha S \subset S$ for all complex numbers α such that $|\alpha| \leqslant 1$. The *absolutely convex hull* of a set S is the smallest convex and balanced set containing S. Alternatively it is the collection of all linear combinations $\alpha_1 x_1 + \ldots + \alpha_n x_n$ of vectors $x_1, \ldots, x_n$ in S such that $|\alpha_1| + \ldots + |\alpha_n| \leqslant 1$. We shall need the following analog of [13, Proposition 2.8].

PROPOSITION 2.2. *Let X be a complex Banach space, let M be a positive number, and let E be a subset of X such that for all φ in X^*,*

$$\|\varphi\| \leqslant M \sup_{x \in E} |\varphi(x)|.$$

Then the closure of the absolutely convex hull of E contains the closed ball $\mathscr{B}_{1/M}$ of radius $1/M$ about the origin in X.

Proof. Let $\mathscr{C}$ be the closed absolutely convex hull of E (that is, the closure of the absolutely convex hull of E). Suppose $\mathscr{B}_{1/M}$ is not contained in $\mathscr{C}$, and let x_0 belong to $\mathscr{B}_{1/M} \setminus \mathscr{C}$. By a standard consequence of the Hahn-Banach theorem (cf. [10, Proposition 14.15]), there exists a linear functional φ in X^* and a real number r such that $\mathrm{Re}(\varphi(x)) \leqslant r$ for all x in $\mathscr{C}$, while $\mathrm{Re}(\varphi(x_0)) > r$. Since $0 \in \mathscr{C}$, the number r is nonnegative. For any x in $\mathscr{C}$, write $|\varphi(x)| = \gamma\varphi(x)$ with $|\gamma| = 1$. We have $|\varphi(x)| = \mathrm{Re}(\gamma\varphi(x)) = \mathrm{Re}(\varphi(\gamma x)) \leqslant r$. Thus $|\varphi(x)| \leqslant r$ for all x in $\mathscr{C}$. But then, by the hypothesis on E, we have $\|\varphi\| \leqslant Mr$, contradicting the fact that

$$\|\varphi\| \geqslant |\varphi(x_0/\|x_0\|)| \geqslant M\,\mathrm{Re}(\varphi(x_0)) > Mr.$$

Therefore $\mathscr{C}$ contains $\mathscr{B}_{1/M}$.

We shall also need the following basic fact from the general theory of Banach spaces. For a proof, see [13, Theorems 2.3 and 2.7]. We write (X^*,w^*) for the topological linear space consisting of the dual of a Banach space X with its weak* topology.

PROPOSITION 2.3. *Let X and Y be complex Banach spaces.*

(i) *If S is a continuous linear map from (X^*, w^*) into (Y^*, w^*) with trivial kernel and norm-closed range, then $S(X^*)$ is weak* closed and S is a weak* homeomorphism of X^* onto $S(X^*)$.*

(ii) *If X is separable, to show that a linear mapping $S:(X^*, w^*) \to (Y^*, w^*)$ is continuous, it suffices to show that whenever a sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges to zero in (X^*, w^*), then so does the sequence $\{S\varphi_n\}_{n=1}^{\infty}$ in (Y^*, w^*).*

We now turn our attention to some preparatory material from the theory of operators on Hilbert space that we shall need. Throughout the paper, we will denote the Banach space of trace-class operators in $\mathscr{L}(\mathscr{H})$ under the trace norm $\|\ \ \|_\tau$ by (τc). Recall from [16, Theorem 8, p. 105] that setting

$$\langle A, T\rangle = \operatorname{tr}(AT), \quad A \in \mathscr{L}(\mathscr{H}),\ T \in (\tau c)$$

induces a bilinear functional on $\mathscr{L}(\mathscr{H}) \times (\tau c)$ that allows us to identify $\mathscr{L}(\mathscr{H})$ with $(\tau c)^*$. This identification, which we use hereafter without additional comment, has the further property that the weak* topology defined on $\mathscr{L}(\mathscr{H}) = (\tau c)^*$ coincides with the ultraweak topology on $\mathscr{L}(\mathscr{H})$. (For more information about the ultraweak and ultrastrong operator topologies, see [16, p. 35]. In particular, recall that a sequence $\{A_n\}$ in $\mathscr{L}(\mathscr{H})$ converges ultraweakly [ultrastrongly] to zero if and only if it converges to zero in the weak [strong] operator topology.) It results easily from this that if $\mathscr{A}$ is any ultraweakly closed subspace of $\mathscr{L}(\mathscr{H})$ and ${}^a\mathscr{A}$ denotes the preannihilator of $\mathscr{A}$ in (τc), then $\mathscr{A}$ becomes the dual space of the quotient space $Q = (\tau c)/{}^a\mathscr{A}$ (see [13, Proposition 2.1 and Corollary 2.2]). Furthermore it is easy to see (cf. [13, Corollary 2.4]) that the relative ultraweak topology induced on $\mathscr{A}$ by $\mathscr{L}(\mathscr{H})$ coincides with the weak* topology that accrues to $\mathscr{A}$ by virtue of its being the dual space of Q. This topology will be called interchangeably the ultraweak or weak* topology on $\mathscr{A}$. Moreover, elements of $Q = (\tau c)/{}^a\mathscr{A}$ will be written as equivalence classes $[T]$, where $T \in (\tau c)$, and the quotient norm on Q will be denoted by $\|\ \ \|_{\mathfrak{Q}}$. Of course the duality between $\mathscr{A}$ and Q is implemented by the bilinear functional

$$\langle A, [T]\rangle = \operatorname{tr}(AT), \quad A \in \mathscr{A},\ [T] \in Q.$$

If x and y are nonzero elements of $\mathscr{H}$, we denote by $x \otimes y$ the rank-one operator $u \to (u, y)x$ in $\mathscr{L}(\mathscr{H})$. It is well-known that $\|x \otimes y\| = \|x \otimes y\|_\tau = \|x\|\cdot\|y\|$. Moreover, easy computations show that $\operatorname{tr}(x \otimes y) = (x, y)$ and that if $B \in \mathscr{L}(\mathscr{H})$, then $B(x \otimes y) = Bx \otimes y$.

Finally we recall that the *left essential spectrum* $\sigma_{le}(A)$ of an operator A in $\mathscr{L}(\mathscr{H})$ can be characterized as the set of all those complex numbers λ for which there exists an orthonormal family $\{x_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ satisfying $\|(A - \lambda)x_i\| \to 0$. If A has the property that $\sigma(A)\diagdown\sigma_{le}(A) \neq \emptyset$, then it is easy to see (cf. [24, p. 47]) that either A or A^* has an eigenvalue, and since A cannot be a scalar, the corresponding eigenspace gives rise to a (nontrivial) hyperinvariant subspace for A. Thus in what follows, we will frequently be able to assume of a given operator A that $\sigma_{le}(A) = \sigma(A)$.

3. REPRESENTATIONS OF $H^{\infty}(G)$

Let G be an arbitrary nonempty bounded domain in $\mathbf{C}$. By a *representation* of $H^{\infty}(G)$ we mean any algebra homomorphism of $H^{\infty}(G)$ into $\mathscr{L}(\mathscr{H})$. (Thus representations are not necessarily unital, and, in fact, need not even be norm-continuous; see §8.) If Φ is a representation of $H^{\infty}(G)$, and $\mathscr{M}$ is a nontrivial subspace of $\mathscr{H}$ that is invariant under the algebra $\Phi(H^{\infty}(G))$, we shall say that $\mathscr{M}$ is an *invariant subspace* of Φ. In this section we study the following problem: under what conditions on a representation Φ will it be true that Φ has an invariant subspace ? This problem is fundamental, because the techniques introduced by Scott Brown in [12] and exploited in the sequence of papers listed in the references (for proving the existence of invariant subspaces for certain classes of operators) involve 1) the construction of a suitable representation Φ of some $H^{\infty}(G)$, and 2) the demonstration that Φ has an invariant subspace. Thus any theorem along the lines of 2) reduces an invariant subspace problem to that of constructing a suitable representation as in 1).

Our program begins with an elementary spectral mapping lemma.

LEMMA 3.1. *Let G be a (nonempty) bounded domain in $\mathbf{C}$, and let Φ be a representation of $H^{\infty}(G)$ such that $\Phi(1) = 1_{\mathscr{H}}$. Suppose that f_0 is the position function $f_0(\zeta) = \zeta$ in $H^{\infty}(G)$ and $\Phi(f_0) = A$. Suppose also that $\lambda \in \sigma(A) \cap G$. Then $f(\lambda) \in \sigma(\Phi(f))$ for every f in $H^{\infty}(G)$.*

Proof. Fix f in $H^{\infty}(G)$ and let $\beta = f(\lambda)$. Then $f(\zeta) - \beta = (\zeta - \lambda)g(\zeta)$ where $g \in H^{\infty}(G)$. Thus $\Phi(f) - \beta I = \Phi(f - \beta) = (A - \lambda I)\Phi(g) = \Phi(g)(A - \lambda I)$. Since $A - \lambda I$ is not invertible, either it is not surjective or it is not one-to-one. In either case we see that $\Phi(f) - \beta$ is not invertible, as was to be shown.

The following is the central result of this section, and might be said to be what the proof of [13, Theorem 4.1] really proves.

THEOREM 3.2. *Let G be a (nonempty) bounded domain in $\mathbf{C}$, and let Φ be a representation of $H^{\infty}(G)$ with the following properties:*

(a) *Φ is norm-continuous;* i.e., *there exists a positive number M such that $\|\Phi(f)\| \leqslant M\|f\|_{\infty}$ for every f in $H^{\infty}(G)$,*

(b) *If* $f_0(\zeta) = \zeta$ *is the position function in* $H^\infty(G)$ *and* $\Phi(f_0) = A$, *then* $\sigma(A) \cap G$ *is a dominating subset of G,*

(c) *If* $\{f_n\}_{n=1}^\infty$ *is any sequence in* $H^\infty(G)$ *that converges weak* to zero, then* $\{\Phi(f_n)\}_{n=1}^\infty$ *converges to zero in the strong (equivalently, ultrastrong) operator topology on* $\mathcal{L}(\mathcal{H})$.

Then Φ *has a nontrivial invariant subspace;* i.e., *there is a nontrivial subspace of* $\mathcal{H}$ *that is invariant under the algebra* $\Phi(H^\infty(G))$.

Proof. We note first that it follows from (b) that Φ is not the zero representation. Since $H^\infty(G)$ is commutative, it is clear that the operator $\Phi(1)$ is a nonzero idempotent that commutes with the range $\mathcal{A}$ of the representation Φ, and thus if $\Phi(1) \neq 1_{\mathcal{H}}$, the range of $\Phi(1)$ is the desired subspace invariant under $\mathcal{A}$. Hence we may assume that $\Phi(1) = 1_{\mathcal{H}}$. From Lemma 3.1 and (b) we deduce that

$$(1) \qquad \|f\|_\infty = \sup_{\lambda \in \sigma(A) \cap G} |f(\lambda)| \leqslant \rho(\Phi(f)) \leqslant \|\Phi(f)\|, \quad f \in H^\infty(G),$$

where as usual, $\rho(\Phi(f))$ denotes the spectral radius of $\Phi(f)$. Thus we see that Φ is bounded below by 1 and that $\mathcal{A}$ is norm-closed.

If the operator A in (b) has a (nontrivial) hyperinvariant subspace, then since $\mathcal{A}$ is commutative, this subspace will be the desired invariant subspace for Φ. Since A cannot be a scalar by (b), we may thus suppose that neither A nor A^* has an eigenvalue. In particular, according to what was said in Section 2, we may suppose throughout the remainder of the proof that $\sigma_{le}(A) = \sigma(A)$.

We note that by virtue of (c), Proposition 2.3, and the fact that the ultrastrong topology on $\mathcal{L}(\mathcal{H})$ is stronger than the ultraweak topology, Φ is continuous if both $H^\infty(G)$ and $\mathcal{A}$ are given their weak* topologies. Thus, by virtue of Proposition 2. again, $\mathcal{A}$ is not only a norm-closed subalgebra of $\mathcal{L}(\mathcal{H})$ but is ultraweakly (weak*) closed as well, and Φ is a weak* homeomorphism. Hence, as noted earlier, $\mathcal{A} = Q^*$ where Q is the Banach space $(\tau c)/{}^a\mathcal{A}$. Furthermore if λ is any fixed point in G, the mapping $f \to f(\lambda)$ is a weak* continuous linear functional on $H^\infty(G)$ (Proposition 2.1), and since Φ^{-1} is a weak* homeomorphism of $\mathcal{A}$ onto $H^\infty(G)$, the map $B \to \to \Phi^{-1}(B)(\lambda)$ is a weak* continuous linear functional on $\mathcal{A}$. But, according to [10, Problem 15J], such linear functionals must arise from the dual action of Q on $\mathcal{A}$, and hence there exists an element $[C_\lambda]$ in Q such that

$$\Phi^{-1}(B)(\lambda) = \langle B, [C_\lambda]\rangle = \mathrm{tr}(BC_\lambda), \quad B \in \mathcal{A}.$$

If we write $\Phi^{-1}(B) = f$, this equation becomes

$$(2) \qquad f(\lambda) = \langle \Phi(f), [C_\lambda]\rangle = \mathrm{tr}(\Phi(f)C_\lambda), \; f \in H^\infty(G).$$

Our goal (following the remarkable idea [12] of S. Brown) will be to show that the coset $[C_\lambda]$ contains a representative that is a rank-one operator $x \otimes y$. (In fact we will show more—namely, that every element of Q is of the form $[x \otimes y]$ for appropriate vectors x and y and that every $[C_\lambda]$ is of the form $[z \otimes z]$.) Suppose, for the moment, that we have shown that $[C_\lambda] = [x \otimes y]$ for some vectors x and y. Then (2) becomes

$$f(\lambda) = \operatorname{tr}(\Phi(f)(x \otimes y)) = (\Phi(f)x, y), \quad f \in H^\infty(G). \tag{3}$$

Taking $f = 1$ we see that $x \neq 0$ and $y \neq 0$. Also if we set $g(\zeta) = \zeta - \lambda$ and consider the product fg, then (3) becomes

$$0 = (\Phi(fg)x, y) = (\Phi(f)(A - \lambda)x, y), \quad f \in H^\infty(G).$$

Since the possibility of $x' = (A - \lambda)x$ being 0 was ruled out earlier, it follows that the linear manifold $\mathscr{A}x'$ is invariant under $\mathscr{A}$ and is orthogonal to y. Since this linear manifold contains $\Phi(1)x' = x'$ and thus is nonzero, the subspace $\mathscr{M} = \{\mathscr{A}x'\}^-$ is the desired invariant subspace of Φ. (To see that $[C_\lambda]$ actually has the form $[z \otimes z]$, note that since $(x, y) = 1$, we know that $x \notin \mathscr{M}$, and thus we may write $x = x_1 + x_2$ where $x_1 \in \mathscr{M}$, $x_2 \in \mathscr{M}^\perp$, and $x_2 \neq 0$. Set $z = x_2/\|x_2\|$. To conclude that $[C_\lambda] = [z \otimes z]$ it suffices to show that $(\Phi(f)z, z) = f(\lambda)$ for all functions f in $H^\infty(G)$. This is obvious if $f = 1$, so we may suppose that f has the form $f(\zeta) = g(\zeta)(\zeta - \lambda)$ and show that $\Phi(f)z \in \mathscr{M}$. But this follows from the equation

$$\Phi(f)x_2 = \Phi(g)(A - \lambda)(x - x_1) = \Phi(g)x' - \Phi(g)(A - \lambda)x_1$$

and the fact that $\mathscr{M}$ is invariant under $\mathscr{A}$.

Thus the remainder of the proof of the theorem is devoted to establishing that every element of Q is of the form $[x \otimes y]$. This will be accomplished by proving a sequence of lemmas that closely resembles the sequences in [12] and [13]. The first lemma shows that for any λ in $\sigma(A) \cap G$, $[C_\lambda]$ is at least the limit (in $\|\ \ \|_{\mathscr{Q}}$) of cosets of the form $[x \otimes y]$.

LEMMA 3.3. *Let $\lambda \in \sigma(A) \cap G$. Then there exists an orthonormal sequence $\{x_n\}_{n=1}^\infty$ in $\mathscr{H}$ such that $\|(A - \lambda)x_n\| \to 0$, and for any such sequence, $\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathscr{Q}} \to 0$.*

Proof. Since $\lambda \in \sigma_{le}(A)$, there exists an orthonormal sequence $\{x_n\}_{n=1}^\infty$ in $\mathscr{H}$ such that $\|(A - \lambda)x_n\| \to 0$ (cf. [24, Proposition 2.15]). Since $\mathscr{A}$ is the conjugate space of the Banach space Q, there exists a sequence $\{\Phi(f_n)\}$ in $\mathscr{A}$ such that for each n, $\|\Phi(f_n)\| = 1$ and

$$\begin{aligned} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathscr{Q}} &= \langle \Phi(f_n), [x_n \otimes x_n] - [C_\lambda] \rangle = \\ &= \operatorname{tr}(\Phi(f_n)(x_n \otimes x_n - C_\lambda)). \end{aligned} \tag{4}$$

Let $\beta_n = f_n(\lambda)$ and write $f_n(\zeta) = \beta_n + (\zeta - \lambda) g_n(\zeta)$. Then, by Proposition 2.1, $g_n \in H^\infty(G)$ and $\|g_n\|_\infty \leqslant 2\|f_n\|_\infty/\text{dist}(\lambda, \partial G)$. By virtue of (1), $\|f_n\| \leqslant 1$ for all n, so the sequence $\{g_n\}$ is bounded. Thus from (4) we obtain

$$\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = \text{tr}(\{\beta_n + \Phi(g_n)(A - \lambda)\}\{x_n \otimes x_n - C_\lambda\}) =$$

$$= \beta_n \text{tr}(x_n \otimes x_n - C_\lambda) + \text{tr}(\Phi(g_n)(A - \lambda)(x_n \otimes x_n)) - \text{tr}(\Phi(g_n)(A - \lambda) C_\lambda) =$$

$$= 0 + \text{tr}(\{\Phi(g_n)(A - \lambda)x_n\} \otimes x_n) - 0 = (\Phi(g_n)(A - \lambda)x_n, x_n) \leqslant$$

$$\leqslant \|\Phi(g_n)\| \, \|(A - \lambda)x_n\| \leqslant M\|g_n\| \, \|(A - \lambda)x_n\| \to 0.$$

We have here used the facts that

$$\text{tr}(x_n \otimes x_n) = (x_n, x_n) = 1,$$

$$\text{tr}(C_\lambda) = \langle 1_{\mathcal{H}}, [C_\lambda]\rangle = \langle \Phi(1), [C_\lambda]\rangle = 1,$$

and

$$\text{tr}(\Phi(g_n)(A - \lambda)C_\lambda) = \langle \Phi(g_n(\zeta - \lambda)), [C_\lambda]\rangle = g_n(\lambda)(\lambda - \lambda) = 0.$$

LEMMA 3.4. *Let $\lambda \in \sigma(A) \cap G$ and let $\{x_n\}_{n=1}^\infty$ be any orthonormal sequence such that $\|(A - \lambda)x_n\| \to 0$. Then for any fixed s in $\mathcal{H}$, $\|[x_n \otimes s]\|_{\mathcal{Q}} \to 0$.*

Proof. Just as in the previous lemma there exists a sequence $\{h_n\}_{n=1}^\infty$ in $H^\infty(G)$ such that $\|\Phi(h_n)\| = 1$ and such that

$$\|[x_n \otimes s]\|_{\mathcal{Q}} = \text{tr}(\Phi(h_n)(x_n \otimes s)) = (\Phi(h_n)x_n, s), \quad n \geqslant 1.$$

As before we write $h_n(\zeta) = \beta_n + (\zeta - \lambda)g_n(\zeta)$. Then

$$\|[x_n \otimes s]\|_{\mathcal{Q}} = \beta_n(x_n, s) + (\Phi(g_n)(A - \lambda)\, x_n, s).$$

The first term on the right tends to zero since the sequence $\{\beta_n = h_n(\lambda)\}$ is bounded and $\{x_n\}$ is an orthonormal sequence, and the second term tends to zero as in the previous lemma.

The next lemma is almost a symmetrical version of Lemma 3.4, but the proof is more difficult. It is principally here that condition (c) in the statement of Theorem 3.2 is used.

LEMMA 3.5. *Let $\{x_n\}_{n=1}^\infty$ be any orthonormal sequence in $\mathcal{H}$. Then for every fixed s in $\mathcal{H}$, $\|[s \otimes x_n]\|_{\mathcal{Q}} \to 0$.*

Proof. As before, let $\{h_n\}_{n=1}^\infty$ be a sequence in $H^\infty(G)$ such that for all $n \geqslant 1$, $\|\Phi(h_n)\| = 1$ and

$$\|[s \otimes x_n]\|_{\mathcal{Q}} = \langle \Phi(h_n), [s \otimes x_n]\rangle = \text{tr}(\Phi(h_n)(s \otimes x_n)) = (\Phi(h_n)\, s, x_n).$$

If this sequence did not tend to zero then there would exist $\delta > 0$ and an increasing sequence $\{n_j\}$ of positive integers such that $\|[s \otimes x_{n_j}]\|_{\mathfrak{Z}} > \delta$ for all j. Since $\{h_{n_j}\}$ is a bounded sequence in $H^\infty(G)$, we may assume, by passing to a further subsequence if necessary, that the sequence $\{h_{n_j}\}$ is weak* convergent—say to g. Let $g_{n_j} = h_{n_j} - g$. Then the sequence $\{g_{n_j}\}$ converges weak* to zero, and we have

$$\delta < (\Phi(h_{n_j})s, x_{n_j}) = (\Phi(g_{n_j})s, x_{n_j}) + (\Phi(g)s, x_{n_j}). \tag{5}$$

But the first term on the right side of (5) tends to zero since the sequence $\{\Phi(g_{n_j})\}$ converges strongly to zero by condition (c), and the second term on the right side tends to zero since the sequence $\{x_{n_j}\}$ is orthonormal. This contradiction completes the proof.

LEMMA 3.6. *Let $\lambda_1, \ldots, \lambda_n$ be any finite sequence of (not necessarily distinct) numbers from $\sigma(A) \cap G$. Then there exists a corresponding family $\{x_i^1\}, \ldots, \{x_i^n\}$ of mutually orthogonal, orthonormal sequences such that $\lim_{i\to\infty} \|(A - \lambda_j)x_i^j\| = 0$ for $1 \leqslant j \leqslant n$ and $\lim_{i\to\infty} \|[x_i^j \otimes x_i^k]\|_{\mathfrak{Z}} = 0$ for all $1 \leqslant j, k \leqslant n$ with $j \neq k$. Furthermore, if $\gamma_1, \ldots, \gamma_n$ is any sequence of n scalars and $u_i = \sum_{j=1}^{n} \gamma_j x_i^j$, $v_i = \sum_{j=1}^{n} \overline{\gamma}_j x_i^j$, then*

$$\lim_{i\to\infty} \|[u_i \otimes v_i] - \sum_{j=1}^{n} \gamma_j^2 [C_{\lambda_j}]\|_{\mathfrak{Z}} = 0.$$

Proof. Consider the operator $\lambda_1 \oplus \ldots \oplus \lambda_n$, where each λ_j acts on an infinite dimensional space. Since the set $\{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{le}(A)$, one can apply [9, Theorem A] with $\lambda_1 \oplus \ldots \oplus \lambda_n = N_\varepsilon$ in the notation of that theorem, and it follows easily that there exists a family $\{x_i^1\}, \ldots, \{x_i^n\}$ of mutually orthogonal, orthonormal sequences (i.e., $(x_i^j, x_l^k) = \delta_{jk}\delta_{il}$) such that

$$\lim_{i\to\infty} \|(A - \lambda_j)x_i^j\| = 0 \text{ for all } 1 \leqslant j \leqslant n.$$

Suppose now that $1 \leqslant j, k \leqslant n$ with $j \neq k$. For clarity of notation we set $\lambda = \lambda_j$, $\mu = \lambda_k$, $z_i = x_i^j$ and $y_i = x_i^k$. As in previous lemmas, there exists a sequence $\{h_i\}_{i=1}^\infty$ in $H^\infty(G)$ such that $\|h_i\| \leqslant \|\Phi(h_i)\| \leqslant 1$ and

$$\|[z_i \otimes y_i]\|_{\mathfrak{Z}} = \langle \Phi(h_i), [z_i \otimes y_i] \rangle = (\Phi(h_i)\, z_i, y_i)$$

for each i. Using the decomposition $h_i(\zeta) = h_i(\lambda) + (\zeta - \lambda)\, g_i(\zeta)$ as before, we obtain

$$\|[z_i \otimes y_i]\|_{\mathfrak{Z}} = h_i(\lambda)(z_i, y_i) + (\Phi(g_i)(A - \lambda)\, z_i, y_i) = (\Phi(g_i)(A - \lambda)\, z_i, y_i),$$

and since the sequence $\{g_i\}$ is bounded (by Proposition 2.1), the right hand side tends to zero.

To prove the last assertion in the statement of the lemma, we write

$$[u_i \otimes v_i] = \sum_{j=1}^{n} \gamma_j^2[x_i^j \otimes x_i^j] + \sum_{1 \leqslant j \neq k \leqslant n} \gamma_j\gamma_k[x_i^j \otimes x_i^k].$$

By what has already been proved, the second summand on the right converges to zero (in $\| \ \|_{\mathscr{Q}}$), and by Lemma 3.3, the first summand converges to $\sum_{j=1}^{n} \gamma_j^2[C_{\lambda_j}]$.

LEMMA 3.7. *Let $S \subset Q$ be the closed absolutely convex hull of the set $\{[C_\lambda]: \lambda \in \sigma(A) \cap G\}$. Then S contains the closed ball $\mathscr{B}_{1/M}$ in Q of radius $1/M$ about the origin.*

Proof. For each f in $H^\infty(G)$ we have

$$\|\Phi(f)\| \leqslant M\|f\|_\infty = M \sup_{\lambda \in \sigma(A) \cap G} |f(\lambda)| = M \sup_{\lambda \in \sigma(A) \cap G} |\langle \Phi(f), [C_\lambda]\rangle|$$

by virtue of (2) and the fact that $\sigma(A) \cap G$ is given to be a dominating set for $H^\infty(G)$. The result follows from Proposition 2.2.

LEMMA 3.8. *Let $[L] \in Q$ and suppose that there exist vectors s and s' in $\mathscr{H}$ such that $\|[s \otimes s'] - [L]\|_{\mathscr{Q}} < \varepsilon < 1$. Then there exist vectors t and t' in $\mathscr{H}$ such that*

$$\|s - t\| < (M\varepsilon)^{1/2},$$

$$\|s' - t'\| < (M\varepsilon)^{1/2},$$

and

$$\|[t \otimes t'] - [L]\|_{\mathscr{Q}} < \varepsilon/4.$$

Proof. Let $[K] = [L] - [s \otimes s']$. The result is trivial if $[K] = 0$, so we assume that $d = \|[K]\|_{\mathscr{Q}} > 0$. By Lemma 3.7 there exist points $\lambda_1, \ldots, \lambda_m$ in $\sigma(A) \cap G$ and scalars $\alpha_1, \ldots, \alpha_m$ such that

$$\|[K/dM] - \sum_{j=1}^{m} \alpha_j[C_{\lambda_j}]\|_{\mathscr{Q}} < \varepsilon/8dM, \quad \sum_{j=1}^{m} |\alpha_j| \leqslant 1.$$

For each j choose γ_j so that $\gamma_j^2 = \alpha_j dM$. Then we have

$$\|[K] - \sum_{j=1}^{m} \gamma_j^2[C_{\lambda_j}]\|_{\mathscr{Q}} < \varepsilon/8. \tag{6}$$

By Lemma 3.6 there exist m mutually orthogonal, orthonormal sequences $\{x_i^j\}_{i=1}^{\infty}$, $1 \leqslant j \leqslant m$, such that $\lim_i \|(A - \lambda_j)x_i^j\| = 0$ for $1 \leqslant j \leqslant m$ and such that, if we set $u_i = \sum_{j=1}^{m} \gamma_j x_i^j$ and $v_i = \sum_{j=1}^{m} \bar{\gamma}_j x_i^j$, then

$$\lim_i \|[u_i \otimes v_i] - \sum_{j=1}^{m} \gamma_j^2 [C_{\lambda_j}]\| = 0.$$

Thus, by virtue of (6), we know that for all i sufficiently large,

$$\|[K] - [u_i \otimes v_i]\|_{\mathcal{Q}} < \varepsilon/8.$$

Define $s_i = s + u_i$ and $s_i' = s' + v_i$. We shall show that we can choose $t = s_{i_0}$ and $t' = s_{i_0}'$ for any integer i_0 sufficiently large. Note first that for each i we have

$$\|u_i\|^2 = \|v_i\|^2 = \sum_{j=1}^{m} |\gamma_j|^2 \leqslant dM = M\|K\|_{\mathcal{Q}} < M\varepsilon.$$

Thus for any choice of i_0, if t and t' are chosen as indicated, we have $\|s - t\| < < (M\varepsilon)^{1/2}$ and $\|s' - t'\| < (M\varepsilon)^{1/2}$. On the other hand,

$$[s_i \otimes s_i'] - [L] = [s \otimes v_i] + [u_i \otimes s'] + [u_i \otimes v_i] - [K],$$

so for sufficiently large i,

$$\|[s_i \otimes s_i'] - [L]\|_{\mathcal{Q}} < \|[s \otimes v_i]\|_{\mathcal{Q}} + \|[u_i \otimes s']\|_{\mathcal{Q}} + \varepsilon/8. \tag{7}$$

Since $[s \otimes v_i] = \sum_{j=1}^{m} \gamma_j [s \otimes x_i^j]$ and similarly for $[u_i \otimes s']$, the first two summands on the right side of (7) tend to zero by Lemmas 3.4 and 3.5, so that we may choose i_0 as desired.

The final lemma shows that each nonzero element in Q has a rank-one representative. In particular, this will apply to the functionals $[C_\lambda]$, and thereby complete the proof of Theorem 3.2.

LEMMA 3.9. *If $[L]$ is an arbitrary element of Q, then there exist vectors x and y in $\mathcal{H}$ such that $[L] = [x \otimes y]$.*

Proof. It obviously suffices to prove the lemma for all $[L]$ in Q with $\|[L]\|_{\mathcal{Q}} \leqslant \leqslant 1/M$. If $[L] = 0$, we set $x = y = 0$, so we may suppose that $[L] \neq 0$. Applying Lemma 3.7 to such an $[L]$, we obtain the existence of a finite set $\lambda_1, \ldots, \lambda_m$ in $\sigma(A) \cap G$ and a finite set $\alpha_1, \ldots, \alpha_m$ of scalars such that $\|[L] - \sum_{j=1}^{m} \alpha_j [C_{\lambda_j}]\|_{\mathcal{Q}} < 1/8$. Moreover, if $\gamma_j^2 = \alpha_j$ for each j, then by Lemma 3.6 there exist vectors u and v such that $\|[u \otimes v] - \sum_{j=1}^{m} \alpha_j [C_{\lambda_j}]\|_{\mathscr{Q}} < 1/8$. Thus $\|[L] - [u \otimes v]\| < 1/4$, and we set $u_0 = u_1 = u$, $v_0 = v_1 = v$. Suppose now, by induction, that u_k and v_k have been chosen for $1 \leqslant k \leqslant n$ so as to satisfy

$$\|[L] - [u_k \otimes v_k]\|_{\mathscr{Q}} < 1/2^{2k}, \quad \|u_k - u_{k-1}\| < M^{1/2}/2^{k-1}$$

$$\|v_k - v_{k-1}\| < M^{1/2}/2^{k-1}.$$

Applying Lemma 3.8 with $\varepsilon = 1/2^{2n}$, $s = u_n$, and $s' = v_n$, we obtain $u_{n+1} = t$ and $v_{n+1} = t'$ satisfying the above inequalities for $k = n + 1$. We have thus constructed by induction two sequences $\{u_n\}$ and $\{v_n\}$ which are obviously Cauchy and satisfy $\|[u_n \otimes v_n] - [L]\|_{\mathscr{Q}} \to 0$. Let $x = \lim u_n$ and $y = \lim v_n$. Since

$$\|[u_n \otimes v_n] - [x \otimes y]\|_{\mathscr{Q}} \leqslant \|u_n \otimes v_n - x \otimes y\|_\tau \leqslant$$

$$\leqslant \|u_n \otimes (v_n - y)\|_\tau + \|(u_n - x) \otimes y\|_\tau =$$

$$= \|u_n\| \, \|v_n - y\| + \|u_n - x\| \, \|y\| \to 0,$$

we have $[x \otimes y] = [L]$, and the proofs of the lemma and of Theorem 3.2 are complete.

The authors conjecture that the version of Theorem 3.2 obtained by dropping hypothesis (c) on Φ is true. In the next section, we prove this in a special case.

4. FINITELY CONNECTED DOMAINS

We consider now the case in which the bounded domain G of Theorem 3.2 is finitely connected (meaning that its boundary has finitely many components). We prove that for such G the hypothesis (c) in Theorem 3.2 can be deleted, thus yielding the following result.

THEOREM 4.1. *Let G be any (nonempty) bounded, finitely connected, domain in* **C**, *and let Φ be a representation of $H^\infty(G)$ into $\mathscr{L}(\mathscr{H})$ having properties* (a) *and* (b) *in the statement of Theorem* 3.2. *Then Φ has a nontrivial invariant subspace.*

Proof. We shall call a *circular domain* any domain that is obtained by removing a finite number of disjoint closed discs (some perhaps of radius zero) from the open unit disc D. Recall from [20, Theorem 2, p. 237] that there exists a conformal mapping θ of G onto some circular domain Ω. The map $\Theta : f \to f \circ \theta$ is easily seen to implement an isometric Banach algebra isomorphism between $H^\infty(\Omega)$ and $H^\infty(G)$. The mapping $\Phi_1 = \Phi \circ \Theta$ is a representation of $H^\infty(\Omega)$ into $\mathscr{L}(\mathscr{H})$ having the same range as Φ, and thus Φ_1 has exactly the same invariant subspaces as Φ. We next show that Φ_1 satisfies hypotheses (a) and (b) in the statement of Theorem 3.2, and therefore that it suffices to prove the present theorem in the special case in which G is a circular domain. As shown at the beginning of the proof of Theorem 3.2, we may assume that $\Phi(1_G) = 1_{\mathscr{H}}$ and thus that $\Phi_1(1_\Omega) = 1_{\mathscr{H}}$. Let $A = \Phi(f_{0,G})$ and $B = \Phi_1(f_{0,\Omega})$, where $f_{0,G}$ and $f_{0,\Omega}$ are the position functions in $H^\infty(G)$ and $H^\infty(\Omega)$, respectively. Since $B = \Phi(\theta)$, we derive from Lemma 3.1 the inclusion $\theta(\sigma(A) \cap G) \subset \subset \sigma(B)$, and since θ must carry dominating sets in G onto dominating sets in Ω, we conclude that $\sigma(B) \cap \Omega$ is dominating in Ω. Furthermore it is clear that Φ_1 is norm-continuous. In other words, Φ_1 satisfies hypotheses (a) and (b) of Theorem 3.2 as was to be shown.

Thus we may now suppose that G is the circular domain $D\diagdown(D_1^- \cup \ldots \cup D_n^-)$ where n is some nonnegative integer, and, in case $n > 0$, the $D_k^- = \{\zeta : |\zeta - \zeta_k| \leqslant r_k\}$, $k = 1, \ldots, n$, are disjoint closed subdiscs of D. Furthermore we may suppose, without loss of generality, that each r_k, $1 \leqslant k \leqslant n$, is a positive number, because if $r_k = 0$ for some k, then ζ_k is a removable singularity for all functions in $H^\infty(G)$, and G may be replaced by $G \cup \{\zeta_k\}$. Of course we may again assume that $\Phi(1_G) = = 1_{\mathscr{H}}$. Next set $G^* = \{\bar{\zeta} : \zeta \in G\}$, and observe that naturally associated with Φ is an "adjoint" representation $\Phi^\sim$ of $H^\infty(G^*)$ defined as follows. For f in $H^\infty(G)$ let f^* be defined on G^* by $f^*(\lambda) = \overline{f(\bar{\lambda})}$. Then f^* belongs to $H^\infty(G^*)$, and the map $f \to f^*$ sets up an isometric Banach algebra anti-isomorphism between $H^\infty(G)$ and $H^\infty(G^*)$. Now set $\Phi^\sim(f^*) = [\Phi(f)]^*$. Then $\Phi^\sim$ is a representation of $H^\infty(G^*)$ which has a nontrivial invariant subspace if and only if Φ does. (More precisely, the ortho-complement of an invariant subspace for $\Phi^\sim$ is invariant for Φ, and vice-versa.) Observe also that $\Phi^\sim$ satisfies hypotheses (a) and (b) of Theorem 3.2 (with G replaced by G^*). Therefore if either Φ or $\Phi^\sim$ satisfies hypothesis (c) of Theorem 3.2, the proof is complete. Suppose then that neither Φ nor $\Phi^\sim$ satisfies that hypothesis. Let $A_0 = A\ (=\Phi(f_0))$ and for $1 \leqslant k \leqslant n$, set $A_k = r_k(A - \zeta_k)^{-1}\ (=\Phi(f_k)$ where $f_k(\lambda) = r_k(\lambda - \zeta_k)^{-1})$. To conclude the proof we will show that one of $A_0, \ldots, A_n$ has a nontrivial hyperinvariant subspace (such a subspace is clearly invariant for Φ). By Theorem 7.2 the fact that Φ does not satisfy hypothesis (c) means that there exists some i, $0 \leqslant i \leqslant n$, such that the sequence $\{A_i^m\}_{m=1}^\infty$ does not tend strongly to 0. Similarly there exists some j, $0 \leqslant j \leqslant n$, such that $\{A_j^{*m}\}_{m=1}^\infty$ does not tend strongly to 0. Thus the subspaces $\mathscr{M} = \{x \in \mathscr{H} : \|A_i^m x\| \to 0\}$ and $\mathscr{N} = \{x \in \mathscr{H} : \|A_j^{*m} x\| \to 0\}$ are both different from $\mathscr{H}$. (Note that $\mathscr{M}$ and $\mathscr{N}$ are closed because the operators A_k, $k = 0, \ldots, n$, are all power bounded, which in turn happens because Φ is norm-continuous and the f_k are bounded by 1.) It is well-known and easy to prove that the subspaces $\mathscr{M}$ and $\mathscr{N}$ are hyperinvariant for A_i and A_j^*, respectively; thus we may assume that $\mathscr{M} = \mathscr{N} = (0)$ (otherwise $\mathscr{M}$ or $\mathscr{N}^\perp$ is the desired nontrivial hyperinvariant subspace). For similar reasons we may assume that all of the operators A_k, $k = 0, \ldots, n$, and their adjoints have trivial kernels. Therefore (see Proposition 5.3 and especially Equation 5.13, p. 80, of [30]), there exist operators X

and Y with trivial kernels and dense ranges and there exist unitary operators U and V such that $XA_i = UX$ and $YA_j^* = VY$. We now show by elimination that $i = j$. Suppose first that $i = 0$ and $j \geqslant 1$. From the equation $YA_j^* = VY$ we obtain successively

$$r_j Y(A^* - \bar{\zeta}_j)^{-1} = VY,$$

$$r_j V^* Y = Y(A^* - \bar{\zeta}_j),$$

$$(\bar{\zeta}_j + r_j V^*)Y = YA^*,$$

$$Y^*(\zeta + r_j V) = AY^*.$$

This last equality combined with $XA = UX$ gives

$$(XY^*)(\zeta_j + r_j V) = U(XY^*).$$

Since XY^* is nonzero (and in fact is a quasiaffinity), the spectra of U and $\zeta_j + r_j V$ must overlap [23]. But $\sigma(U) \subset \partial D$, $\sigma(\zeta_j + r_j V) \subset \partial D_j$, and these two circles are disjoint. Thus we cannot have $i = 0$ and $j \neq 0$. By an entirely analogous argument, we eliminate the possibility that $j = 0$ and $i \neq 0$. Suppose now that i and j are distinct and nonzero. As before, $YA_j^* = VY$ leads to

$$Y^*(\zeta_j + r_j V) = AY^*$$

and $XA_i = UX$ yields successively

$$r_i X(A - \zeta_i)^{-1} = UX,$$

$$r_i U^* X = X(A - \zeta_i),$$

$$(\zeta_i + r_i U^*)X = XA.$$

Combining the first and last equations, we obtain

$$(XY^*)(\zeta_j + r_j V) = (\zeta_i + r_i U^*)(XY^*)$$

and a contradiction results from the disjointness of the spectra of $\zeta_j + r_j V$ and $\zeta_i + r_i U^*$, which are contained in ∂D_j and ∂D_i, respectively. Therefore we have, necessarily, $i = j$. But in this case, we obtain, again by [30, Proposition 5.3, p. 79], that A_i is quasisimilar to a unitary operator. Since A_i cannot be scalar because of the richness of its spectrum, A_i must have a nontrivial hyperinvariant subspace, and the proof is complete.

5. INVARIANT SUBSPACES FOR SINGLE OPERATORS

In this section we will apply Theorem 4.1 to obtain the existence of invariant subspaces for a class of operators that is much larger than the class treated in [13]. We begin by recalling some notation and terminology. If X is a nonempty compact subset of $\mathbf{C}$, we denote as usual by $\mathrm{C}(X)$ the Banach algebra of all continuous, complex-valued functions on X under the supremum norm. Moreover, we denote by $\mathrm{R}(X)$ the closure in $\mathrm{C}(X)$ of the subalgebra of all rational functions with poles off X. If A is an operator in $\mathscr{L}(\mathscr{H})$, $\sigma(A) \subset X$, and r is any rational function with poles off X, then, of course, $r(A)$ is well-defined as the quotient of polynomials. (It is well-known that this definition of $r(A)$ coincides with the definition of $r(A)$ given by the Riesz-Dunford functional calculus; cf. Example Q, p. 393, of [10].) We shall say of an operator A satisfying $\sigma(A) \subset X$ that X is an M-spectral set for A if M is a positive number such that

$$\|r(A)\| \leqslant M \sup_{\lambda \in X} |r(\lambda)| \tag{8}$$

for all rational functions r with poles off X. If X is an M-spectral set for A, then the rational functional calculus $r \to r(A)$ has a unique continuous extension to $\mathrm{R}(X)$ which comes about as follows: if $f \in \mathrm{R}(X)$, then there exists a sequence $\{r_n\}_{n=1}^{\infty}$ of rational functions with poles off X converging to f uniformly on X. It follows from (8) that the sequence $\{r_n(A)\}$ is Cauchy in the uniform topology on $\mathscr{L}(\mathscr{H})$ and that its limit is independent of the particular sequence $\{r_n\}$. Thus $f(A)$ may be defined to be this uniform limit, and it is an easy exercise using (8) to show that the mapping $f \to f(A)$ is a Banach algebra homomorphism of $\mathrm{R}(X)$ into $\mathscr{L}(\mathscr{H})$ of norm no greater than M (in other words, (8) remains valid for all the functions in $\mathrm{R}(X)$). It will be convenient to denote the range of this homomorphism by $\mathscr{R}_X(A)$. Note that a function f in $\mathrm{R}(X)$ need not be analytic on a neighborhood of $\sigma(A)$, and thus that the $\mathrm{R}(X)$-functional calculus is not subsumed by the Riesz-Dunford functional calculus. The following proposition shows that, nevertheless, the $\mathrm{R}(X)$-functional calculus enjoys some of the properties of the more familiar Riesz-Dunford calculus.

Proposition 5.1. *Let A be an operator in $\mathscr{L}(\mathscr{H})$ and suppose that X is an M-spectral set for A. Then for any f in* $\mathrm{R}(X)$ *we have*

a) $f(\sigma(A)) \subset \sigma(f(A)) \subset f(X)$,

b) $f(X)$ *is an M-spectral set for* $f(A)$,

c) $(g \circ f)(A) = g(f(A))$ *for all g in* $\mathrm{R}(f(X))$,

and

d) $\mathscr{R}_{f(X)}(f(A)) \subset \mathscr{R}_X(A)$.

Proof. If $f \in \mathrm{R}(X)$ and if ζ does not belong to $f(X)$, then the function $f(\lambda) - \zeta$ is invertible in $\mathrm{R}(X)$. (This follows, for example, from the fact that the maximal ideal space of $\mathrm{R}(X)$ can be identified with X; cf. [29, p. 136].) Therefore the operator

$f(A) - \zeta$ is invertible in $\mathscr{L}(\mathscr{H})$. This proves the second inclusion in a). Let now r_n be a sequence of rational functions with poles off X converging uniformly to f and let ζ be in $\sigma(A)$. It follows from the spectral mapping theorem for rational functions that, for each n, $r_n(A) - r_n(\zeta)$ is not invertible. Since $f(A) - f(\zeta) = \lim_{n\to\infty} (r_n(A) - r_n(\zeta))$, the operator $f(A) - f(\zeta)$ is not invertible; thus $f(\zeta) \in \sigma(f(A))$. This proves the first inclusion and completes the proof of a).

Next suppose that s is a rational function with poles off $f(X)$, and let J be a neighborhood of $f(X)$ such that J^- does not contain any poles of s. There exists a sequence $\{r_n\}$ of rational functions with poles off X that converges uniformly to f on X. For n large enough the sets $r_n(X)$ are contained in J, and therefore without loss of generality we may assume that $r_n(X) \subset J$ for all n. Since s is uniformly continuous on J^-, the sequence $\{s \circ r_n\}$ converges uniformly to $s \circ f$ on X. Hence $s \circ f$ belongs to $\mathrm{R}(X)$ and $\|(s\circ f)(A) - (s \circ r_n)(A)\| \to 0$. But $(s \circ r_n)(A) = s(r_n(A))$ by a well-known property of the Riesz-Dunford functional calculus, cf. [10, Proposition 17.28, p. 394]. Furthermore the continuity of this functional calculus ([10, Proposition 17.26, p. 393]), together with the fact that $\|r_n(A) - f(A)\| \to 0$, implies that $\|s(r_n(A)) - s(f(A))\| \to 0$. Thus we have $(s\circ f)(A) = s(f(A))$ and

$$\|s(f(A))\| = \|(s\circ f)(A)\| \leqslant M \sup_{\zeta\in X} |(s\circ f)(\zeta)| = M \sup_{\lambda\in f(X)} |s(\lambda)|.$$

Therefore $f(X)$ is an M-spectral set for $f(A)$, and b) is proved.

The equality to be proved in c) was established in b) in case g is a rational function with poles off $f(X)$. Suppose now that g is an arbitrary function in $\mathrm{R}(f(X))$, and let s_n be a sequence of rational functions in $\mathrm{R}(f(X))$ converging uniformly to g on $f(X)$. Clearly $\{s_n \circ f\}$ converges uniformly to $g \circ f$ on X; thus $g \circ f \in R(X)$ and

$$(g\circ f)(A) = \lim_n (s_n \circ f)(A) = \lim_n s_n(f(A)) = g(f(A)),$$

where the last equality follows from the fact (proved in b)) that $f(X)$ is an M-spectral set for $f(A)$ and the definition of $g(f(A))$. This completes the proof of c).

Finally, d) is an immediate consequence of c), and thus the proof is complete.

The following is our main result concerning invariant subspaces of single operators. Recall that a *finitely connected Jordan domain* is a finitely connected domain whose (entire) boundary consists of the union of a finite number of pairwise disjoint Jordan loops.

THEOREM 5.2. *Let A be an operator in $\mathscr{L}(\mathscr{H})$ for which there exist a finitely connected Jordan domain G in $\mathbf{C}$ and a positive number M such that G^- is an M-spectral set for A and such that $\sigma(A) \cap G$ is dominating in G. Then there is a nontrivial subspace $\mathscr{M}$ of $\mathscr{H}$ that is invariant under the algebra $\mathscr{R}_{G^-}(A)$ (consisting of all operators $f(A)$ where f runs over* $\mathrm{R}(G^-)$).

Proof. Let θ be a conformal mapping of G onto a circular domain Ω. Since G is a finitely connected Jordan domain, an argument similar to that in the simply connected case (cf. [20, p. 44]) shows that θ extends to a homeomorphism (which we continue to call θ) of G^- onto Ω^-. Therefore θ belongs to $\mathrm{R}(G^-)$ [18, Theorem 2.10.4], and we set $B = \theta(A)$. Since $\theta^{-1} \circ \theta = f_{0,G^-}$, the position function on G^-, it follows from c) of Proposition 5.1 that $A = \theta^{-1}(B)$, from d) of the same proposition that $\mathscr{R}_{G^-}(A) = \mathscr{R}_{\Omega^-}(B)$, and from b) that Ω^- is an M-spectral set for B. Finally, an argument entirely similar to that used in the proof of Theorem 4.1 shows that $\Omega \cap \sigma(B)$ is dominating in Ω. We conclude from these considerations that it suffices to prove Theorem 5.2 in the case in which G is a circular domain. Suppose then that $G = D \setminus (D_1^- \cup \ldots \cup D_n^-)$ where n is a nonnegative integer and where, if $n > 0$, $D_k^- = \{\zeta : |\zeta - \zeta_k| \leqslant r_k\}$ for $k = 1, \ldots, n$. We may suppose, without loss of generality, that all of the r_k, $k = 1, \ldots, n$, are positive, because if some $D_k^- = \{\zeta_k\}$, then we may replace G by $G \cup \{\zeta_k\}$ and all of the hypotheses remain valid on this larger domain. Set $A_0 = A$, and if $n > 0$, set $A_k = r_k(A - \zeta_k)^{-1}$, $k = 1, \ldots, n$. It is easy to see that all of the A_k, $0 \leqslant k \leqslant n$, are power bounded, and the argument used at the end of the proof of Theorem 4.1 shows that at least one of the following three statements is true:

(i) $\{A_k^m\}_{m=1}^{\infty}$ tends strongly to 0 for $k = 0, \ldots, n$,

(ii) $\{A_k^{*m}\}_{m=1}^{\infty}$ tends strongly to 0 for $k = 0, \ldots, n$,

(iii) there exists some i, $0 \leqslant i \leqslant n$, such that A_i has a nontrivial hyperinvariant subspace.

Of course if (iii) holds, the proof is complete, since the algebra $\mathscr{R}_{G^-}(A)$ commutes with A_i. On the other hand, (ii) is (i) where A has been replaced by A^*, and clearly the hypotheses of Theorem 5.1 hold for A^* (with G replaced by G^*). Moreover the orthocomplement of a subspace invariant under $\mathscr{R}_{(G^*)^-}(A^*)$ is invariant under $\mathscr{R}_{G^-}(A)$. Therefore we may assume that (i) holds. By Theorem 7.3 there exists a norm-continuous representation $\Phi : H^\infty(G) \to \mathscr{L}(\mathscr{H})$ that satisfies $\Phi(r) = r(A)$ for all r in $\mathrm{R}(G^-)$. Since $\sigma(A) \cap G$ is dominating in G, Φ satisfies the hypotheses of Theorem 4.1 and therefore, by that theorem, Φ has a nontrivial subspace which is certainly invariant under the smaller algebra $\mathscr{R}_{G^-}(A)$. Thus the proof is complete.

As a special case of the above theorem we obtain a generalization of the main result of [13] due to Apostol [5]. Recall that an operator A is said to be *polynomially bounded* if there exists a positive number M such that the closed unit disc D^- is an M-spectral set for A.

COROLLARY 5.3 ([5]). *If A is any polynomially bounded operator such that $\sigma(A) \cap D$ is a dominating subset of D, then A has a nontrivial invariant subspace.*

It is worthwhile to compare Theorem 5.2 with the main result of [28], which says that if $A \in \mathscr{L}(\mathscr{H})$ and $\sigma(A)$ is an M-spectral set for A, then A has a nontrivial invariant subspace. Neither theorem implies the other, since each covers cases to which the other is not applicable. However, in case both results can be applied, that is, if $\sigma(A)$ is not only an M-spectral set for A, but in addition, is so situated with respect to some finitely connected Jordan domain G that $\sigma(A) \cap G$ is a dominating subset of G, then Theorem 5.2 yields the existence of a subspace $\mathcal{M}$ that is invariant under all rational functions of A with poles off G^-.

We also remark that a forthcoming paper [6], which continues the study of representations of $H^\infty(G)$, contains results, based on Theorem 3.2, that generalize Theorems 4.1 and 5.2 of the present paper as well as the main theorem of [28] mentioned above.

6. BILATERAL WEIGHTED SHIFTS

In this section we apply Theorem 5.2 to obtain invariant subspaces for some special classes of operators—in particular, we establish the existence of hyperinvariant subspaces for a large class of invertible bilateral weighted shift operators.

If A is any invertible operator in $\mathcal{L}(\mathcal{H})$, then we may associate with A two annuli (possibly circles): the *norm annulus*

$$\mathrm{N}(A) = \{\zeta \in \mathbf{C} : 1/\|A^{-1}\| \leqslant |\zeta| \leqslant \|A\|\}$$

and the *spectral annulus*

$$\mathrm{S}(A) = \{|\zeta| : 1/\rho(A^{-1}) \leqslant |\zeta| \leqslant \rho(A)\},$$

where, as usual, $\rho(B)$ denotes the spectral radius of an operator B. In general, all that can be said about the relationship between $\sigma(A)$, $\mathrm{S}(A)$, and $\mathrm{N}(A)$ is that

$$\sigma(A) \subset \mathrm{S}(A) \subset \mathrm{N}(A).$$

However, if $\mathrm{N}(A)$ is a circle, then $\mathrm{S}(A) = \mathrm{N}(A)$ and A must be a scalar multiple of a unitary operator, so A is either a scalar itself or A has a good supply of nontrivial hyperinvariant subspaces. Thus, when looking for invariant subspaces for A, no generality is lost by assuming that $\mathrm{N}(A)$ is an annulus with nonvoid interior. In this case, it follows from [27, Proposition 23] that there exists a positive number M such that $\mathrm{N}(A)$ is an M-spectral set for A. These considerations, together with Theorem 5.2, immediately yield the following result.

THEOREM 6.1. *Let A be an invertible operator in $\mathcal{L}(\mathcal{H})$, and suppose that the interior $\mathrm{N}(A)^\circ$ of $\mathrm{N}(A)$ is nonvoid and that $\sigma(A) \cap \mathrm{N}(A)^\circ$ is a dominating subset of $\mathrm{N}(A)^\circ$. Then the algebra $\mathcal{R}_{\mathrm{N}(A)}(A)$ has a nontrivial invariant subspace. In particular, if A is any invertible operator such that $\sigma(A) = \mathrm{N}(A)$, then there is a nontrivial subspace $\mathcal{M}$ of $\mathcal{H}$ invariant under both A and A^{-1}.*

Recall next that a *bilateral weighted shift* is an operator A in $\mathscr{L}(\mathscr{H})$ that maps each vector in some orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$ of $\mathscr{H}$ to a scalar multiple of the next vector:

(9) $$Ae_n = w_n e_{n+1}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

There is no loss of generality in assuming that the weight sequence $\{w_n\}$ consists of nonnegative numbers [27, p. 51], and if this sequence is not bounded away from zero (i.e., if A is not invertible), it is known [27, p. 91] that every invariant subspace of A is hyperinvariant for A. Thus the hyperinvariant subspace problem for bilateral weighted shift operators is unresolved only for invertible shifts. Recall that if A is any invertible bilateral weighted shift, then $\sigma(A) = \mathrm{S}(A)$ [27, p. 67].

THEOREM 6.2. *Every invertible bilateral weighted shift operator A for which $\sigma(A)$ is an M-spectral set for A has a nontrivial hyperinvariant subspace. In particular, every invertible bilateral weighted shift A satisfying $\sigma(A) = \mathrm{N}(A)$ has a nontrivial hyperinvariant subspace.*

Proof. If A is any invertible bilateral weighted shift, one knows from [27, p. 91] that the commutant of A is the strongly closed algebra generated by A and A^{-1}. Thus, to show that a subspace $\mathscr{M}$ is hyperinvariant for A, it suffices to show that $\mathscr{M}$ is invariant under A and A^{-1}. Consider first the case in which $\sigma(A)$ is a circle. Then, upon multiplication of A by a scalar, we may suppose that $\sigma(A)$ is the unit circle. In this case, the assumption that $\sigma(A)$ is an M-spectral set for A (for some $M > 0$) tells us that both A and A^{-1} are power bounded, and hence by a theorem of Sz.-Nagy (or, alternatively, by Theorem 6.3), A is similar to a unitary operator. Thus we may suppose that $\sigma(A)$ has nonvoid interior. We may now apply Theorem 5.2 with $G = \sigma(A)^\circ$ to conclude that there is a nontrivial subspace $\mathscr{M}$ of $\mathscr{H}$ that is invariant under the algebra $\mathscr{R}_{\sigma(A)}(A)$, and therefore invariant under A and A^{-1}, as was to be shown. The final assertion now follows from the fact, noted above, that $\mathrm{N}(A)$ is always an M-spectral set for A.

We remark that it is easy to give examples of invertible bilateral weighted shifts A for which $\sigma(A)$ is an M-spectral set for A but $\sigma(A) \neq \mathrm{N}(A)$. Perhaps the easiest way to see this is to begin with a shift A satisfying $\sigma(A) = \mathrm{N}(A)$ and increase one weight dramatically, thereby obtaining a shift A' satisfying $\mathrm{N}(A') \neq \mathrm{N}(A)$. One knows from the criterion for similarity of such shifts [25, p. 54] that A' is similar to A, and thus $\sigma(A') = \sigma(A) = \mathrm{N}(A) \neq \mathrm{N}(A')$. Furthermore, $\sigma(A')$ is easily seen to be an M'-spectral set for A' (for some $M' > 0$) because of this similarity.

The following result shows that the above construction exemplifies the only way such a situation can occur. In other words, more precisely: Up to similarity, the class of invertible bilateral weighted shifts A for which there exists an $M > 0$ such that $\sigma(A)$ is an M-spectral set for A coincides exactly with the class of invertible bilateral shifts A for which $\sigma(A) = \mathrm{N}(A)$.

THEOREM 6.3. *Let A be a bilateral weighted shift such that $\sigma(A) = \{\zeta : 0 < r \leqslant |\zeta| \leqslant R\}$ and suppose that there exists an $M > 0$ such that*

$$\|A^n\| \leqslant MR^n, \quad n = 1, 2, \ldots,$$

and

$$\|A^{-n}\| \leqslant M\left(\frac{1}{r}\right)^n, \qquad n = 1, 2, \ldots .$$

Then A is similar to a bilateral weighted shift B whose weight sequence $\{v_n\}$ satisfies $r \leqslant v_n \leqslant R$ for all integers n.

The argument is based on the following proposition, which is what the proof of [27, Corollary, p. 75] actually proves.

PROPOSITION 6.4. *Let T be a power bounded bilateral weighted shift with positive weight sequence $\{w_n\}$. Then there is a bilateral weighted shift S which is similar to T and whose weight sequence $\{v_n\}$ satisfies $\inf_{k \in \mathbf{Z}} w_k \leqslant v_n \leqslant 1$ for all n.*

Proof of Theorem 6.3. As usual, we may suppose that the weight sequence $\{w_n\}$ of A consists of positive numbers. We apply Proposition 6.4 to the power bounded shift A/R and conclude that A/R is similar to a bilateral weighted shift whose weight sequence $\{u_n\}$ satisfies $(\inf_{k \in \mathbf{Z}} w_n)/R \leqslant u_n \leqslant 1$. In other words, A is similar to a shift B whose weight sequence $\{v_n\}$ satisfies $\inf_{n \in \mathbf{Z}} w_n \leqslant v_n \leqslant R$. Note that the shift B^{-1} (whose weight sequence is $1/v_{n-1}$) satisfies $\|B^{-n}\| \leqslant M'\left(\frac{1}{r}\right)^n$ for some $M' > 0$ since B^{-1} is similar to A^{-1}. Applying Proposition 6.4 again (to rB^{-1}), we see that B^{-1} is similar to a bilateral weighted shift C whose weights lie between $1/R$ and $1/r$. Hence A is similar to C^{-1}, whose weight sequence satisfies the desired inequalities.

We now give an example to show that there are shifts A that satisfy the hypotheses of Theorem 6.2 and also have the property that neither A nor A^* has point spectrum (so Theorem 6.2 is not trivial). We arrange the example to show also that Theorem 6.2 is not covered by Atzmon's result [7, Theorem 5.1] on hyperinvariant subspaces for bilateral weighted shifts.

EXAMPLE 6.5. Let A be a bilateral weighted shift with weight sequence $\{w_n\}$ defined as follows:

$$w_0 = 1,$$

$$w_n = 1 \quad \text{for } 2^{2i} \leqslant n < 2^{2i+1}, \ i = 0, 1, 2, \ldots$$

$$w_n = 2 \quad \text{for } 2^{2i+1} \leqslant n < 2^{2i+2}, \ i = 0, 1, 2, \ldots$$

$$w_{-n} = w_{n-1}, \quad n = 1, 2, \ldots .$$

Clearly $\mathrm{N}(A) = \{\zeta : 1 \leqslant |\zeta| \leqslant 2\}$, and since there are arbitrarily long strings of 1's and arbitrarily long strings of 2's in the weight sequence, $\|A^n\| = 2^n$ and $\|A^{-n}\| = 1$ for all positive integers n. Thus $\sigma(A) = \mathrm{N}(A)$ and Theorem 6.2 is applicable. To show that the point spectra of A and A^* are empty, we must show, in the notation of [27, p. 71], that $r_2^- < r_3^+$ and $r_2^+ < r_3^-$. Because of the symmetry of the weight sequence ($w_{-n} = w_{n-1}$), we have $r_2^- = r_2^+$ and $r_3^- = r_3^+$. Furthermore, a straightforward calculation, which we omit, shows that $r_3^+ = 2^{2/3}$ and $r_2^+ = 2^{1/3}$. Thus the desired inequalities hold and the point spectra of both A and A^* are empty. Next set $\alpha_n = w_0 \cdot w_1 \ldots w_{n-1}$ for all positive integers n. Then, since $2^{1/3} = r_2^+ = \liminf_n \alpha_n^{1/n}$ (by definition), we see that if ε is any small positive number, there are only a finite number of positive integers n such that $\alpha_n^{1/n} < 2^{1/3} - \varepsilon$. Thus eventually we have $\alpha_n \geqslant (2^{1/3} - \varepsilon)^n$, and hence the series $\sum_{n=0}^{\infty} \frac{\log \alpha_n}{n^2 + 1}$ diverges. This shows that the operator A does not satisfy the hypotheses of [7, Theorem 5.1], and therefore that Theorem 6.2 is not a consequence of Atzmon's result. (On the other hand, his result applies to certain bilateral weighted shifts A such that $\sigma(A) = \partial D$ and such that A^n is not power bounded, and such operators clearly do not satisfy the hypotheses of Theorem 6.2.)

We remark, finally, that Herrero's result [21, Theorem 3] that obtains the existence of hyperinvariant subspaces for certain invertible bilateral weighted shifts is an immediate corollary of Theorem 6.2.

7. AN $H^\infty(\Omega)$ FUNCTIONAL CALCULUS

This section, which is independent of Sections 4, 5, and 6, contains material on the construction and continuity properties of certain representations that is needed in the proofs of Theorems 4.1 and 5.2. The reason for presenting this material separately was precisely to facilitate the exposition of those proofs.

We begin by establishing a decomposition theorem for spaces of the form $H^\infty(\Omega)$ where Ω is a circular domain (as defined in §4). If U is any unbounded domain in $\mathbf{C}$ whose complement is bounded, we denote by $H^\infty(U)$ the Banach algebra of all bounded analytic functions on U under the supremum norm and by $H_0^\infty(U)$ the closed subalgebra of $H^\infty(U)$ consisting of those functions that vanish at ∞. It is easily verified that $H^\infty(U)$ is a weak* closed subspace of $L^\infty(U)$ (where the measure on U is planar Lebesgue measure) and that a sequence in $H^\infty(U)$ is weak* convergent to zero if and only if it is bounded and converges pointwise to zero on U.

Suppose now that $\Omega = D \setminus (D_1^- \ldots D_n^-)$ is a circular domain in $\mathbf{C}$, where n is a positive integer and for $i = 1, \ldots, n$, $D_i = \{\zeta : |\zeta - \zeta_i| < r_i\}$ ($r_i > 0$). We set $\Omega_0 = D$, $\Omega_i = \mathbf{C} \setminus D_i^-$, $i = 1, \ldots, n$, and observe that there is a natural isometric embedding of each of the spaces $H^\infty(\Omega_i)$, $i = 0, \ldots, n$, into $H^\infty(\Omega)$ obtained simply by restricting a function f in $H^\infty(\Omega_i)$ to the domain Ω. The isometric character of these embeddings follows immediately from the maximum modulus principle. Moreover, a sequence $\{g_k\}$ converges pointwise boundedly to 0 in $H^\infty(\Omega_i)$ for some $i = 0, \ldots, n$, if and only if the restricted sequence $\{g_k|\Omega\}$ converges pointwise boundedly to 0 in $H^\infty(\Omega)$; see [20, Theorem 2, p. 18]. Thus the embeddings are weak* homeomorphisms (Proposition 2.3). Henceforth in this section we shall frequently regard $H^\infty(\Omega_i)$, $i = 0, \ldots, n$, as subspaces of $H^\infty(\Omega)$ without further comment.

THEOREM 7.1. *Let $\Omega = D\setminus(D_1^- \cup \ldots \cup D_n^-)$ be a circular domain in $\mathbf{C}$, where n is a positive integer and for $i = 1, \ldots, n$, $D_i = \{\zeta : |\zeta - \zeta_i| < r_i\}$ $(r_i > 0)$. Then there exist norm-continuous and weak* continuous projections (that is, idempotents) $P_i : H^\infty(\Omega) \to H^\infty(\Omega)$, $i = 0, \ldots, n$, such that*

i) $\sum_{i=0}^{n} P_i = 1$,

ii) *the range of P_0 is $H^\infty(D)$ (regarded as a subspace of $H^\infty(\Omega)$),*

and

iii) *for $i=1, \ldots, n$, the range of P_i is $H_0^\infty(\Omega_i)$ (where $\Omega_i = \mathbf{C}\setminus D_i^-$ and $H^\infty(\Omega_i)$ is regarded as a subspace of $H^\infty(\Omega)$).*

Proof. Let ε be a positive number chosen sufficiently small that $\varepsilon < \inf\{\text{dist}(\partial D_i, \partial D_j) : 0 \leqslant i, j \leqslant n;\ i \neq j\}$, where we set $D_0 = D$. Also let Γ_0 denote the circle $\left\{\zeta : |\zeta| = 1 - \frac{\varepsilon}{2}\right\}$ parameterized so as to be positively oriented, and for $i=1, \ldots, n$, let Γ_i denote the circle $\left\{\zeta : |\zeta - \zeta_i| = r_i + \frac{\varepsilon}{2}\right\}$ parameterized so as to be negatively oriented. Furthermore let $\text{Int}(\Gamma_i)$ and $\text{Ext}(\Gamma_i)$ denote the bounded and unbounded components of $\mathbf{C}\setminus\Gamma_i$ respectively. Finally, for any h in $H^\infty(\Omega)$, set

$$h_0(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_0} \frac{h(\zeta)}{\zeta - \lambda}\, d\zeta, \quad \lambda \in \text{Int}(\Gamma_0), \tag{10}$$

and

$$h_i(\lambda) = \frac{1}{2\pi i}\int_{\Gamma_i} \frac{h(\zeta)}{\zeta - \lambda}\, d\zeta, \quad \lambda \in \text{Ext}(\Gamma_i),\ i = 1, \ldots, n. \tag{11}$$

It is any easy consequence of the definition of the h_i and the Cauchy integral formula that h_0 is analytic in $\text{Int}(\Gamma_0)$ and can be extended to be analytic on $D_0 = D$, while for $i = 1, \ldots, n$, h_i is analytic on $\text{Ext}(\Gamma_i)$, satisfies $h_i(\infty) = 0$, and can be extended to be analytic on Ω_i. Moreover it follows from the Cauchy integral formula for h computed on $\Gamma_0 + \Gamma_1 \ldots + \Gamma_n$ that

$$h(\lambda) = h_0(\lambda) + \ldots + h_n(\lambda), \quad \lambda \in \Omega,\ h \in H^\infty(\Omega). \tag{12}$$

Since each h_i, $i = 1, \ldots, n$, is obviously bounded near ∂D and h is bounded on Ω, it follows from (12) and the maximum modulus principle that $h_0 \in H^\infty(D)$. Using this fact and (12) again, one sees easily that each $h_i \in H_0^\infty(\Omega_i)$, $i = 1, \ldots, n$. Furthermore the Cauchy integral theorem shows that if $h \in H^\infty(D)$ [respectively, $h \in H_0^\infty(\Omega_i)$ for some $i = 1, \ldots, n$], then $h = h_0$ [respectively, $h = h_i$]. This proves that the maps $P_i : H^\infty(\Omega) \to H^\infty(\Omega)$ defined by $P_i h = h_i$ (where h and the h_i are as in (10) and (11)), $i = 0, \ldots, n$, are idempotents satisfying i), ii), and iii) in the statement of the theorem.

To see that P_0 is norm-continuous we must show that there exists $M_0 > 0$ such that $|h_0(\lambda)| \leqslant M_0 \|h\|$, $\lambda \in D$, for all h in $H^\infty(\Omega)$. By the maximum modulus principle it suffices to prove this for λ close to ∂D. For such λ, and for ε sufficiently small, we have $\lambda \in \operatorname{Int}(\Gamma_0) \cap \operatorname{Ext}(\Gamma_1) \cap \ldots \cap \operatorname{Ext}(\Gamma_n)$ and

$$(P_0 h)(\lambda) = h_0(\lambda) = h(\lambda) - \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{\Gamma_i} \frac{h(\zeta)}{\zeta - \lambda} \, d\zeta. \tag{13}$$

The desired constant M_0 (depending only on the geometry of Ω) is now obtained easily from standard integral estimates upon letting ε tend to 0. Thus P_0 is norm-continuous, and similar considerations show that all of the P_i, $i = 1, \ldots, n$, are also norm-continuous. To prove the weak* continuity of the P_i, it suffices to prove sequential weak* continuity since $H^\infty(\Omega)$ is the dual of a separable Banach space (Proposition 2.3). Thus suppose that $\{g_k\}_{k=1}^\infty$ is a bounded sequence of functions in $H^\infty(\Omega)$ that converges pointwise to zero on Ω (and therefore uniformly to zero on every compact subset of Ω, since the family $\{g_k\}$ is normal). Considering the case $i = 0$ and writing $g_{k,0} = P_0(g_k)$, we see that the sequence $\{g_{k,0}\}$ is bounded in $H^\infty(D)$ since P_0 is norm-continuous. To prove that $g_{k,0}(\lambda) \to 0$ for all λ in D, it is enough to prove this for λ near ∂D [20, Theorem 2, p. 18]. For such λ, and for all sufficiently small ε, we have $\lambda \in \operatorname{Int}(\Gamma_0) \cap \operatorname{Ext}(\Gamma_1) \cap \ldots \cap \operatorname{Ext}(\Gamma_n)$, and we may represent $g_{k,0}(\lambda)$ by (13). The result now follows from the fact that $g_k \to 0$ uniformly on each $\Gamma_i (i = 1, \ldots, n)$. Thus P_0 is weak* continuous, and similar considerations show the same is true of the other P_i, $i = 1, \ldots, n$. Thus the proof is complete.

We observe that the above theorem essentially proves that there is a norm--bicontinuous linear isomorphism between $H^\infty(\Omega)$ and the direct sum

$$H^\infty(D) \oplus H_0^\infty(\Omega_1) \oplus \ldots \oplus H_0^\infty(\Omega_n)$$

(given the (ℓ_1) norm) that is also a weak* homeomorphism.

We turn now to applications of Theorem 7.1 to the theory of representations. The first result gives a necessary and sufficient condition for a norm-continuous representation of $H^\infty(\Omega)$ (where Ω is either D or as in Theorem 7.1) to map sequences converging weak* to zero in $H^\infty(\Omega)$ to sequences converging strongly to zero in

$\mathcal{L}(\mathcal{H})$. This result was used in the proof of Theorem 4.1. We state and prove it in a slightly more general form that will enable us to avoid repeating a nearly identical argument later.

THEOREM 7.2. *Let Ω be either D or the circular domain $D\setminus(D_1^- \cup \ldots \cup D_n^-)$ in the statement of Theorem 7.1, and let L be a subalgebra of $H^\infty(\Omega)$ which is either $H^\infty(\Omega)$ or the algebra R of all rational functions whose poles belong to the set $\{\infty, \zeta_1, \ldots, \zeta_n\}$ ($= \{\infty\}$ in case $\Omega = D$). Furthermore, let Φ be a norm-continuous algebra homomorphism of L into $\mathcal{L}(\mathcal{H})$ such that $\Phi(1_\Omega) = 1_\mathcal{H}$, and let $f_0, f_1, \ldots, f_n$ be the functions in L defined by $f_0(\zeta) = \zeta$ and (in case $\Omega \neq D$) $f_j(\zeta) = r_j(\zeta-\zeta_j)^{-1}$, $j = 1, \ldots, n$. Then the following conditions are equivalent:*

i) *Each of the sequences $\{\Phi(f_0^k)\}$, (and in case $\Omega \neq D$) $\{\Phi(f_1^k)\}, \ldots, \{\Phi(f_n^k)\}$ converges strongly to 0 in $\mathcal{L}(\mathcal{H})$,*

ii) *If $\{g_k\}$ is any sequence in L converging weak* to 0, then the sequence $\{\Phi(g_k)\}$ converges strongly to 0 in $\mathcal{L}(\mathcal{H})$.*

Proof. Since each of the sequences $\{f_j^k\}_{k=1}^\infty, j=0, \ldots, n$, is bounded and converges pointwise to 0 on Ω, it is clear that ii) implies i).

To show that i) implies ii), suppose first that $\Omega = D$, and note that R is the algebra of polynomials in this case. Let $\{g_k\}$ be a bounded sequence in L converging pointwise to 0 on D, and let $\varepsilon > 0$ and $x \neq 0$ in $\mathcal{H}$ be given. By hypothesis we may choose N large enough so that $\|\Phi(f_0^N)x\| < \varepsilon/4MC$ where $M=\|\Phi\|$ and $C > > \sup_k\|g_k\|$. We write

$$g_k(\lambda) = p_k(\lambda) + s_k(\lambda)\lambda^N, \qquad \lambda \in D, \tag{14}$$

where p_k is a polynomial of degree at most $N-1$ and $s_k \in L$. More precisely,

$$p_k(\lambda) = \sum_{j=0}^{N=1} \frac{g_k^{(j)}(0)}{j!} \lambda^j,$$

and since $\{g_k\}$ converges weak* to 0, each sequence of Taylor coefficients $\{g_k^{(j)}(0)/j!\}_{k=0}^\infty$ tends to zero also. (This can be seen by using the Cauchy integral formula for derivatives.) Thus $\|p_k\| \to 0$, and for k large enough we have $\|p_k\| \leqslant \inf\{\varepsilon/2M\|x\|, C\}$. For these values of k we obtain from (14) that $\|s_k\| \leqslant 2C$ and consequently

$$\|\Phi(g_k)x\| \leqslant \|\Phi(p_k)\|\,\|x\| + \|\Phi(s_k)\|\,\|\Phi(f_0^N)x\| \leqslant$$
$$\leqslant M(\varepsilon/2M\|x\|)\,\|x\| + M2C(\varepsilon/4MC) = \varepsilon.$$

Thus $\lim_k \|\Phi(g_k)x\| = 0$, as was to be proved.

We turn now to the case in which $\Omega=D\setminus(D_1^- \cup \ldots \cup D_n^-)$, and we will show that, roughly speaking, the argument can be reduced to the case already treated. To this end, we will associate with Φ $n+1$ representations $\Phi_j, j = 0, \ldots, n$, of L_0

into $\mathscr{L}(\mathscr{H})$ where L_0 is the subalgebra of $H^\infty(D)$ equal to $H^\infty(D)$ if $L=H^\infty(G)$ and equal to the algebra of polynomials if $L = R$. This is done as follows. Set $\Omega_0 = D$ and $\Omega_j=\mathbf{C}\setminus D_j^-$ for $j=1, \ldots, n$, and recall that we are regarding the spaces $H^\infty(\Omega_j)$, $j = 0, \ldots, n$, as subspaces of $H^\infty(\Omega)$. Since for each $j=0, \ldots, n$, f_j is a conformal mapping of Ω_j onto D, the map $h \to h\circ f_j$ is an isometric Banach algebra isomorphism of $H^\infty(D)$ onto $H^\infty(\Omega_j)$ that is also a weak* homeomorphism. We define Φ_j, $j = 0, \ldots, n$, by $\Phi_j(h) = \Phi(h\circ f_j)$ for all h in L_0. (Observe that if $L = R$, then $h\circ f_j$ belongs to L for every h in L_0, and thus $\Phi(h\circ f_j)$ is defined.) It is easily seen that each Φ_j is an algebra homomorphism of L_0 into $\mathscr{L}(\mathscr{H})$ such that $\Phi_j(1_D)= 1_{\mathscr{H}}$, and we have

$$\|\Phi_j(h)\| = \|\Phi(h\circ f_j)\| \leqslant \|\Phi\|\,\|h\circ f_j\|_\Omega = \|\Phi\|\|h\|_D, \quad j = 0, \ldots, n,$$

so all of the Φ_j are norm-continuous. Observe also that $\Phi_j(f_0) = \Phi(f_j)$, $j= 0, \ldots, n$.

Let now $\{g_k\}$ be a sequence in L converging weak* to 0, and let $h_{k,j} = (P_j g_k)\circ f_j^{-1}$, $0 \leqslant j \leqslant n$, $1 \leqslant k < +\infty$, where the P_j are the projections given by Theorem 7.1. It is easily checked, using the partial fraction decomposition of a rational function and the definition of the P_j, that $P_j(R)$ is the set of rational functions of the form $p\circ f_j$, where p is a polynomial vanishing at 0. Thus, whether L is $H^\infty(\Omega)$ or R, it is still the case that the functions $h_{k,j}$ all belong to L_0. Since the P_j as well as the maps $h \to h\circ f_j^{-1}$ (from $H^\infty(\Omega_j)$ onto $H^\infty(D)$) are weak* continuous, each sequence $\{h_{k,j}\}_{k=1}^\infty$, $0 \leqslant j \leqslant n$, converges weak* to 0 in $H^\infty(D)$. Moreover, by definition of the Φ_j, we have $\Phi_j(h_{k,j}) = \Phi(P_j g_k)$ and therefore $\Phi(g_k) = \sum_{j=0}^{n} \Phi_j(h_{k,j})$. Thus it suffices to show that each sequence $\{\Phi_j(h_{k,j})\}_{k=1}^\infty$, $0 \leqslant j\leqslant n$, converges strongly to 0 in $\mathscr{L}(\mathscr{H})$. But this follows from what was already proved in the case $\Omega =D$ treated earlier, since the sequence $\{\Phi_j(f_0^k)(=\Phi(f_j^k))\}$ converges strongly to 0. Thus the proof is complete.

We are now ready to construct the functional calculus promised by the title of this section and used in Theorem 5.2.

THEOREM 7.3. *Let Ω be either D or the circular domain $\Omega=D\setminus(D_1^- \cup \ldots \cup D_n^-)$ in the statement of Theorem 7.1. Furthermore, let A be an operator in $\mathscr{L}(\mathscr{H})$ such that Ω^- is an M-spectral set for A for some positive number M. Suppose, finally, that the sequence of powers $\{A_j^m\}_{m=1}^\infty$ of each operator $A_0 = A$, (and in case $\Omega \neq D$) $A_j = r_j(A - \zeta_j)^{-1}$, $1 \leqslant j \leqslant n$, converges strongly to 0 in $\mathscr{L}(\mathscr{H})$. Then there exists a unique norm-continuous representation Φ of $H^\infty(\Omega)$ such that $\Phi(1) = 1_{\mathscr{H}}$ and $\Phi(f_0) = A$. Moreover Φ has the property that whenever a sequence $\{g_k\}$ converges weak* to 0 in $H^\infty(\Omega)$, then $\{\Phi(g_k)\}$ converges strongly to 0 in $\mathscr{L}(\mathscr{H})$.*

Proof. Let R denote the subalgebra of $H^\infty(\Omega)$ consisting of those rational functions whose poles belong to $\{\infty, \zeta_1, \ldots, \zeta_n\}$ ($= \{\infty\}$ in case $\Omega = D$). The map $\Phi_1 : R\to \mathscr{L}(\mathscr{H})$ defined by $\Phi_1(s) = s(A)$ is a norm-continuous algebra homomorphism of norm at most M, since Ω^- is an M-spectral set for A. Moreover, $\Phi_1(1) = 1_{\mathscr{H}}$, $\Phi_1(f_0) = A$, and, for $j = 1, \ldots, n$, $\Phi_1(f_j) = A_j$ where the f_j are defined as in Theorem 7.2. We now apply Theorem 7.2 with $L = R$ to conclude that Φ_1 satisfies condition ii) in the statement of that theorem.

Note that by Runge's theorem (cf. [10, p. 421]), R is norm-dense in $\mathrm{R}(\Omega^-)$, and by [19], for any g in $H^\infty(\Omega)$, there exists a sequence $\{s_k\}$ in $\mathrm{R}(\Omega^-)$ such that $\{s_k\}$ converges pointwise to g on Ω and $\|s_k\| \leqslant \|g\|$ for all k. Thus the weak* closure of the unit ball in R is the unit ball in $H^\infty(\Omega)$.

We shall now use these facts to extend Φ_1 to a representation Φ defined on all of $H^\infty(\Omega)$. To this end, let $g \in H^\infty(\Omega)$, and let $\{r_n\}$ be a bounded sequence of functions from R that converges pointwise to g on Ω. Since $r_n - r_m \to 0$ pointwise on Ω as $m, n \to \infty$, it follows easily from the continuity property of Φ mentioned above (property ii) in Theorem 7.2) that the sequence $\{\Phi(r_n)\}$ is Cauchy in the strong operator topology on $\mathscr{L}(\mathscr{H})$. Since $\mathscr{L}(\mathscr{H})$ is sequentially complete in this topology, the sequence $\{\Phi(r_n)\}$ is strongly convergent to some operator B in $\mathscr{L}(\mathscr{H})$, and we define $\Phi(g) = B$. If $\{s_n\}$ is any other bounded sequence of functions in R that converges pointwise to g on Ω, then $\{r_n - s_n\}$ converges weak* to 0 in $H^\infty(\Omega)$, so $\{\Phi(s_n)\}$ also converges to B in the strong operator topology. Thus the definition of $\Phi(g)$ is independent of which bounded sequence of rational functions in R converging pointwise to g is used. In particular this shows that Φ extends Φ_1, i.e., $\Phi(s) = \Phi_1(s)$ for every rational function s in R, since s can be approached by the constant sequence $\{s, s, s, \ldots\}$.

That Φ is linear now follows trivially from its definition and the linearity of Φ_1, and that Φ is multiplicative follows from a similar argument and the well-known fact that if $\{B_n\}$ and $\{C_n\}$ are sequences of operators in $\mathscr{L}(\mathscr{H})$ converging strongly to B_0 and C_0, respectively, then $\{B_nC_n\}$ converges strongly to B_0C_0. Thus Φ is a representation of $H^\infty(\Omega)$ such that $\Phi(1) = 1_{\mathscr{H}}$ and $\Phi(f_0) = A$ (where f_0 is the position function on Ω). To see that Φ is bounded by M and thus is norm-continuous, recall from above that if $g \in H^\infty(\Omega)$, then there exists a sequence $\{r_n\}$ of functions from R converging weak* to g and satisfying $\sup_n \|r_n\| = \|g\|$. Thus

$$\|\Phi(r_n)\| = \|\Phi_1(r_n)\| \leqslant M\|r_n\| \leqslant M\|g\|,$$

and since closed balls in $\mathscr{L}(\mathscr{H})$ are also closed in the strong operator topology, we have $\|\Phi(g)\| \leqslant M\|g\|$.

That Φ maps weak* convergent sequences in $H^\infty(\Omega)$ to strongly convergent sequences in $\mathscr{L}(\mathscr{H})$ now is an immediate consequence of the hypothesis and Theorem 7.2 (applied in the case in which $L = H^\infty(\Omega)$).

Finally, the fact that there is a unique norm-continuous representation Φ of $H^\infty(\Omega)$ mapping 1 to $1_{\mathscr{H}}$ and f_0 to A now follows immediately from the following facts: Φ is uniquely determined on R, Φ must have property ii) of Theorem 7.2, and R is sequentially weak* dense in $H^\infty(\Omega)$. Thus the proof is complete.

We close this section with three remarks concerning Theorem 7.3.

REMARK 7.4. Note that, in particular, Theorem 7.3 produces an $H^\infty(D)$ functional calculus for any polynomially bounded operator A in $\mathcal{L}(\mathcal{H})$ whose powers tend strongly to 0.

REMARK 7.5. There is a composition theorem (whose proof we omit) for the functional calculus developed in Theorem 7.3 analogous to Proposition 5.1 (c) and [30, Chapter III, Theorem 2.1 (e)] that goes as follows. If $h \in H^\infty(\Omega)$ and $h(\Omega) \subset \subset \Omega$, then Ω^- is an M-spectral set for $B = h(A)$ $(=\Phi(h))$, and it is easy to see that all the hypotheses of Theorem 7.3 are satisfied with A replaced by B, so that $f(B)$ is defined for all f in $H^\infty(\Omega)$. Furthermore $(f \circ h)(A) = f(B)$ for all such f.

REMARK 7.6. If in the hypothesis of Theorem 7.3 we replace the assumption that for each $j = 0, \ldots, n$, $\{A_j^k\}_{k=1}^\infty$ converges strongly to 0 by the weaker assumption that for each $j=0, \ldots, n$, either $\{A_j^k\}_{k=1}^\infty$ tends strongly to 0 or $\{A_j^{*k}\}_{k=1}^\infty$ tends strongly to 0, one can prove the existence of a unique norm-continuous representation Φ of $H^\infty(\Omega)$ such that $\Phi(1) = 1_{\mathcal{H}}$ and $\Phi(f_0) = A$, but the last conclusion is weakened as follows: If $\{g_k\}$ converges weak* to 0 in $H^\infty(\Omega)$, then $\{\Phi(g_k)\}$ converges to 0 in the weak operator topology.

8. CONCLUDING REMARKS

We would like to thank Professor Garth Dales for pointing out to us that there exist norm-discontinuous representations of $H^\infty(D)$ (into $\mathcal{L}(\mathcal{H})$). Such representations may even be taken to be one-to-one. Indeed Corollary 2 on page 339 of [2] remains valid when the disc algebra is replaced by $H^\infty(D)$ and the convolution algebra $L^1[0, 1]$ is replaced by the algebra $L^2[0,1]$ with the same convolution multiplication. Thus there exists a norm-discontinuous one-to-one homomorphism of $H^\infty(D)$ into the algebra obtained by adjoining a unit to the convolution algebra $L^2[0,1]$. Thus, considering the elements of $L^2[0,1]$ as operators (acting by convolution) on the Hilbert space $L^2[0,1]$, we obtain a discontinuous one-to-one representation of $H^\infty(D)$ into $\mathcal{L}(L^2[0,1])$. For further details on discontinuous homomorphisms and on situations where continuity is automatic, see, for example, [14] and [15].

In a different direction, it is interesting to inquire whether another variant of Theorem 7.3 analogous to that mentioned in Remark 7.6 is valid. More precisely, if one assumes of a polynomially bounded operator A that the sequence $\{A^k\}$ converges to 0 in the weak operator topology, can one hope to construct a representation Φ of $H^\infty(D)$ that maps 1 to $1_{\mathcal{H}}$ and f_0 to A and has the additional property that Φ maps weak* convergent sequences in $H^\infty(\Omega)$ to sequences in $\mathcal{L}(\mathcal{H})$ converging in the weak operator topology? The following example shows that this is not always possible.

EXAMPLE 8.1. We will establish the existence of a unitary operator A and a sequence of polynomials $\{p_k\}$ such that $A^k \to 0$ in the weak operator topology, $p_k \to 0$ weak* in $H^\infty(D)$, but $p_k(A) \to 1_{\mathscr{H}}$ in the weak operator topology. Since every unitary operator is polynomially bounded, and all representations Φ of $H^\infty(D)$ that map 1 to $1_{\mathscr{H}}$ and f_0 to A must take the same value at any polynomial, this will provide the desired counterexample. It is a result of Menšov (see [32, Theorem IX.6.14]) that there exists a probability measure μ supported on a subset of ∂D that is singular with respect to arclength measure on ∂D and has the further property that $\int \zeta^k d\mu \to 0$. It follows at once that $\int \zeta^k t(\zeta) d\mu \to 0$ for all trigonometric polynomials t, and since these functions are dense in $L^1(\mu)$, it follows easily that $\int \zeta^k h(\zeta) d\mu \to 0$ for every function h in $L^1(\mu)$. If $A = M_\zeta$, multiplication by the position function on $L^2(\mu)$, then A is obviously unitary and that $A^k \to 0$ in the weak operator topology follows from what was shown above. On the other hand, a result of Sarason [26] shows that there exists a bounded sequence of polynomials $\{p_n\}$ in $H^\infty(D)$ that converges pointwise to 0 on D but also converges weak* to 1 in $L^\infty(\mu)$. Thus $p_k(A) \to 1_{L^2(\mu)}$ in the weak operator topology, and the argument is complete.

Another, related question, to which we do not know the answer, is the following: If B is a polynomially bounded operator such that $B^k \to 0$ in the weak operator topology, can there exist two different norm-continuous representations of $H^\infty(D)$ each of which sends 1 to $1_{\mathscr{H}}$ and the position function f_0 to B ?

D. Sarason has pointed out to the authors that if one drops the requirement that $\{B^k\}$ converges weakly to 0, then the answer is affirmative. Indeed, let μ be a nontrivial representing measure for $H^\infty(D)$, supported in the fiber over the point $\zeta = 1$ in the maximal ideal space $\mathscr{M}$ of $H^\infty(D)$. (See the last chapter of [22] for a discussion of the maximal ideal space of $H^\infty(D)$.) Let $\mathscr{H} = L^2(\mu)$. To construct the first representation, we let the functions in $H^\infty(D)$ act on $L^2(\mu)$ by multiplication. Then the range of this representation is an infinite dimensional subalgebra of $\mathscr{L}(\mathscr{H})$. However the functions in the disc algebra are constant on each fiber, and so each such function f corresponds to a scalar multiple $f(1)1_{\mathscr{H}}$ of the identity operator. In particular, the operator B corresponding to f_0 is the identity operator, which is polynomially bounded but whose powers do not tend weakly to 0.

To construct the second representation, we let each f in $H^\infty(D)$ correspond to a scalar multiple of the identity operator: $f \to \left(\int \hat{f} d\mu\right) 1_{\mathscr{H}}$. (The multiplicativity of this map follows from the fact that μ is a representing measure.) This representation is different from the previous one but agrees with it on the polynomials.

One might think that another way to produce two different representations of $H^\infty(D)$ that agree on the polynomials would be to start with one representation and compose it with an algebra endomorphism of $H^\infty(D)$ that leaves each polynomial fixed. However, it will be a consequence of the following result that any such endomorphism is the identity. We regard the open unit disc as being embedded in the maximal ideal space $\mathcal{M}$ of $H^\infty(D)$.

PROPOSITION 8.2. *Let Ψ be a nonzero algebra endomorphism of $H^\infty(D)$. Then there exists a continuous map F of $\mathcal{M}$ into itself such that*

$$\widehat{\Psi(f)} = \hat{f}\circ F, \quad f \in H^\infty(D), \tag{14}$$

where $\hat{}$ *denotes the Gelfand transform. Furthermore Ψ is norm-continuous. Finally, we have the following dichotomy: Either*

a) $F(D) \subset D$, *in which case $F|D$ belongs to $H^\infty(D)$, and Ψ is weak* sequentially continuous,*

or

b) $F(\mathcal{M})$ *is contained in the fiber over some boundary point. In this case $\Psi(f)$ is a constant function for each f in the disc algebra, and Ψ is not weak* sequentially continuous.*

Proof. First observe that $\Psi(1) = 1$. (Indeed, $\Psi(1)$ must be an idempotent; hence it must be either zero or the identity, and if it were zero then Ψ would be identically zero.) Since $\Psi(1) = 1$ and $\lambda(1) = 1$ for any $\lambda \in \mathcal{M}$, $\lambda\circ\Psi$ is a nonzero multiplicative linear functional on $H^\infty(D)$, and thus belongs to $\mathcal{M}$. We define $F: \mathcal{M} \to \mathcal{M}$ by $F(\lambda) = \lambda\circ\Psi$. In terms of the Gelfand transform, this means that $\hat{f}(F(\lambda)) = \widehat{\Psi(f)}(\lambda)$; in other words, (14) is valid. That F is continuous is an immediate consequence of the definition of the topology on $\mathcal{M}$. (Recall that a net λ_α in $\mathcal{M}$ converges to λ if and only if $\hat{f}(\lambda_\alpha)$ converges to $\hat{f}(\lambda)$ for all f in $H^\infty(D)$). We have

$$\|\Psi(f)\| = \|\widehat{\Psi(f)}\|_{\mathcal{M}} = \|\hat{f}\circ F|_{\mathcal{M}} \leqslant \|\hat{f}\|_{\mathcal{M}} = \|\_\|,$$

so Ψ is norm-continuous and $\|\Psi\| \leqslant 1$. In fact, $\|\Psi\| = 1$ since $\Psi(1) = 1$. Thus the first part of the theorem is proved.

Observe next that for any g in $H^\infty(D)$, $\hat{g}|D = g$. Since $\Psi(f)$ belongs to $H^\infty(D)$ we obtain from (14) that $(\hat{f}\circ F)|D$ belongs to $H^\infty(D)$ for any f in $H^\infty(D)$. Suppose first that $F(D) \subset D$. Then, since $\hat{f}_0|D = f_0$, we obtain $(\hat{f}_0\circ F)|D = (\hat{f}_0|D)\circ(F|D) = F|D$ and hence $F|D \in H^\infty(D)$. Let now $\{g_k\}$ be a sequence (in $H^\infty(D)$) converging weak* to 0. Since Ψ is norm-continuous, the sequence $\{\Psi(g_k)\}$ is bounded, and from the equalities

$$\Psi(g_k) = (\hat{g}_k\circ F)|D = g_k\circ(F|D)$$

we conclude that $\{\Psi(g_k)\}$ converges pointwise to zero. This proves a).

If $F(D) \not\subset D$, then there exists $\lambda_0 \in D$ such that $F(\lambda_0)$ belongs to the fiber over some boundary point w_0, i.e., $(\hat{f}_0 \circ F)(\lambda_0) = w_0$. Since $(\hat{f}_0 \circ F)|D$ is analytic and bounded by 1, we must have (by the maximum modulus principle) $(\hat{f}_0 \circ F)(\lambda) = w_0$ for all λ in D. In other words, $F(D)$ is contained in the fiber over w_0. The Carleson corona theorem says that D is dense in $\mathcal{M}$; therefore $F(\mathcal{M})$ lies in the fiber over w_0. For any f in the disc algebra, $\hat{f}$ is constant on each fiber, and so $\Psi(f)(=\widehat{\Psi(f)}|D = (\hat{f} \circ F)|D)$ is a constant function. It follows from this that Ψ cannot be sequentially weak* continuous, and the proposition is proved.

Suppose now that Ψ is an endomorphism of $H^\infty(D)$ such that $\Psi(p) = p$ whenever p is a polynomial. Let $F: \mathcal{M} \to \mathcal{M}$ be the continuous map given by Proposition 8.1 such that $\widehat{\Psi(f)} = \hat{f} \circ F$. Since $\Psi(p)$ is not a constant function when p is a polynomial we are in case a) of Proposition 8.1. Therefore Ψ is weak* sequentially continuous and we have $\Psi(h) = h$ for any h in $H^\infty(D)$ because the polynomials are weak* sequentially dense in $H^\infty(D)$. Thus we have proved the following result.

COROLLARY 8.3. *If Ψ is an algebra endomorphism of $H^\infty(D)$ that leaves each polynomial fixed, then Ψ is the identity.*

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