J. OPERATOR THEORY
7(1982), 303–305



# QUADRATIC INTERPOLATION

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A long-known interpolation theorem [2, Theorem 5.4.1] has recently played an important role [3] in the solution of the multiplicativity problem for Toeplitz operators [1]. The purpose of this note is to make that solution more accessible by offering a simple interpretation and proof of the pertinent special case. Neither interpolation theory nor Toeplitz operators are needed in what follows.

The theorem has to do with what happens to an operator on an $L^2$ space when the underlying measure is multiplied by a positive function. Suppose, to be specific, that $X$ is a measure space, with measure $\mu$, and that $w$ is a positive function on $X$. (Positive means strictly positive here; $w(x) > 0$ for almost every $x$. All the functions that enter the discussion are required to be measurable.) Both $L^2(\mathrm{d}\mu)$ and $L^2(w\mathrm{d}\mu)$ are Hilbert spaces. What sense does it make to say that an operator (a bounded linear transformation) acts on both these spaces? One possible interpretation of such an expression, in harmony with the point of view of classical analysis, is that an operator on $L^2(\mathrm{d}\mu)$ and an operator on $L^2(w\mathrm{d}\mu)$ agree on the elements common to their domains (or equivalently, if preferred, agree on all functions that are finite linear combinations of characteristic functions of sets whose measure is finite in both senses).

Suppose now that $w_0$ and $w_1$ are positive functions and that $w$ is their geometric mean ($w^2 = w_0 w_1$). Suppose that an operator $T$ acts on both $H_0$ ($= L^2(w_0\mathrm{d}\mu)$) and $H_1$ ($= L^2(w_1\mathrm{d}\mu)$). Does it then act on $H_w$ ($= L^2(w\mathrm{d}\mu)$), and, if so, what are the relations among the norms $\|T\|_0$, $\|T\|_1$, and $\|T\|_w$ on $H_0$, $H_1$, and $H_w$ respectively? The answer is

$$(1) \qquad \|T\|_w^2 \leqslant \|T\|_0 \cdot \|T\|_1 ,$$

or, in other words, $\|T\|_w$ is dominated by the geometric mean of the norms $\|T\|_0$ and $\|T\|_1$. This statement is the special case of the interpolation theorem (the quadratic special case) that was mentioned above and that is to be proved below. (The theorem, in exactly this form, was called to my attention by Sheldon Axler.)

Begin with a very special case: assume that $w_0$ is bounded away from 0 and $\infty$ and that $w_1 = 1/w_0$. In that case $w$ is the constant function 1, and, in that case, the subscript $w$ can safely be omitted (from $H_w$ and $\|T\|_w$). The inequality to be proved becomes

$$(2) \qquad \|T\|^2 \leqslant \|T\|_0 \cdot \|T\|_1 .$$

Observe that in the case at hand the spaces $H_0$, $H_1$, and $H$ consist of exactly the same elements; what changes is the inner product. The multiplication operator $W_0$ on $H$, defined by $W_0 f = w_0^{1/2} f$, is a positive invertible operator, and inner product and norm in $H_0$ can be expressed in terms of $W_0$:

$$(f,g)_0 = (W_0 f, W_0 g), \quad \|f\|_0 = \|W_0 f\|.$$

It follows that

$$\|T\|_0 = \sup\{\|Tf\|_0 : \|f\|_0 \leqslant 1\} =$$
$$= \sup\{\|W_0 Tf\| : \|W_0 f\| \leqslant 1\} =$$
$$= \sup\{\|W_0 T W_0^{-1} g\| : \|g\| \leqslant 1\} =$$
$$= \|W_0 T W_0^{-1}\| .$$

Similarly, of course,

$$\|T\|_1 = \|W_1 T W_1^{-1}\| = \|W_0^{-1} T W_0\| ,$$

where $W_1 f = w_1^{1/2} f$. The desired inequality therefore takes the form

$$(3) \qquad \|T\|^2 \leqslant \|PTP^{-1}\| \cdot \|P^{-1}TP\| ,$$

or, equivalently,

$$(3) \qquad \|TT^*\| \leqslant \|PTP^{-1}\| \cdot \|PT^*P^{-1}\| .$$

In this form the assertion is a general theorem about abstract Hilbert spaces from which all traces of the interpolation mystique have been removed: if $T$ is an arbitrary operator and $P$ is an invertible Hermitian operator (even positiveness is no longer relevant), then (3) is true.

The proof is shorter than the preface. If $T$ happens to be such that its norm is equal to its spectral radius $\mathrm{r}(T)$, then

$$\|T\| = \mathrm{r}(T) = \mathrm{r}(PTP^{-1}) \leqslant \|PTP^{-1}\| .$$

Since $TT^*$ is Hermitian, and, therefore, has its norm equal to its spectral radius, it follows that

$$\|TT^*\| \leqslant \|PTT^*P^{-1}\| = \|PTP^{-1} \cdot PT^*P^{-1}\| ,$$

and the proof is complete. (I thank John B. Conway for reminding me of the trick of replacing $T$ by $TT^*$.) Comment: (3) remains true for every normal operator $P$; for not necessarily normal $P$ it is false, even on 2-dimensional spaces.

Consider next the assertion (1) under the assumption that both $w_0$ and $w_1$ are bounded away from both 0 and $\infty$. The multiplication operators $W_0$, $W_1$, and $W$ induced by $w_0^{1/2}$, $w_1^{1/2}$, and $w^{1/2}$ on $H_0$, $H_1$, and $H_w$ respectively are positive invertible operators. (Observe that $W_0$ and $W_1$ commute and $W^2 = W_0W_1$.) The desired inequality takes the form

$$\|(W_0W_1)^{1/2}T(W_0W_1)^{-1/2}\| \leqslant \|W_0TW_0^{-1}\|\cdot\|W_1TW_1^{-1}\| . \tag{4}$$

This too is an inequality true for abstract Hilbert spaces: $T$ is an arbitrary operator and $W_0$ and $W_1$ are commuting positive invertible operators. The proof is just a matter of a trivial substitution. Indeed: if

$$S = (W_0W_1)^{1/2}T(W_0W_1)^{-1/2}$$

and

$$P = W_0^{1/2}W_1^{-1/2},$$

then

$$PSP^{-1} = W_0TW_0^{-1}$$

and

$$P^{-1}SP = W_1TW_1^{-1},$$

so that the desired inequality follows from (3) (with $S$ in place of $T$).

The original inequality (1), with no boundedness restriction, is not a statement about (bounded) operators: multiplication by an unbounded function does not map $L^2$ into itself. Suppose, however, that $w$ is an arbitrary positive function, and let $v_n$ be a truncated version of $w$; that is, write

$$v_n(x) = w(x) \quad \text{when } \frac{1}{n} \leqslant w(x) \leqslant n,$$

and $v_n(x) = 1$ otherwise ($n = 1,2,3, \ldots$). Since

$$\|T\|_w = \lim_{n\to\infty}\|T\|_{v_n},$$

the truth of (1) in the general case follows from its truth in the bounded case.

*Research supported in part by a grant from the National Science Foundation.*

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Received April 21, 1981.