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OPERATOR DECOMPOSABILITY AND WEAKLY CONTINUOUS REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

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0. INTRODUCTION

Let G be a locally compact abelian group, Γ its dual group, $L^1(G)$ the space of complex valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We show (Corollary 3) that for non-discrete G there exists a measure $\mu \in M(G)$ such that the convolution operator

$$T_\mu : L^1(G) \to L^1(G); \quad g \to \mu * g$$

is *not* decomposable; thus answering in the negative a question of I. Colojoară and C. Foiaş ([5], p. 218). On the other hand for any measure $\mu \in M_a + M_d$, the subalgebra of $M(G)$ consisting of all measures μ whose continuous part μ_a belongs to $L^1(G)$, the operator T_μ is strongly decomposable. More generally, using recent results of D'Antoni, R. Longo and L. Zsidó ([6]) we obtain that for any weakly continuous representation U of G by isometries on a Banach space X and for any $\mu \in M_a + M_d$ the generalized convolution operator $\pi(\mu) \in B(X)$ defined by

$$\pi(\mu) = \int_G U(s)\, d\mu(s)$$

is strongly decomposable. For the notion of decomposable and strongly decomposable operators we refer the reader to [3], [5], [7]. I want to thank Professor L. Zsidó for inspiring this work.

1. THE COUNTEREXAMPLE

Let X be a Banach space and $B(X)$ the space of continuous linear operators in X. For any $T \in B(X)$ we denote by $\sigma(T)$ the spectrum of T and by $\mathrm{Inv}(T)$ the system of all closed linear T-invariant subspaces of X. We say an operator $T \in B(X)$

has the weak 2-SDP (Spectral Decomposition Property) if for any open covering $\sigma(T) \subset U_1 \cup U_2$ there are spaces $X_1, X_2 \in \mathrm{Inv}(T)$ such that $X = \overline{X_1 + X_2}$; $\sigma(T|X_i) \subset U_i$, $i = 1, 2$.

LEMMA 1. *Let X, Y be Banach spaces, $T \in B(X)$, $S \in B(Y)$, and $A \in B(X, Y)$ injective such that $AT = SA$. If T has the weak 2-SDP we have*

$$\sigma(T) \subset \sigma(S).$$

Proof. Let T have the weak 2-SDP. For $\lambda_0 \notin \sigma(S)$ we can find $X_1, X_2 \in \mathrm{Inv}(T)$ such that $X = \overline{X_1 + X_2}$, $\sigma(T|X_1) \subset \rho(S)$, $\sigma(T|X_2) \subset \mathbf{C} \setminus \{\lambda_0\}$. If Γ is a cycle surrounding $\sigma(S)$ in $\rho(T|X_1)$ such that $\sigma(S)$ and $\sigma(T|X_1)$ are separated by Γ, we get for any $x \in X_1$:

$$Ax = \frac{1}{2\pi i}\int_\Gamma \mathrm{R}(\lambda, S)\, Ax \,\mathrm{d}\lambda = \frac{1}{2\pi i}\int_\Gamma \mathrm{R}(\lambda, S)\, A(\lambda - T)\, \mathrm{R}(\lambda, T|X_1)\, x \,\mathrm{d}\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma A\mathrm{R}(\lambda, T|X_1)\, x \,\mathrm{d}\lambda = 0.$$

Since A is injective we have $X_1 = \{0\}$ and therefore

$$\lambda_0 \in \rho(T|X_2) = \rho(T).$$

As in [8] we denote by $A(\Gamma) = \{\hat{f}, f \in L^1(G)\}$, $B(\Gamma) = \{\hat{\mu}, \mu \in M(G)\}$ the Banach algebras of all Fourier transforms of functions $f \in L^1(G)$ respectively measures $\mu \in M(G)$ with norms $\|\hat{f}\| = \|f\|$, $\|\hat{\mu}\| = \|\mu\|$. The canonical embeddings

$$\imath_A(\imath_B) : A(\Gamma)\ (B(\Gamma)) \to C_b(\Gamma); \quad f \to f$$

into the space of all bounded, continuous functions on Γ equipped with sup-norm are injective, continuous algebra homomorphisms. For $h \in C_b(\Gamma)$ the multiplication operator

$$S = S_h : C_b(\Gamma) \to C_b(\Gamma), \quad f \to hf$$

is continuous with $\sigma(S_h) = \overline{h(\Gamma)}$.

THEOREM 2. *Let G be a locally compact abelian group, $\mu \in M(G)$ and $X = L^1(G)$ or $X = M(G)$. If the convolution operator $T : X \to X$; $T(k) = \mu * k$ has the weak 2-SDP, its spectrum is given by*

$$\sigma(T) = \overline{\hat{\mu}(\Gamma)}.$$

Proof. Define $A: X \to C_b(\Gamma)$; $k \to \hat{k}$, $S: C_b(\Gamma) \to C_b(\Gamma)$; $f \to \hat{\mu}f$.

Now we are in the setting of Lemma 1. $A \in B(X, C_b(\Gamma))$ is injective such that

$$AT(k) = \widehat{\mu * k} = \hat{\mu}\hat{k} = SA(k)$$

for all $k \in X$. If T has the weak 2-SDP we obtain $\sigma(T) \subset \sigma(S) = \hat{\mu}(\Gamma)$. On the other hand if $\lambda = \hat{\mu}(\gamma)$, $\gamma \in \Gamma$, we have $\widehat{(\lambda - \mu) * k}(\gamma) = 0$ for all $k \in X$ showing $\overline{(\lambda - T)X} \subsetneqq X$. Consequently we obtain $\overline{\hat{\mu}(\Gamma)} \subset \sigma(T)$.

COROLLARY 3. *If G is a non-discrete locally compact abelian group there is a measure $\mu \in M(G)$ such that the convolution operator T_μ does not possess the weak* 2-SDP *(in particular is not decomposable) neither as a multiplication operator in $M(G)$ nor considered as a multiplier in $L^1(G)$.*

Proof. It is well known (see for example [8], p. 107, Theorem 5.3.4.) that for non-discrete G there exists a measure $\mu \in M(G)$ not invertible in $M(G)$ such that $\hat{\mu}(\gamma) \geqslant 1$ for all $\gamma \in \Gamma$. If T_μ denotes multiplication by μ in $M(G)$ we have $\overline{\hat{\mu}(\Gamma)} \subsetneqq \sigma(T_\mu) \subset \sigma(T_\mu | L^1(G))$. For the last inclusion see [8], p. 73, Theorem 3.8.1. Now Corollary 3 is a consequence of the preceding theorem.

If G is discrete every convolution operator is decomposable. More precisely, in this case it follows $L^1(G) = M(G)$ and Theorem 2.11, Ch. 6 in [5] is applicable.

2. GENERALIZED CONVOLUTIONS

Let U be a weakly continuous representation of a locally compact abelian group G by means of isometries on a Banach space X, i.e. a map $U: G \to B(X)$ satisfying

(i) $U(s + t) = U(s)\,U(t)$ for all $s, t \in G$, $U(0) = I$,

(ii) $\|U(s)\,x\| = \|x\|$ for $s \in G$, $x \in X$,

(iii) $G \to X$; $s \to U(s)x$ is weakly continuous for each $x \in X$.

The following lemma is well known (see for example Arveson [4]) and it is used to construct a representation of $M(G)$.

LEMMA 4. *Let S be a locally compact Hausdorff space and μ a regular complex Borel measure on S. Then each norm-bounded weakly continuous function $f: S \to X$ into a Banach space X is weakly μ-integrable, i.e.*

(a) *$\langle f, x' \rangle$ is μ-integrable for each $x' \in X'$.*

(b) *There is a unique vector $y \in X$ such that*

$$\langle y, x' \rangle = \int_S \langle f, x' \rangle \, d\mu \quad \textit{for each } x' \in X'.$$

In this sense we get a continuous algebra homomorphism

$$\pi\colon M(G) \to B(X);\ \pi(\mu) = \int_G U(s)\mathrm{d}\mu(s)$$

with $\|\pi\| = 1$.

W. Arverson introduces in [4] spectra and spectral subspaces of U:

$$\mathrm{sp}(U) = \cap\,(\mathrm{Z}(f),\ f \in \mathrm{Ker}(\pi|L^1(G))) = \mathrm{Z}(\mathrm{Ker}\,\pi|L^1(G));$$

$$\mathrm{sp}_U(x) = \cap\,(\mathrm{Z}(f);\ f \in L^1(G)\colon \pi(f)\,x = 0) =$$

$$= \mathrm{Z}(\{f \in L^1(G),\ \pi(f)\,x = 0\});$$

$$M^U(E) = \{x \in X;\ \mathrm{sp}_U(x) \subset E\}$$

for E closed in Γ. As in [8] $\mathrm{Z}(f)$ is the set of zeros of $\hat{f}$ and $\mathrm{Z}(I) = \cap\,(\mathrm{Z}(f);\ f \in I)$ denotes the hull for an arbitrary ideal $I \subset L^1(G)$.

Corollary 7.2.5. a) in [8] shows that for $E = \bar{E} \subset \Gamma$:

$$M^U(E) = \cap\,(\mathrm{Ker}\,\pi(f);\ f \in L^1(G)\colon \mathrm{supp}(\hat{f}) \cap E = \emptyset).$$

Hence each $M^U(E)$ is a closed linear subspace invariant with respect to π. For any $Y \in \mathrm{Inv}(\pi)$, the system of closed linear subspaces of X invariant with respect to π, we get a new representation by restriction

$$U|Y\colon G \to B(Y);\ s \to U(s)|Y$$

satisfying conditions (i)—(iii) and inducing the representation

$$\pi|Y\colon M(G) \to B(Y);\ \mu \to \pi(\mu)|Y.$$

LEMMA 5. a) *For every closed set* $E \subset \Gamma$, $M^U(E)$ *is maximal among all* $Y \in \mathrm{Inv}(\pi)$ *with* $\mathrm{sp}(U|Y) \subset E$.

b) $\mathrm{sp}_U(x) = \emptyset$ *if and only if* $x = 0$;

c) $\mathrm{sp}_U(\pi(\mu)\,x) \subset \mathrm{supp}(\hat{\mu}) \cap \mathrm{sp}_U(x)$ *for every* $x \in X$, $\mu \in M(G)$;

d) *if* $x \in X$ *and* $\hat{\mu} \in B(\Gamma)$ *is equal to* 1 *in a neighbourhood of* $\mathrm{sp}_U(x)$ *then* $\pi(\mu)\,x = x$.

Proof. a) For each $\gamma \notin E$ we can choose a function $f \in L^1(G)$ such that $\hat{f}(\gamma) \neq 0$, $\mathrm{supp}(\hat{f}) \cap E = \emptyset$. Because $(\pi|M^U(E))\,(f) = \pi(f)|M^U(E) = 0$ and $\gamma \notin \mathrm{Z}(f)$ we obtain $\gamma \notin \mathrm{sp}(U|M^U(E))$. If Y is any π-invariant subspace with $\mathrm{sp}(U|Y) \subset E$ we have $\mathrm{sp}_U(y) = \mathrm{sp}_{U|Y}(y) \subset \mathrm{sp}(U|Y) \subset E$ for every $y \in Y$.

b) On account of [8], Corollary 7.2.5. (c), $\mathrm{sp}_U(x) = \emptyset$ if and only if $\pi(f)\, x = = \int_G f(s)\, U(s) x \mathrm{d}s = 0$ for every $f \in L^1(G)$. But this is equivalent to $x = 0$.

c) Obviously $\pi(\mu)\, X \subset \cap\, (\mathrm{Ker}\, \pi(f);\ f \in L^1(G),\ \ \mathrm{supp}(\hat{f}) \cap \mathrm{supp}(\hat{\mu}) = \emptyset) = = M^U(\mathrm{supp}(\hat{\mu}))$ and $\pi(\mu)\, M^U(\mathrm{sp}_U(x)) \subset M^U(\mathrm{sp}_U(x))$.

d) Combining b) and c) we conclude $x - \pi(\mu)\, x = \pi(\delta_0 - \mu)\, x = 0$, where δ_0 denotes the Dirac measure concentrated in 0.

In [6] D'Antoni, R. Longo and L. Zsidó obtain the following spectral mapping theorem:

THEOREM 6. *For every measure* $\mu \in M_a + M_d$ *we have* $\sigma(\pi(\mu)) = \overline{\hat{\mu}(\mathrm{sp}(U))}$.

If we define $X_\mu(F) = M^U(\overset{-1}{\hat{\mu}}(F))$ for every closed set $F \subset \mathbf{C}$ we obtain:

COROLLARY 7. *For every measure* $\mu \in M_a + M_d$ *and every closed set* $F \subset \mathbf{C}$ *we have*

$$\sigma(\pi(\mu)|X_\mu(F)) \subset F.$$

Proof. The above spectral mapping theorem applied to the representation $U|X_\mu(F)$ yields:

$$\sigma(\pi(\mu)|X_\mu(F)) = \overline{\hat{\mu}(\mathrm{sp}(U|X_\mu(F)))} \subset F.$$

An operator $T \in B(X)$ is *decomposable* if and only if there is a spectral capacity for T, i.e. a map $E\colon \mathscr{F}(\mathbf{C}) = \{F;\, F = \overline{F} \subset \mathbf{C}\} \to \mathrm{Inv}(T)$ with:

(i) $E(\emptyset) = \{0\}$, $E(\mathbf{C}) = X$,

(ii) $E(\bigcap_{n \in \mathbf{N}} F_n) = \bigcap_{n \in \mathbf{N}} E(F_n)$ for any sequence $(F_n)_{n \in \mathbf{N}}$ in $\mathscr{F}(\mathbf{C})$,

(iii) $X = \sum_{i=1}^{n} E(\overline{U_i})$ for any open covering $\sigma(T) \subset U_1 \cup \ldots \cup U_n$,

(iv) $\sigma(T|E(F)) \subset F$ for all $F \in \mathscr{F}(\mathbf{C})$.

Compare [7] for this characterization. In this case each $E(F)$ is hyperinvariant for T and maximal among all $Y \in \mathrm{Inv}(T)$ with $\sigma(T|Y) \subset F$, in particular uniquely determined.

PROPOSITION 8. *If* $\pi(\mu)$, $\mu \in M_a + M_d$, *is decomposable then its spectral capacity is given by*

$$E(F) = X_\mu(F) \quad \textit{for } F \in \mathscr{F}(\mathbf{C}).$$

Proof. Let E be the spectral capacity of $\pi(\mu)$. Corollary 7 shows that $\sigma(\pi(\mu)|X_\mu(F)) \subset F$ and therefore $X_\mu(F) \subset E(F)$. $E(F)$ being hyperinvariant for $\pi(\mu)$ we get

$$\hat{\mu}(\mathrm{sp}_U(y)) \subset \hat{\mu}(\mathrm{sp}(U|E(F))) \subset \sigma(\pi(\mu)|E(F)) \subset F$$

for any $y \in E(F)$, $F = \overline{F} \subset \mathbf{C}$. Since y is arbitrary it follows that $E(F) \subset M^U(\overset{-1}{\hat{\mu}}(F)) = = X_\mu(F)$.

How can we find measures $\mu \in M_a \dotplus M_d$ satisfying Proposition 8? First we notice

$$\left\| \prod_{j=1}^{n} U(t_j)^{m_j} \right\| = \left\| U\left(\sum_{j=1}^{n} m_j t_j \right) \right\| = 1$$

for $t_1, \ldots, t_n \in G$, $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$. Fix $t_1, \ldots, t_n \in G$, $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$. By [1], Theorem 3.14,

$$(\pi(\delta_{t_1}), \ldots, \pi(\delta_{t_n})) = (U(t_1), \ldots, U(t_n)) \in B(X)^n$$

is a $C^p(\Gamma^n)$-scalar-n-system for $p > \frac{n}{2}$. Consequently

$$\pi\left(\sum_{j=1}^{n} \alpha_j \delta_{t_j} \right) = \sum_{j=1}^{n} \alpha_j \pi(\delta_{t_j})$$

possesses a spectral capacity (see [2], Corollary 9).

Now we are able to prove our main result:

THEOREM 9. *For any measure $\mu \in M_a \dotplus M_d$, $\pi(\mu)$ is decomposable and its spectral capacity is given by $E(F) = M^U(\overset{-1}{\hat{\mu}}(F))$ $(F = \bar{F} \subset \mathbf{C})$.*

Proof. In order to show that $E(F) = M^U(\overset{-1}{\hat{\mu}}(F))$ defines a spectral capacity for $\pi(\mu)$ the only thing left is the proof of condition (iii). So, let $\overline{\hat{\mu}(\mathrm{sp}(U))} \subset U_1 \cup \ldots \cup U_n$ be an open covering and $x \in X$. Choose open sets $V_1, \ldots, V_n \subset \mathbf{C}$, $\varepsilon > 0$ such that

$$\bar{K}_\varepsilon(V_i) = \{z \in \mathbf{C};\ \inf_{w \in \bar{V}_i} |z - w| \leqslant \varepsilon\} \subset U_i,\ 1 \leqslant i \leqslant n,$$

$$\bar{K}_\varepsilon(\overline{\hat{\mu}(\mathrm{sp}(U))}) \subset V_1 \cup \ldots \cup V_n.$$

Now, if $\mu = \mu_a + \mu_d$, μ_a absolutely continuous, μ_d discrete, we can find $K \subset \Gamma$ compact and $\delta \in M(G)$ having finite support such that

$$\hat{\mu}_a(\Gamma \setminus K) \cup (\hat{\delta} - \hat{\mu}_d)(\Gamma) \subset K_{\frac{\varepsilon}{2}}(0) = \left\{ z \in \mathbf{C};\ |z| < \frac{\varepsilon}{2} \right\}.$$

Since $K \subset \Gamma$ is compact there is a function $\hat{\theta} \in A(\Gamma)$ with compact support which is equal to 1 in a neighbourhood of K. Setting $y = \pi(\theta)\,x$, $z = \pi(\delta_0 - \theta)\,x$ we have $x = y + z$, $\mathrm{sp}_U(y) \subset \mathrm{supp}(\hat{\theta}) \cap \mathrm{sp}_U(x)$; $\mathrm{sp}_U(z) \subset \mathrm{supp}(1 - \hat{\theta}) \subset \Gamma \setminus K$.

Let us show how to decompose y and z. Because $\mathrm{sp}_U(y)$ is compact and $\mathrm{sp}_U(y) \subset \overset{-1}{\hat{\mu}}(U_1) \cup \ldots \cup \overset{-1}{\hat{\mu}}(U_n)$ there are $\varkappa_1, \ldots, \varkappa_n \in L^1(G)$ with $\hat{\varkappa}_1 + \ldots + \hat{\varkappa}_n = 1$ in a neighbourhood of $\mathrm{sp}_U(y)$ and $\mathrm{supp}(\hat{\varkappa}_i) \subset \overset{-1}{\hat{\mu}}(U_i)$, $i = 1, \ldots, n$. Therefore y has the desired form:

$$y = \pi(\varkappa_1)\, y + \ldots + \pi(\varkappa_n)\, y; \quad \mathrm{sp}_U(\pi(\varkappa_i)\, y) \subset \overset{-1}{\hat{\mu}}(U_i), \quad i = 1, \ldots, n.$$

If E_δ denotes the spectral capacity of $\pi(\delta)$ and $F = \overline{\hat{\delta}((\Gamma \setminus K) \cap \mathrm{sp}(U))}$ we get

$$z \in M^U(\overset{-1}{\hat{\delta}}(F)) = E_\delta(F), \quad F \subset V_1 \cup \ldots \cup V_n.$$

$\pi(\delta)|E_\delta(F)$ belonging to the representation induced by $U|E_\delta(F)$ is decomposable. So we find $z_1, \ldots, z_n \in E_\delta(F)$ satisfying $z = z_1 + \ldots + z_n$, and

$$\mathrm{sp}_U(z_i) = \mathrm{sp}_{U|E_\delta(F)}(z_i) \subset \overset{-1}{\hat{\delta}}(\overline{V_i})$$

for $1 \leqslant i \leqslant n$. Without loss of generality we may assume $\mathrm{sp}_U(z_i) \subset \Gamma \setminus K$ for $1 \leqslant i \leqslant n$. Otherwise choose $\hat{\varkappa} \in B(\Gamma)$ equal to 1 in a neighbourhood of $\mathrm{sp}_U(z)$, equal to 0 in a neighbourhood of K and replace z_i by $\pi(\varkappa)\, z_i$. Finally we obtain $\hat{\mu}(\mathrm{sp}_U(z_i)) \subset U_i$, $1 \leqslant i \leqslant n$, and

$$x \in M^U(\overset{-1}{\hat{\mu}}(\overline{U_1})) + \ldots + M^U(\overset{-1}{\hat{\mu}}(\overline{U_n})) = E(\overline{U_1}) + \ldots + E(\overline{U_n})$$

as was to be shown.

We conclude by listing some interesting consequences of Theorem 9.

COROLLARY 10. *For every measure $\mu \in M_a + M_d$ the operator $\pi(\mu)$ is strongly decomposable.*

Proof. We recall (see [3]) that it is sufficient to show that $\pi(\mu)$ restricted to every spectral maximal space is decomposable. Since spectral maximal spaces Y are hyperinvariant ([5], p. 18) they are invariant with respect to π and everything follows from Theorem 9.

COROLLARY 11. *For every measure $\mu \in M_a + M_d$, $x \in X$ we have the local spectral mapping theorem:*

$$\sigma_{\pi(\mu)}(x) = \overline{\hat{\mu}(\mathrm{sp}_U(x))}.$$

Proof. Let us recall that for $x \in X$ we have

$$\sigma_{\pi(\mu)}(x) = \mathbf{C} \setminus \{z_0 \in \mathbf{C};\ \text{there is an analytic } f\colon U \to X \text{ defined in a neigh-bourhood } U \text{ of } z_0 \text{ such that } (z - \pi(\mu))f(z) = x \text{ for all } z \in U\}$$

and that the spectral capacity of $\pi(\mu)$ is given by $E(F) = \{x \in X;\ \sigma_{\pi(\mu)}(x) \subset F\}$. (Compare [7].)

Now Corollary 11 is a consequence of

$$x \in E(\sigma_{\pi(\mu)}(x)) = \overset{-1}{M^U}(\hat{\mu}(\sigma_{\pi(\mu)}(x))),$$

$$x \in \overset{-1}{M^U}(\hat{\mu}(\overline{\hat{\mu}(\mathrm{sp}_U(x))})) = E(\overline{\hat{\mu}(\mathrm{sp}_U(x))}) \quad (x \in X).$$

COROLLARY 12. *The multiplier* $T_\mu \colon L^1(G) \to L^1(G); f \to \mu * f$ *is strongly decomposable for every measure* $\mu \in M_a + M_d$.

Proof. Apply Corollary 10 to the special group representation

$$U(t) \colon L^1(G) \to L^1(G);\ (U(t)f)(s) = f(s - t).$$

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