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WEAKLY CLOSED IDEALS OF NEST ALGEBRAS

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INTRODUCTION

Let $\mathscr{A}$ be a nest algebra of operators on a Hilbert space H. We obtain a description of all weakly closed (two-sided) ideals of $\mathscr{A}$; more generally, we describe all weakly closed (two-sided) $\mathscr{A}$-submodules of $\mathscr{L}(H)$. In the course of the proof, we identify the finite rank operators of any norm-closed $\mathscr{A}$-submodule of $\mathscr{L}(H)$ and show that such modules have the same closure in the weak, strong, ultraweak or ultrastrong topologies. A simple and more direct proof of the main result, for the special case of the Volterra nest algebra, is sketched at the end of Section 1.

Some of the properties of such ideals and modules are investigated. The predual of any such weakly closed module is found. For certain cases, the perturbed modules $\mathscr{U} + \mathscr{K}$ (where $\mathscr{K}$ denotes the compact operators on H) are described in a fashion analogous to the description of $\mathscr{A} + \mathscr{K}$ in [3]. For any module $\mathscr{U}$ of this type, the commutant $C(\mathscr{A}, \mathscr{U})$ of $\mathscr{A}$ modulo $\mathscr{U}$ is described thus giving a description of the first Hochschild cohomology spaces with coefficients in these modules. We find that $C(\mathscr{A}, \mathscr{U})$ is of the form $\mathscr{C}_{\mathscr{U}} \oplus \mathscr{U}$ where $\mathscr{C}_{\mathscr{U}}$ is a subalgebra of the core of $\mathscr{A}$. For the case when $\mathscr{U}$ is an algebra, we show that $\mathscr{C}_{\mathscr{U}} \oplus \mathscr{U} = \mathrm{AlgLat}\mathscr{U}$. This appears to be a new example of a reflexive algebra.

Throughout, $\mathscr{E}$ will denote a complete nest of projections on a Hilbert space H. The nest algebra $\mathrm{Alg}\mathscr{E}$ of $\mathscr{E}$ is denoted by $\mathscr{A}$. The terminology and notation concerning nest algebras used in this paper are standard and may be found in [7, 2, 6]. All Hilbert spaces considered will be complex and the term projection will always mean orthogonal projection. The rank one operator $x \mapsto \langle x, e\rangle f$ will be denoted by $e \otimes f$. The terms module and ideal will be used to mean two-sided module and two-sided ideal.

1. MODULES AND IDEALS

Suppose that $E \mapsto \tilde{E}$ is an order homomorphism of $\mathscr{E}$ into itself (that is, $E \leqslant F$ implies that $\tilde{E} \leqslant \tilde{F}$). Then the set

$$\mathscr{U} = \{X \in \mathscr{L}(H) : (I - \tilde{E})XE = 0 \text{ for all } E \in \mathscr{E}\}$$

is clearly a weakly closed subset of $\mathscr{L}(H)$ and is easily seen to be an $\mathscr{A}$-module; for example, if $A \in \mathscr{A}$, $X \in \mathscr{U}$ then $(I - \tilde{E})XAE = (I - \tilde{E})XEAE = 0$.

In this section it will be shown that every weakly closed subset of $\mathscr{L}(H)$ which is an $\mathscr{A}$-module under operator multiplication is of the above form. To this end we analyse the finite rank operators in $\mathscr{U}$. For each $E \in \mathscr{E}$, define the projection E_* by

$$E_* = \bigwedge \{\tilde{F} : F > E\}.$$

Since $\mathscr{E}$ is complete, E_* is in $\mathscr{E}$ but E_* need not be of the form $\tilde{F}$ for any F in $\mathscr{E}$.

LEMMA 1.1. *Let $\mathscr{U}$ be as above. Then a non-zero operator of rank* 1, *$e \otimes f$, is in $\mathscr{U}$ if and only if, for some $E \in \mathscr{E}$, $E_* f = f$ and $(I - E)\, e = e$.*

Proof. If $R = e \otimes f$ is of the given form, then $R = E_* R(I - E)$ and so, for $F \in \mathscr{E}$

$$(I - \tilde{F})RF = (I - \tilde{F})\, E_* R(I - E)\, F.$$

If $F \leqslant E$, $(I - E)\, F = 0$ and, if $F > E$ it follows from the definition of E_* that $\tilde{F} \geqslant E_*$ so that $(I - \tilde{F})\, E_* = 0$. Thus, for all $F \in \mathscr{E}$,

$$(I - \tilde{F})\, RF = 0$$

showing that $R \in \mathscr{U}$.

Conversely, if $R = e \otimes f \in \mathscr{U}$, let

$$E = \bigvee \{F \in \mathscr{E} : (I - F)\, e = e\}.$$

Then, if $F > E$, we have $R(I - F) \neq R$ and so $RF = \tilde{F}RF \neq 0$. That is, $Fe \otimes f = Fe \otimes \tilde{F}f \neq 0$ and therefore $\tilde{F}f = f$ for all $F > E$. Thus $E_* f = f$ and, since $(I - E)\, e = e$ the proof is complete.

If $E_{\sim}$ is defined by

$$E_{\sim} = \bigvee \{F : \tilde{F} < E\},$$

the above lemma may be restated as follows:

$e \otimes f$ is in $\mathscr{U}$ if and only if, for some $E \in \mathscr{E}$, $Ef = f$ and $(I - E_{\sim})\, e = e$.

Note that $\mathscr{U} = \mathscr{A}$, when the homomorphism is the identity. In this case $E_* = E_+$ and $E_{\sim} = E_-$ so Lemma 1.1 reduces to Lemma 3.3 of [7].

The proof of the lemma below is modelled on the proof of Theorem 1 of [2].

LEMMA 1.2. *Let $R \in \mathscr{U}$ have rank n. Then R may be written as the sum of n elements of $\mathscr{U}$ each having rank* 1.

Proof. Suppose $R = \sum_{i=1}^{n} x_i \otimes y_i$ belongs to $\mathscr{U}$. Then $(I - \tilde{E})RE = 0$ for all $E \in \mathscr{E}$ and so for any $x \in H$

$$\sum_{i=1}^{n} \langle x, Ex_i \rangle (I - \tilde{E}) y_i = 0.$$

Thus, for any $E \in \mathscr{E}$, either $Ex_i = 0$ for $1 \leqslant i \leqslant n$ or $\{(I - \tilde{E}) y_i : 1 \leqslant i \leqslant n\}$ is a linearly dependent set (or both). Let

$$F = \bigvee \{E \in \mathscr{E} : Ex_i = 0 \text{ for } 1 \leqslant i \leqslant n\}.$$

Then $Fx_i = 0$ $(1 \leqslant i \leqslant n)$ and for all $G \in \mathscr{E}$ with $G > F$, Gx_i does not vanish for some i. So, from above, $\{(I - \tilde{G}) y_i : 1 \leqslant i \leqslant n\}$ is a linearly dependent set. Hence, for all $G > F$, the Grammian determinant

$$\det[\langle (I - \tilde{G})y_i, \ (I - \tilde{G})y_j \rangle] = 0.$$

By taking the infimum over all $G > F$, it follows that

$$\det[\langle (I - F_*) y_i, (I - F_*) y_j \rangle] = 0$$

and so $\{(I - F_*) y_i : 1 \leqslant i \leqslant n\}$ is linearly dependent.

After re-indexing, if needed, we have that

$$(I - F_*) y_1 = \sum_{i=2}^{n} \alpha_i (I - F_*) y_i$$

where α_i are scalars. Thus

$$R = x_1 \otimes [F_* y_1 + (I - F_*) y_1] + \sum_{i=2}^{n} x_i \otimes y_i =$$

$$= x_1 \otimes \left[F_* y_1 + \sum_{i=2}^{n} \alpha_i (I - F_*) y_i \right] + \sum_{i=2}^{n} x_i \otimes y_i =$$

$$= x_1 \otimes F_* \left[y_1 - \sum_{i=2}^{n} \alpha_i y_i \right] + \sum_{i=2}^{n} (x_i + \overline{\alpha}_i x_1) \otimes y_i.$$

From Lemma 1.1, since $Fx_1 = 0$, the first term is a rank 1 element of $\mathscr{U}$. An obvious induction completes the proof.

Suppose now that $\mathscr{V}$ is any *norm* closed subset of $\mathscr{L}(H)$ which is an $\mathscr{A}$-module. For each $E \in \mathscr{E}$, let $\tilde{E}$ be the projection onto

$$\bigwedge \{\operatorname{ran} XE : X \in \mathscr{V}\}.$$

Since $\mathscr{V}$ is a module, $\tilde{E}$ is invariant under $\mathscr{A}$ and so, by the reflexivity of complete nests, $\tilde{E} \in \mathscr{E}$. Clearly, $E \mapsto \tilde{E}$ is an order homomorphism. Further, $E \mapsto \tilde{E}$ is left order continuous in the sense that

$$\lim_{E \uparrow F} \tilde{E} = [\lim_{E \uparrow F} F]^{\sim} = \tilde{F}_{-},$$

(recall that $F_{-} = \bigvee \{E \in \mathscr{E} : E < F\}$). This follows easily since if $x \perp \operatorname{ran} XE$ for all $E < F$ then $x \perp \operatorname{ran} XF_{-}$. Thus, if $\tilde{E}x = 0$ for all $E < F$ then $\tilde{F}_{-}x = 0$ showing that $\lim_{E \uparrow F} \tilde{E} \geqslant \tilde{F}_{-}$. The opposite inequality is trivial.

LEMMA 1.3. *Let $\mathscr{V}$ and $E \mapsto \tilde{E}$ be as above and let*

$$\mathscr{U} = \{X \in \mathscr{L}(H) : (I - \tilde{E})XE = 0\}.$$

Then $\mathscr{U}$ and $\mathscr{V}$ contain the same set of rank 1 operators.

Proof. Let $R \in \mathscr{U}$ have rank 1. Then, from Lemma 1.1, $R = e \otimes f$ with $(I - E)\, e = e$ and $E_{*}f = f$ for some $E \in \mathscr{E}$. It is easy to see from the proof of Lemma 1.1 that we may choose E such that $(I - F)\, e \neq e$ for all $F > E$.

Now choose $F > E$ so that, for some given $\varepsilon > 0$,

$$\|e - (I - F_{-})\, e\| < \varepsilon. \tag{1}$$

(If $E \neq E_{+}$ then take $F = E_{+}$). Let G be the projection onto the closure of $\{A(F - E)\, e : A \in \mathscr{A}\}$. It follows from the reflexivity of nests that $G \in \mathscr{E}$ and, as $(F - E)\, e \neq 0$ we have that $G > E$. Since $\mathscr{V}$ is a module,

$$\{X(F - E)e : X \in \mathscr{V}\}$$

is dense in the range of $\tilde{G}$. Using the definition of E_{*}, $\tilde{G} \geqslant E_{*}$ and so, as $E_{*}f = f$, it follows that for some X in $\mathscr{V}$,

$$\|X(F - E)\, e - f\| < \varepsilon. \tag{2}$$

Now

$$(I - F_{-})\, e \otimes (F - E)\, e \in \mathscr{A};$$

this follows from Lemma 3.3 of [7] or from the special case of Lemma 1.1 above (i.e. the case when $\mathscr{U} = \mathscr{A}$). Therefore

$$R_0 = X[(I - F_{-})\, e \otimes (F - E)\, e] =$$

$$= (I - F_{-})\, e \otimes X(F - E)\, e \in \mathscr{V}.$$

It is clear from (1) and (2) that R_0 may be made arbitrarily close to R and, since $\mathscr{V}$ is norm closed, the lemma follows.

COROLLARY 1.4. *$\mathscr{U}$ and $\mathscr{V}$ contain the same set of operators of finite rank.*

Proof. Immediate from Lemma 1.2.

The following theorem is the main result of this section.

THEOREM 1.5. *Let $\mathscr{V}$ be an $\mathscr{A}$-submodule of $\mathscr{L}(H)$ that is closed in any one of the weak, strong, ultraweak or ultrastrong operator topologies. Then $\mathscr{V}$ is of the form*

$$\{X \in \mathscr{L}(H) : (I - \tilde{E}) XE = 0\}$$

for some left order continuous order homomorphism $E \mapsto \tilde{E}$ of $\mathscr{E}$ into $\mathscr{E}$.

Proof. Let $\mathscr{A}_1$ be the unit ball of $\mathscr{A}$. It is proved in [2] that the finite rank operators of $\mathscr{A}_1$ are strongly dense in $\mathscr{A}_1$. Thus there exists a net (R_γ) of finite rank operators of $\mathscr{A}_1$ converging strongly to the identity I. Clearly (R_γ) converges to I also in the weak operator topology. Further, since the strong and weak topologies coincide on bounded sets with the ultrastrong and ultraweak topologies respectively, it follows that (R_γ) converges to I in any of the four topologies mentioned.

Now let $\mathscr{V}$ be the given module and let $E \mapsto \tilde{E}$ be the left order continuous homomorphism determined by $\mathscr{V}$ as in the above discussion. Put

$$\mathscr{U} = \{X \in \mathscr{L}(H) : (I - \tilde{E}) XE = 0\}.$$

Then, for any $X \in \mathscr{U}$, XR_γ is a finite rank element of $\mathscr{U}$ and so, from Corollary 1.4, $XR_\gamma \in \mathscr{V}$. Since $\mathscr{V}$ is closed in one of the four topologies and $(R_\gamma) \to I$ in that topology, it follows that $X \in \mathscr{V}$ and so $\mathscr{V} = \mathscr{U}$.

COROLLARY 1.6. *A norm closed $\mathscr{A}$-submodule $\mathscr{V}$ of $\mathscr{L}(H)$ has the same closure $\mathscr{U}$ in any one of the weak, strong, ultraweak or ultrastrong topologies. The module $\mathscr{U}$ is the closure, in any of these topologies, of the finite rank elements of $\mathscr{V}$.*

COROLLARY 1.7. *Any ideal $\mathscr{J}$ of $\mathscr{A}$ that is closed in any one of the weak, strong, ultraweak or ultrastrong topologies is of the form*

$$\mathscr{J} = \{X \in \mathscr{L}(H) : (I - \tilde{E}) XE = 0\}$$

where $E \mapsto \tilde{E}$ is a left order continuous order homomorphism of $\mathscr{E}$ into $\mathscr{E}$ such that $\tilde{E} \leqslant E$ for each $E \in \mathscr{E}$.

Proof. This is immediate from the theorem and the fact that $\mathscr{U} \subseteq \mathscr{A}$ if and only if $\tilde{E} \leqslant E$ for all $E \in \mathscr{E}$.

REMARK. Every order homomorphism of $\mathscr{E}$ into $\mathscr{E}$ determines a weakly closed $\mathscr{A}$-module. Clearly the correspondence is not bijective since every module is determined by a left order continuous homomorphism. Even the imposition of left continuity does not produce a bijection as the following example shows: let $\mathscr{E}$ be any continuous nest (i.e. $E = E_-$ for all $E \in \mathscr{E}$) and define $\tilde{E} = I$ for all $E \in \mathscr{E}$. Define $\hat{E}$ by $\hat{0} = 0$ and $\hat{E} = I$ for $E \in \mathscr{E}\diagdown\{0\}$. It is easy to see that both homomorphisms are left continuous and both determine the $\mathscr{A}$-module $\mathscr{L}(H)$. However all one can show for the general case is that, if $E \mapsto \tilde{E}$ and $E \mapsto \hat{E}$ determine the same module $\mathscr{U}$ and are both left continuous then $\hat{F} = \tilde{F}$ for all F in $\mathscr{E}$ such that $F = F_- \neq 0$. To see this, note than $\mathscr{U}$ contains every operator of the form $S = (\hat{E} - \tilde{E})X(I - E)$ for every $E \in \mathscr{E}$. Thus $(I - \tilde{F})SF = 0$ for all $F \in \mathscr{E}$ and so it follows that $\tilde{F} \geqslant \hat{E}$ whenever $F > E$. Therefore, if $0 \neq F = F_-$

$$\hat{F} = \lim_{E \uparrow F} \hat{E} \leqslant \tilde{F}$$

and so $\hat{F} = \tilde{F}$ by symmetry. It follows that for continuous nests there is a bijection between weakly closed modules and left continuous homomorphisms satisfying $\tilde{0} = 0$.

In the sequel we shall always assume that the weakly closed module $\mathscr{U}$ is given by the homomorphism $E \mapsto \tilde{E}$ such that $\tilde{E}$ is the projection onto

$$\bigvee \{\operatorname{ran} XE : X \in \mathscr{U}\}.$$

Then $E \mapsto \tilde{E}$ is left continuous and $\tilde{0} = 0$.

We now examine when the module is an algebra and determine the algebra generated by any module.

LEMMA 1.8. *The module determined by the homomorphism $E \mapsto \tilde{E}$ is an algebra if and only if $\tilde{\tilde{E}} \leqslant \tilde{E}$ for all $E \in \mathscr{E}$.*

Proof. For any $X, Y \in \mathscr{U}$,

$$XYE = X\tilde{E}YE = \tilde{\tilde{E}}X\tilde{E}YE = \tilde{\tilde{E}}XYE.$$

Thus, if the given condition is satisfied,

$$(I - \tilde{E})XYE = 0$$

and so $XY \in \mathscr{U}$. Since $\mathscr{U}$ is always a linear space, this shows that $\mathscr{U}$ is an algebra.

Conversely, if $\tilde{\tilde{E}} > \tilde{E}$ for some $E \in \mathscr{E}$, then, since the map is order preserving, $\tilde{E} > E$. Since $\tilde{E}$ is the projection onto $\bigvee \{\operatorname{ran} XE : X \in \mathscr{U}\}$ there exists $X \in \mathscr{U}$ such that

$$(\tilde{E} - E)XE \neq 0.$$

Hence, for some $x \in H$, $(\tilde{E} - E)XEx = y \neq 0$ and $(I - E)y = y$. Similarly, since $\tilde{\tilde{E}} > \tilde{E}$, there exists $Y \in \mathscr{U}$ and $z \in H$ such that $(\tilde{\tilde{E}} - \tilde{E})Y\tilde{E}z \neq 0$ and $\tilde{E}z = z$. Then

$$y \otimes z = \tilde{E}(y \otimes z)(I - E) \in \mathscr{U}$$

and also

$$(I - \tilde{E})Y(y \otimes z)XE \neq 0.$$

Hence $Y(y \otimes z)X \notin \mathscr{U}$. Thus, if $\tilde{\tilde{E}} > \tilde{E}$ then $\mathscr{U}$ is not an algebra. This completes the proof.

Note that the condition $\tilde{\tilde{E}} \leqslant \tilde{E}$ for all $E \in \mathscr{E}$ may be restated as: for all $E \in \mathscr{E}$ either $\tilde{E} < E$ or $\tilde{\tilde{E}} = \tilde{E}$.

COROLLARY 1.9. *If the module $\mathscr{U}$ determined by the homomorphism $E \mapsto \tilde{E}$ is an algebra then $\mathscr{U}$ is an ideal of the nest algebra* Alg$\mathscr{F}$ *where* $\mathscr{F} = \{E \in \mathscr{E} : \tilde{E} \leqslant E\}$.

Proof. Clearly $\mathscr{U} \subseteq \operatorname{Alg}\mathscr{F}$. We must show that if $X \in \mathscr{L}(H)$ is such that $XE = \tilde{E}XE$ for $E \in \mathscr{F}$ then $X \in \mathscr{U}$. Let $E \in \mathscr{E} \setminus \mathscr{F}$. Then $\tilde{E} > E$. Now $\tilde{\tilde{E}} = \tilde{E} \in \mathscr{F}$ and so

$$(I - \tilde{\tilde{E}})X\tilde{E} = (I - \tilde{E})X\tilde{E} = 0.$$

Therefore, since $\tilde{E}E = E$,

$$(I - \tilde{E})XE = 0$$

and $X \in \mathscr{U}$.

In the following lemma the symbol $\tilde{E}^{(n)}$ denotes the result of applying the homomorphism $\tilde{}$ to E, n times.

LEMMA 1.10. *Let $\mathscr{U}$ be the module determined by the homomorphism $E \mapsto \tilde{E}$. Then the weakly closed algebra generated by $\mathscr{U}$ is the module determined by $E \mapsto \hat{E}$ where*

$$\hat{E} = \begin{cases} \tilde{E} & \text{if } \tilde{E} \leqslant E \\ \bigvee\{\tilde{E}^{(n)} : n \geqslant 0\} & \text{if } \tilde{E} > E. \end{cases}$$

Proof. Let $\mathscr{W}$ be the module corresponding to the homomorphism $E \mapsto \hat{E}$. Then, if $\tilde{E} \leqslant E$ we have $\hat{E} \leqslant E$ and if $\tilde{E} > E$ we have $\tilde{\hat{E}} = \hat{E}$ and so by the definition of $\hat{}$, $\hat{\hat{E}} = \tilde{\hat{E}} = \hat{E}$. Thus, from Lemma 1.8, $\mathscr{W}$ is an algebra and clearly $\mathscr{W} \supseteq \mathscr{U}$.

Now the weakly closed algebra $\mathscr{W}_1$ generated by $\mathscr{U}$ is clearly a weakly closed $\mathscr{A}$-submodule of $\mathscr{L}(H)$. Thus $\mathscr{W}_1 \subseteq \mathscr{W}$. Suppose $\mathscr{W}_1$ is determined by the homomorphism $E \mapsto \check{E}$. Clearly, since $\mathscr{W}_1 \supseteq \mathscr{U}$, $\check{E} \geqslant \tilde{E}$ for any $E \in \mathscr{E}$. Thus $\check{\tilde{E}} \geqslant \tilde{\tilde{E}}$. Also, since $\check{}$ is an order homomorphism,

$$\check{\check{E}} \geqslant \check{\tilde{E}} \geqslant \tilde{\tilde{E}}.$$

But, since $\mathscr{W}_1$ is an algebra, Lemma 1.8 shows that $\check{\check{E}} = \check{E}$ and so, repeating the above argument

$$\check{E} \geqslant \tilde{E}^{(n)}$$

for each n. Therefore $\check{E} \geqslant \hat{E}$ and so $\mathscr{W}_1 \supseteq \mathscr{W}$.

THE VOLTERRA NEST. For the special case of the Volterra nest $\mathscr{E} = \{E_t : 0 \leqslant t \leqslant 1\}$ where E_t is the projection onto $L^2[0, t]$ (considered as a subspace of $H = L^2[0, 1]$), a simple proof of Theorem 1.5 can be given. This proof is independent of the description of the finite rank operators and does not use the results of [2].

We give a very brief outline of this proof. Given any module $\mathscr{U}$, it determines a homomorphism $E_t \mapsto \tilde{E}_t$ as above. Now, if $\mathscr{U}$ is weakly closed then $\tilde{E}_t\mathscr{L}(H)(I - E_t) \subseteq \mathscr{U}$; this is a routine argument using the definition of the homomorphism $\tilde{}$. Now define $T_n \in \operatorname{Alg}\mathscr{E}$ by

$$(T_n f)(t) = \begin{cases} f\left(t + \dfrac{1}{n}\right) & t + \dfrac{1}{n} \leqslant 1 \\[2ex] 0 & t + \dfrac{1}{n} > 1. \end{cases}$$

Then $\|T_n\| \leqslant 1$ and $T_n \to I$ strongly as $n \to \infty$. If X satisfies $(I - \tilde{E}_t) XE_t = 0$ for all $E_t \in \mathscr{E}$, a calculation shows that

$$XT_n = \sum_{i=1}^{n-1} \tilde{E}_{\frac{i}{n}} XT_n(E_{\frac{i+1}{n}} - E_{\frac{i}{n}})$$

and so $XT_n \in \mathscr{U}$. Thus taking strong limits shows that $X \in \mathscr{U}$. The same argument works for the other topologies.

2. COMPACT PERTURBATIONS AND PREDUALS

In many respects the weakly closed modules discussed in Section 1 have properties similar to those of nest algebras. For example, if $\mathscr{U}$ is the $\mathscr{A}$-module determined by the homomorphism $E \mapsto \tilde{E}$ (which, according to our standing assumption, is left continuous and satisfies $\tilde{0} = 0$) then the distance formula

$$\operatorname{dist}(X, \mathscr{U}) = \sup_{E \in \mathscr{E}} \|(I - \tilde{E})XE\|$$

follows from the generalization given in [5]. One can make use of this to obtain analogues of the results in [3] on compact perturbation of nest algebras. In the sequel, $\mathcal{K}$ denotes the set of all compact operators on H.

DEFINITION. Let $\mathcal{U}$ be the weakly closed module determined by the homomorphism $E \mapsto \tilde{E}\,(\mathcal{E} \to \mathcal{E})$. Define $Q\mathcal{U}$ to be the set of all A in $\mathcal{L}(H)$ such that

(i) $(I - \tilde{E})\,AE \in \mathcal{K}$

(ii) the map $E \mapsto (I - \tilde{E})\,AE$ is continuous from $\mathcal{E}$ with the strong topology to $\mathcal{L}(H)$ with the norm topology.

Note that the topology induced on $\mathcal{E}$ by the strong operator topology coincides with the order topology on $\mathcal{E}$ (that is the topology generated by the sub-base $\{E \in \mathcal{E} : E > E_0\}$, $\{E \in \mathcal{E},\ E < E_0\}$ with E_0 ranging over $\mathcal{E}$).

THEOREM 2.1. *If the homomorphism $E \mapsto \tilde{E}$ is continuous as a map from ($\mathcal{E}$, strong topology) to itself and determines the module $\mathcal{U}$ then*

$$Q\mathcal{U} = \mathcal{U} + \mathcal{K}.$$

Proof. Since the proof is along the same lines as the proof of Theorem 2.3] of [3], full details will not be given here.

The fact that $\mathcal{U} + \mathcal{K}$ is closed follows from Corollary 1 of Theorem 1.1 of [3 and Corollary 1.6 above. Now if $X \in Q\mathcal{U}$, since $E \mapsto \tilde{E}$ is continuous, it follows as in Proposition 2.2 of [3] that $\{(I - \tilde{E})\,XE : E \in \mathcal{E}\}$ is a normcompact set of operators. The argument given in the proof of Theorem 2.3 of [3] together with the generalized distance formula shows that $X \in \mathcal{U} + \mathcal{K}$. Thus $Q\mathcal{U} \subseteq \mathcal{U} + \mathcal{K}$.

To prove the opposite inclusion, if $X \in \mathcal{U} + \mathcal{K}$, condition (i) is clearly satisfied. Also, since $(I - \tilde{E})\,AE = 0$ for all $A \in \mathcal{U}$, it is clearly enough to show that $E \mapsto (I - \tilde{E})KE$ is continuous (strong $\to$ norm) for any compact operator K. Now if $K = e \otimes f$ is any operator of rank 1 an easy calculation yields

$$\|(I - \tilde{E})KE - (I - \tilde{F})KF\| \leqslant \|(E - F)\,e\| \cdot \|f\| + \|(\tilde{E} - \tilde{F})f\| \cdot \|e\|$$

showing the required continuity in this case. Now taking finite linear combinations and then norm limits of these verifies condition (ii) in all cases.

COROLLARY 2.2. *If the map $E \mapsto \tilde{E}$ is surjective then*

$$Q\mathcal{U} = \mathcal{U} + \mathcal{K}.$$

Proof. It follows easily from considering the order topology that in this case $E \mapsto \tilde{E}$ is continuous.

COROLLARY 2.3. *If, for each $E \in \mathcal{E} \setminus \{I\}$, $E \neq E_+$ then for any weakly closed $\mathcal{A}$-submodule $\mathcal{U}$ of $\mathcal{L}(H)$,*

$$Q\mathcal{U} = \mathcal{U} + \mathcal{K}.$$

Proof. Since, in general, the module $\mathscr{U}$ is determined by a left continuous homomorphism in this case the homomorphism must be continuous.

It is of interest to point out that the condition of Corollary 2.3 is satisfied when each element of $\mathscr{E}\setminus\{I\}$ has finite rank. It is such nests that are connected with quasitriangular operators in the "classical" case (see [6], Chapter 5).

A description of the predual of a nest algebra appears in [3] (Proposition A). An analogous description of the predual of the module $\mathscr{U}$ appears below. A proof could be given along the lines of the proof in [3]. However, the proof below using the results of Section 1 is shorter. Note that it includes Proposition A of [3] and the proof for this case uses only the results of [2].

We use the following standard notation. The Banach space $\mathscr{L}(H)_*$ consists of all ultraweakly continuous linear functionals ρ on $\mathscr{L}(H)$ and is identified with the space $\mathscr{L}^1(H)$ of trace class operators X (with norm $\|X\|_1 = \mathrm{tr}[(X^*X)^{\frac{1}{2}}]$), via the formula

$$\rho(T) = \mathrm{tr}(XT), \quad T \in \mathscr{L}(H).$$

THEOREM 2.4. *Let $\mathscr{U}$ be the module determined by the homomorphism $E \mapsto \tilde{E}$ and let*

$$\mathscr{W} = \{X \in \mathscr{L}(H) : (I - E)XE_* = 0 \text{ for all } E \in \mathscr{E}\}.$$

Then $\rho \in \mathscr{L}(H)_*$ annihilates $\mathscr{U}$ if and only if ρ is of the form

$$\rho(T) = \mathrm{tr}(XT)$$

where X is a trace class operator in $\mathscr{W}$.

Proof. Recall that $E_* = \wedge\{\tilde{F} : F > E\}$. Thus, for any $Y \in \mathscr{L}(H)$, and any $E \in \mathscr{E}$, we have that $E_*Y(I - E) \in \mathscr{U}$. Hence if $\rho \in \mathscr{L}(H)_*$ is identified with the trace class operator X and ρ annihilates $\mathscr{U}$, then for any $Y \in \mathscr{L}(H)$

$$\mathrm{tr}[(I - E)XE_*Y] = \mathrm{tr}[XE_*Y(I - E)] = 0$$

and thus $(I - E)XE_* = 0$.

Conversely, if $X \in \mathscr{W}$ is trace class, let $e \otimes f$ be any rank 1 element of $\mathscr{U}$. Then, for some $F \in \mathscr{E}$, $(I - F)e = e$ and $F_*f = f$. Therefore,

$$e \otimes f = F_*(e \otimes f)(I - F)$$

and so

$$\mathrm{tr}[X(e \otimes f)] = \mathrm{tr}[XF_*(e \otimes f)(I - F)] =$$

$$= \mathrm{tr}[(I - F)XF_*(e \otimes f)] = 0$$

since $X \in \mathscr{W}$. Since the map $A \mapsto \operatorname{tr}(XA)$ is linear and ultraweakly continuous, it follows from Lemma 1.2 and Corollary 1.6 that, for all $A \in \mathscr{U}$,

$$\operatorname{tr}(XA) = 0.$$

This completes the proof.

COROLLARY 2.5. *(i) $\mathscr{U}$ is isomorphic to the dual space of $\mathscr{L}^1(H)/\mathscr{L}^1(H) \cap \mathscr{W}$.*
(ii) *$\mathscr{L}^1(H)/\mathscr{L}^1(H) \cap \mathscr{W}$ is isomorphic to the dual space of $\mathscr{U} \cap \mathscr{K}$.*

Proof. (i) This is an immediate consequence of Theorem 2.4 and elementary duality.

(ii) Since $\mathscr{L}^1(H)$ is the dual space of $\mathscr{K}$ (under the standard duality) it follows that the dual of $\mathscr{U} \cap \mathscr{K}$ is isomorphic to $\mathscr{L}^1(H)/\mathscr{R}$ where $\mathscr{R}$ is the annihilator of $\mathscr{U} \cap \mathscr{K}$ in $\mathscr{L}^1(H)$. However, by Corollary 1.6 and the ultraweak continuity of functionals in $\mathscr{L}^1(H)$, this annihilator is equal to the annihilator of $\mathscr{U}$, which is $\mathscr{L}^1(H) \cap \mathscr{W}$.

3. COMMUTANTS RELATIVE TO WEAKLY CLOSED MODULES

Let $\mathscr{U}$ be the $\mathscr{A}$-module determined by the homomorphism $E \mapsto \tilde{E}$. (Recall that, by assumption the homomorphism is left continuous and $\tilde{0} = 0$.) In this section we shall give a description of the commutant of $\mathscr{A}$ modulo $\mathscr{U}$; that is the set $C(\mathscr{A}, \mathscr{U})$ defined by

$$C(\mathscr{A}, \mathscr{U}) = \{X \in \mathscr{L}(H) : XA - AX \in \mathscr{U} \text{ for all } A \in \mathscr{A}\}.$$

The results determine the first Hochschild cohomology spaces with coefficients in $\mathscr{U}$. We enlarge on this point at the end of the section.

LEMMA 3.1. *Let $X \in C(\mathscr{A}, \mathscr{U})$. Then*
(i) *if $\tilde{E} \geq E$,*

$$(I - \tilde{E})XE = 0,$$

(ii) *if $\tilde{E} < E$,*

$$(I - \tilde{E})XE = \lambda_E(E - \tilde{E})$$

for some constant λ_E,
(iii) *if $F > \tilde{E} > \tilde{F} \geq \tilde{E}$,*

$$\lambda_E = \lambda_F$$

where λ_E, λ_F are as in (ii).

Proof. For any $E \in \mathscr{E}$, since $E \in \mathscr{A}$ and $X \in C(\mathscr{A}, \mathscr{U})$

$$XE - EX \in \mathscr{U}.$$

Thus

$$(I - \tilde{E})(XE - EX)E = 0.$$

If $\tilde{E} \geqslant E$, this reduces to $(I - \tilde{E})XE = 0$ and proves (i).

If $\tilde{E} < E$, we have

$$(I - \tilde{E})XE = (E - \tilde{E})XE.$$

Now also $I - \tilde{E} \in \mathscr{A}$ so

$$(I - \tilde{E})[X(I - \tilde{E}) - (I - \tilde{E})X]E = 0.$$

Therefore

$$(I - \tilde{E})X(E - \tilde{E}) = (I - \tilde{E})XE = (E - \tilde{E})XE$$

and hence

$$(I - \tilde{E})XE = (E - \tilde{E})X(E - \tilde{E}).$$

Now for any $A \in \mathscr{A}$,

$$(I - \tilde{E})[XA - AX]E = 0$$

and since E and $\tilde{E}$ are invariant projections of $\mathscr{A}$,

$$(I - \tilde{E})XE[(E - \tilde{E})A(E - \tilde{E})] - [(E - \tilde{E})A(E - \tilde{E})](I - \tilde{E})XE.$$

This means that $(I - \tilde{E})XE = (E - \tilde{E})X(E - \tilde{E})$ commutes with the compression of $\mathscr{A}$ to the range of $E - \tilde{E}$. Since this compression in a nest algebra and the commutant of any nest algebra is trivial, it follows that

$$(I - \tilde{E})XE = \lambda_E(E - \tilde{E})$$

for some constant λ_E, proving (ii). For (iii) multiply the above on either side by $E - \tilde{F}$ to give

$$(E - \tilde{F})X(E - \tilde{F}) = \lambda_E(E - \tilde{F}).$$

The same process on the result (ii) for F yields

$$(E - \tilde{F})X(E - \tilde{F}) = \lambda_F(E - \tilde{F})$$

and (iii) follows.

Our aim is to show that $C(\mathscr{A}, \mathscr{U})$ is the direct sum of $\mathscr{U}$ and a certain subalgebra $\mathscr{C}_{\mathscr{U}}$ of the core $\mathscr{C}$. The nature of $\mathscr{C}_{\mathscr{U}}$ depends on the way λ_E of Lemma 3.1 (ii) varies with E. Part (iii) of Lemma 3.1 shows that λ_E has a constant value on $[E_1, E_2]$

if the intervals $(\tilde{E}_1, E_1)$ and $(\tilde{E}_2, E_2)$ overlap (by convention, $(F, G) = \emptyset$ when $F \geqslant G$). It follows easily that λ_E is constant on each maximal connected component of $\cup \{(\tilde{E}, E) : E \in \mathscr{E}\}$. However, it is possible to link points in different components by intervals of the form $(\tilde{E}_1, E_1)$ and $(\tilde{E}_2, E_2)$. It is crucial that this link be overlapping as the following contrasting cases indicate.

EXAMPLE. Let $\mathscr{E}$ be a finite nest $\{0 = E_0, E_1, E_2, \ldots, E_{n-1}, E_n = I\}$.

1. The map $E_i \to \tilde{E}_i = E_{i-1}$ determines an ideal $\mathscr{U}$ of strictly upper triangular block matrices. For this case one can verify that the full core $\mathscr{C}$ lies in $C(\mathscr{A}, \mathscr{U})$. (Thus λ_E can vary.)

2. The map $E_i \to \tilde{E}_i = E_{i-2}$ determines an ideal $\mathscr{U}$ for which $C(\mathscr{A}, \mathscr{U}) \cap \mathscr{C} = \mathbf{C}I$. (Thus λ_E is constant.)

The above discussion motivates our next definition.

DEFINITION. The element F of $\mathscr{E}$ is $\tilde{\ }$ *-connected* to E (notation $E \gtrsim F$ or $F \lesssim E$) if $E < F$ and there exists a finite chain $F = E_0 > E_1 > \ldots > E_n$ with $\tilde{E}_i < E_{i+1}$ $(0 \leqslant i \leqslant n-1)$ and $\tilde{E}_n < E$.

Let $\mathscr{E}_0 = \{E \in \mathscr{E} : \tilde{E} < E\}$. Clearly $\mathscr{E}_0 = \cup \{(\tilde{E}, E] : E \in \mathscr{E}\}$, (where the notation $(\tilde{E}, E]$ is for order intervals on $\mathscr{E}$). It is easy to verify that $\gtrsim$ is reflexive and transitive on $\mathscr{E}_0$ and that no element of the complement of $\mathscr{E}_0$ is related to any element (either by $\gtrsim$ or by $\lesssim$). For each $E \in \mathscr{E}_0$ we define the $\tilde{\ }$-component $\gamma(E)$ of E by

$$\gamma(E) = \{F \in \mathscr{E} : F \gtrsim E\} \cup \{F \in \mathscr{E} : F \lesssim E\}.$$

Clearly, $\mathscr{E}_0$ is a disjoint union of $\tilde{\ }$ -components and it is easy to see that $\tilde{\ }$ -components are intervals.

Note that if $E = E_-$ and $E \gtrsim F$, it follows that $F \lesssim G$ for some $G < E$; (any G with $\tilde{E}_n < G < E$ will serve — where $\tilde{E}_n$ is as in the definition of $\gtrsim$). Thus for any $\tilde{\ }$ -component γ, if $P = \inf\gamma$ and $P \in \gamma$ then $P \neq P_-$. Thus γ may be written either in the form $(E, F]$ or (E, F) where E may be P or P_-. Hence we may write

$$\mathscr{E}_0 = \bigcup_{\omega \in \Omega} \gamma_\omega$$

where Ω is some index set and $\{\gamma_\omega : \omega \in \Omega\}$ are pairwise disjoint left open intervals with end points E_ω and F_ω $(E_\omega < F_\omega)$. Let $\mathscr{C}_{\mathscr{U}}$ be the weakly closed algebra generated by the projections $\{F_\omega - E_\omega : \omega \in \Omega\}$. Then $\mathscr{C}_{\mathscr{U}}$ is a subalgebra of the core $\mathscr{C}$ of $\mathscr{A}$ (where $\mathscr{C} = \mathscr{E}''$).

In the case of a continuous nest (i.e., $E = E_-$ for all $E \in \mathscr{E}$) the sets (E_ω, F_ω) are precisely the maximal connected components of $\cup \{(\tilde{E}, E) : E \in \mathscr{E}\}$. The above definition is required in order to deal with nests having an "atomic" part for which topological connectedness is not the concept appropriate to this situation.

THEOREM 3.2.

$$C(\mathscr{A}, \mathscr{U}) = \mathscr{C}_{\mathscr{U}} \oplus \mathscr{U}$$

where the sum is a direct sum of vector spaces.

Proof. Suppose $X \in \mathscr{C}_{\mathscr{U}}$. If $\tilde{E} \geqslant E$ then $(I - \tilde{E})XE = 0$. If $\tilde{E} < E$ then $(\tilde{E}, E]$ is contained in a single $\tilde{}$ -component of $\mathscr{E}_0$ and so, for some constant λ,

$$(I - \tilde{E})XE = \lambda(E - \tilde{E}).$$

Thus, if $A \in \mathscr{A}$, for all cases

$$(I - \tilde{E})[XA - AX]E = (I - \tilde{E})XEAE - (I - \tilde{E})A(I - \tilde{E})XE = 0.$$

This shows that $X \in C(\mathscr{A}, \mathscr{U})$ and so the inclusion $\mathscr{C}_{\mathscr{U}} + \mathscr{U} \subseteq C(\mathscr{A}, \mathscr{U})$ follows.

Now suppose $X \in C(\mathscr{A}, \mathscr{U})$. Then it follows from Lemma 3.1. that if γ_ω is any $\tilde{}$-component of $\mathscr{E}_0$ and $E \in \gamma_\omega$, we have

$$(E - \tilde{E})X(E - \tilde{E}) = \lambda_\omega(E - \tilde{E}),$$

where λ_ω is a constant depending only on the $\tilde{}$ -component γ_ω. Define $X_{\mathscr{U}}$ by

$$X_{\mathscr{U}} = \sum \lambda_\omega(F_\omega - E_\omega).$$

Since $|\lambda_\omega| \leqslant \|X\|$, the series converges in the strong operator topology and $X_{\mathscr{U}} \in \mathscr{C}_{\mathscr{U}}$.

If $E \in \mathscr{E}$ and $\tilde{E} < E$ then E is in some component γ_ω of $\mathscr{E}_0$ and $\tilde{E} \geqslant E_\omega$. Thus

$$(I - \tilde{E})(X - X_{\mathscr{U}})E = \lambda_\omega(E - \tilde{E}) - \lambda_\omega(E - \tilde{E}) = 0.$$

Also, if $\tilde{E} \geqslant E$ then $(I - \tilde{E})X_{\mathscr{U}}E = 0$ and also, from Lemma 3.1, $(I - \tilde{E})XE = 0$. Thus in all cases

$$(I - \tilde{E})(X - X_{\mathscr{U}})E = 0$$

and $X - X_{\mathscr{U}} \in \mathscr{U}$. Therefore $X \in \mathscr{C}_{\mathscr{U}} + \mathscr{U}$.

To prove that the sum is direct, if $T \in \mathscr{C}_{\mathscr{U}}$ then for each $\omega \in \Omega$,

$$T(F_\omega - E_\omega) = \lambda_\omega(F_\omega - E_\omega).$$

If also $T \in \mathscr{U}$, choose $E \in \gamma_\omega$. Then $E_\omega < \tilde{E} \leqslant F_\omega$ and

$$0 = (I - \tilde{E})TE = \lambda_\omega(E - \tilde{E}).$$

Thus $\lambda_\omega = 0$ for each $\omega \in \Omega$ and therefore $T = 0$.

REMARK. The first Hochschild cohomology space $H^1(\mathscr{A}, \mathscr{M})$ of $\mathscr{A}$ with coefficients in the module $\mathscr{M}$ is defined to be the difference space $Z^1(\mathscr{A}, \mathscr{M})/B^1(\mathscr{A}, \mathscr{M})$ where $Z^1(\mathscr{A}, \mathscr{M})$ is the space of continuous derivations of $\mathscr{A}$ into $\mathscr{M}$ and $B^1(\mathscr{A}, \mathscr{M})$ is the space of all derivations of the form $a \mapsto am - ma$ where $m \in \mathscr{M}$.

It is a fact that every continuous derivation of a nest algebra $\mathscr{A}$ into $\mathscr{L}(H)$ is spatial; that is, it is of the form $A \mapsto XA - AX$ for some $X \in \mathscr{L}(H)$. In other words $H^1(\mathscr{A}, \mathscr{L}(H))=(0)$. This fact was first proved by Christensen [1]. (Note that [1] also contains a proof of the fact that every derivation is continuous.) The proof in [1] that $H^1(\mathscr{A}, \mathscr{L}(H)) = (0)$ comes fom combining two results:

(i) $H^1(\mathscr{A}, \mathscr{A}) = H^1(\mathscr{A}, \mathscr{B})$ for any ultraweakly closed subalgebra of $\mathscr{L}(H)$ containing $\mathscr{A}$ and

(ii) $H^1(\mathscr{A}, \mathscr{A}) = (0)$.

Recently, Lance [4] has given a simple proof that for every positive integer n, $H^n(\mathscr{A}, \mathscr{L}(H)) = (0)$. For the reader's convenience we give a version of this proof for the case $n = 1$.

Suppose $D: \mathscr{A} \to \mathscr{L}(H)$ is a continuous derivation. For each $E \in \mathscr{E}$ with $E_- \neq I$, choose a unit vector ξ in the range of $I - E_-$. Define X_E by $X_E\eta = 0$ if $E\eta = 0$ and $X_E\eta = D(\xi \otimes \eta)\,\xi$ if $E\eta = \eta$ (note that $\xi \otimes \eta \in \mathscr{A}$ by Lemma 3.3 of [7]). Then if $A \in \mathscr{A}$ and $E\eta = \eta$,

$$X_E A\eta = D[A(\xi \otimes \eta)]\,\xi =$$

$$= AD(\xi \otimes \eta)\,\xi + D(A)\,(\xi \otimes \eta)\,\xi =$$

$$= AX_E\eta + D(A)\eta.$$

Thus $D(A)\,E = X_E A - AX_E$. Since $\|X_E\| \leqslant \|D\|$, the family $\{X_E : E_- \neq I\}$ forms a bounded net as E increases in the set $\mathscr{F} = \{E : E_- \neq I\}$. Since $\sup_{E\in\mathscr{F}} E = I$, it follows easily that if X is the limit of a weakly convergent subnet,

$$D(A) = XA - AX \qquad A \in \mathscr{A}.$$

Using this result and the fact that the commutant of any nest algebra is trivial it follows from Theorem 3.2 that for any (ultra)-weakly closed module $\mathscr{U}$,

$$H^1(\mathscr{A}, \mathscr{U}) = C(\mathscr{A}, \mathscr{U})/\mathbf{C}I + \mathscr{U} = \mathscr{C}_{\mathscr{U}}/\mathbf{C}I_1,$$

where I_1 is the $\mathscr{C}_{\mathscr{U}}$ component of I.

In the case when $\mathscr{U} \supseteq \mathscr{A}$ we have that $\mathscr{E}_0 = \emptyset$ and $\mathscr{C}_{\mathscr{U}} = (0)$. Thus we recover Christensen's result (actually a slight extension since $\mathscr{U}$ need not be an algebra) that

$$H^1(\mathscr{A}, \mathscr{U}) = (0).$$

4. THE INVARIANT LATTICE OF A MODULE

Let $\mathscr{V}$ be any $\mathscr{A}$-submodule of $\mathscr{L}(H)$. Clearly the norm closure of $\mathscr{V}$ and the weak closure of $\mathscr{V}$ have the same invariant subspaces as does $\mathscr{V}$. Now, for each $E \in \mathscr{E}$, let $\tilde{E}$ be the projection onto

$$\bigvee \{\operatorname{ran} XE : X \in \mathscr{V}\}.$$

The results of Section 1 show that the weak closure of $\mathscr{V}$ is the module determined by the homomorphism $E \mapsto \tilde{E}$. Thus, for invariant lattices, we need only consider weakly closed modules.

THEOREM 4.1. *Let $\mathscr{U}$ be the module determined by the homomorphism $E \mapsto \tilde{E}$. Then P is an invariant projection for $\mathscr{U}$ if and only if, for some $E \in \mathscr{E}$,*

$$\tilde{E} \leqslant P \leqslant E.$$

Proof. If $\tilde{E} \leqslant P \leqslant E$ for some $E \in \mathscr{E}$ and $X \in \mathscr{U}$ then, since $XE = \tilde{E}XE$ we have that $XP = XEP = \tilde{E}XEP = P\tilde{E}XEP = PXP$. Thus P is an invariant projection for $\mathscr{U}$.

For the converse, suppose $P \in \operatorname{Lat}\mathscr{U}$ and define E by

$$E = \bigvee \{F : \tilde{F} \leqslant P\}.$$

The left continuity of $E \mapsto \tilde{E}$ shows that $\tilde{E} \leqslant P$. Suppose that $P \nleqslant E$. Then there exists f in the range of P such that $(I - E)f \neq 0$. Now, there exists $F \in \mathscr{E}$ with $F > E$ (and $(I - F_-)f \neq 0$ (if $E = E_+$, $\lim_{F \downarrow E}(I - F)f \neq 0$ and $F > E \Rightarrow F_- > E$; if $E \neq E_+$ take $F = E_+$). Now, for any $X \in \mathscr{L}(H)$, $\tilde{F}X(I - F_-) \in \mathscr{U}$ and $\bigvee\{\tilde{F}X(I - F_-)f : X \in \mathscr{L}(H)\} = \operatorname{ran}\tilde{F}$. Thus $\tilde{F} \leqslant P$ which (since $F > E$) contradicts the definition of E. Thus $P \leqslant E$ and the theorem is proved.

COROLLARY 4.2. *The von Neumann algebra generated by $\mathscr{U}$ is equal to $\mathscr{L}(H)$ if and only if*

$$\tilde{I} \geqslant \bigvee \{E : \tilde{E} = 0\}.$$

Proof. Let $\mathscr{W}^*(\mathscr{U})$ denote the generated von Neumann algebra. Since $\mathscr{U}$ (non-zero) contains finite rank operators, it suffices, by a well known theorem (see [6] Theorem 8.12) to show that $\mathscr{W}^*(\mathscr{U})$ is a transitive algebra; or equivalently that $\operatorname{Lat}\mathscr{U} \cap \operatorname{Lat}\mathscr{U}^* = \{(0), I\}$. Suppose then, that G is a projection in this intersection. By Theorem 4.1, it follows that for some $E \in \mathscr{E}, G = \tilde{E} + G_1$ with $G_1 \leqslant E - \tilde{E}$ and also that, for some $F \in \mathscr{E}$, $G = (I - F) + G_2$ with $G_2 \leqslant F - \tilde{F}$. In particular, $I - F \leqslant E$ which implies that $F = I$ or $E = I$. But, if $F = I$ then $G \leqslant I - \tilde{I}$ which, being non-trivial, is not in $\operatorname{Lat}\mathscr{U}$. Hence $E = I$. Also, since $\tilde{E} \leqslant I - \tilde{F}$, it follows that $\tilde{F} = 0$ or $\tilde{E} = 0$. But if $\tilde{E} = 0$ then $E < I$ and $G \leqslant E$ which is impossible for $G \in \operatorname{Lat}\mathscr{U}^*$. Hence $\tilde{F} = 0$. Thus, $G \in \operatorname{Lat}\mathscr{W}^*(\mathscr{U})$ if and only if

$$G = (I - F) + G_2 = \tilde{I} + G_1$$

where $F \in \mathscr{E}$ is such that $\tilde{F} = 0$, $G_2 < F$ and $G_1 < (I - \tilde{I})$. It follows that all such projections G are trivial if and only if $\tilde{I} \geqslant F$ whenever $\tilde{F} = 0$. This completes the proof.

The theorem below enables us to describe a new class of reflexive algebras.

THEOREM 4.3. *Let $\mathscr{U}$ be the module determined by the homomorphism $E \mapsto \tilde{E}$. Then*

$$\operatorname{AlgLat}\mathscr{U} = \mathscr{C}_{\mathscr{U}} + \mathscr{W}$$

where $\mathscr{C}_{\mathscr{U}}$ is the weakly closed algebra generated by $\{F_\omega - E_\omega : \omega \in \Omega\}$ where F_ω and E_ω are the end points of the $\sim$ -components of $\mathscr{E}$ and $\mathscr{W}$ is the weakly closed algebra generated by $\mathscr{U}$.

Proof. Clearly $\operatorname{AlgLat}\mathscr{U} = \operatorname{AlgLat}\mathscr{W}$. Also, since $\mathscr{C}_{\mathscr{U}}$ depends only on the elements E of $\mathscr{E}$ such that $\tilde{E} < E$, reference to Lemma 1.10 shows that $\mathscr{C}_{\mathscr{U}} = \mathscr{C}_{\mathscr{W}}$. Thus we need only consider the case when $\mathscr{W} = \mathscr{U}$ is an algebra.

That $\operatorname{AlgLat}\mathscr{W} \supseteq \mathscr{C}_{\mathscr{W}} + \mathscr{W}$ is obvious. Let $T \in \operatorname{AlgLat}\mathscr{W}$. Then Theorem 4.1 shows that for all $E \in \mathscr{E}$ with $\tilde{E} \leqslant E$ and any $G \leqslant E - \tilde{E}$, $\tilde{E} \oplus G$ is invariant for T. Thus $(E - \tilde{E})T(E - \tilde{E})$ leaves every subprojection of $E - \tilde{E}$ invariant. This shows that

$$(E - \tilde{E})T(E - \tilde{E}) = \lambda_E(E - \tilde{E})$$

for some scalar λ_E. If E and F are in the same $\sim$ -component it follows as in Lemma 3.1 that $\lambda_E = \lambda_F$. The proof is now completed as in Theorem 3.2.

COROLLARY 4.4. *If $E \mapsto \tilde{E}$ determines an algebra $\mathscr{W}$ (that is, if $\tilde{\tilde{E}} \leqslant \tilde{E}$) then*

$$\mathscr{C}_{\mathscr{W}} \oplus \mathscr{W} = C(\mathscr{A}, \mathscr{W})$$

is a reflexive algebra.

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