J. OPERATOR THEORY
7(1982), 341–372

ESTIMATES OF FUNCTIONS OF POWER BOUNDED OPERATORS ON HILBERT SPACES

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1. INTRODUCTION

One of the basic methods in the spectral theory of operators is the construction of a rich functional calculus for the class of operators under investigation. The main difficulty to construct such a calculus is to obtain sharp estimates of norms of functions of operators. The most famous and important inequality of this type is J. von Neumann's inequality [14]:

$$\|\varphi(T)\| \leqslant \sup\{|\varphi(\zeta)| : \zeta \in \mathbf{D}\} \stackrel{\text{def}}{=} \|\varphi\|_{H^\infty}$$

for any contraction T (i.e. $\|T\| \leqslant 1$) on a Hilbert space and for any complex polynomial φ.

The main problem treated in this paper is the investigation of the class of *power bounded operators* on a Hilbert space, i.e. such operators T that $\|T^n\| \leqslant \text{const}$, $n \geqslant 0$. In other words the problem is to calculate explicitly the norm

$$|||\varphi|||_c \stackrel{\text{def}}{=} \sup\{\|\varphi(T)\| : \|T^n\| \leqslant c,\ n \geqslant 0\},\ \ c > 1,$$

on the set of polynomials.

If T is an invertible operator with $\|T^n\| \leqslant \text{const}$, $n \in \mathbf{Z}$, it follows from B.Sz.-Nagy's theorem [18] that T is similar to a unitary operator and thus for any trigonometric polynomial φ the following inequality holds

$$\|\varphi(T)\| \leqslant \text{const}\, \|\varphi\|_{L^\infty}.$$

However S. R. Foguel [8] constructed an operator T on a Hilbert space such that $\|T^n\| \leqslant \text{const}$, $n \geqslant 0$, but T is not similar to a contraction. This operator is defined on $\ell^2 \oplus \ell^2$ by the operator matrix

$$T = \begin{pmatrix} S^* & Q \\ 0 & S \end{pmatrix},$$

where S is the shift operator on ℓ^2 $(S(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots))$ and Q is the orthogonal projection onto the subspace of ℓ^2 spanned by $\{e_k : k = 3^n, n \in \mathbf{Z}_+\}$. A. Lebow [11] showed that this operator is not polynomially bounded, i.e. for any $k > 0$ there exists a polynomial φ such that

$$\|\varphi(T)\| > k\|\varphi\|_{H^\infty}.$$

Besides the power bounded operators we consider the operators with the *growth of powers of order* α, $\alpha > 0$, i.e. operators T satisfying $\|T^n\| \leqslant \text{const}(1 + n^\alpha)$, $n \in \mathbf{Z}_+$. It is evident that for any operator T on a Banach space such that $\|T^n\| \leqslant \text{const}(1 + n)^\alpha$, $n \in \mathbf{Z}_+$, the following inequality holds

$$(1) \qquad \|\varphi(T)\| \leqslant \text{const} \sum_{n \geqslant 0} |\hat{\varphi}(n)| (1 + n)^\alpha, \quad \varphi \in \mathscr{P}_A,$$

($\mathscr{P}_A$ stands for the set of complex polynomials). It is impossible to improve this inequality on the class of all Banach spaces because if T is multiplication by z in the algebra

$$\mathscr{F}\ell^1_{(\alpha)} = \{f = \sum_{n \geqslant 0} \hat{f}(n) z^n : \|f\|_{\mathscr{F}\ell^1_{(\alpha)}} = \sum_{n \geqslant 0} |\hat{f}(n)|(1 + n)^\alpha < +\infty\}$$

then $\|T^n\| \leqslant \text{const}(1 + n)^\alpha$ and $\|\varphi(T)\| \geqslant \text{const} \sum_{n \geqslant 0} |\hat{\varphi}(n)| (n + 1)^\alpha$. We are interested in the following question. Is it possible to improve the inequality (1) for operators on Hilbert spaces? This is equivalent to the question of whether $\mathscr{F}\ell^1_{(\alpha)}$ is an operator algebra. (Recall that a Banach algebra is called an *operator algebra* if it is isomorphic to a subalgebra of the algebra of bounded operators on a Hilbert space.)

Indeed if $\mathscr{F}\ell^1_{(\alpha)}$ is an operator algebra, there exists an operator T on a Hilbert space such that $\|\varphi(T)\| \asymp \|\varphi\|_{\mathscr{F}\ell^1_{(\alpha)}}$ and so the estimate (1) cannot be strengthened. Conversely if (1) cannot be improved, there exists $\delta > 0$ such that for any polynomial φ there exists an operator Q_φ such that $\|Q_\varphi^n\| \leqslant \text{const}(1 + n)^\alpha, n \geqslant 0$, and $\|\varphi(Q_\varphi)\| \geqslant \delta\|\varphi\|_{\mathscr{F}\ell^1_{(\alpha)}}$. Put

$$R = \bigoplus_\varphi Q_\varphi,$$

the orthogonal sum being taken over a countable dense subset of the unit ball of $\mathscr{F}\ell^1_{(\alpha)}$. It is evident that the mapping $\varphi \to \varphi(R)$ is an embedding of $\mathscr{F}\ell^1_{(\alpha)}$ into the algebra of operators on a Hilbert space.

It was N. Th. Varopoulos [20] who firstly obtained non-trivial estimates of $\|\varphi(T)\|$ for operators satisfying $\|T^n\| \leqslant \text{const}(1 + n)^\alpha$, $\alpha \geqslant 0$. Namely he showed that

$$\|\varphi(T)\| \leqslant k_\varepsilon \sup_{|\zeta| < 1} \left|\sum_{n \geqslant 0} \hat{\varphi}(n)\, n^{2\alpha+\varepsilon} \zeta^n\right|$$

for any $\varepsilon > 0$. Moreover he proved that

$$\|\varphi(T)\| \leqslant k'_\varepsilon \|\sum_{n \geqslant 0} \hat{\varphi}(n)\, n^{2\alpha} (\log(2+n))^{1+\varepsilon} z^n\|_{L^\infty}, \quad \varepsilon > 0,$$

$$\|\varphi(T)\| \leqslant k''_\varepsilon \|\sum_{n \geqslant 0} \hat{\varphi}(n)\, (1+n)^{2\alpha} \log(2+n)\, (\log\log(n+3))^{1+\varepsilon} z^n\|_{L^\infty}, \quad \varepsilon > 0,$$

etc. He deduced from these estimates that for $\alpha < 1/2$ the algebra $\mathscr{F}\ell^1_{(\alpha)}$ is not an operator algebra.

It turned out (N. Th. Varopoulos [20]) that for $\alpha > 1/2$ $\mathscr{F}\ell^1_{(\alpha)}$ is a Q-algebra. Recall that a Banach algebra B is called an *isometric Q-algebra* if there exists a compact K, a subalgebra A of the algebra $C(K)$ of continuous functions on K and an ideal I of A such that B is isometric to the algebra A/I. An algebra isomorphic to an isometric Q-algebra is called a *Q-algebra.* By B. Cole's theorem (see [23]) each Q-algebra is an operator algebra. The converse is false as N. Th. Varopoulos showed [22].

Therefore for $\alpha > 1/2$, $\mathscr{F}\ell^1_{(\alpha)}$ is an operator algebra and thus the inequality (1) cannot be improved. In N. Th. Varopoulos's papers [20], [21] it is shown that $\mathscr{F}\ell^1_{(1/2)}$ is not a Q-algebra. Applying a result of Ph. Charpentier [4] it follows that $\mathscr{F}\ell^1_{(1/2)}$ is not an operator algebra.

In § 3 we obtain estimates of operator polynomials for operators satisfying $\|T^n\| \leqslant \text{const}\,(1+n)^\alpha$, $n \in \mathbf{Z}_+$, $0 \leqslant \alpha \leqslant 1/2$. These estimates are more precise than those of N. Th. Varopoulos [20]. If $\alpha = 0$, the estimate is given in terms of the Hankel matrices of the class $\ell^\infty \hat{\otimes} \ell^\infty$ and in terms of multipliers of the Hardy class H^1. The main tool for obtaining these estimates is the Grothendieck inequality [9].

In [7] A. M. Davie constructed an example of a power bounded element in a Q-algebra which is not polynomially bounded. The calculation of norms of its functions yields estimates from bellow for $|||\cdot|||_c$. In § 4 we compare these estimates from bellow with those from above obtained in § 3. We also show that A. M. Davie's example is extremal on the class of all Q-algebras. We give in § 4 a sufficient condition for a Hankel matrix to belong to $\ell^\infty \hat{\otimes} \ell^\infty$. In conclusion of the section we present an analogue of A. M. Davie's example for the case of the growth of powers of order α.

In § 5 we describe the Hankel matrices of the class $\ell^1 \check{\otimes} \ell^1$ (the bounded Hankel operators from c_0 to ℓ^1). As a consequence of this we prove that the natural projection (averaging projection) onto the set of Hankel matrices is unbounded on $\ell^1 \check{\otimes} \ell^1$ and $\ell^\infty \hat{\otimes} \ell^\infty$.

In § 6 we present new examples of power bounded non-polynomially bounded operators. These operators are constructed using the Hankel operators. The functions of these operators can be calculated explicitly. This permits us to obtain explicit estimates from bellow for $|||\cdot|||_c$ and to construct examples of operator algebras.

Moreover applying estimates from above obtained in § 3 to these operators we obtain some embedding theorems. In conclusion we consider the case of the growth of powers of order α.

In § 7 we state some open problems.

Acknowledgement. I would like to express my deep gratitude to S. V. Hruščev S. V. Kisliakov, N. K. Nikol'skiĭ and S. A. Vinogradov for numerous helpful, discussions.

2. PRELIMINARIES

TENSOR ALGEBRAS OVER DISCRETE SPACES. We denote by $\underbrace{\ell^\infty \hat{\otimes} \ell^\infty \hat{\otimes} \dots \hat{\otimes} \ell^\infty}_{n}$ the set of tensors $\{\gamma(k_1, \dots, k_n)\}_{k_i \geqslant 0}$ which admit a representation

$$\gamma(k_1, \dots, k_n) = \sum_{j \geqslant 0} f_j^{(1)}(k_1) f_j^{(2)}(k_2) \dots f_j^{(n)}(k_n),$$

where $f_j^{(r)} \in \ell^\infty$, $j \geqslant 0$, $1 \leqslant r \leqslant n$, and $\sum_{j \geqslant 0} \|f_j^{(1)}\|_{\ell^\infty} \dots \|f_j^{(n)}\|_{\ell^\infty} < +\infty$.

The space $\underbrace{\ell^\infty \hat{\otimes} \ell^\infty \hat{\otimes} \dots \hat{\otimes} \ell^\infty}_{n}$ is supplied with the following norm

$$\|\gamma\| = \inf\{\sum_{j \geqslant 0} \|f_j^{(1)}\|_{\ell^\infty} \dots \|f_j^{(n)}\|_{\ell^\infty} : \sum_{j \geqslant 0} f_j^{(1)}(k_1) \dots f_j^{(n)}(k_n) = \gamma(k_1, \dots, k_n)\}.$$

Denote by V^n the set of tensors $\gamma = \{\gamma(k_1, \dots, k_n)\}_{k_i \geqslant 0}$ such that $\sup_N \|P_N \gamma\| < \infty$, where

$$(P_N\gamma)(k_1, \dots, k_n) = \begin{cases} \gamma(k_1, \dots, k_n), & k_1 \leqslant N, \dots, k_n \leqslant N \\ 0 & \text{otherwise.} \end{cases}$$

Denote by $V_*^n = \underbrace{\ell^1 \check{\otimes} \ell^1 \check{\otimes} \dots \check{\otimes} \ell^1}_{n}$ the completion of the tensor product $\underbrace{\ell^1 \otimes \ell^1 \otimes \dots \otimes \ell^1}_{n}$ in the norm

$$\|\{\gamma(k_1, \dots, k_n)\}_{k_i \geqslant 0}\|_{V_*^n} =$$

$$= \sup\{|\sum_{k_i \geqslant 0} \gamma(k_1, \dots, k_n) f^{(1)}(k_1) \dots f^{(n)}(k_n)| : \|f^{(j)}\|_{\ell^\infty} \leqslant 1, \ 1 \leqslant j \leqslant n\}.$$

Then $(V^n_*)^* = V^n$ with respect to the duality

$$(\gamma, \beta) = \sum_{k_i \geqslant 0} \gamma(k_1, \ldots, k_n)\, \beta(k_1, \ldots, k_n);$$

here γ is a tensor in V^n_* whose entries vanish except a finite number and $\beta \in V^n$.

For $n = 2$ the space $\ell^1 \check{\otimes} \ell^1$ can be naturally identified with the space of all bounded operators from c_0 to ℓ^1. Note that each bounded operator from c_0 to ℓ^1 is compact. The espace $\ell^\infty \hat{\otimes} \ell^\infty$ is identified with the space of nuclear operators from ℓ^1 to ℓ^∞. For the properties of V^n, V^n_* see [19].

We need the following apparently known assertion.

LEMMA 2.1. *Let S_i, $1 \leqslant i \leqslant M$, be sets in $\mathbf{Z}^n_+$ whose projections to the coordinate axes are disjoint. Let $\gamma_i \in \underbrace{\ell^\infty \hat{\otimes} \ldots \hat{\otimes} \ell^\infty}_{n}$ and* $\operatorname{supp}\gamma_i \subset S_i$. *Then*

$$\Big\| \sum_{1 \leqslant i \leqslant M} \gamma_i \Big\|_{\ell^\infty \hat{\otimes} \ell^\infty \hat{\otimes} \ldots \ell^\infty} = \sup_{1 \leqslant i \leqslant M} \|\gamma_i\|_{\ell^\infty \hat{\otimes} \ldots \hat{\otimes} \ell^\infty}.$$

Proof. To simplify notations we suppose $n = 2$. Clearly we can suppose $M = 2$. Let $S_1 \subset I_1 \times J_1$, $S_2 \subset I_2 \times J_2$, where $I_1 \cap I_2 = \varnothing$, $J_1 \cap J_2 = \varnothing$. Let

$$\gamma_1(k, m) = \sum_{j \geqslant 0} \delta_j f_j^{(1)}(k) g_j^{(1)}(m), \qquad \gamma_2(k, m) = \sum_{j \geqslant 0} \delta_j f_j^{(2)}(k) g_j^{(2)}(m),$$

where

$$\operatorname{supp} f_j^{(1)} \subset I_1, \quad \operatorname{supp} g_j^{(1)} \subset J_1, \quad \operatorname{supp} f_j^{(2)} \subset I_2, \ \operatorname{supp} g_j^{(2)} \subset J_2,$$

$$\|f_j^{(1)}\|_{\ell^\infty} \leqslant 1, \quad \|g_j^{(1)}\|_{\ell^\infty} \leqslant 1, \quad \|f_j^{(2)}\|_{\ell^\infty} \leqslant 1, \quad \|g_j^{(2)}\|_{\ell^\infty} \leqslant 1.$$

Let us show that $\|\gamma_1 + \gamma_2\|_{\ell^\infty \hat{\otimes} \ell^\infty} \leqslant \sum_{j \geqslant 0} \delta_j$. Define functions $\varphi_j^{(1)}$, $\varphi_j^{(2)}$, $\psi_j^{(1)}$, $\psi_j^{(2)}$ by

$$\varphi_j^{(1)}(k) = \begin{cases} 1/2\, f_j^{(1)}(k), & k \in I_1 \\ 1/2\, f_j^{(2)}(k), & k \in I_2 \\ 0 \quad , & k \notin I_1 \cup I_2 \end{cases}; \qquad \varphi_j^{(2)}(k) = \begin{cases} 1/2 f_j^{(1)}(k), & k \in I_1 \\ -1/2 f_j^{(2)}(k), & k \in I_2 \\ 0 \quad , & k \notin I_1 \cup I_2 \end{cases};$$

$$\psi_j^{(1)}(m) = \begin{cases} g_j^{(1)}(m), & m \in J_1 \\ g_j^{(2)}(m), & m \in J_2 \\ 0 \quad , & m \notin J_1 \cup J_2 \end{cases}; \qquad \psi_j^{(2)}(m) = \begin{cases} g_j^{(1)}(m), & m \in J_1 \\ -g_j^{(2)}(m), & m \in J_2 \\ 0 \quad , & m \notin J_1 \cup J_2. \end{cases}$$

It is easy to check that

$$(\gamma_1 + \gamma_2)(k,m) = \sum_{j\geqslant 0} \delta_j\varphi_j^{(1)}(k)\,\psi_j^{(1)}(m) + \sum_{j\geqslant 0} \delta_j\varphi_j^{(2)}(k)\,\psi_j^{(2)}(m)$$

and

$$\|\gamma_1 + \gamma_2\|_{\ell^\infty \hat{\otimes} \ell^\infty} \leqslant \sum_{j\geqslant 0} \delta_j .$$

SCHUR MULTIPLIERS. A matrix $A = \{a_{nk}\}_{n,k\geqslant 0}$ is called a *Schur multiplier* of the space $\mathscr{B}(\ell^2)$ of bounded operators on ℓ^2 if for any bounded operator $B = \{b_{nk}\}_{n,k\geqslant 0}$ on ℓ^2 the matrix

$$A * B = \{a_{nk}b_{nk}\}_{n,k\geqslant 0}$$

generates a bounded operator on ℓ^2. It is easy to see that if $A \in V^2$ then A is a Schur multiplier. Indeed, it is sufficient to verify this for the case $A = \{a_{nk}\}_{n,k\geqslant 0}$, $a_{nk} = f_n g_k$, where $\{f_n\}_{n\geqslant 0}$, $\{g_k\}_{k\geqslant 0} \in \ell^\infty$. But in this case $A * B = \mathscr{D}_g B \mathscr{D}_f$, where $\mathscr{D}_f$ and $\mathscr{D}_g$ are diagonal operators on ℓ^2 generated by $\{f_n\}_{n\geqslant 0}$ and $\{g_n\}_{n\geqslant 0}$. It turns out that the converse is also true: each Schur multiplier of the space of bounded operators on ℓ^2 belongs to V^2 [2].

GROTHENDIECK INEQUALITY. Let $\gamma = \{\gamma_{nk}\}_{n,k\geqslant 0} \in \ell^1 \check{\otimes} \ell^1$. Suppose that $\{x_n\}_{n\geqslant 0}$, $\{y_n\}_{n\geqslant 0}$ are sequences of elements of the unit ball of a Hilbert space. Then

$$\Big|\sum_{n,k\geqslant 0} \gamma_{nk}(x_n,y_k)\Big| \leqslant k_G \|\gamma\|_{\ell^1 \check{\otimes} \ell^1}$$

for a constant $k_G > 0$. This inequality is called the Grothendieck inequality ([9], [12]). In other words Grothendieck inequality means that the matrix $\{(x_n, y_k)\}_{n,k\geqslant 0}$ belongs to V^2.

FUNCTION CLASSES. Denote by H^p, $1 \leqslant p \leqslant +\infty$, the Hardy class of functions analytic in the unit disc $\mathbf{D} = \{\zeta : |\zeta| < 1\}$ and such that

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_{\mathbf{T}} |f(r\zeta)|^p \,\mathrm{d}m(\zeta)\right)^{1/p} < +\infty, \quad p < +\infty,$$

$$\|f\|_{H^\infty} = \sup_{\zeta\in\mathbf{D}} |f(\zeta)| < +\infty,$$

where m is the normalized Lebesgue measure on $\mathbf{T}$. We do not distinguish notationally between the H^p functions and their boundary values on $\mathbf{T}$. Thus the following equality holds

$$H^p = \{f \in L^p : \hat{f}(n) = 0 \text{ if } n < 0\}, \quad 1 \leqslant p \leqslant \infty,$$

where $\hat{f}(n)$ is the n-th Fourier coefficient of f, $f \in L^p = L^p(\mathbf{T}, m)$.

If $f \in L^1$ then $\mathbf{P}_+ f \stackrel{\text{def}}{=} \sum_{n \geq 0} \hat{f}(n) z^n$, $\mathbf{P}_- f \stackrel{\text{def}}{=} f - \mathbf{P}_+ f$. Denote by $BMO_A(VMO_A)$ the space of functions φ analytic in $\mathbf{D}$ and such that $\varphi = \mathbf{P}_+ g$ for a function g in L^∞ (in $C(\mathbf{T})$),

$$\|\varphi\|_{BMO_A} \stackrel{\text{def}}{=} \inf\{\|g\|_{L^\infty} : \mathbf{P}_+ g = \varphi\}.$$

By the Hahn-Banach theorem $(VMO_A)^* = H^1$, $(H^1)^* = BMO_A$ with respect to the natural duality

$$(f, g) = \lim_{r \to 1} \int_{\mathbf{T}} f(r\zeta)\, g(r\bar{\zeta})\, \mathrm{d}m(\zeta)$$

or $(f, g) = \sum_{n \geq 0} \hat{f}(n)\, \hat{g}(n)$ if one of the functions is a polynomial.

The kernels W_N, $N \geq 0$, are defined as follows. If $N > 0$ then $\hat{W}_N(2^N) = 1$, $\hat{W}_N \equiv 0$ outside $(2^{N-1}, 2^{N+1})$ and $\hat{W}_N$ is a linear function on the intervals $[2^{N-1}, 2^N]$ and $[2^N, 2^{N+1}]$, $W_0 \stackrel{\text{def}}{=} 1 + z$. It is easy to see that $\|W_N\|_{L^1} \leq 3/2$, $N \geq 0$.

The Besov classes of analytic functions B^s_{pq}, $1 \leq p \leq +\infty$, $1 \leq q \leq +\infty$, $s \in \mathbf{R}$ are defined as follows

$$f \in B^s_{pq} \Leftrightarrow \Big(\sum_{n \geq 0} (2^{ns} \|W_n * f\|_{L^p})^q\Big)^{1/q} < +\infty, \quad q < +\infty,$$

$$f \in B^s_{p\infty} \Leftrightarrow \sup_{n \geq 0} \|W_n * f\|_{L^p} < +\infty.$$

These classes have also the following description.

$$f \in B^s_{pq} \Leftrightarrow \int_0^1 (1-r)^{nq - sq - 1} \|f_r^{(n)}\|^q_{L^p}\, \mathrm{d}r < +\infty, \quad q < +\infty,$$

$$f \in B^s_{p\infty} \Leftrightarrow \sup_{0 < r < 1} (1-r)^{n-s} \|f_r^{(n)}\|_{L^p} < +\infty,$$

where n is an integer such that $n > s$ and $f_r^{(n)}(\zeta) \stackrel{\text{def}}{=} f^{(n)}(r\zeta)$, $\zeta \in \mathbf{T}$.

If $s > 0$ then the classes B^s_{pq} can be defined by

$$f \in B^s_{pq} \Leftrightarrow \int_{-\pi}^{\pi} \frac{\|\Delta_t^n f\|^q_{L^p}}{|t|^{1+sq}}\, \mathrm{d}t < +\infty, \quad q < +\infty,$$

$$f \in B^s_{p\infty} \Leftrightarrow \sup_{t \neq 0} \frac{\|\Delta_t^n f\|_{L^p}}{|t|^s} < +\infty,$$

where n is an integer such that $n > 0$, $\Delta_t^n = \Delta_t \Delta_t^{n-1}$ and $(\Delta_t f)(e^{ix}) = f(e^{i(x+t)}) - f(e^{ix})$.

We shall use the following notation: $B^s_{pp} = B^s_p$, $B^s_{\infty\infty} = \Lambda_s$. If $0 < s < 1$, the class Λ_s consists of analytic in $\mathbf{D}$ functions satisfying the Hölder condition of order s, the class Λ_1 coincides with the Zygmund class. The closure of the set of polynomials in Λ_s is denoted by λ_s.

The dual spaces can be described as follows.

$$(B^s_{pq})^* = B^{-s}_{p'q'} \text{ if } q < +\infty, \text{ where } p' = \frac{p}{p-1}, \ q' = \frac{q}{q-1},$$

$$(\lambda_s)^* = B^{-s}_1$$

with respect to the natural duality.

If $I_\alpha f = \sum_{n \geqslant 0} \hat{f}(n)(1+n)^{-\alpha} z^n$ then $I_\alpha B^s_{pq} = B^{s+\alpha}_{pq}$, $\alpha \in \mathbf{R}$. For the detailed information on the Besov classes see [1], [17].

HANKEL OPERATORS. If f is an analytic function in $\mathbf{D}$, the Hankel matrix Γ_f is defined by

$$\Gamma_f = \{\hat{f}(n+k)\}_{n, k \geqslant 0}.$$

Z. Nehari's theorem [13] yields a criterion for Γ_f to be bounded on ℓ^2: Γ_f is a matrix of a bounded operator on ℓ^2 if and only if $f \in BMO_A$; $\|\Gamma_f\| = \|f\|_{BMO_A}$.

3. ESTIMATES FROM ABOVE

In this section using the Grothendieck inequality we obtain some estimates from above of polynomials of powers bounded operators. The obtained estimates permit us to construct a functional calculus for such operators. Further we consider the case of the growth of powers of orders α.

Denote by $\mathscr{L}$ the class of functions f analytic in $\mathbf{D}$ for which there exists $\{\gamma_{mk}\}_{m,k \geqslant 0} \in \ell^1 \check{\otimes} \ell^1$ such that $\hat{f}(n) = \sum_{m+k=n} \gamma_{mk}$, $n \geqslant 0$. Put

$$\|f\|_{\mathscr{L}} = \inf\{\|\{\gamma_{mk}\}_{m,k \geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1} : \sum_{m+k=n} \gamma_{mk} = \hat{\varphi}(n)\}.$$

THEOREM 3.1. *Let T be an operator on a Hilbert space satisfying $\|T^n\| \leqslant c$, $n \in Z_+$. Then*

$$\|\varphi(T)\| \leqslant c^2 k_G \|\varphi\|_{\mathscr{L}}$$

for any polynomial φ.

Proof. Let $x, y \in \mathscr{H}$, $\|x\| = \|y\| = 1$ and $\{\gamma_{mk}\}_{m,k\geqslant 0}$ be a complex matrix whose entries vanish except a finite number. Suppose that $\sum_{m+k=n} \gamma_{mk} = \hat{\varphi}(n)$. We have

$$(\varphi(T)x, y) = \sum_{n\geqslant 0} \hat{\varphi}(n)(T^n x, y) = \sum_{m,k\geqslant 0} \gamma_{mk}(T^m T^k x, y) =$$

$$= \sum_{m,k\geqslant 0} \gamma_{mk}(T^k x, (T^*)^m y).$$

Since $\|T^k x\| \leqslant c$, $\|(T^*)^m y\| \leqslant c$, it follows from the Grothendieck inequality (see § 2) that

$$|(\varphi(T)\, x, y)| \leqslant k_G c^2 \|\{\gamma_{mk}\}_{m,k\geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1}.$$

Therefore $\|\varphi(T)\| \leqslant k_G c^2 \|\varphi\|_{\mathscr{L}}$. ▨

Recall that a matrix $\{\beta_{mk}\}_{m,k\geqslant 0}$ is called a Hankel matrix if there exists a sequence $\{x_n\}_{n\geqslant 0}$ satisfying $\beta_{mk} = x_{m+k}$, $m, k \in \mathbf{Z}_+$. If φ is an analytic function in the unit disc, we denote by Γ_φ the Hankel matrix

$$\Gamma_\varphi = \{\hat{\varphi}(m+k)\}_{m,k\geqslant 0}.$$

Using the duality between $\ell^1 \check{\otimes} \ell^1$ and V^2, Theorem 3.1 can be stated as follows.

COROLLARY 3.2. *Let T be an operator on a Hilbert space satisfying $\|T^n\| \leqslant c$, $n \in \mathbf{Z}_+$. Then for any polynomial φ*

$$\|\varphi(T)\| \leqslant c^2 k_G \sup\{|(\varphi, \psi)| : \|\Gamma_\psi\|_{V^2} \leqslant 1\}.$$ ▨

Recall that $(\varphi, \psi) = \sum_{n\geqslant 0} \hat{\varphi}(n)\hat{\psi}(n)$.

The obtained results permit us to find more explicit estimates of polynomials of power bounded operators.

DEFINITION. An analytic function ψ in $\mathbf{D}$ is called a *multiplier* of H^1 if

$$f \in H^1 \Rightarrow f * \psi = \sum_{n\geqslant 0} \hat{f}(n)\hat{\psi}(n) z^n \in H^1.$$

We denote by $\mathscr{M}H^1$ the space of all multipliers of H^1, the norm of ψ in $\mathscr{M}H^1$ being defined as the norm of the operator $f \mapsto f * \psi$ on H^1.

The following assertion seems to be well-known (see for example [2]).

LEMMA 3.3. *If $\Gamma_\psi \in V^2$ then $\psi \in \mathscr{M}H^1$.*

Proof. Since $\Gamma_\psi \in V^2$, Γ_ψ is a Schur multiplier of the space of bounded operators on ℓ^2 (see § 2). Therefore for any bounded operator Γ_f on ℓ^2 the Schur product $\Gamma_\psi * \Gamma_f$ is also a bounded operator. Obviously $\Gamma_\psi * \Gamma_f = \Gamma_{\psi * f}$. By Z. Nehari's theorem Γ_f is bounded if and only if $f \in BMO_A$ (see § 2). It follows that ψ is a multiplier of BMO_A, i.e.

$$f \in BMO_A \Rightarrow \psi * f \in BMO_A .$$

Since $(H^1)^* = BMO_A$, we have $\psi \in \mathcal{M}H^1$. ▧

COROLLARY 3.4. *Let T be a power bounded operator on a Hilbert space. Then*

$$\|\varphi(T)\| \leqslant \text{const}\cdot\sup\{|(\varphi, \psi)| : \|\psi\|_{\mathcal{M}H^1} \leqslant 1\}$$

for any polynomial φ. ▧

Consider the following function space.

$$VMO_A \hat{\circledast} H^1 \overset{\text{def}}{=} \{\varphi = \sum_{m\geqslant 0} f_m * g_m : f_m \in VMO_A,\ g_m \in H^1,\ \sum_{m\geqslant 0} \|f_m\|_{VMO_A} \|g_m\|_{H^1} < \infty\}.$$

Define the norm in $VMO_A \hat{\circledast} H_1$ by

$$\|\varphi\|_{VMO_A \hat{\circledast} H^1} = \inf\{\sum_{m\geqslant 0} \|f_m\|_{VMO_A} \|g_m\|_{H^1} : \sum_{m\geqslant 0} f_m * g_m = \varphi\}.$$

It is easy to see that $(VMO_A \hat{\circledast} H^1)^* = \mathcal{M}H^1$ with respect to the natural duality. Indeed, let $\varphi = \sum_{m\geqslant 0} f_m * g_m$, $\psi \in \mathcal{M}H^1$. Then

$$|(\varphi, \psi)| = |\sum_m (f_m * g_m, \psi)| = |\sum_m (f_m, g_m * \psi)| \leqslant$$

$$\leqslant \sum_m \|f_m\|_{VMO_A} \cdot \|g_m * \psi\|_{H^1} \leqslant \text{const} \sum_m \|f_m\|_{VMO_A} \|g_m\|_{H^1}.$$

Hence $\psi \in (VMO_A \hat{\circledast} H^1)^*$.

Conversely, let $\psi \in (VMO_A \hat{\circledast} H^1)^*$ and g be a polynomial. We have

$$\|g * \psi\|_{H^1} = \sup\{|(f, g * \psi)| : \|f\|_{VMO_A} \leqslant 1\} =$$

$$= \sup\{|(f * g, \psi)| : \|f\|_{VMO_A} \leqslant 1\} \leqslant \|\psi\|_{(VMO_A \hat{\circledast} H^1)^*} \cdot \|g\|_{H^1}.$$

So $\psi \in \mathcal{M}H^1$.

Thus the following assertion is valid.

THEOREM 3.5. *Let T be a power bounded operator on a Hilbert space. Then*

$$\|\varphi(T)\| \leqslant \text{const}\|\varphi\|_{VMO_A \hat{\circledast} H^1}$$

for any polynomial φ. ▧

Now we are going to show that $VMO_A \hat{\circledast} H^1$ is a Banach algebra with respect to the pointwise multiplication and so Theorem 3.5 enables us to construct a functional calculus on the class $VMO_A \hat{\circledast} H^1$ for the power bounded operators.

LEMMA 3.6. *$VMO_A \hat{\circledast} H^1$ is a Banach algebra with respect to the pointwise multiplication.*

Proof. It is sufficient to prove the following assertion. Let f_1, f_2, g_1, g_2 be polynomials, then

$$\|(f_1 * g_1)\cdot(f_2 * g_2)\|_{VMO_A \hat{\otimes} H^1} \leqslant \|f_1\|_{VMO_A}\|f_2\|_{VMO_A}\|g_1\|_{H^1}\|g_2\|_{H^1}.$$

Choose trigonometric polynomials $\tilde{f}_1, \tilde{f}_2$ such that $\mathbf{P}_+\tilde{f}_i = f_i$, $\|\tilde{f}_i\|_{L^\infty} \leqslant \leqslant (1+\varepsilon)\|f_i\|_{VMO_A}$, $i = 1, 2$. Obviously $f_i * g_i = \tilde{f}_i * g_i$, $i = 1, 2$.

For $s \in \mathbf{T}$ put $\tilde{\varphi}_s(\theta) = \tilde{f}_1(\theta)\tilde{f}_2(s\bar{\theta})$, $\theta \in \mathbf{T}$, $\varphi_s \stackrel{\text{def}}{=} \mathbf{P}_+\tilde{\varphi}_s$, $\psi_s(\tau) = g_1(\tau)g_2(\tau\bar{s})$, $\tau \in \mathbf{T}$. Let us show that

$$(f_1 * g_1)\cdot(f_2 * g_2) = \int_{\mathbf{T}} \varphi_s * \psi_s \mathrm{d}m(s)$$

and

$$\int_{\mathbf{T}} \|\varphi_s\|_{VMO_A}\|\psi_s\|_{H^1}\mathrm{d}m(s) \leqslant \|\tilde{f}_1\|_{L^\infty}\|\tilde{f}_2\|_{L^\infty}\|g_1\|_{H^1}\|g_2\|_{H^1}.$$

After that, the proof will be finished because the integral $\int_{\mathbf{T}} \varphi_s * \psi_s \mathrm{d}m(s)$ can be approximated by Riemann sums. We have $\|\varphi_s\|_{VMO_A} \leqslant \|\tilde{\varphi}_s\|_{L^\infty} \leqslant \|\tilde{f}_1\|_{L^\infty}\|\tilde{f}_2\|_{L^\infty}$. Therefore

$$\int_{\mathbf{T}} \|\varphi_s\|_{VMO_A}\|\psi_s\|_{H^1}\mathrm{d}m(s) \leqslant \|\tilde{f}_1\|_{L^\infty}\|\tilde{f}_2\|_{L^\infty} \int_{\mathbf{T}}\int_{\mathbf{T}} |g_1(\tau)|\cdot|g_2(\tau\bar{s})|\,\mathrm{d}\tau\,\mathrm{d}s =$$

$$= \|\tilde{f}_1\|_{L^\infty}\|\tilde{f}\|_{L^\infty}\|g_1\|_{H^1}\|g_2\|_{H^1}.$$

It remains to show that $(f_1 * g_1)(f_2 * g_2) = \int_{\mathbf{T}} \varphi_s * \psi_s \mathrm{d}m(s)$ or equivalently $(\tilde{f}_1 * g_1)(\tilde{f}_2 * g_2) = \int_{\mathbf{T}} \tilde{\varphi}_s * \psi_s\,\mathrm{d}m(s)$. We have

$$\left(\int_{\mathbf{T}} \tilde{\varphi}_s * \psi_s\,\mathrm{d}m(s)\right)(\zeta) = \int_{\mathbf{T}}\int_{\mathbf{T}} \tilde{\varphi}_s(\theta)\psi_s(\zeta\bar{\theta})\,\mathrm{d}m(\theta)\,\mathrm{d}m(s) =$$

$$= \int_{\mathbf{T}}\int_{\mathbf{T}} \tilde{f}_1(\theta)\,\tilde{f}_2(s\bar{\theta})\,g_1(\zeta\bar{\theta})\,g_2(\zeta\bar{\theta}\bar{s})\,\mathrm{d}m(\theta)\,\mathrm{d}m(s) =$$

$$= \int_{\mathbf{T}} \tilde{f}_1(\theta)\,g_1(\zeta\bar{\theta})\,\mathrm{d}m(\theta)\int_{\mathbf{T}} \tilde{f}_2(s)\,g_2(\zeta\bar{s})\,\mathrm{d}m(s) =$$

$$= ((\tilde{f}_1 * g_1)(\zeta))((\tilde{f}_2 * g_2)(\zeta)).$$

▨

COROLLARY 3.7. *Let T be a power bounded operator on a Hilbert space. Then the mapping $\varphi \to \varphi(T)$ defined on the set of polynomials can be extended to a bounded linear homomorphism from $VMO_A \hat{\circledast} H^1$ into the algebra of bounded operators.*

The obtained results enable us to construct the functional calculus on the class $B^0_{\infty 1}$. Recall (see § 2) that

$$\varphi \in B^0_{\infty 1} \Leftrightarrow \sum_{N \geqslant 0} \|\varphi * W_N\|_{L^\infty} < +\infty \Leftrightarrow \int_0^1 \|\varphi'_r\|_{L^\infty} dr < +\infty.$$

It is evident that $B^0_{\infty 1} \subset H^\infty$ and that $B^0_{\infty 1}$ is a Banach algebra with respect to the pointwise multiplication because

$$\int_0^1 \|(fg)'_r\|_{L^\infty} dr \leqslant \|f\|_{L^\infty} \int_0^1 \|g'_r\|_{L^\infty} dr + \|g\|_{L^\infty} \int_0^1 \|f'_r\|_{L^\infty} dr.$$

THEOREM 3.8. *Let T be a power bounded operator on a Hilbert space. Then*

$$\|\varphi(T)\| \leqslant \text{const}\|\varphi\|_{B^0_{\infty 1}}, \quad \varphi \in \mathcal{P}_A,$$

and the mapping $\varphi \to \varphi(T)$ can be extended to a representation of the algebra $B^0_{\infty 1}$ into the algebra of bounded operators.

Proof. Let us show that $B^0_{\infty 1} \subset VMO_A \hat{\circledast} H^1$ and that this embedding is continuous. Put $Q_N = W_{N-1} + W_N + W_{N+1}$, $N > 0$. Clearly $\|Q_N\|_{L^1} \leqslant 9/2$ and $Q_N * W_N = W_N$.

Let $\varphi \in B^0_{\infty 1}$. We have

$$\varphi = \sum_{N \geqslant 0} \varphi * W_N = \varphi * W_0 + \sum_{N \geqslant 1} (\varphi * W_N) * Q_N.$$

Therefore

$$\|\varphi\|_{VMO_A \hat{\circledast} H^1} \leqslant |\hat{\varphi}(0)| + |\hat{\varphi}(1)| + \sum_{N \geqslant 1} \|\varphi * W_N\|_{VMO_A} \|Q_N\|_{L^1} \leqslant$$

$$\leqslant |\hat{\varphi}(0)| + |\hat{\varphi}(1)| + \frac{9}{2} \sum_{N \geqslant 1} \|\varphi * W_N\|_{L^\infty} \leqslant \text{const}\|\varphi\|_{B^0_{\infty 1}}. \qquad ▨$$

Note that the estimate of $\|\varphi(T)\|$ given in Theorem 3.8 is better than that of N. Th. Varopoulos [20].

COROLLARY 3.9. *Let T be a power bounded operator on a Hilbert space. Then there exists a positive M such that for any polynomial φ of degree n the following inequality holds*

$$\|\varphi(T)\| \leqslant M\log(n+2)\,\|\varphi\|_{L^\infty}.$$

▨

Note that the trivial estimate $\|\varphi(T)\| \leqslant \text{const} \sum_{n\geqslant 0} |\hat{\varphi}(n)|$ yields the inequality $\|\varphi(T)\| \leqslant \text{const}\,\sqrt{\deg \varphi}$.

We have obtained the following inequalities

$$|||\varphi|||_c \leqslant \text{const}\,\|\varphi\|_{\mathscr{L}} \leqslant \text{const}\,\|\varphi\|_{VMO_A \hat{\circledast} H^1} \leqslant \text{const}\,\|\varphi\|_{B^0_{\infty 1}}.$$

Naturally arises the question of whether these estimates are precise. In other words, does there exist $c > 1$ such that the norm $|||\cdot|||_c$ is equivalent to one of the norms $\|\cdot\|_{\mathscr{L}}$, $\|\cdot\|_{VMO_A \hat{\circledast} H^1}$, $\|\cdot\|_{B^0_{\infty 1}}$? As we pointed out in § 1 this question is equivalent to the following one:

Is it true that $\mathscr{L}$, $VMO_A \hat{\circledast} H^1$, $B^0_{\infty 1}$ are operator algebras (with respect to the pointwise multiplication)? For the classes $\mathscr{L}$, $VMO_A \hat{\circledast} H^1$ the answer is unknown. Moreover, I do not know whether $\mathscr{L}$ is a Banach algebra. Let us show that the answer is negative for the class $B^0_{\infty 1}$.

THEOREM 3.10. *The Banach algebra $B^0_{\infty 1}$ is not an operator algebra.*

Proof. Suppose $B^0_{\infty 1}$ is an operator algebra. Then there exists an operator T on a Hilbert space such that

$$\|\varphi(T)\| \asymp \|\varphi\|_{B^0_{\infty 1}}, \quad \varphi \in \mathscr{P}_A.$$

Since $\|z^n\|_{B^0_{\infty 1}} \leqslant \text{const}$, we have $\|T^n\| \leqslant \text{const}$, $n \geqslant 0$. It follows that $\|\cdot\|_{\mathscr{L}} \asymp \|\cdot\|_{VMO_A \hat{\circledast} H^1} \asymp \|\cdot\|_{B^0_{\infty 1}}$. Hence $B^0_{\infty 1} = VMO_A \hat{\circledast} H^1$. Since $(VMO_A \hat{\circledast} H^1)^* = \mathscr{M}H^1$ and $(B^0_{\infty 1})^* = B^0_{1\infty}$ (see § 2), it follows that $B^0_{1\infty} = \mathscr{M}H^1$. Recall that

$$\varphi \in B^0_{1\infty} \Leftrightarrow \sup_N \|\varphi * W_N\|_{L^1} < +\infty \Leftrightarrow \|\varphi_r'\|_{L^1} \leqslant \frac{\text{const}}{1-r}, \quad 0 < r < 1.$$

But it is known (see [10], [16]) that there exists an analytic function φ such that $\|\varphi_r'\|_{L^1} \leqslant \dfrac{\text{const}}{1-r}$ and $\varphi \notin \mathscr{M}H^1$. ▨

Now we proceed to the operators satisfying $\|T^n\| \leqslant c(1+n)^\alpha$, $n \in \mathbb{Z}_+$. The following assertion can be proved just as in Theorem 3.1.

THEOREM 3.11. *Let T be an operator on a Hilbert space such that $\|T^n\| \leqslant c(1+n)^\alpha$, $n \in \mathbf{Z}_+$, $\alpha > 0$. Then*

$$\|\varphi(T)\| \leqslant k_G c^2 \inf\{\|\{\gamma_{mk}(1+m)^\alpha (1+k)^\alpha\}_{m,k\geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1} : \sum_{m+k=n} \gamma_{mk} = \hat{\varphi}(n)\}$$

for any polynomial φ. ▨

Given an analytic in $\mathbf{D}$ function φ, $\beta \in \mathbf{R}$, we denote by Γ_ψ^β the matrix

$$\Gamma_\psi^\beta = \{\hat{\psi}(m+k)(1+m)^\beta (1+k)^\beta\}_{m,k\geqslant 0}.$$

Using the duality arguments Theorem 3.11 can be stated as follows.

$$\|\varphi(T)\| \leqslant k_G c^2 \sup\{|(\varphi, \psi)| : \|\Gamma_\psi^{-\alpha}\|_{V^2} \leqslant 2\}, \ \varphi \in \mathscr{P}_A.$$

It is well-known that Γ_φ^α, $\alpha > 0$, determines a bounded operator on ℓ^2 if and only if $\varphi \in \Lambda_{2\alpha}$. We need only the following implication which can be easily proved.

$$\varphi \in \Lambda_{2\alpha} \Rightarrow \Gamma_\varphi^\alpha \in \mathscr{B}(\ell^2).$$

Indeed let $x = \{x_n\}_{n\geqslant 0}$, $y = \{y_n\}_{n\geqslant 0} \in \ell^2$. Then

$$(\Gamma_\varphi^\alpha x, y) = (\varphi, FG),$$

where $F = \sum_{n\geqslant 0} (1+n)^\alpha x_n z^n$, $G = \sum_{n\geqslant 0} (1+n)^\alpha y_n z^n$. It is evident that $\iint_{\mathbf{D}} |F|^2(1-|z|)^{2\alpha-1}\,dx\,dy < +\infty$ and $\iint_{\mathbf{D}} |G|^2(1-|z|)^{2\alpha-1}dx\,dy < +\infty$. Therefore $FG \in B_1^{-2\alpha}$ (see § 2). The result follows from the fact that $(B_1^{-2\alpha})^* = \Lambda_{2\alpha}$ (see § 2).

If $\Gamma_\psi^{-\alpha} \in V^2$ then $\Gamma_\psi^{-\alpha}$ is a Schur multiplier of the space of bounded operators and since $\Gamma_\psi^{-\alpha} * \Gamma_\varphi^\alpha = \Gamma_{\varphi * \psi}$, it follows that φ is a multiplier from $\Lambda_{2\alpha}$ to BMO_A, i.e.

$$f \in \Lambda_{2\alpha} \Rightarrow f * \psi \in BMO_A.$$

Therefore

$$\|\varphi(T)\| \leqslant \text{const}\|\varphi\|_{H^1 \hat{\otimes} \Lambda_{2\alpha}} \stackrel{\text{def}}{=} \inf\{\sum_{n\geqslant 0} \|f_n\|_{H^1}\|g_n\|_{\Lambda_{2\alpha}} : \sum_{n\geqslant 0} f_n * g_n = \varphi\}.$$

As in the case $\alpha = 0$ it is easy to show that

$$\|\varphi(T)\| \leqslant \text{const}\|\varphi\|_{B^{2\alpha}_{\infty 1}},$$

where $B^{2\alpha}_{\infty 1} = \{\varphi : \sum_{n \geqslant 0} \|\varphi * W_n\|_{L^\infty} 2^{N\alpha} < +\infty\}$. This estimate strengthens the results of N. Th. Varopoulos [20]. In particular it follows that

$$\mathscr{F}\ell^1_{(\alpha)} = \{\varphi : \sum_{n \geqslant 0} |\hat{\varphi}(n)|(1+n)^\alpha < +\infty\}$$

is not an operator algebra.

It is possible to obtain one more sufficient condition for $\Gamma_{\psi}^{-\alpha} \in V^2$. Namely if $\varphi \in BMO$ then $\Gamma_{\varphi * \psi}^{-\alpha}$ is a bounded operator on ℓ^2. It follows that

$$\|\varphi(T)\| \leqslant \text{const} \cdot \inf\{\sum_{n \geqslant 0} \|f_n\|_{VMO_A} \sum_{k \geqslant 0} \|g_{n,k}\|_{W_2^\alpha} \|h_{n,k}\|_{W_2^\alpha} : \varphi = \sum_{n \geqslant 0} f_n * \sum_{k \geqslant 0} g_{nk} \cdot h_{nk}\},$$

where $W_2^\alpha = \{g : \sum_{n \geqslant 0} |\hat{g}(n)|^2 n^{2\alpha} < +\infty\}$.

4. A. M. DAVIE'S EXAMPLE

In [7] A. M. Davie constructed an example of a power bounded element in a Q-algebra which is not polynomially bounded. In this section we show that this example is extremal on the class of all Q-algebras. A. M. Davie's example yields an estimate from below of the norm $|||\cdot|||_c$. Here we discuss the relations between this estimate and those from above obtained in § 3. Roughly speaking the distinction between these estimates is that in the case of A. M. Davie's example besides the Hankel matrices in V^2 it is necessary to consider the Hankel tensors $\overset{M}{\Gamma}_\psi$ in V^M, where

$$\overset{M}{\Gamma}_\psi = \{\hat{\psi}(k_1 + \ldots + k_M)\}_{k_1, \ldots, k_M \geqslant 0}.$$

We obtain a sufficient condition on ψ for $\overset{M}{\Gamma}_\psi \in V^M$. In conclusion we give an analogue of A. M. Davie's example for the case of the growth of powers of order α.

A. M. DAVIE'S EXAMPLE. Let $\mathbf{D}_c = \{\zeta : |\zeta| \leqslant c\}$, $c > 1$, $K = \mathbf{D}_c^\infty$ be the space of sequences $(\zeta_1, \zeta_2, \ldots)$, $\zeta_k \in \mathbf{D}_c$, endowed with the product topology. Denote by B_c the closed subalgebra of $C(K)$ generated by the coordinate functions $z_k(\zeta_1, \zeta_2, \ldots) = \zeta_k$. Put

$$i : \mathbf{D} \to X,\ i(\zeta) = (\zeta, \zeta^2, \ldots).$$

If $f \in C(K)$, set $\rho f \overset{\text{def}}{=} f \circ i$. The algebra $\mathscr{D}_c$ consists of those functions g analytic in $\mathbf{D}$ for which there exists $f \in B_c$ such that $\rho f = g$. Define

$$\|g\|_{\mathscr{D}_c} \overset{\text{def}}{=} \inf\{\|f\|_{B_c} : \rho f = g\}.$$

Then $\mathscr{D}_c$ is an isometric Q-algebra and its element z is power bounded. Indeed $z^n = \rho z_n$ and so $\|z^n\|_{\mathscr{D}_c} \leqslant c$. A. M. Davie [7] showed that for $c > 4e$ this element is not polynomially bounded or more precisely

$$\|\varphi\|_{\mathscr{D}_c} \geqslant \text{const} \sum_{n\geqslant 0} |\hat{\varphi}(2^n)|.$$

The following assertion says that A. M. Davie's example is extremal on the class of all Q-algebras.

LEMMA 4.1. *Let A be an isometric Q-algebra, $x \in A$, $c > 1$, $\|x^n\|_A \leqslant c$, $n \geqslant 0$. Then $\|\varphi(x)\| \leqslant \|\varphi\|_{\mathscr{D}_c}$ for any polynomial φ.*

Proof. Let f be a polynomial in B_c such that $\rho f = \varphi$ and $\|f\|_{B_c} \leqslant \|\varphi\|_{\mathscr{D}_c} + \varepsilon$, $\varepsilon > 0$. We have

$$\varphi(x) = f(x, x^2, \ldots).$$

Since $\mathscr{D}_c$ is an isometric Q-algebra, $\|\varphi(x)\| \leqslant \|f\|_{B_c} \leqslant \|\varphi\|_{\mathscr{D}_c} + \varepsilon$. ▧

The following assertions permit us to estimate $\|\cdot\|_{\mathscr{D}_c}$ in terms of the Hankel tensors. Recall that $V_*^M = \underbrace{\ell^1 \check{\otimes} \ell^1 \check{\otimes} \ldots \check{\otimes} \ell^1}_{M}$.

PROPOSITION 4.2. 1) *Let φ be a polynomial. Then there exists a sequence $\{\varphi_n\}_{n\geqslant 0}$ of polynomials such that $\varphi = \sum_{n\geqslant 0} \varphi_n$,*

$$\hat{\varphi}_M(k) = \sum_{k_1+\cdots+k_M=k} a^{(M)}_{k_1,\ldots,k_M},$$

where $a^{(M)} \in V_*^M$ and $\|a^{(M)}\|_{V_*^M} \leqslant \left(\frac{2e}{c}\right)^M \|\varphi\|_{\mathscr{D}_c}$.

2) *Let $L > 1$, $a^{(M)} \in V_*^M$, $\|a^{(M)}\|_{V_*^M} \leqslant L^{-M}$. Let $\varphi = \sum_{M\geqslant 0} \varphi_M$, where $\hat{\varphi}_M(k) = \sum_{k_1+\cdots+k_M=k} a^{(M)}_{k_1,\ldots,k_M}$. Then $\varphi \in \mathscr{D}_c$ for $c < L$.*

Proof. 1) Suppose that $\|\varphi\|_{\mathscr{D}_c} = 1$. Let f be a polynomial in B_c such that $\rho f = \varphi$ and $\|\varphi\|_{B_c} \leqslant 1 + \varepsilon$, $\varepsilon > 0$. We have $f = \sum_{M\geqslant 0} P_M$, where P_M is a homogeneous polynomial of degree M. Since $\|f\|_{B_c} \leqslant 1+\varepsilon$ it follows that $\|P_M\|_{B_c} \leqslant 1 + \varepsilon$ (to prove this well-known assertion (see for example [6]) it is sufficient to consider the function $F(\zeta) = f(\zeta z_1, \zeta z_2, \ldots)$ and to estimate $\hat{F}(M)$). Therefore

$$\sup_{|\zeta_k| \leqslant 1} |P_M(\zeta_1, \zeta_2, \ldots)| \leqslant (1 + \varepsilon)c^{-M}.$$

Put

$$P_M(\zeta_1, \zeta_2, \ldots) = \sum_{k=\{k_1,\ldots,k_M\}} a_k^{(M)} \zeta_{k_1} \ldots \zeta_{k_M},$$

where $a^{(M)}$ is a tensor symmetric with respect to permutations of the coordinates. Then by A. M. Davie's Lemma ([6], Lemma 2.1) we have

$$\|a^{(M)}\|_{V_*^M} \leqslant (2e)^M \sup_{|\zeta_k|<1} |P_M(\zeta_1, \zeta_2, \ldots)| \leqslant (2e)^M(1+\varepsilon).$$

Put $\varphi_M = \rho P_M$. Obviously $\hat{\varphi}_M(k) = \sum_{k_1+\ldots+k_M=k} a^{(M)}_{k_1,\ldots,k_M}$.

2) It is easy to see that $\varphi_M = \rho \sum_{k=\{k_1,\ldots,k_M\}} a_k^{(M)} z_{k_1} \ldots z_{k_M}$.

Therefore

$$\|\varphi\|_{\mathscr{D}_c} \leqslant \sum_{M\geqslant 0} \|\sum_k a_k^{(M)} z_{k_1} \ldots z_{k_M}\|_{B_c} \leqslant \sum_M c^M \|a^{(M)}\|_{V_*^M} \leqslant \sum_M \left(\frac{c}{L}\right)^M < +\infty. \blacksquare$$

Using the duality arguments we can obtain the following version of Proposition 4.2.

PROPOSITION 4.3. *Let ψ be an analytic function in* $\mathbf{D}$.

1) *If* $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant L^M$ *then* $\psi \in (\mathscr{D}_c)^*$ *for* $c > 2Le$.

2) *If* $\varphi \in (\mathscr{D}_c)^*$ *and* $L > c$ *then* $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant \text{const} \cdot L^M$. ▧

Thus the problem of description of $\mathscr{D}_c$ is related to that of description of the Hankel tensors $\overset{M}{\Gamma}_\psi$ of the class V^M. For $M = 2$ this is equivalent to the question of characterizing the Hankel-Schur multipliers of the space of bounded operators on ℓ^2. The latter problem was posed by G. Bennett [2]. In § 3 we mentioned that if $\Gamma_\psi \in V^2$ then ψ is a multiplier of H^1. N. Th. Varopoulos's theorem [19] gives a sufficient condition for $\Gamma_\psi \in V^2$: if $\hat{\psi}(n) = \hat{\mu}(n)$, $n \in \mathbf{Z}_+$, for a complex measure μ on $\mathbf{T}$ then $\Gamma_\psi \in V^2$ and $\|\Gamma_\psi\|_{V^2} \leqslant \text{Var}\mu$. This assertion can be proved as follows. If $\mu = \delta_\zeta$ is the δ-measure at $\zeta \in \mathbf{T}$ then $\hat{\mu}(n+k) = e^{-in\zeta}e^{-ik\zeta}$ and so $\|\{\hat{\mu}(n+k)\}_{n,k\geqslant 0}\|_{V^2} \leqslant 1$. Now the result follows from the fact that the convex combinations of the measures $\lambda\delta_\zeta$, $|\lambda| \leqslant 1$, $\zeta \in \mathbf{T}$, are weakly dense in the unit ball of the space of borelian measures. Clearly the same arguments show that $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant 1$ if $\hat{\psi}(n) = \hat{\mu}(n)$, $n \in \mathbf{Z}_+$, and $\text{Var}\mu \leqslant 1$.

We give another sufficient condition for $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant L^M$.

THEOREM 4.4. *Let $\{s_j\}_{j\geqslant 1}$ be a sequence of positive numbers such that* $\frac{s_{j+1}}{s_j} \geqslant \delta > 1$ *and $\{F_j\}_{j\geqslant 1}$ be a sequence of polynomials such that* $\text{supp}\hat{F}_j \subset [s_j, s_{j+1})$

and $F = \sum_{j\geq 1} F_j$. *Suppose that*

$$\int_{\mathbf{T}} \sup_j |F_j(\zeta)| \, dm(\zeta) < +\infty.$$

Then $\overset{M}{\Gamma}_F \in V^M$ *and* $\|\overset{M}{\Gamma}_F\|_{V^M} \leqslant \text{const} \cdot 2^M$.

Proof. For the sake of simplicity we give the proof for the case $M = 2$. Moreover we suppose $s_j = 2^j$.

We have $F = F^{(1)} + F^{(2)}$, where $F^{(1)} = \sum_{j\geqslant 1} F_{2j}$, $F^{(2)} = \sum_{j\geqslant 0} F_{2j+1}$. Let us show that $\Gamma_{F^{(1)}} \in V^2$. The fact that $\Gamma_{F^{(2)}} \in V^2$ can be proved by the same arguments.

Put

$$S_1 = \bigcup_{j\in\mathbf{Z}_+} \{(m, k)\colon 2^{2j} \leqslant m + k < 2^{2j+1},\ m \geqslant 2^{2j-1}\},$$

$$S_2 = \bigcup_{j\in\mathbf{Z}_+} \{(m, k)\colon 2^{2j} \leqslant m + k < 2^{2j+1},\ k \geqslant 2^{2j-1}\},$$

$$S_3 = S_1 \cap S_2.$$

Clearly

$$\Gamma_{F^{(1)}} = \Gamma_{F^{(1)}}\chi_{S_1} + \Gamma_{F^{(1)}}\chi_{S_2} - \Gamma_{F^{(1)}}\chi_{S_3}.$$

By Lemma 2.1

$$\|\Gamma_{F^{(1)}}\chi_{S_3}\|_{V^2} \leqslant \|\Gamma_{F^{(1)}}\chi_{S_1}\|_{V^2}.$$

Obviously

$$\|\Gamma_{F^{(1)}}\chi_{S_1}\|_{V^2} = \|\Gamma_{F^{(1)}}\chi_{S_2}\|_{V^2}.$$

It remains to estimate $\|\Gamma_{F^{(1)}}\chi_{S_1}\|_{V^2}$. Suppose that $F_j = \mathbf{O}$ except a finite number of j's. We are going to obtain an estimate depending only on $\int_{\mathbf{T}} \sup_j |F_j(\zeta)| \mathrm{d}m(\zeta)$.

Put

$$f_\theta(m) = F_{2j}(e^{i\theta})\, e^{-im\theta}, \quad 2^{2j-1} \leqslant m < 2^{2j+1},$$
$$g_\theta(k) = e^{-ik\theta}.$$

It is easy to check that $\Gamma_{F^{(1)}}\chi_{S_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta \otimes g_\theta \, \mathrm{d}\theta$. Indeed if $(m, k) \in S_1$ then $\hat{F}^{(1)}(m + k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta(m) g_\theta(k) \, \mathrm{d}\theta$. If $(m, k) \notin S_1$ then $\frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta(m) g_\theta(k) \, \mathrm{d}\theta = \hat{F}_{2j}(m+k)$, where j satisfies the relation $2^{2j-1} \leqslant m < 2^{2j+1}$. But in this case $\hat{F}_{2j}(m+k)=0$. Therefore $\Gamma_{F^{(1)}}\chi_{S_1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_\theta \otimes g_0 \, d\theta$. Approximating this integral by Riemann sums we obtain that

$$\|\Gamma_{F^{(1)}}\chi_{S_1}\|_{V^2} \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \|f_\theta\|_{\ell^\infty}\|g_0\|_{\ell^\infty} \, d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \sup_j |F_{2j}(e^{i\theta})| \, d\theta.$$

COROLLARY 4.5. *If*

$$\int_{\mathbf{T}} \sup_N |(F * W_N)(\zeta)| \, dm(\zeta) < +\infty$$

then

$$\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant \text{const}\cdot 2^M.$$

This implies that $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant \text{const}\cdot 2^M$ if $\psi = \sum_{n\geqslant 0} c_n z^{2^n}$ and $\{c_n\}_{n\geqslant 0} \in \ell^\infty$. Thus we obtain A. M. Davie's Theorem: if $c > 4e$ then $\|\varphi\|_{\mathscr{D}_c} \geqslant \text{const} \sum_{n\geqslant 0} |\hat{\varphi}(2^n)|$.

COROLLARY 4.6. *If F satisfies the hypothesis of Theorem 4.4 or Corollary 4.5 then $F \in \mathscr{M}H^1$.*

The latter assertion can be proved by another way using the following description of H^1 (see [5]):

$$f \in H^1 \Leftrightarrow \int_{\mathbf{T}} \Big(\sum_{N\geqslant 0} |(f * W_N)(\zeta)|^2\Big)^{1/2} dm(\zeta) < +\infty .$$

The following question seems to be of interest. Suppose that $\Gamma_\psi \in V^2$, does it follow that $\overset{M}{\Gamma}_\psi \in V^M$ and $\|\overset{M}{\Gamma}_\psi\|_{V^M} \leqslant \text{const}\cdot L^M$? The affirmative answer to this question would imply that the estimate obtained in Theorem 3.1 is exact, i.e. for some $c > 1$

$$\|\varphi\|_{\mathscr{D}_c} \asymp \|\varphi\|_c \asymp \inf\{\|\{\gamma_{mk}\}_{m,k\geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1} : \sum_{m+k=n} \gamma_{mk} = \hat{\varphi}(n)\}, \ \varphi \in \mathscr{P}_A.$$

If we consider the compact $\mathbf{D}_c \times \mathbf{D}_{c\cdot 2^\alpha} \times \ldots \times \mathbf{D}_{c\cdot(1+n)^\alpha} \times \ldots$ instead of $\mathbf{D}_c^\infty$ we can construct analogously the Q-algebra $\mathscr{D}_c^{(\alpha)}$ for which $\|z^n\|_{\mathscr{D}_c^{(\alpha)}} \leqslant c(n+1)^\alpha$. Just as in Lemma 4.1 it can be proved that this example is extremal on the class of all elements x in Q-algebras satisfying $\|x^n\| \leqslant c(n+1)^\alpha$. It is not difficult to obtain analogues of Propositions 4.2 and 4.3. The analogue of Theorem 4.4 can be stated as follows. If $\operatorname{supp} F_j \subset [2^j, 2^{j+1})$ and $\int_{\mathbf{T}} \sup_N \frac{|f_N(\zeta)|}{2^{N\alpha}} \, dm(\zeta) < +\infty$ then $\overset{M}{\Gamma}{}_{\psi}^{-\alpha} \in V^M$ and $\|\overset{M}{\Gamma}{}_{\psi}^{-\alpha}\|_{V^M} \leqslant \text{const} \cdot 2^M$, where

$$\overset{M}{\Gamma}{}_{\psi}^{-\alpha}(k_1, \ldots, k_M) = \{\hat{\psi}(k_1 + \ldots + k_M)(1 + k_1)^{-\alpha} \ldots (1 + k_M)^{-\alpha}\}_{k_i \geqslant 0}.$$

It follows that for $c > 4e$, $\|\varphi\|_{\mathscr{D}_c^{(\alpha)}} \geqslant \text{const} \sum_{n \geqslant 0} 2^{n\alpha} |\hat{\varphi}(2^n)|$, $\varphi \in \mathscr{P}_A$. Since $\mathscr{F}\ell^1_{(\alpha)}$ is a Q-algebra for $\alpha > 1/2$, it follows from the extremal property of $z \in \mathscr{D}_c^{(\alpha)}$ that for $\alpha > 1/2$

$$\|\varphi\|_{\mathscr{D}_c^{(\alpha)}} \asymp \sum_{n \geqslant 0} |\hat{\varphi}(n)|(1 + n)^\alpha$$

for some $c > 1$.

5. HANKEL MATRICES IN $\ell^1 \check{\otimes} \ell^1$

In § 3 we showed that for a power bounded operator T on a Hilbert space the following inequality holds

$$\|\varphi(T)\| \leqslant \text{const} \cdot \inf\{\|\{\gamma_{mk}\}_{m,k \geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1} : \sum_{m+k=n} \gamma_{mk} = \hat{\varphi}(n)\}, \; \varphi \in \mathscr{P}_A.$$

The question of minimizing the norm $\|\{\gamma_{mk}\}_{m,k \geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1}$ arises in a natural way. In particular, which norm on the set of polynomials will be induced if we choose $\{\gamma_{mk}\}_{m,k \geqslant 0}$ putting $\gamma_{mk} = \dfrac{\hat{\varphi}(m + k)}{m + k}$? The latter question is obviously equivalent to that of description of the Hankel matrices in $\ell^1 \check{\otimes} \ell^1$. In this section we obtain such a description from which it follows that this choice of the matrix $\{\gamma_{mk}\}_{m,k \geqslant 0}$ is not optimal. We deduce from this fact that the natural projection (averaging projection) on the set of Hankel matrices is unbounded on $\ell^1 \check{\otimes} \ell^1$, $\ell^\infty \hat{\otimes} \ell^\infty$ and V^2. We also describe the space of analytic functions generated by the products FG where F and G are analytic functions in $\mathbf{D}$ such that $\lim_{n \to +\infty} \hat{F}(n) = \lim_{n \to +\infty} \hat{G}(n) = 0$.

Let $\mathscr{A}$ and $\mathscr{B}$ be operators from the set of matrices to the space of analytic functions in $\mathbf{D}$ defined by

$$(\widehat{\mathscr{A}x})(n) = \frac{1}{n + 1} \sum_{m+k=n} x_{mk}, \qquad x = \{x_{mk}\}_{m,k \geqslant 0}, \; n \geqslant 0;$$

$$(\widehat{\mathscr{B}x})(n) = \sum_{m+k=n} x_{mk}, \qquad n \geqslant 0.$$

THEOREM 5.1. $\mathscr{A}V^2 = B^0_{1\infty}$, $\mathscr{B}V^2 = B^{-1}_{1\infty}$.

Recall that

$$B^0_{1\infty} = \{f : \sup_{N\geqslant 0}\|f * W_N\|_{L^1} < +\infty\};\ B^{-1}_{1\infty} = \{f: \|f*W_N\|_{L^1} \leqslant \mathrm{const}\cdot 2^N\}.$$

Proof. Clearly it is sufficient to show that $\mathscr{B}V^2 = B^{-1}_{1\infty}$. To prove that $\mathscr{B}V^2 = B^{-1}_{1\infty}$ it suffices to establish that for $f, g \in \ell^\infty$

$$\|\mathscr{B}(f \otimes g)\|_{B^{-1}_{1\infty}} \leqslant \mathrm{const}\, \|f\|_{\ell^\infty}\|g\|_{\ell^\infty}.$$

Put

$$F = \sum_{k\geqslant 0} f_k z^k,\ G = \sum_{k\geqslant 0} g_k z^k.$$

Then $\mathscr{B}(f \otimes g) = F\cdot G$. We have

$$W_N * (F\cdot G) = W_N * \sum_{0\leqslant m,k\leqslant 2^{N+1}} f_m g_k z^{m+k}.$$

Let

$$F_N = \sum_{m=0}^{2^{N+1}} f_m z^m,\ G_N = \sum_{k=0}^{2^{N+1}} g_k z^k.$$

Then

$$\|W_N * (F\cdot G)\|_{L^1} \leqslant \frac{3}{2}\left\| \sum_{0\leqslant m,k\leqslant 2^{N+1}} f_m g_k z^{m+k}\right\|_{L^1} =$$

$$= \frac{3}{2}\|F_N G_N\|_{L^1} \leqslant \frac{3}{2}\|F_N\|_{L^2}\cdot\|G_N\|_{L^2} \leqslant 3\cdot 2^N\|f\|_{\ell^\infty}\cdot\|g\|_{\ell^\infty}.$$

Thus

$$\|FG\|_{B^{-1}_{1\infty}} \leqslant \mathrm{const}\, \|f\|_{\ell^\infty}\|g\|_{\ell^\infty}.$$

Let us show that $\mathscr{B}V^2 \supset B^{-1}_{1\infty}$. For this it is sufficient to prove that for any sequence of functions $\{f_N\}_{N\geqslant 0}$ satisfying $\sup_N\|f * W_N\|_{L^1} < +\infty$ there exists $x \in V^2$ such that

$$\mathscr{B}x = \sum_{N\geqslant 0} 2^N f_N * W_N.$$

Define kernels L_N and R_N by

$$\hat{L}_N(k) = \begin{cases} 0, & k \leqslant 2^{N-1} \\ 1 - \dfrac{|k-2^N|}{2^{N-1}}, & 2^{N-1} \leqslant k \leqslant 2^N + 2^{N-1} \\ 0, & k \geqslant 2^N + 2^{N-1} \end{cases}$$

$$\hat{R}_N(k) = \begin{cases} 0, & k \leqslant 2^N \\ 1 - \dfrac{|k-2^N-2^{N-1}|}{2^{N-1}}, & 2^N \leqslant k \leqslant 2^{N+1} \\ 0, & k \geqslant 2^{N+1}. \end{cases}$$

Then $W_N = L_N + \frac{1}{2} R_N$.

We are going to find $x \in V^2$ such that $\mathscr{B}x = \sum_{N\geqslant 1} 2^{2N} f_{2N+1} * L_{2N+1}$. By the same way it can be proved that $\sum_{N\geqslant 1} 2^{2N} f_{2N+1} * R_{2N+1} \in \mathscr{B}V^2$, $\sum_{N\geqslant 1} 2^{2N} f_{2N} * L_{2N} \in \mathscr{B}V^2$ and $\sum_{N\geqslant 1} 2^{2N} f_{2N} * R_{2N} \in \mathscr{B}V^2$.

Let

$$S_N = [2^{2N-1},\ 2^{2N-1} + 2^{2N} - 1] \times [2^{2N-1} - 1,\ 2^{2N-1} + 2^{2N}]$$

be the sets on the plane. Note that their projections to the coordinate axes are pairwise disjoint. Define $x = \{x_{mk}\}_{m,k\geqslant 0}$ by

$$x_{mk} = \begin{cases} \hat{f}_{2N+1}(m+k), & (m,k) \in S_N \\ 0, & (m,k) \notin \bigcup_{N\geqslant 1} S_N. \end{cases}$$

Then $x = \sum_{N\geqslant 1} \Gamma_{f_{2N+1}} \cdot \chi_{S_N}$. By N. Th. Varopoulos's theorem [14] (see also § 4) $\|\Gamma_{f_{2N+1}}\|_{V^2} \leqslant$ const and by Lemma 2.1

$$\|x\|_{V^2} = \sup_{N\geqslant 1} \|\Gamma_{f_{2N+1}} \chi_{S_N}\|_{V^2} \leqslant \sup_{N\geqslant 1} \|\Gamma_{f_{2N+1}}\|_{V^2}.$$

It remains to note that $\mathscr{B}x = \sum_{N\geqslant 1} 2^{2N} f_{2N+1} * L_{2N+1}$. ▨

The following assertion which describes the Hankel matrices of the class $\ell^1 \check{\otimes} \ell^1$ (the bounded Hankel operators from c_0 to ℓ^1) is a dual version of Theorem 5.1.

THEOREM 5.2. *Let φ be an analytic function in* **D**. *Then $\Gamma_\varphi \in \ell^1 \check{\otimes} \ell^1$ if and only if $\varphi \in B^1_{\infty 1}$.* ▨

Recall that

$$B^1_{\infty 1} = \{\varphi : \sum_{n\geqslant 0} 2^N \|\varphi * W_N\|_{L^\infty} < +\infty\} = \left\{\varphi : \iint_{\mathbf{D}} \|\varphi_r''\|_{L^\infty} dr < +\infty\right\}.$$

It follows that if φ is a polynomial and we choose the matrix $\{\gamma_{mk}\}_{m,k\geqslant 0}$ such that $\gamma_{mk} = \dfrac{\hat{\varphi}(m+k)}{m+k}$ then

$$\|\{\gamma_{mk}\}_{m,k\geqslant 0}\|_{\ell^1 \check{\otimes} \ell^1} \asymp \|\varphi\|_{B^0_{\infty 1}}.$$

As we pointed out in § 3 this estimate is worse than that in Theorem 3.1. On the set of matrices the averaging projection $\mathscr{P}$ onto the set of Hankel matrices is defined by

$$(\mathscr{P}\alpha)_{st} = \frac{1}{s+t+1} \sum_{m+k=s+t} \alpha_{mk}, \quad \alpha = \{\alpha_{mk}\}_{m,k \geqslant 0}.$$

COROLLARY 5.3. *$\mathscr{P}$ is unbounded on $\ell^1 \check{\otimes} \ell^1$, $\ell^\infty \hat{\otimes} \ell^\infty$ and V^2.*

Proof. Clearly it is sufficient to consider the case of V^2. By Theorem 5.1 we have

$$\mathscr{P} V^2 = \{\Gamma_\varphi : \varphi \in B^0_{1\infty}\}.$$

If $\mathscr{P}$ would be bounded it would follow that

$$\varphi \in B^0_{1\infty} \Rightarrow \Gamma_\varphi \in V^2 \Rightarrow \varphi \in \mathscr{M}H^1$$

by Lemma 3.3. But as we pointed out in Theorem 3.10 $B^0_{1\infty} \setminus \mathscr{M}H^1 \neq \emptyset$. ▨

Define the space $\mathscr{F}c_0$ of analytic functions in $\mathbf{D}$ by

$$\mathscr{F}c_0 = \{\varphi : \lim_{n \to +\infty} \hat{\varphi}(n) = 0\}, \quad \|\varphi\|_{\mathscr{F}c_0} \stackrel{\text{def}}{=} \sup_{n \geqslant 0} |\hat{\varphi}(n)|.$$

COROLLARY 5.4. *Let B be the space of functions F analytic in $\mathbf{D}$ and such that*

$$F = \sum_{j \geqslant 0} \varphi_j \psi_j, \ \sum_{j \geqslant 0} \|\psi_j\|_{\mathscr{F}c_0} \|\psi_j\|_{\mathscr{F}c_0} < +\infty.$$

Then

$$B = \{F : \lim_{N \to \infty} 2^{-N} \|F * W_N\|_{L^1} = 0\} = \{F : \lim_{r \to 1} \|F_r\|_{L^1} (1-r) = 0\}.$$

This assertion easily follows from Theorem 5.1. ▨

6. NON-POLYNOMIALLY BOUNDED POWER BOUNDED OPERATORS ORIGINATED FROM THE HANKEL OPERATORS AND RELATED OPERATOR ALGEBRAS

In this section we find one more example of non-polynomially bounded power bounded operators. This operator is constructed using the Hankel operators and we explicitly calculate norms of its functions. This permits us to obtain estimates from below of $|||\cdot|||_c$. Using estimates from above given in § 3 we obtain some embedding theorems. Moreover this example permits us to construct new operator algebras. In conclusion we give an analogous example for the operators with the growth of powers of order α.

We show that for some $g \in L^\infty$ the operator T on L^2 defined by

$$TF = zF - \hat{F}(-1) + \mathbf{P}_- g \mathbf{P}_+ F, \quad F \in L^2,$$

is power bounded but not polynomially bounded.

Instead of this operator we shall consider the following unitary equivalent operator R_f on $\ell^2 \oplus \ell^2$ defined by the operator matrix

$$R_f = \begin{pmatrix} S^* & \Gamma_f \\ \mathbf{O} & S \end{pmatrix},$$

where S is the shift operator on ℓ^2.

It is easy to see that

$$R_f^n = \begin{pmatrix} (S^*)^n & \sum_{k=0}^{n-1} (S^*)^k \Gamma_f S^{n-k-1} \\ \mathbf{O} & S^n \end{pmatrix} = \begin{pmatrix} (S^*)^n & n\Gamma_{(S^*)^{n-1}f} \\ \mathbf{O} & S^n \end{pmatrix}$$

because $(S^*)^k \Gamma_f S^{n-k-1} = \Gamma_{(S^*)^{n-1}f}$ for $0 \leqslant k \leqslant n-1$. Therefore

$$\varphi(R_f) = \begin{pmatrix} \varphi(S^*) & \Gamma_{\varphi'(S^*)f} \\ \mathbf{O} & \varphi(S) \end{pmatrix}, \quad \varphi \in \mathscr{P}_A.$$

It follows that

$$\|\varphi(R_f)\| \asymp \|\varphi\|_{H^\infty} + \|\varphi'(S^*)f\|_{BMO_A}.$$

By Z. Nehari's theorem [13], $\|R_f^n\| \leqslant \text{const}$ if and only if $\|(S^*)^n f\|_{BMO_A} \leqslant$ $\leqslant \dfrac{\text{const}}{1+n}$, $n \geqslant 0$.

LEMMA 6.1. *Let $f \in BMO_A$, $\alpha > 0$. Then $f \in \Lambda_\alpha$ if and only if* $\|(S^*)^n f\|_{BMO_A} \leqslant$ $\leqslant \dfrac{\text{const}}{(1+n)^\alpha}$, $n \geqslant 0$.

Proof. Suppose first $f \in \Lambda_\alpha$. Then obviously

$$\|(S^*)^{2^N-1} f\|_{BMO_A} \leqslant \sum_{k \geqslant N} \|f * W_k\|_{L^\infty} \leqslant \text{const} \cdot 2^{-N\alpha}$$

(see § 2). Therefore

$$\|(S^*)^n f\|_{BMO_A} \leqslant \frac{\text{const}}{(1+n)^\alpha}.$$

Let now

$$\|(S^*)^{2^{N-1}}f\|_{BMO_A} \leqslant \text{const}\cdot 2^{-N\alpha}.$$

We have

$$\|f * W_N\|_{L^\infty} \leqslant \|(S^*)^{2^{N-1}}f\|_{BMO_A} \leqslant \text{const}\cdot 2^{-\alpha N}.$$

Therefore $f \in \Lambda_\alpha$. ▨

This lemma implies that $\|R_f^n\| \leqslant$const if and only if $f \in \Lambda_1$.

Consider an operator R defined by

$$R = \bigoplus_f R_f,$$

the orthogonal sum being taken over a countable dense subset of the unit ball of the space λ_1. Clearly $\|R^n\| \leqslant$const.

Recall that a positive measure μ in $\mathbf{D}$ is called a Carleson measure if

$$g \in H^1 \Rightarrow \int_{\mathbf{D}} |g|\,\mathrm{d}\mu < +\infty.$$

We denote by $(\mathscr{C})$ the the set of all Carleson measures.

By L. Carleson's Theorem ([3], see also [15] where another proof due to S. A. Vinogradov is given) $\mu \in (\mathscr{C})$ if and only

$$\mu(C_\xi^{(r)}) \leqslant \text{const}\cdot r, \quad \xi \in \mathbf{T},\ r > 0,$$

where

$$C_\xi^{(r)} = \{\zeta \in \mathbf{D} : |\zeta - \xi| < r\}.$$

Define a class X of analytic functions in $\mathbf{D}$ by

$$\varphi \in X \Leftrightarrow \varphi \in H^\infty \text{ and } |\varphi'(z)|\,\mathrm{d}x\mathrm{d}y \in (\mathscr{C}),$$

$$\|\varphi\|_X \stackrel{\text{def}}{=} \|\varphi\|_{H^\infty} + \sup_{\|g\|_{H^1} \leqslant 1} \iint_{\mathbf{D}} |\varphi' g|\,\mathrm{d}x\mathrm{d}y.$$

THEOREM 6.2. *X is a Banach algebra with respect to the pointwise multiplication. The mapping*

$$\varphi \to \varphi(R)$$

can be extended from $\mathscr{P}_A$ *to a representation of* X *such that* $\|\varphi(R)\| \asymp \|\varphi\|_X$.

Proof. If $\varphi \in X$, $f \in \lambda_1$ we define $\varphi(R_f)$ by

$$\varphi(R_f) = \begin{pmatrix} \varphi(S^*) & \Gamma_{\varphi'(S^*)f} \\ \mathbf{O} & \varphi(S) \end{pmatrix},$$

where the function $\varphi'(S^*)f \in BMO_A$ is defined by

$$(\varphi'(S^*)f, g) = (f, \varphi' g), \quad g \in H^1.$$

(Note that since $\varphi \in X$, it follows that

$$\varphi' g \in \lambda_1^* = B_1^{-1} = \left\{ h : \iint_{\mathbf{D}} |h| \, dx \, dy < +\infty \right\}.)$$

Let us check that for $\varphi, \psi \in X$, $(\varphi\psi)(R_f) = \varphi(R_f)\psi(R_f)$. Indeed

$$\varphi(R_f)\psi(R_f) = \begin{pmatrix} (\varphi\psi)(S^*) & \varphi(S^*)\Gamma_{\psi'(S^*)f} + \Gamma_{\varphi'(S^*)f}\psi(S) \\ \mathbf{O} & (\varphi\psi)(S) \end{pmatrix}.$$

It is easy to see that

$$\varphi(S^*)\Gamma_{\psi'(S^*)f} = \Gamma_{\varphi(S^*)\psi'(S^*)f},$$

$$\Gamma_{\varphi'(S^*)f}\psi(S) = \Gamma_{\varphi'(S^*)\psi(S^*)f}.$$

Thus

$$\varphi(S^*)\Gamma_{\psi'(S^*)f} + \Gamma_{\varphi'(S^*)f}\psi(S) = \Gamma_{(\varphi\psi)'(S^*)f},$$

Put now $\varphi(R) = \bigoplus_f \varphi(R_f)$. It remains to prove that $\|\varphi(R)\| \asymp \|\varphi\|_X$. We have

$$\|\varphi(R)\| \asymp \|\varphi\|_{H^\infty} + \sup_{\|f\|_{\lambda_1} \leqslant 1} \|\varphi'(S^*)f\|_{BMO_A} =$$

$$= \|\varphi\|_{H^\infty} + \sup_{\|f\|_{\lambda_1} \leqslant 1, \|g\|_{H^1} \leqslant 1} |(\varphi'(S^*)f, g)| = \|\varphi\|_{H^\infty} + \sup_{\|f\|_{\lambda_1} \leqslant 1, \|g\|_{H^1} \leqslant 1} |(f, \varphi' g)|.$$

Since $(\lambda_1)^* = B_1^{-1}$ it follows that

$$\|\varphi(R)\| \asymp \|\varphi\|_{H^\infty} + \sup_{\|g\|_{H^1} \leqslant 1} \iint_{\mathbf{D}} |\varphi' g| \, dx \, dy = \|\varphi\|_X. \qquad ▨$$

COROLLARY 6.3. $\|\varphi(R)\| \geqslant \text{const } \|\varphi\|_{B_1^0} \geqslant \text{const} \sum_{n \geqslant 0} |\hat{\varphi}(2^n)|.$

Proof. Indeed,

$$\|\varphi(R)\| \geqslant \sup_{\|g\|_{H^1} \leqslant 1} \iint_{\mathbf{D}} |\varphi' g| \, \mathrm{d}x \, \mathrm{d}y \geqslant \iint_{\mathbf{D}} |\varphi'| \, \mathrm{d}x \, \mathrm{d}y.$$

Therefore

$$\|\varphi(R)\| \geqslant \text{const } \|\varphi\|_{B_1^0} \asymp \sum_{N \geqslant 0} \|\varphi * W_N\|_{L^1} \geqslant \sum_{n \geqslant 0} |\hat{\varphi}(2^n)|. \quad ▨$$

Thus R is not a polynomially bounded operator.

From Theorem 6.2 and the estimates from above obtained in § 3 the following embedding theorems easily follow.

COROLLARY 6.4. *If* $\{\gamma_{mk}\}_{m,k \geqslant 0} \in \ell^1 \check{\otimes} \ell^1$ *and an analytic function in* $\mathbf{D}$ φ *is defined by* $\hat{\varphi}(n) = \sum_{m+k=n} \gamma_{mk}$ *then* $\varphi \in X$. ▨

The following assertion seems to be of independent interest.

COROLLARY 6.5. *If* $f \in H^1$, $g \in BMO_A$, $\varphi = f * g$ *then* $|\varphi'(z)| \mathrm{d}x\mathrm{d}y$ *is a Carleson measure.*

Proof. It follows immediately from Theorem 6.2 and Theorem 3.5 that $|\varphi'(z)| \, \mathrm{d}x \, \mathrm{d}y \in (\mathscr{C})$ if $f \in H^1$ and $g \in VMO_A$. Let now $g \in BMO_A$. Put $\varphi_r = f * g_r$, where $g_r(\zeta) \stackrel{\text{def}}{=} g(r\zeta)$, $r < 1$. We have $\varphi_r \in X$ and $\|\varphi_r\|_X \leqslant$ const. It follows that $\varphi \in X$. ▨

COROLLARY 6.6. *X is an operator algebra.*

This follows from Theorem 6.2. ▨

Note that Theorem 6.2 gives an explicit embedding of X into the algebra of bounded operators on a Hilbert space.

S. A. Vinogradov called my attention to the fact that the algebra X admits the following description.

PROPOSITION 6.7. *Let* φ *be an analytic function in* $\mathbf{D}$. *Then* $\varphi \in X$ *if and only if*

$$f \in B_1^0 \Rightarrow \varphi f \in B_1^0.$$

In other words X coincides with the set of multipliers (with respect to the pointwise multiplication) of the class B_1^0.

Proof. Let $\varphi \in X$, $f \in B_1^0$. We have

$$\iint_{\mathbf{D}} |(\varphi f)'| \, \mathrm{d}x \, \mathrm{d}y \leqslant \iint_{\mathbf{D}} |f'\varphi| \, \mathrm{d}x \, \mathrm{d}y + \iint_{\mathbf{D}} |f\varphi'| \, \mathrm{d}x \, \mathrm{d}y \leqslant$$

$$\leqslant \|\varphi\|_{H^\infty} \iint_{\mathbf{D}} |f'| \, \mathrm{d}x \, \mathrm{d}y + \text{const} \|f\|_{H^1} \|\varphi\|_X \leqslant \text{const} \|f\|_{B_1^0} \|\varphi\|_X$$

because obviously $B_1^0 \subset H^1$.

Conversely, suppose that φ is a multiplier of X. Then $\varphi \in H^\infty$. Let us show that $|\varphi'(z)|\, dx\, dy \in (\mathscr{C})$. To do this we use the following characterization of the Carleson measures due to S. A. Vinogradov (see [15], Lecture VII):

$$\mu \in (\mathscr{C}) \Leftrightarrow \sup_{\zeta \in \mathbf{D}} (1 - |\zeta|) \left\| \frac{1}{(1 - \zeta z)^2} \right\|_{L^1(\mu)} < +\infty. \tag{2}$$

It is not difficult to check that $\left\| \frac{1}{(1 - \zeta z)^2} \right\|_{B_1^0} \asymp \frac{1}{1 - |\zeta|}$, $\zeta \in \mathbf{D}$. Put $f_\zeta = \frac{1}{(1 - \zeta z)^2}$. In view of (2) it is sufficient to check that

$$\iint_{\mathbf{D}} |f_\zeta(z)|\, |\varphi'(z)|\, dx\, dy \leqslant \text{const} \|f_\zeta\|_{B_1^0}, \quad \zeta \in \mathbf{D}.$$

We have

$$\iint_{\mathbf{D}} |f_\zeta \varphi'|\, dx\, dy \leqslant \iint_{\mathbf{D}} |(f_\zeta \varphi)'|\, dx\, dy + \iint_{\mathbf{D}} |f_\zeta' \varphi|\, dx\, dy.$$

Since φ is a multiplier of B_1^0, it follows from the closed graph theorem that

$$\|f_\zeta \varphi\|_{B_1^0} \leqslant \text{const} \|f_\zeta\|_{B_1^0}$$

and so

$$\iint_{\mathbf{D}} |(f_\zeta \varphi)'|\, dx\, dy \leqslant \text{const} \|f_\zeta\|_{B_1^0}, \quad \zeta \in \mathbf{D}.$$

Since $\varphi \in H^\infty$, we have

$$\iint_{\mathbf{D}} |f_\zeta' \varphi|\, dx\, dy \leqslant \text{const} \|f_\zeta\|_{B_1^0} \cdot$$ ▨

The following characterization of X is also due to S. A. Vinogradov.

PROPOSITION 6.8. *Let φ be an analytic function in $\mathbf{D}$. Then $\varphi \in X$ if and only if $\varphi \in B_1^0$ and*

$$\sup_\omega \|\varphi \circ \omega\|_{B_1^0} < +\infty,$$

the supremum being taken over all conformal mappings of $\mathbf{D}$ onto itself.

This assertion can be proved using (2). ▨

Consider now the case of the growth of powers of order α. It follows from Lemma 6.1 that $\|R_f^n\| \leqslant \text{const}\,(1+n)^\alpha$ if and only if $f \in \Lambda_{1-\alpha}$. Put

$$R^{(\alpha)} = \bigoplus_f R_f,$$

the orthogonal sum being taken over a countable dense subset of the unit ball of $\lambda_{1-\alpha}$.

Clearly $\|(R^{(\alpha)})^n\| \leqslant \text{const}\,(1+n)^\alpha$. Let X_α be the space of functions φ analytic in $\mathbf{D}$ such that $\varphi \in H^\infty$ and $|\varphi'(z)|\,(1-|z|)^{-\alpha}\,\mathrm{d}x\,\mathrm{d}y \in (\mathscr{C})$. Just as in Theorem 6.2 it can be proved that

$$\|\varphi(R^{(\alpha)})\| \asymp \|\varphi\|_{X_\alpha} \geqslant \text{const}\,\|\varphi\|_{B_1^\alpha} \asymp \sum_{n\geqslant 0} \|\varphi * W_n\|_{L^1} 2^{n\alpha} \geqslant \text{const} \sum_{n\geqslant 0} 2^{n\alpha}|\hat{\varphi}(2^n)|.$$

Corollary 6.9. *X_α is an operator algebra.* ▨

Note that it is possible to calculate explicitly the functions of R_f and this permits us to obtain examples of operator algebras. For example, let Z be a Banach space of analytic functions in $\mathbf{D}$ such that $Z \subset BMO_A$ and $\mathscr{P}_A$ is dense in Z. Define an algebra X_Z as the set of bounded analytic functions φ such that φ' is a multiplier (with respect to the pointwise multiplication) from H^1 to Z^*.

Proposition 6.10. *X_Z is an operator algebra.*

Proof. Let $\mathscr{D}$ be a countable dense subset of the unit ball of Z. Then the mapping

$$\varphi \to \bigoplus_{f\in\mathscr{D}} \varphi(R_f)$$

is an embedding of X_Z into the algebra of bounded operators. This can be proved in the same way as in Theorem 6.2. ▨

In particular if $Z = VMO_A$ then $X_Z = \{\varphi : \varphi' \in H^\infty\}$, if $Z = \lambda_1$ then $X_Z = X$ and if $Z = \lambda_\alpha$, $0 < \alpha < 1$, then $X_Z = X_\alpha$.

7. CONCLUDING REMARKS

In this section we state open problems related to the subject of the article (some of them were mentioned in the previous sections).

1) Recall that

$$\mathscr{L} = \{\varphi : \exists\, \{\gamma_{mk}\}_{m,k\geqslant 0} \in \ell^1 \check{\otimes}\, \ell^1,\ \hat{\varphi}(n) = \sum_{m+k=n} \gamma_{mk},\quad n \in \mathbf{Z}_+\}.$$

Is it true that $\mathscr{L}$ is a Banach algebra with respect to the pointwise multiplication?

2) Is $H^1 \hat{\circledast} VMO_A$ an operator algebra?

If the answer to the first question is negative, the estimate in Theorem 3.1 could be strengthened because

$$\|(\varphi\psi)(T)\| \leqslant \|\varphi(T)\|\cdot\|\psi(T)\|.$$

Thus the negative answer to the first question would imply negative answer to the second one.

If the second question has the positive answer, it would follow that

$$|||\varphi|||_c \asymp \|\varphi\|_{H^1 \hat{\circledast} VMO_A}$$

for some $c > 1$.

3) Is it true that

$$\psi \in \mathcal{M}H^1 \Rightarrow \Gamma_\psi \in V^2?$$

4) Is it true that

$$\psi \in \mathcal{M}H^1 \Rightarrow \overset{M}{\Gamma_\psi} \in V^M \text{ and } \|\overset{M}{\Gamma_\psi}\|_{V^M} \leqslant L^M?$$

5) Is it true that

$$\Gamma_\psi \in V^2 \Rightarrow \overset{M}{\Gamma_\psi} \in V^M \text{ and } \|\overset{M}{\Gamma_\psi}\|_{V^M} \leqslant L^M?$$

If the answer to the question 4) is positive then

$$|||\varphi|||_c \asymp \|\varphi\|_{H^1 \hat{\circledast} VMO_A}$$

for some $c > 1$. If the answer to 5) is positive then for some $c > 1$

$$|||\varphi|||_c \asymp \|\varphi\|_{\mathcal{D}_c}.$$

6) Let $X = \{\varphi : \varphi \in H^\infty \text{ and } |\varphi'(Z)| \, dx\, dy \in (\mathscr{C})\}$. Is it true that for some $c > 1$

$$\|\varphi\|_X \asymp |||\varphi|||_c\,?$$

7) Are the norms $|||\cdot|||_c$ equivalent for all $c > 1$?

Note that from the results of § 6 it follows that for all $c > 1$

$$|||\varphi|||_c \geqslant \text{const}\, \|\varphi\|_X\,, \quad \varphi \in \mathscr{P}_A.$$

8) Let T be a power bounded operator on a Hilbert space and $c > 1$. Does there exist an operator T_1 similar to T satisfying $\|T_1^n\| \leqslant c$, $n \in \mathbf{Z}_+$?

It is clear that the positive answer to 8) would imply the positive answer to 7).

9) Is it true that for all $c > 1$ the element z in $\mathscr{D}_c$ is not polynomially bounded?

10) Are the norms $\|\cdot\|_{\mathscr{D}_c}$ equivalent for all $c > 1$?

Obviously the positive answer to the question 10) would imply the positive answer to the question 9).

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Received June 1, 1981.