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SUB-JORDAN OPERATORS: BISHOP'S THEOREM, SPECTRAL INCLUSION, AND SPECTRAL SETS

JIM AGLER

0. INTRODUCTION

In this paper we are interested in the following condition on an operator $J \in \mathcal{L}(\mathcal{H})$, the algebra of bounded linear transformations on a complex Hilbert space $\mathcal{H}$.

(0.1) There exist $N, Q \in \mathcal{L}(\mathcal{H})$ such $J = N + Q$, N is normal, $QN = NQ$, and $Q^n = 0$.

An operator $J \in \mathcal{L}(\mathcal{H})$ is a *Jordan operator* if it satisfies (0.1) for some positive integer n. We say $J \in \mathcal{L}(\mathcal{H})$ is an *n-c Jordan operator* if J satisfies (0.1) with $\|Q\| \leqslant c$. The collection of *n-c* Jordan operators on a space $\mathcal{H}$ will be denoted, $\mathcal{J}_c^n(\mathcal{H})$. $S \in \mathcal{L}(\mathcal{H})$ is *sub n-c Jordan* if there is a Hilbert space $\mathcal{K}$ and $J \in \mathcal{J}_c^n(\mathcal{K})$ such that $\mathcal{H}$ is invariant for J and $S = J \mid \mathcal{H}$. The collection of *sub n-c Jordan* operators on a space $\mathcal{H}$ will be denoted by sub $\mathcal{J}_c^n(\mathcal{H})$.

By a *Jordan block* we mean any operator which is unitarily equivalent to an operator of the form

$$(0.2) \qquad \begin{bmatrix} N & & & & & 0 \\ c & N & & & & \\ & c & \ddots & & & \\ & & \ddots & \ddots & & \\ & & & & N & \\ 0 & & & & c & N \end{bmatrix},$$

on $\bigoplus_{k=1}^{n} \mathcal{M}$ where $\mathcal{M}$ is a Hilbert space (of arbitrary dimension), $N \in \mathcal{L}(\mathcal{M})$ is normal, and $0 \leqslant c \in \mathbf{R}$. If J is a Jordan block of the form (2), then the *order* of J

is n and the *constant* of J is c. We set $\mathscr{B}_c^n(\mathscr{H})$ equal to the set of Jordan blocks in $\mathscr{L}(\mathscr{H})$ with order n and constant c. Finally, sub $\mathscr{B}_c^n(\mathscr{H})$ is the set of $S \in \mathscr{L}(\mathscr{H})$ such that there exists a Hilbert space $\mathscr{K}$ and a $J \in \mathscr{B}_c^n(\mathscr{K})$ such that $\mathscr{H}$ is invariant for J and $S = J \mid \mathscr{H}$.

We can now state the principal result of this paper. Recall that the *strong topology* of $\mathscr{L}(\mathscr{H})$ (which we will denote in the sequel with the letter s) is that topology defined by the family of seminorms, $\{\rho_x \mid x \in \mathscr{H}\}$, where $\rho_x(T) = \|Tx\|$ for $T \in \mathscr{L}(\mathscr{H})$.

THEOREM A.

$$(\mathscr{J}_c^n(\mathscr{H}))^{-s} = \operatorname{sub}\mathscr{J}_c^n(\mathscr{H}) = \operatorname{sub}\mathscr{B}_c^n(\mathscr{H}) = (\mathscr{B}_c^n(\mathscr{H}))^{-s}.$$

Setting $n = 1$ or $c = 0$ reduces Theorem A to Bishop's Theorem for subnormal operators: the strong closure of the normal operators on a Hilbert space is the subnormals on that space (see [4]). The $\operatorname{sub}\mathscr{J}_c^n(\mathscr{H}) = \operatorname{sub}\mathscr{B}_c^n(\mathscr{H})$ part of Theorem A represents a generalization of the equivalence of conditions (2) and (3) in Theorem 5.26 of [1].

As a byproduct of the proof of Theorem A we obtain the appropriate extension to the class of sub-Jordan operators of the spectral inclusion relations between a subnormal and its minimal normal extension.

THEOREM B. *If $S \in \operatorname{sub}\mathscr{J}_c^n(\mathscr{H})$ then S has an extension to $J \in \mathscr{J}_c^n(\mathscr{K})$ where $\sigma(J) \subseteq S$. In this event $\sigma(S)\setminus\sigma(J)$ is a union of components of $\mathbf{C}\setminus\sigma(N)$.*

Setting $n = 1$ or $c = 0$ in Theorem B yield a weakened version of theorems due to Halmos and Bram ([9] and [5]; or see [10], problems 157 and 158) in which J explicitly appears as the minimal normal extension of S. Theorem B should also be contrasted with Lemma 6.7 in [3] which states that if $S \in \bigcup_{c>0} \operatorname{sub}\mathscr{J}_c^n(\mathscr{H})$ and $J \in \bigcup_{c>0} \mathscr{J}_c^n(\mathscr{K})$ is a minimal Jordan extension for S then $\sigma(J) \subseteq \sigma(S)$. While the notion of a "minimal Jordan extension" can easily be defined, that the resulting structure of the spatial relationships between a sub-Jordan and its minimal extensions is quite complicated is not to be denied (see [3], in particular, the interesting Theorem 1.4).

Once Theorem B is established it becomes possible to develop an analog of von Neumann's theory of spectral sets. Thus, in a properly interpreted sense, the matricial spectrum of a sub-Jordan operator, S, is a matricial spectral set for S. This is a generalization of the fact that subnormal operators are von Neumann operators. Natural proofs of the following two theorems result.

THEOREM C. *If $S \in \operatorname{sub}\mathscr{J}_c^n(\mathscr{H})$ and S is compact then $S \in \mathscr{J}_c^n(\mathscr{H})$.*

THEOREM D. *If $T \in \mathscr{L}(\mathscr{H})$ satisfies the equation,*

$$T^{*3} - 3T^{*2}T + 3T^*T^2 - T^3 = 0, \tag{0.2}$$

and $\sigma(T)$ (which by consequence of the above equation is contained in the real numbers) has empty interior (in the real numbers) then $T \in \mathcal{J}_c^2(\mathcal{H})$ for c appropriately chosen.

Theorem C, of course, corresponds to the well known fact that a compact subnormal (indeed, hyponormal) is normal. Theorem D constitues a generalization of Theorem II (page 221) of [3] which stated that Theorem D holds with $\dim\mathcal{H}$ is finite.

The study of sub-Jordan operators was initiated by J. W. Helton in [11] (Helton actually studied (0.2) but subsequently it has been learned that T satisfies (0.2) if and only if for some c, T has an extension to a 2-c Jordan operator with real spectrum (see [1])) and has continued in [12], [13], [3], and [1]. The many points of contact between the theory of sub-Jordan operators and other areas of operator theory and of analysis in general isolated in these papers belie the notion that the theory of sub-Jordan operators is an idle generalization of the immensely successful theory of subnormal operators.

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1. THE cp CONDITION

In this section we adapt techniques from Chapter II of [1] (which involve the Stinespring Representation Theorem and the Arveson Extension Theorem) to replace the geometrical problem of constructing Jordan extensions with an equivalent problem more amenable to analysis. In addition to deriving a number of useful lemmas for application in succeeding sections we obtain an interesting characterization of subnormality and the result that the classes of sub-Jordan and sub-Jordan block operators are identical.

Let $\mathcal{P}$ denote the set of polynomials in two noncommuting variables x and y of the form

$$p(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{m} c_{kl} y^l x^k, \quad c_{kl} \in \mathbf{C}. \tag{1.1}$$

If a is any element in a C^*-algebra and $p \in \mathcal{P}$ is as in (1.1) define $p(a)$ by

$$p(a) = \sum_{k=0}^{m} \sum_{l=0}^{m} c_{kl} a^{*l} a^k.$$

If $p \in \mathcal{P}$ is as in (1.1), define $\mathrm{D}_x p$, $\mathrm{D}_y p \in \mathcal{P}$ by

$$\mathrm{D}_x p = \sum_{k=1}^{m} \sum_{l=0}^{m} k c_{kl} y^l x^{k-1},$$

and

$$\mathrm{D}_y p = \sum_{k=0}^{m} \sum_{l=1}^{m} l c_{kl} y^{l-1} x^k.$$

Finally, if $p \in \mathscr{P}$ is as in (1.1), define $p^* \in \mathscr{P}$ by

$$p^*(x, y) = \sum_{k=0}^{m} \sum_{l=0}^{m} \bar{c}_{kl} y^k x^l .$$

Evidently, if $p \in \mathscr{P}$ and a is any element in a C^*-algebra, then $p^*(a) = p(a)^*$.

LEMMA 1.2. *If $p \in \mathscr{P}$ and J, N, and Q satisfy* (0.1), *then*

$$p(J) = \sum_{i,j=0}^{n-1} \frac{1}{j!i!} Q^{*j}(\mathrm{D}_y^j \mathrm{D}_x^i p)(N) Q^i. \tag{1.3}$$

Proof. Let $p(x, y) = y^l x^k$. By the Binomial Theorem,

$$J^k = \sum_{i=0}^{k} \binom{k}{i} N^{m-i} Q^i$$

and

$$J^{*l} = \sum_{j=0}^{l} \binom{l}{j} Q^{*j} N^{*l-j}.$$

Thus,

$$p(J) = J^{*l} J^k =$$

$$= \sum_{j=0}^{l} \sum_{i=0}^{k} Q^{*j} \binom{l}{j} \binom{k}{i} N^{*l-j} N^{m-i} Q^i =$$

$$= \sum_{j=0}^{l} \sum_{i=0}^{k} Q^{*j} \frac{1}{j!} \frac{1}{i!} (\mathrm{D}_y^j \mathrm{D}_x^i p)(N) Q^i =$$

$$= \sum_{i,j=0}^{n-1} \frac{1}{j!i!} Q^{*j} (\mathrm{D}_y^j \mathrm{D}_x^i p)(N) Q^i.$$

Thus (1.3) holds for $p(x, y) = y^l x^k$. The proof of Lemma 1.2 is then completed by taking linear combinations.

For K a compact set in the plane let $\mathscr{C}_n(K) = C(K, \mathscr{M}_n)$, the C^*-algebra of $n \times n$ matrix valued continuous functions defined on K. Let $\{e_i : 0 \leqslant i \leqslant n-1\}$ be the standard basis for $\mathbf{C}^n$ and define $v_n \in \mathscr{M}_n$ by,

$$(v_n e_j, e_i) = \begin{cases} 1 & \text{if } i = j+1 \text{ and } 0 \leqslant j \leqslant n-2 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\chi_{n,c,K} \in \mathscr{C}_n(K)$ by,

$$\chi_{n,c,K}(z) = z + v_n, \quad z \in K.$$

Let $B_0(n, c, K) = \{p(\chi_{n,c,K}) : p \in \mathscr{P}\} \subseteq \mathscr{C}_n(K)$ and set $B(n, c, K) =$ the closure of $B_0(n, c, K)$ in $\mathscr{C}_n(K)$.

PROPOSITION 1.4. *$J \in \mathscr{B}_c^n(\mathscr{H})$ if and only if there exists a representation $\pi: \mathscr{C}_n(\sigma(J)) \to \mathscr{L}(\mathscr{H})$ with $\pi(1) = 1$ and $\pi(\chi_{n,c,\sigma(J)}) = J$.*

Proof. See Lemma 4.1 of [16].

DEFINITION 1.5. $S \in \mathscr{L}(\mathscr{H})$ is cp(n, c, K) if for every positive integer m, $[p_{ij}(\chi_{n,c,K})] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, K)$ implies $[p_{ij}(S)] \geqslant 0$ in $\mathscr{M}_m \otimes \mathscr{L}(\mathscr{H})$.

The significance of Definition 1.5 in the study of sub-Jordan operators rests in the following theorem and in the fact that the cp(n, c, K) condition is analyzable in many situations where geometric intuition is absent.

THEOREM 1.6. *If $K \subseteq \mathbf{C}$ is compact, $c \geqslant 0$, and $\mathscr{H}$ is a Hilbert space then the following are equivalent:*

(a) *There is a Hilbert space $\mathscr{K} \supseteq \mathscr{H}$ and a $J \in \mathscr{B}_c^n(\mathscr{K})$ such that $J\mathscr{H} \subseteq \mathscr{H}$, $\sigma(J) \subseteq K$ and $S = J \mid \mathscr{H}$.*

(b) *$S \in \mathscr{L}(\mathscr{H})$ is* cp(n, c, K).

Proof. (cf. Theorem 2.7 of [1]; Remark 4.3 of [6]). Suppose (a). Then there exists $\mathscr{K} \supseteq \mathscr{H}$ and a Jordan block $J \in \mathscr{L}(\mathscr{K})$ with constant c and order n such that $S = J \mid \mathscr{H}$. If we let $V: \mathscr{H} \to \mathscr{K}$ be the canonical inclusion then we have

$$p(S) = V^* p(J) V$$

for all $p \in \mathscr{P}$. By Proposition 1.4 there is a representation $\pi: \mathscr{C}_n(\sigma(J)) \to \mathscr{L}(\mathscr{K})$ with $\pi(1) = 1$ and $\pi(\chi_{n,c,\sigma(J)}) = J$. Thus if $p \in \mathscr{P}$,

$$p(S) = V^* p(J) V = V^* \pi(p(\chi_{n,c,\sigma(J)})) V. \tag{1.7}$$

Define $\rho: \mathscr{C}_n(K) \to \mathscr{C}_n(\sigma(J))$ by

$$\rho(x) = x \mid \sigma(J), \quad x \in \mathscr{C}_n(K).$$

Since $\sigma(J) \subseteq K$, ρ is a representation. If we set $\pi_1 = \pi\rho$ and $\chi = \chi_{n,c,K}$ (1.7) becomes,

$$p(S) = V^* \pi_1(p(\chi)) V,$$

for all $p(\chi) \in B_0(n, c, K)$.

Consequently, if m is a positive integer and $0 \leqslant [p_{ij}(\chi)] \in \mathscr{M}_m \otimes B_0(n, c, K)$ then

$$[p_{ij}(S)] = (1_m \otimes V)^* (\mathrm{id}_m \otimes \pi_1)([p_{ij}(\chi)])(1_m \otimes V) \geqslant 0$$

in $\mathcal{M}_m \otimes \mathcal{L}(\mathcal{H})$. Thus (b) holds. Conversely, assume that (b) holds. Then in particular (set $m = 1$ in Definition 1.5), we can unambiguously define a map $\Phi: B_0(n, c, K) \to \mathcal{L}(\mathcal{H})$ by

$$\Phi(p(\chi)) = p(S), \quad p \in \mathcal{P}.$$

In the formula above we have abreviated $\chi_{n,c,K}$ by simply, χ, as we often will do in the sequel when no danger of confusion exists. The condition of Definition 1.5 with $m = 1$ says that Φ is positive. Since $B_0(n, c, K)$ is self-adjoint it follows that Φ is continuous and that Φ has a continuous extension, Φ_1, to $B(n, c, K)$. That S is cp(n, c, K) translates into the fact that Φ_1 is completely positive. By Theorem 1.2.3 in [2] Φ_1 has a completely positive extension, Φ_2, to $\mathcal{C}_n(K)$. By Stinespring's theorem (Theorem 1.1.1 in [2]), there is a Hilbert space $\mathcal{K}$, a representation π, and an isometry, $V: \mathcal{H} \to \mathcal{K}$, such that,

$$\Phi_2(x) = V^*\pi(x) V$$

for all $x \in \mathcal{C}_n(K)$. Thus in particular if $p \in \mathcal{P}$,

$$\begin{aligned} p(S) &= \Phi(p(\chi)) = \Phi_2(p(\chi)) = \\ &= V^*\pi(p(\chi)) V = V^*p(\pi(\chi)) V. \end{aligned} \tag{1.8}$$

We now use a trick in [6] (Remark 4.3) to show that range V is invariant for $\pi(\chi)$. Thus was done in [1] (Lemma 2.9) by a more laborious means using a well known Lemma of Sarason. If we let $p(x, y) = yx$ notice that (1.8) becomes

$$S^*S = V^*\pi(\chi)^*\pi(\chi) V.$$

Also from (1.8) ($p(x, y) = x$) we see that

$$S = V^*\pi(\chi) V.$$

Thus,

$$\begin{aligned} &(VV^*\pi(\chi) V - \pi(\chi) V)^* (VV^*\pi(\chi) V - \pi(\chi) V) = \\ &= V^*\pi(\chi)^*\pi(\chi) V - V^*\pi(\chi)^* VV^*\pi(\chi) V = 0, \end{aligned}$$

and range V is invariant for $\pi(\chi)$. (1.8) now implies that $S \cong \pi(\chi) \mid V\mathcal{H}$. Since (Proposition 1.4) $\pi(\chi) \in \mathcal{B}_c^n(\mathcal{K})$ and $\sigma(\pi(\chi)) \subseteq K$ we see that (a) holds. This concludes the proof of Theorem 1.6.

The following corollary to Theorem 1.6 will not be used in the sequel.

COROLLARY 1.9. *The following are equivalent.*

(a) $S \in \mathcal{L}(\mathcal{H})$ *is subnormal.*

(b) *There exists a compact* $K \subseteq \mathbf{C}$ *such that* $p(S) \geqslant 0$ *whenever* $p \in \mathscr{P}$ *and* $p(z, \bar{z}) \geqslant 0$ *for all* $z \in K$.

(c) *There exists a compact* $K \subseteq \mathbf{C}$ *and a positive linear map* $\Phi: C(K) \to \mathscr{L}(\mathscr{H})$ *such that* $\Phi(z) = S$ *and* $\Phi(|x|^2) = S^*S$.

That (b) implies (a) in Corollary 1.9 follows by noticing that the map,

$$C(K) \ni p(z, \bar{z}) \to p(S),$$

which is positive by assumption, is, in fact, completely positive (see [14]). (a) implies (b) is obtained by compressing the spectral representation of the minimal normal extension of S to $\mathscr{H}$. The equivalence of (a) and (c) (which was pointed out to the author by J. B. Conway) follows similarly.

LEMMA 1.10. *Let* $z \in \mathbf{C}$ *and* $Q \in \mathscr{L}(\mathscr{H})$ *with* $Q^n = 0$. *Then* $z + Q$ *is* cp$(n, \|Q\|, \{z\})$.

Proof. By Theorem 1.6, the conclusion of the lemma is equivalent $z + Q$ having an extension to a $J \in \mathscr{B}^n_{\|Q\|}(\mathscr{K})$ where $\sigma(J) = \{z\}$. Clearly without loss of generality we may assume $z = 0$ and $\|Q\| = 1$. Thus by Theorem II, Section 2 of [8], $Q = S^* \mid \mathscr{M}$ where S is a unilateral shift of infinite multiplicity and $\mathscr{M}$ is invariant for S^*. Since $Q^n = 0$, $\mathscr{M} \subseteq \operatorname{Ker} S^{*n}$. But $S^* \mid \operatorname{Ker} S^{*n} \in \mathscr{B}^n_1(\operatorname{Ker} S^{*n})$ and $\sigma(S^* \mid \operatorname{Ker} S^{*n}) = \{0\}$. Since $Q = (S^* \mid \operatorname{Ker} S^{*n}) \mid \mathscr{M}$ this concludes the proof of (1.10).

We record the following simple exercises without proof.

LEMMA 1.11. *Let* $m \geqslant 1$ *and let* $K \subseteq \mathbf{C}$ *be compact. Then* $[p_{ij}(\chi_{n,c,K})] \geqslant 0$ *in* $\mathscr{M}_m \otimes B_0(n, c, K)$ *if and only if* $[p_{ij}(\chi_{n,c,\{z\}})] \geqslant 0$ *in* $\mathscr{M}_m \otimes B_0(n, c, \{z\})$ *for every* $z \in K$. *If* $\Phi_{ij} \in \mathscr{C}_n(K)$ *for* $i, j \leqslant m$ *then* $[\Phi_{ij}] \geqslant 0$ *in* $\mathscr{M}_m \otimes \mathscr{C}_n(K)$ *if and only if* $[\Phi_{ij}(z)] \geqslant 0$ *in* $\mathscr{M}_m \otimes \mathscr{M}_n$ *for every* $z \in K$.

We now are ready to show that Jordan operators have Jordan block extensions.

THEOREM 1.12. *Let* J, N, *and* Q *satisfy* (0.1). *Then* J *has an extension to* $H \in \mathscr{B}^n_{\|Q\|}(\mathscr{K})$ *where* $\sigma(H) \subseteq \sigma(J)$.

Proof. Set $K = \sigma(J)$ and $c = \|Q\|$. By Theorem 1.6 it is enough to show that J is cp(n, c, K). Let

$$N \cong \bigoplus_{k=1}^{\infty} (M_z^{(k)} \text{ on } L^2(\mu_k)^{(k)}) \tag{1.13}$$

where μ_k, $1 \leqslant k \leqslant \infty$, is a sequence of pairwise singular, positive, finite, compactly supported Borel measures in the plane. Since $NQ = QN$,

$$Q = \bigoplus_{k=1}^{\infty} Q_k$$

where for each k, $1 \leqslant k \leqslant \infty$, $Q_k = Q_k(z)$ is a μ_k a.e. defined $k \times k$ matrix valued function. Since $Q^n = 0$ and $\|Q\| = c$, for each k,

$$Q_k(z)^n = 0 \quad \text{and} \quad \|Q_k(z)\| \leqslant c$$

for μ_k a.e. z.

We now show that J is cp(n, c, K). Notice that $\left[\bigcup_{k=1}^{\infty} \operatorname{spt}\mu_k\right]^- = \sigma(N) = \sigma(N + Q) = \sigma(J) = K$. Let $[p_{ij}(\chi)] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, K)$. Then for every $z \in K$, $[p_{ij}(\chi)] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, \{z\})$. Hence by Lemma 1.10, $[p_{ij}(z + Q_k(z))] \geqslant 0$ in $\mathscr{M}_m \otimes \mathscr{L}(L^2(\mu)^{(k)})$ for every $k \geqslant 1$ and μ_k a.e. z. From the representation (1.13) we conclude that $[p_{ij}(N + Q)] \geqslant 0$ in $\mathscr{M}_m \otimes \mathscr{L}(\mathscr{H})$. Thus J is cp (n, c, K) and the proof of Theorem 1.12 is complete.

We close this section with a technical lemma (cf. Lemma 2.26 of [1]).

LEMMA 1.13. *Let K be a compact subset of* $\mathbf{C}$ *and suppose that* $S \in \mathscr{L}(\mathscr{H})$ *is* cp(n, c, G^-) *for every bounded open subset of* $\mathbf{C}$ *such that* $K \subseteq G$. *Then S is* cp(n, c, K).

Proof. Let $G_k = \left\{z \in \mathbf{C}\colon \operatorname{dist}(z, K) < \frac{1}{k}\right\}$. Fix $[p_{ij}] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, K)$. Let $q_{ij} \in \mathscr{P}$ be defined by $q_{ij} = \frac{1}{2}(p_{ij} + p_{ij}^*)$. Since S is cp(n, c, G_k^-),

$$\|[p_{ij}(S)] - [q_{ij}(S)]\| \leqslant \|[(p_{ij} - q_{ij})(\chi_{n,c,G_k^-})]\|.$$

By the continuity of the polynomials p_{ij} and q_{ij}, Lemma 1.11, and our assumption that $[p_{ij}] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, K)$ we obtain, by letting $k \to \infty$ in this last inequality, that

$$[p_{ij}(S)] = [q_{ij}(S)]. \tag{1.14}$$

Now fix $\varepsilon > 0$. For k sufficiently large $[q_{ij}] + \varepsilon \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, G_k^-)$. Hence, since S is cp(n, c, G_k^-), $[q_{ij}(S)] + \varepsilon \geqslant 0$. Since ε is arbitrary, $[q_{ij}(S)] \geqslant 0$. By (1.14), $[p_{ij}(S)] \geqslant 0$ which establishes Lemma 1.13.

2. THE STRONG LIMIT OF A NET OF JORDAN BLOCKS

This section will be devoted to proving the following theorem.

THEOREM 2.1. *If* $S \in (\mathscr{B}_c^n(\mathscr{H}))^{-s}$ *then S has an extension to a* $J \in \mathscr{B}_c^n(\mathscr{H})$ *with* $\sigma(J) \subseteq \sigma(S)$.

Accordingly, we shall consider fixed throughout this section a net, $\{J_\gamma | \gamma \in \Gamma\} \subseteq \mathscr{B}_c^n(\mathscr{H})$, and a $S \in \mathscr{L}(\mathscr{H})$ with $J_\gamma \to S$ in the strong topology on $\mathscr{L}(\mathscr{H})$. By Proposition 1.4, for each γ there is a representation, $\pi_\gamma\colon \mathscr{C}_n(\sigma(J_\gamma)) \to \mathscr{L}(\mathscr{H})$, with

$\pi_\gamma(\chi_{n,c,\sigma(J_\gamma)}) = J_\gamma$ and $\pi_\gamma(1) = 1$. We assume for each γ such a representation is picked and fixed once and for all in this section. Recall that $\chi_{n,c,\sigma(J_\gamma)}(z) = z + cv_n$. Set $N_\gamma = \pi_\gamma(z)$ and $Q_\gamma = \pi_\gamma(cv_n)$. Thus J_γ, N_γ, Q_γ satisfy (0.1).

Let $N_\gamma = \int z dE_\gamma$ be the spectral resolution of N_γ. For each $\gamma \in \Gamma$, $x, y \in \mathscr{H}$ and i, j in the range $0 \leqslant i, j \leqslant n-1$ define a Borel measure $\mu_{x,y}^{i,j,\gamma}$ by:

$$\mu_{x,y}^{i,j,\gamma}(\Delta) = \langle Q^{j*} E_\gamma(\Delta) Q^i x, y \rangle, \tag{2.2}$$

for all Borel subsets, Δ, of the plane. If we set $D = \max\{c^{n-1}, c^{-(n-1)}\}$ then clearly $\|\mu_{x,y}^{i,j,\gamma}\| \leqslant D\|x\|\,\|y\|$ for all i, j, γ, x, and y. Recall that $M(\mathbf{C})$, the space of finite Borel measures on $\mathbf{C}$, is the dual of $C_0(\mathbf{C})$, the space of continuous functions on $\mathbf{C}$ with compact support. Let

$$\Xi = \{(i, j, x, y) \mid 0 \leqslant i, j \leqslant n-1; \quad x, y \in \mathscr{H}\}.$$

If $\xi \in \Xi$ define

$$M_\xi = \{\mu \in M(\mathbf{C}) \colon \|\mu\| \leqslant D\|x\|\,\|y\|\}.$$

The w* topology on $M(\mathbf{C})$ relativized to M_ξ makes M_ξ into a compact space. Hence,

$$\Pi = \prod_{\xi \in \Xi} M_\xi$$

is compact. For $\gamma \in \Gamma$ let $\tau_\gamma \in \Pi$ be defined by,

$$\tau_\gamma((i, j, x, y)) = \mu_{x,y}^{i,j,\gamma}, \quad (i, j, x, y) \in \Xi.$$

Let $\tau \in \Pi$ be a cluster point of $\{\tau_\gamma \colon \gamma \in \Gamma\}$ and let $\mu_{x,y}^{i,j} = \tau((i, j, x, y))$.

From the construction in the preceeding paragraph the following lemma is immediate.

LEMMA 2.3. *For every finite set, $\{(i_k, j_k, x_k, y_k) \colon 1 \leqslant k \leqslant p\} \subseteq \Xi$, every finite set $\{\varphi_l \colon 1 \leqslant l \leqslant q\} \subseteq C_0(\mathbf{C})$, every $\varepsilon > 0$, and every $\gamma \in \Gamma$, there exists a $\beta \in \Gamma$ such that $\beta > \gamma$ and*

$$\left| \int \varphi_l \, d\mu_{x_k, y_k}^{i_k, j_k} - \int \varphi_l \, d\mu_{x_k, y_k}^{i_k, j_k, \beta} \right| < \varepsilon$$

for all $k \leqslant p$ and all $l \leqslant q$.

COROLLARY 2.4. *If $0 \leqslant i, j \leqslant n-1$ and $x, y \in \mathscr{H}$ then $\mu_{x,y}^{i,j} = \overline{\mu_{y,x}^{j,i}}$.*

Proof. Notice that if $\gamma \in \Gamma$ and $\Delta \subseteq \mathbf{C}$ is a Borel set then

$$\begin{aligned} \mu_{x,y}^{i,j,\gamma}(\Delta) &= \langle Q^{*j} E_\gamma(\Delta) Q^i x, y \rangle = \\ &= \overline{\langle Q^{*i} E_\gamma(\Delta)\, Q^j y, x \rangle} = \overline{\mu_{y,x}^{j,i,\gamma}(\Delta)}. \end{aligned} \tag{2.5}$$

Fix $\varphi \in C_0(\mathbf{C})$. Let $\varepsilon > 0$. By Lemma 2.3 there exists a $\beta \in \Gamma$ such that

$$\left|\int \varphi \, \mathrm{d}\mu_{x,y}^{i,j} - \int \varphi \, \mathrm{d}\mu_{x,y}^{i,j,\beta}\right| < \frac{\varepsilon}{2}$$

and

$$\left|\int \bar{\varphi} \, \mathrm{d}\mu_{y,x}^{j,i} - \int \bar{\varphi} \, \mathrm{d}\mu_{y,x}^{j,i,\beta}\right| < \frac{\varepsilon}{2}.$$

But (2.5) implies $\int \varphi \, \mathrm{d}\mu_{x,y}^{i,j,\beta} = \overline{\int \bar{\varphi} \, \mathrm{d}\mu_{y,x}^{j,i,\beta}}$.

Thus,

$$\left|\int \varphi \, \mathrm{d}\mu_{x,y}^{i,j} - \int \varphi \, \overline{\mathrm{d}\mu_{y,x}^{j,i}}\right| < \varepsilon.$$

Since ε is arbitrary,

$$\int \varphi \, \mathrm{d}\mu_{x,y}^{i,j} = \int \varphi \, \overline{\mathrm{d}\mu_{y,x}^{j,i}}.$$

Since φ is arbitrary, $\mu_{x,y}^{i,j} = \overline{\mu_{y,x}^{j,i}}$ and Corollary 2.4 is proven.

The following corollary is proven in much the same way that Lemma 2.3 was just used to derive Corollary 2.4.

COROLLARY 2.6. *Fix i, j, and a Borel set $\Delta \subseteq \mathbf{C}$. Then $[\cdot,\cdot]$ defined on $\mathscr{H} \times \mathscr{H}$ by,*

$$[x, y] = \mu_{x,y}^{i,j}(\Delta),$$

is a bounded sesquilinear form.

We now use Corollary 2.6 and Riesz's Theorem on the representation of bounded sesquilinear forms to define a matricial functional calculus. It will soon become apparent that after a "change of variable" this calculus does for S what the representation of Proposition 1.4 does for a Jordan block.

DEFINITION 2.7. For $[\varphi_{ij}]$ an $n \times n$ matrix $(0 \leqslant i, j \leqslant n-1)$ with entries $\varphi_{ij} \in B_c(\mathbf{C})$, the space of compactly supported Borel functions on $\mathbf{C}$, define $\{\varphi_{ij}\} \in \mathscr{L}(\mathscr{H})$ by,

(a) $$\langle \{\varphi_{ij}\} x, y \rangle = \sum_{i,j=0}^{n-1} \int \varphi_{ij} \, \mathrm{d}\mu_{x,y}^{i,j},$$

for all $x, y \in \mathscr{H}$. Similarly, for $\gamma \in \Gamma$ define $\{\varphi_{ij}\}_\gamma$ by,

(b) $$\langle \{\varphi_{ij}\}_\gamma x, y \rangle = \sum_{i,j=0}^{n-1} \int \varphi_{ij} \, \mathrm{d}\mu_{x,y}^{i,j,\gamma}.$$

For $[\varphi_{ij}]$ as above let $[\tilde{\varphi}_{ij}]$ denote the $n \times n$ matrix with entries in $B_c(\mathbf{C})$ defined by (cf. (2.19)),

$$\text{(c)} \quad \begin{aligned} \tilde{\varphi}_{ij}(z) &= \bar{z}\varphi_{ij} + \varphi_{i\,j-1} \quad 0 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n-1 \\ \tilde{\varphi}_{i0}(z) &= \bar{z}\varphi_{i0} \quad 0 \leqslant i \leqslant n-1 . \end{aligned}$$

Several computational facts involving the notations introduced in Definition 2.7 are summarized in the following lemma.

LEMMA 2.8. *Let* $[\varphi_{ij}]$ *be as in Definition* 2.6. *Then*

(a) $\{\varphi_{ij}\}^* = \{\overline{\varphi}_{ji}\}$;

(b) *for every* $\gamma \in \Gamma$, $\{\varphi_{ij}\}_\gamma = \sum_{i,j=0}^{n-1} Q_\gamma^{*j}\varphi_{ij}(N_\gamma)\, Q_\gamma^i$;

(c) *for every* $\gamma \in \Gamma$, $J_\gamma^*\{\varphi_{ij}\}_\gamma = \{\tilde{\varphi}_{ij}\}_\gamma$.

Proof. To prove (a) we apply Corollary 2.3. Fix x and y in $\mathscr{H}$. Then.

$$\langle\{\varphi_{ij}\}^*x, y\rangle = \overline{\langle\{\varphi_{ij}\}\, y, x\rangle} = \sum_{i,j=0}^{n-1}\int \overline{\varphi}_{ij}\mathrm{d}\overline{\mu_{y,x}^{i,j}} =$$

$$= \sum_{i,j=0}^{n-1}\int \overline{\varphi}_{ij}\,\mathrm{d}\mu_{x,y}^{i,j} = \langle\{\overline{\varphi}_{ji}\}x, y\rangle ,$$

which proves (a). (b) is an immediate consequence of (2.2). Finally, to prove (c we use (b).

$$J_\gamma^*\{\varphi_{ij}\}_\gamma = (N_\gamma^* + Q_\gamma^*)\sum_{i,j=0}^{n-1} Q_\gamma^{*j}\varphi_{ij}(N_\gamma)Q_\gamma^i =$$

$$= \sum_{i,j=0}^{n-1} Q_\gamma^{*j}(\bar{z}\varphi_{ij} + \varphi_{ij-1})(N_\gamma)\, Q_\gamma^i = \qquad (\varphi_{i-1} = 0.)$$

$$= \{\tilde{\varphi}_{ij}\}_\gamma .$$

This concludes the proof of Lemma 2.8.

The key fact about the brace matricial functional calculus for S that we shall need is expressed in Lemma 2.16. The purpose of the next four lemmas (which follow the course of John Conway's simplification to the context of subnormal operators of Theorems 2.4 and 3.2 in [4]) is to obtain Lemma 2.16 by taking a limit in γ of the formula

$$p(J) = \left\{\frac{1}{j!i!}\,\Delta_\gamma(\mathrm{D}_y^j\mathrm{D}_x^i p)\ (z, \bar{z})\right\}_\gamma$$

(here, $\Delta_\gamma \in B_c(\mathbf{C})$ is the characteristic function of $\sigma(J_\gamma)$), which follows from Lemma 1.2 and Definition 2.8 (b). In the remainder of this section η will denote the $n \times n$ matrix defined by the relations $\langle \eta e_i, e_j \rangle = 1$ if $i = j = 0$, $\langle \eta e_i, e_j \rangle = 0$ otherwise with respect to the standard basis, $\{e_i : 0 \leqslant i \leqslant n-1\}$, of $\mathbf{C}^n$. If $\varphi \in B_c(\mathbf{C})$ then $\varphi\eta$ will denote the matrix $[\varphi_{ij}]$, with entries in $B_c(\mathbf{C})$ defined by

$$\varphi_{ij} = \begin{cases} \varphi & \text{if } i = j = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Finally, $C_c(\mathbf{C}) = C(\mathbf{C}) \cap B_c(\mathbf{C})$, the space of compactly supported continuous functions on $\mathbf{C}$.

LEMMA 2.9. *If x and y are in $\mathscr{H}$ then $\mu_{x,y}^{0,0}(\mathbf{C}) = \langle x, y \rangle$.*

Proof. Note that (2.2) implies that $\mu_{x,y}^{0,0,\gamma}(\mathbf{C}) = \langle x, y \rangle$. As our proof shows the act of taking the limit of this expression (and thereby obtaining Lemma 2.9) is subtle matter, the difficulty being that the constant function 1 is not of compact support. It is perhaps while surmounting this difficulty that Bishop's original argument is at its most surprising and beautiful point. Fix $r, r > 1$. Let $\omega_r = \mathbf{C}/r\mathbf{D}$. Let $\varphi \in C_c(\mathbf{C})$ with $\varphi \equiv 0$ off ω_r. For each i, $0 \leqslant i \leqslant n$, and each j, $0 \leqslant j \leqslant n-1$ define $\varphi_{ij} \in C(\mathbf{C})$ by,

$$\begin{aligned} &\varphi_{ij} \equiv 0 && \text{if } i \geqslant 1 \\ &\varphi_{0j}(z) = 0 && \text{if } 0 \leqslant j \leqslant n-1 \text{ and } |z| < r \\ &\varphi_{0j}(z) = (-1)^j \bar{z}^{-(j+1)}\varphi && \text{if } 0 \leqslant j \leqslant n-1 \text{ and } z \in \omega_r. \end{aligned}$$

Using Definition 2.7 (c) it is easy to see that

$$[\tilde{\varphi}_{ij}] = \varphi\eta.$$

Hence by Lemma 2.8 (c),

$$\langle \{\varphi\eta\}_\gamma x, y \rangle = \langle \{\tilde{\varphi}_{ij}\}_\gamma x, y \rangle = \langle \{\varphi_{ij}\}_\gamma x, J_\gamma y \rangle.$$

We conclude that,

$$\int \varphi \, d\mu_{x,y}^{0,0,\gamma} = \langle \{\varphi\eta\}_\gamma x, y \rangle = \langle \{\varphi_{ij}\}_\gamma x, J_\gamma y \rangle = \\ = \sum_{j=0}^{n-1} (-1)^j \int \bar{z}^{-(j+1)} \varphi \, d\mu_{x,J_\gamma y}^{0,j,\gamma}.$$

Recalling that $r > 1$ and that $\varphi \equiv 0$ off ω_r, and then crashing through with absolute values in this last expression yields the estimate,

$$\left| \int \varphi \, d\mu_{x,y}^{0,0,\gamma} \right| \leqslant nD \|\varphi\| \|x\| \|J_\gamma y\| r^{-1}.$$

Finally, since,

$$|\mu_{x,y}^{0,0,\gamma}|(\omega_r) = \sup\left\{ \left| \int \varphi \, d\mu_{x,y}^{0,0,\gamma} \right| \quad \varphi \in C_c(\mathbf{C}),\ \|\varphi\| \leqslant 1, \text{ and } \varphi \equiv 0 \text{ off } \omega_r \right\},$$

we see that,

$$(2.10) \qquad |\mu_{x,y}^{0,0,\gamma}|(\omega_r) \leqslant nD\|x\|\,\|J_\gamma y\|r^{-1}.$$

Now fix $\varepsilon > 0$. Choose $r > \max\left\{\frac{1}{\varepsilon}, 1\right\}$ and $\psi \in C_0(\mathbf{C})$ so that $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $r\mathbf{D}$ and

$$\left|\mu_{x,y}^{0,0}(\mathbf{C}) - \int \psi \mathrm{d}\mu_{x,y}^{0,0}\right| < \varepsilon.$$

By (2.10) and the fact that $J_\gamma \to S$ strongly in $\mathscr{L}(\mathscr{H})$, there is a constant d and a $\gamma_0 \in \Gamma$ such that,

$$|\mu_{x,y}^{0,0,\gamma}|(\omega_r) \leqslant d_\gamma^{-1} < d\varepsilon,$$

for all $\gamma > \gamma_0$. Using Lemma 2.3 pick $\gamma_1 > \gamma_0$ so that,

$$\left|\int \psi \mathrm{d}\mu_{x,y}^{0,0,\gamma_1} - \int \psi \mathrm{d}\mu_{x,y}^{0,0}\right| < \varepsilon.$$

Finally, notice that as an immediate consequence of (2.2) we have, $\mu_{x,y}^{0,0,\gamma_1}(\mathbf{C}) = \langle x, y\rangle$. Combining the last three inequalities,

$$\begin{aligned}
&|\mu_{x,y}^{0,0}(\mathbf{C}) - \langle x, y\rangle| < \\
&< \varepsilon + \left|\int \psi \,\mathrm{d}\mu_{x,y}^{0,0} - \langle x, y\rangle\right| < \\
&< 2\varepsilon + \left|\int \psi \,\mathrm{d}\mu_{x,y}^{0,0,\gamma_1} - \langle x, y\rangle\right| = \\
&= 2\varepsilon + \left|\int (\psi - 1)\,\mathrm{d}\mu_{x,y}^{0,0,\gamma_1}\right| \leqslant \\
&\leqslant 2\varepsilon + |\mu_{x,y}^{0,0,\gamma_1}|(\omega_r) < \\
&< 2\varepsilon + \mathrm{d}\varepsilon.
\end{aligned}$$

Lemma 2.9 is now established by letting ε pass to 0.

LEMMA 2.11. *If* $[\varphi_{ij}]$ *is an* $n \times n$ *matrix with entries in* $C_c(\mathbf{C})$ *then* $S^*\{\varphi_{ij}\} = \{\tilde{\varphi}_{ij}\}$.

Proof. Fix $x, y \in \mathscr{H}$. For $\gamma \in \Gamma$ define complex numbers $A_\gamma, B_\gamma, C_\gamma$, and D_γ in the following way:

$$A_\gamma = \langle \{\varphi_{ij}\} x, Sy \rangle - \langle \{\varphi_{ij}\}_\gamma x, Sy \rangle,$$

$$B_\gamma = \langle \{\varphi_{ij}\}_\gamma x, Sy \rangle - \langle \{\varphi_{ij}\}_\gamma x, J_\gamma y \rangle,$$

$$C_\gamma = \langle \{\varphi_{ij}\}_\gamma x, J_\gamma y \rangle - \langle \{\tilde{\varphi}_{ij}\}_\gamma x, y \rangle,$$

and

$$D_\gamma = \langle \{\tilde{\varphi}_{ij}\}_\gamma x, y \rangle - \langle \{\tilde{\varphi}_{ij}\} x, y \rangle.$$

Evidently,

$$|\langle S^*\{\varphi_{ij}\} x, y \rangle - \langle \{\tilde{\varphi}_{ij}\} x, y \rangle| =$$

$$= |A_\gamma + B_\gamma + C_\gamma + D_\gamma| \leqslant$$

$$\leqslant |A_\gamma| + |B_\gamma| + |C_\gamma| + |D_\gamma|.$$

Fix $\varepsilon > 0$. By Lemma 2.8 (c), $C_\gamma = 0$. From Definition 2.7 (b) it is clear that $\{\|\{\varphi_{ij}\}_\gamma x\| : \gamma \in \Gamma\}$ is bounded. Thus since $J_\gamma \to S$ strongly, there exists a $\gamma_0 \in \Gamma$ such that $\gamma > \gamma_0$ implies $|B_\gamma| < \frac{\varepsilon}{3}$. Finally, by Definition 2.7 (b) and Lemma 2.3 there exists a $\gamma_1 > \gamma_0$ such that $|A_{\gamma_1}| < \frac{\varepsilon}{3}$ and $|D_{\gamma_1}| < \frac{\varepsilon}{3}$. Thus,

$$|\langle S^*\{\varphi_{ij}\} x, y \rangle - \langle \{\tilde{\varphi}_{ij}\} x, y \rangle| < \varepsilon,$$

and since ε is arbitrary we have established Lemma 2.11.

The following corollary to Lemma 2.11 is proven by a routine approximation argument.

COROLLARY 2.12. *If* $[\varphi_{ij}]$ *is an* $n \times n$ *matrix with entries in* $B_c(\mathbf{C})$ *then* $S^*\{\varphi_{ij}\} = \{\tilde{\varphi}_{ij}\}$.

LEMMA 2.13. *If* $x, y \in \mathscr{H}$ *then* $|\mu_{x,y}^{0;0}|(\mathbf{C} \setminus \sigma(S)) = 0$.

Proof. Let K be a compact subset of $\mathbf{C} \setminus \sigma(S)$. We want to show that $\mu_{x,y}^{0;0}(K) = 0$. For $\lambda \in \mathbf{C} \setminus \bar{K}$ define $\{\psi_{ij}^\lambda\}$, an $n \times n$ matrix with entries in $B_c(\mathbf{C})$, by

$$\psi_{ij}^\lambda \equiv 0 \qquad \text{if } 1 \leqslant i \leqslant n-1,\ 0 \leqslant j \leqslant n-1$$

$$\psi_{0j}^\lambda(z) = \begin{cases} (-1)^j(\bar{z} - \lambda)^{-(j+1)} & \text{if } 0 \leqslant j \leqslant n-1 \text{ and } z \in K \\ 0 & \text{if } 0 \leqslant j \leqslant n-1 \text{ and } z \notin K. \end{cases}$$

Then using Definition 2.7 (c) (with the understanding that χ_{ij} is zero if j is negative) we see that

$$[\tilde{\psi}^{\lambda}_{ij}] - \lambda[\psi^{\lambda}_{ij}] =$$

$$= (\bar{z} - \lambda)[\psi^{\lambda}_{ij}] + [\psi^{\lambda}_{ij-1}] = [\Delta_K \eta],$$

where $\Delta_K \in B_c(\mathbf{C})$ is the characteristic function of K. Hence by Corollary 2.12,

$$(S^* - \lambda)\{\psi^{\lambda}_{ij}\} = S^*\{\psi^{\lambda}_{ij}\} - \lambda\{\psi^{\lambda}_{ij}\} =$$

$$= \{\tilde{\psi}^{\lambda}_{ij}\} - \lambda\{\psi^{\lambda}_{ik}\} = \{\Delta_K \eta\}.$$

Hence if $\lambda \notin (\bar{K} \cup \sigma(S^*))$ we see that

$$\{\psi^{\lambda}_{ij}\} = (S^* - \lambda)^{-1}\{\Delta_K \eta\}. \tag{2.14}$$

Now, fix x and y in $\mathscr{H}$ and define a function $F : \mathbf{C} \to \mathbf{C}$ by

$$F(\lambda) = \begin{cases} \langle (S^* - \lambda)^{-1}\{\Delta_K \eta\}x, y\rangle & \text{if } \lambda \notin \sigma(S^*) \\ \langle \{\psi^{\lambda}_{ij}\}x, y\rangle & \text{if } \lambda \notin \bar{K}. \end{cases}$$

That F is in fact defined on all of $\mathbf{C}$ follows from the fact that $K \subseteq \mathbf{C} \setminus \sigma(S)$. That F is *well* defined follows from (2.14). Finally, it is clear that F is holomorphic and vanishes at ∞. Hence $F \equiv 0$. Then

$$\langle (S^* - \lambda)^{-1}\{\Delta_K \eta\}x, y\rangle = 0, \quad \text{all } \lambda \notin \sigma(S^*).$$

Since x and y are arbitrary we conclude that $\{\Delta_K \eta\} = 0$. But then (Definition 2.7)

$$\mu^{0;0}_{x,y}(K) = \langle \{\Delta_K \eta\}x, y\rangle = 0.$$

This concludes the proof of Lemma 2.13.

LEMMA 2.14. *If* $0 \leqslant i, j \leqslant n-1$ *and* $x, y \in \mathscr{H}$ *then* $|\mu^{i;j}_{x,y}|(\mathbf{C} \setminus \sigma(S)) = 0$.

Proof. The lemma will be proved if we can show that,

$$\int \varphi \mathrm{d}\mu^{i;j}_{x,y} = 0,$$

whenever $\varphi \in C_c(\mathbf{C})$ and $\varphi \equiv 0$ on $\sigma(S)$. Let $\varepsilon > 0$. Use Lemma 2.3 to find $\gamma \in \Gamma$ such that

$$\left| \int \varphi \mathrm{d}\mu^{i;j}_{x,y} - \int \varphi \mathrm{d}\mu^{i;j;\gamma}_{x,y} \right| < \varepsilon,$$

$$\left| \int |\varphi| \mathrm{d}\mu^{0;0}_{x,x} - \int |\varphi| \mathrm{d}\mu^{0;0;\gamma}_{x,x} \right| < \varepsilon, \tag{2.15}$$

and

$$\left|\int |\varphi| \mathrm{d}\mu_{y,y}^{0,0} - \int |\varphi| \mathrm{d}\mu_{y,y}^{0,0,\gamma}\right| < \varepsilon .$$

Now notice that since $N_\gamma Q_\gamma = Q_\gamma N_\gamma$ we also have $E_\gamma(\Delta) Q_\gamma = Q_\gamma E_\gamma(\Delta)$ whenever $\Delta \subseteq \mathbf{C}$ is a Borel set. Thus,

$$|\mu_{x,y}^{i,j,\gamma}(\Delta)| = |\langle Q_\gamma^{*j} E_\gamma(\Delta) Q_\gamma^i x, y\rangle| \leqslant$$

$$\leqslant \|Q_\gamma^i E_\gamma(\Delta)x\| \, \|Q_\gamma^j E_\gamma(\Delta)y\| \leqslant$$

$$\leqslant c^{i+j} \|E_\gamma(\Delta)x\| \, \|E_\gamma(\Delta)y\| \leqslant$$

$$\leqslant \frac{c^{i+j}}{2} (\|E_\gamma(\Delta)x\|^2 + \|E_\gamma(\Delta)y\|^2) =$$

$$= \frac{c^{i+j}}{2} (\mu_{x,x}^{0,0,\gamma}(\Delta) + \mu_{y,y}^{0,0,\gamma}(\Delta)) .$$

Hence, setting $d = \dfrac{c^{i+j}}{2}$, we have,

$$\left|\int \varphi \mathrm{d}\mu_{x,y}^{i,j,\gamma}\right| \leqslant \int |\varphi| \, \mathrm{d}|\mu_{x,y}^{i,j,\gamma}| \leqslant$$

$$\leqslant d\int |\varphi| \mathrm{d}\mu_{x,x}^{0,0,\gamma} + d\int |\varphi| \, \mathrm{d}\mu_{y,y}^{0,0,\gamma} .$$

Since Lemma 2.13 implies $\int |\varphi| \mathrm{d}\mu_{x,x}^{0,0} = 0$ we conclude from (2.15) that

$$\left|\int \varphi \mathrm{d}\mu_{x,y}^{i,j,\gamma}\right| \leqslant 2d\varepsilon .$$

Hence using (2.15) again we obtain,

$$\left|\int \varphi \mathrm{d}\mu_{x,y}^{i,j}\right| \leqslant \varepsilon + \left|\int \varphi \mathrm{d}\mu_{x,y}^{i,j,\gamma}\right| \leqslant \varepsilon + 2d\varepsilon .$$

Since d does not depend on ε and ε is arbitrary we conclude that $\int \varphi \mathrm{d}\mu_{x,y}^{i,j} = 0$ which establishes Lemma 2.14.

The following lemma is the means we will use to link up with the results of Section 1. In what follows let $\Delta = \Delta(z) \in B_c(\mathbf{C})$ denote the characteristic function of $\sigma(S)$.

LEMMA 2.16. *If* $p \in \mathscr{P}$ *then* $p(S) = \left\{\frac{1}{j!i!} \Delta(D_y^j D_x^i p)(z, \bar{z})\right\}$.

Proof. By Lemma 2.9 and 2.13 we have, $\langle\{\Delta\eta\}x, y\rangle = \mu_{x,y}^{0;0}(\sigma(S)) = \mu_{x,y}^{0;0}(\mathbf{C}) = \langle x, y\rangle$. Thus,

$$\{\Delta\eta\} = 1. \tag{2.17}$$

The following statement follows by a simple mathematical induction which we omit.

$$(z + v_n)^k \eta(z + v_n)^{*l} = \left[\frac{1}{j!i!}(D_y^j D_x^i y^l x^k)(z, \bar{z})\right], \tag{2.18}$$

for every $z \in \mathbf{C}$. Also, if $[\varphi_{ij}]$ is an $n \times n$ matrix with entries in $B_c(\mathbf{C})$ we have that

$$[\tilde{\varphi}_{ij}(z)] = [\varphi_{ij}(z)](z + v_n)^* \tag{2.19}$$

for every $z \in \mathbf{C}$. Using (2.17), (2.19) and iterating Corollary 2.12, we obtain

$$S^{*k} = \{\Delta\eta(z + v_n)^{*k}\}.$$

Thus, by Lemma 2.8 (a),

$$S^k = \{(z + v_n)^k \Delta\eta\}.$$

Using this last fact, (2.19), and iterating Corollary 2.12 we obtain

$$S^{*l}S^k = \{(z + v_n)^k \Delta\eta(z + v_n)^{*l}\}.$$

Thus by (2.18),

$$S^{*l}S^k = \left\{\frac{1}{j!i!} \Delta(D_y^j D_x^i y^l x^k)(z, \bar{z})\right\}.$$

Taking linear combinations of this last statement completes the proof of Lemma 2.16.

Finally, we are ready to prove Theorem 2.1. Thus, we wish to show that S has an extension to a Jordan block $J \in \mathscr{B}_c^n(\mathscr{H})$ where $\sigma(J) \subseteq \sigma(S)$. By Theorem 1.6 this is equivalent to showing that S is cp$(n, c, \sigma(S))$. We do this by an application of Lemma 1.13. Accordingly, let $G \subseteq \mathbf{C}$ be open with $\sigma(S) \subseteq G$ and G^- compact. Let $\omega \in C_c(\mathbf{C})$ have the properties, $\omega(z) = 1$ if $z \in \sigma(S)$, $\omega(z) = 0$ if $z \notin G$, and $0 \leqslant \omega(z) \leqslant 1$ for all $z \in \mathbf{C}$. We now show that S is cp(n, c, G^-).

Let $[p_{kl}(\chi)] \geqslant 0$ in $\mathscr{M}_m \otimes B_0(n, c, G^-)$. By Lemma 1.11 this means that

$$[p_{kl}(\chi_{n,c,\{z\}})] \geqslant 0 \quad \text{in } \mathscr{M}_m \otimes B_0(n, c, \{z\})$$

for every $z \in G^-$. Since $\chi_{n,c,\{z\}}(z) = z + v_n$ for $z \in G^-$ we see by

Lemma 1.2 that,

$$\left[\sum_{i,j=0}^{n-1} \frac{1}{j!i!} v_n^{*j}(\mathrm{D}_y^j \mathrm{D}_x^i p_{kl})(z,\bar{z}) v_n^i\right] \geqslant 0 \quad \text{in } \mathcal{M}_m \otimes \mathcal{M}_n \tag{2.20}$$

for every $z \in G^-$. Now let

$$\varphi_{i,j}^{k,l}(z) = \frac{1}{j!i!}\omega(z)(\mathrm{D}_y^j \mathrm{D}_x^i p_{kl})(z,\bar{z}).$$

The since $\omega \geqslant 0$ and $\omega \equiv 0$ off G it is clear from (2.20) that

$$\left[\sum_{i,j=0}^{n-1} v_n^{*j}\varphi_{i,j}^{k,l}(z) v_n^i\right] \geqslant 0 \quad \text{in } \mathcal{M}_m \otimes \mathcal{M}_n \tag{2.21}$$

for every $z \in \mathbf{C}$. Also, $\varphi_{i,j}^{k,l} \in C_c(\mathbf{C})$.

Fix k and l. Then by restriction if $\gamma \in \Gamma$, $\sum_{i,j=0}^{n-1} v_n^{*j}\varphi_{ij}^{kl}(z) v_n^i \in \mathscr{C}_n(\sigma(J_\gamma))$. Since $\pi_\gamma(z) = N_\gamma$ and $\pi_\gamma(v_n) = Q_\gamma$ we have

$$\pi_\gamma\left(\sum_{i,j}^{n-1} v_n^{*j}\varphi_{ij}^{kl} v_n^i\right) = \sum_{i,j} Q_\gamma^{*j}\varphi_{ij}^{kl}(N_\gamma)Q_\gamma^i = \{\varphi_{ij}^{kl}\}_\gamma.$$

Thus,

$$\mathrm{id}_m \otimes \pi_\gamma\left(\left[\sum_{i,j}^{n-1} v_n^{*j}\varphi_{ij}^{kl} v_n^i\right]\right) = [\{\varphi_{ij}^{kl}\}_\gamma]. \tag{2.22}$$

Since by (2.21) and Lemma 1.11,

$$\left[\sum_{i,j}^{n-1} v_n^{*j}\varphi_{ij}^{kl} v_n^i\right] \geqslant 0 \quad \text{in } \mathcal{M}_m \otimes \mathscr{C}_n(\sigma(N_\gamma)),$$

we conclude from (2.22) that

$$[\{\varphi_{ij}^{kl}\}_\gamma] \geqslant 0 \quad \text{in } \mathcal{M}_m \otimes \mathscr{L}(\mathscr{H}) \text{ for all } \gamma \in \Gamma. \tag{2.23}$$

We now show that

$$[p_{kl}(S)] \geqslant 0 \quad \text{in } \mathcal{M}_m \otimes \mathscr{L}(\mathscr{H}).$$

Let $x_1, \ldots, x_m \in \mathscr{H}$ and fix $\varepsilon > 0$. By Lemmas 2.14 and 2.15 we have

$$\begin{aligned} \langle [p_{kl}(S)](x_l),(x_l)\rangle &= \sum_{k,l=1}^{m} \langle p_{kl}(S)\, x_l, x_k\rangle = \\ &= \sum_{k,l=1}^{m} \left\langle \left\{ \frac{1}{i!j!} \Delta(\mathrm{D}_y^j \mathrm{D}_x^i p_{kl})(z,\bar{z}) \right\} x_l,\, x_k \right\rangle = \\ &= \sum_{k,l}\sum_{i,j} \frac{1}{i!j!} \int_\Delta (\mathrm{D}_y^j \mathrm{D}_x^i p_{kl})(z,\bar{z})\, \mathrm{d}\mu_{x_l,x_k}^{i,j} = \\ &= \sum_{k,l}\sum_{i,j} \int \varphi_{i,j}^{k,l}\, \mathrm{d}\mu_{x_l,x_k}^{i,j}. \end{aligned} \tag{2.24}$$

Let $\alpha_\gamma = \langle [\{\varphi_{ij}^{kl}\}_\gamma](x_l), (x_l) \rangle$. Then by 2.7(b),

$$\alpha_\gamma = \sum_{k,l} \sum_{i,j} \int \varphi_{i,j}^{k,l} \mathrm{d}\mu_{x_l, x_k}^{i,j,\gamma}.$$

Thus, by Lemma 2.3 and (2.24) there exists a $\gamma \in \Gamma$ such that

$$|\langle [p_{kl}(S)](x_l), (x_l) \rangle - \alpha_\gamma| < \varepsilon.$$

But ε is arbitrary and $\alpha_\gamma \geqslant 0$ by (2.23). Hence

$$\langle [p_{kl}(S)](x_l), (x_l) \rangle \geqslant 0.$$

This establishes that $[p_{kl}(S)] \geqslant 0$ and concludes the proof of Theorem 2.1.

3. SOME THEOREMS

In this section we prove the theorems alluded to in the introduction.

LEMMA 3.1. *If $\mathscr{H}$ is infinite dimensional and separable then* $\mathrm{sub}\mathscr{J}_c^n(\mathscr{H}) \subseteq (\mathscr{J}_c^n(\mathscr{H}))^{-s}$ *and* $\mathrm{sub}\mathscr{B}_c^n(\mathscr{H}) \subseteq (\mathscr{B}_c^n(\mathscr{H}))^{-s}$.

Proof. Let $S \in \mathrm{sub}\mathscr{J}_c^n(\mathscr{H})$ with $S = J|\mathscr{H}$ and $J \in \mathscr{J}_c^n(\mathscr{K})$. Clearly we may assume that $\mathscr{K}$ is separable. Let V denote the inclusion map of $\mathscr{H}$ into $\mathscr{K}$. Choose a sequence of Hilbert space isomorphisms, $V_j : \mathscr{H} \to \mathscr{K}$, such that V_j tends to V and V_j^* tends to V^* strongly as $j \to \infty$. Then $V_j^* J V_j \to S$ strongly as $j \to \infty$ and $V_j^* J V_j \in \mathscr{J}_c^n(\mathscr{H})$. We conclude that $S \in (\mathscr{J}_c^n(\mathscr{H}))^{-s}$. Thus $\mathrm{sub}\mathscr{J}_c^n(\mathscr{H}) \subset (\mathscr{J}_c^n(\mathscr{H}))^{-s}$. An argument similar to the one just executed shows that $\mathrm{sub}\mathscr{B}_c^n(\mathscr{H}) \subseteq (\mathscr{B}_c^n(\mathscr{H}))^{-s}$ and concludes the proof of Lemma 3.1.

THEOREM 3.2. *Let $\mathscr{H}$ be infinite dimensional. Then*

$$\mathrm{sub}\mathscr{J}_c^n(\mathscr{H}) = (\mathscr{J}_c^n(\mathscr{H}))^{-s} = (\mathscr{B}_c^n(\mathscr{H}))^{-s} = \mathrm{sub}\mathscr{B}_c^n(\mathscr{H}).$$

Proof. A routine argument shows that we may assume $\mathscr{H}$ is separable. For convenience we omit the symbol $\mathscr{H}$ from the four collections of operators referred to in Theorem 3.2. Note first as an immediate consequence of their definitions that $\mathscr{B}_c^n \subseteq \mathscr{J}_c^n$. Thus,

$$\mathrm{sub}\mathscr{B}_c^n \subseteq \mathrm{sub}\mathscr{J}_c^n. \tag{3.3}$$

By Lemma 3.1,

$$\mathrm{sub}\mathscr{J}_c^n \subseteq (\mathscr{J}_c^n)^{-s}. \tag{3.4}$$

By Theorem 1.12,

$$(\mathscr{J}_c^n)^{-s} \subseteq (\mathrm{sub}\mathscr{B}_c^n)^{-s}. \tag{3.5}$$

By Lemma 3.1, $\mathrm{sub}\mathscr{B}_c^n \subseteq (\mathscr{B}_c^n)^{-s}$, and thus,

$$(\mathrm{sub}\mathscr{B}_c^n)^{-s} \subseteq (\mathscr{B}_c^n)^{-s}. \tag{3.6}$$

Finally, by Theorem 2.1,

$$(\mathscr{B}_c^n)^{-s} \subseteq \mathrm{sub}\mathscr{B}_c^n. \tag{3.7}$$

Combining (3.3), (3.4), (3.5), (3.6), and (3.7) establishes Theorem 3.2.

One might reasonably inquire what happens to Theorem 3.2 when the constant c is omitted. That it fails dramatically can be seen using the result that the strong closure of the order two nilpotents on an infinite dimensional $\mathscr{H}$ is $\mathscr{L}(\mathscr{H})$ ([10], Problem 91).

THEOREM 3.8. *If $S \in \mathrm{sub}\mathscr{J}_c^n(\mathscr{H})$ then S has an extension to $J \in \mathscr{B}_c^n(\mathscr{H})$ where $\sigma(J) \subseteq \sigma(S)$ and $\sigma(S) \setminus \sigma(J)$ is a union of components of $\mathbf{C} \setminus \sigma(J)$.*

Proof. By Theorem 3.2, $S \in (\mathscr{B}_c^n(\mathscr{H}))^{-s}$. Hence by Theorem 2.1, S has an extension to $J \in \mathscr{B}_c^n(\mathscr{H})$ with $\sigma(J) \subseteq \sigma(S)$. That $\sigma(S) \setminus \sigma(J)$ is a union of components of $\mathbf{C} \setminus \sigma(J)$ follows as in the solution to Problem 158 in [10].

We now introduce a matricial function algebra which can be used to generalize a theorem of von Neumann (see [16]).

DEFINITION 3.9. Let K be a compact set in $\mathbf{C}$, n a positive integer, and $c \geqslant 0$. $R_{n,c}(K)$ is the closure in $\mathscr{C}_n(K)$ of the functions of the form,

$$g(z) = f(z + c v_n) \quad z \in K,$$

where f is a rational function with poles off K.

Of course, if $n = 1$ then the above definition reduces to the usual definition of $R(K)$. Similarly the following definition generalizes von Neumann's notion of a spectral set.

DEFINITION 3.10. Let $T \in \mathscr{L}(\mathscr{H})$ and suppose $K \subseteq \mathbf{C}$ is compact with $\sigma(T) \subseteq K$. Then K is a *(n, c)-spectral set* for T if

$$\| f(T) \| \leqslant \max_{z \in K} \| f(z + c v_n) \|$$

for every rational function f with poles off K.

We shall be interested in the situation where $R_{n,c}(K)$ is as large as it could possibly be. Accordingly,

DEFINITION 3.11. $C_{n,c}(K)$ is the set of all f in $\mathscr{C}_n(K)$ of the form,

$$f(z) = \sum_{j=0}^{n-1} \varphi_j(z) v_n^j \quad z \in K,$$

where $\varphi_j \in C(K)$ for $0 \leqslant j \leqslant n-1$.

LEMMA 3.12. *The following are equivalent:*

(a) $R_{n,c}(K) = C_{n,c}(K)$,

(b) $R(K) = C(K)$ *and* $z \in R_{n,c}(K)$,

(c) *For every* $\varphi_0, \varphi_1, \ldots, \varphi_{n-1} \in C(K)$ *there exists a sequence* $\{f_i\}$ *of rational functions with poles off* K *such that* $\lim_{i\to\infty} \| f_i^{(j)} - \varphi_j \|_K = 0$.

Proof. (a) implies (b) and (c) implies (a) are clear from the definitions. To see that (b) implies (c) simply notice that $R_{n,c}(K)$ is an algebra.

LEMMA 3.13. *If* K *is an* (n, c)*-spectral set for* T *then the map,* φ*, densely defined on* $R_{n,c}(K)$ *by*

$$\varphi(f(z + c\,v_n)) = f(T), \quad f \in R(K),$$

extends to a contractive algebra homomorphism.

Proof. Immediate from the definitions.

The following proposition is the generalization to the sub-Jordan context of the familiar fact that sub-normal operators are von Neumann operators.

PROPOSITION 3.14. *If* $S \in \operatorname{sub} \mathscr{J}_c^n(\mathscr{H})$ *then* $\sigma(S)$ *is an* (n, c)*-spectral set for* S.

Proof. By Theorem 3.8 there is a $J \in \mathscr{B}_c^n(\mathscr{K})$ with $S = J \mid \mathscr{H}$ and $\sigma(J) \subseteq \sigma(S)$. By Proposition 1.4 there is a representation $\pi_1 : \mathscr{C}_n(\sigma(J)) \to \mathscr{L}(\mathscr{K})$ with $\pi_1(\chi_{n,c,\sigma(J)}) = J$. Let $\pi_0 : \mathscr{C}_n(\sigma(S)) \to \mathscr{C}_n(\sigma(J))$ denote the restriction representation and set $\pi = \pi_1 \pi_0$. Then $\pi(\chi_{n,c,\sigma(S)}) = J$. Thus, if f is a rational function with poles off $\sigma(S)$, then

$$\| f(S) \| = \| f(J) | \mathscr{M} \| \leqslant \| f(J) \| =$$

$$= \| f(\pi(\chi_{n,c,\sigma(S)})) \| = \| \pi(f(\chi_{n,c,\sigma(S)})) \| \leqslant$$

$$\leqslant \| f(\chi_{n,c,\sigma(S)}) \| = \max_{z \in \sigma(S)} \| f(z + v_n) \|,$$

and we see that the assertion of Proposition 3.14 is proved.

A well known theorem of von Neumann says that if K a spectral set for T and $R(K) = C(K)$ then T is normal.

PROPOSITION 3.15. *If* K *is an* (n, c)*-spectral set for* $T \in \mathscr{L}(\mathscr{H})$ *and* $R_{n,c}(K) = C_{n,c}(K)$ *then* $T \in \mathscr{J}_c^n(\mathscr{H})$.

Proof. Let $\varphi : R_{n,c}(K) \to \mathscr{L}(\mathscr{H})$ be the algebra homomorphism of Lemma 3.13. Evidently,

$$T = \varphi(z + c\,v_n) = \varphi(z) + \varphi(c\,v_n).$$

By von Neumann's Theorem $\varphi(z)$ is normal. Since φ is a homomorphism $\varphi(v_n)^n = 0$ and $\varphi(z)\,\varphi(cv_n) = \varphi(cv_n)\,\varphi(z)$. Finally, $\|\varphi(cv_n)\| \leqslant c$, since φ is a contraction. Thus $T \in \mathscr{J}_c^n(\mathscr{H})$ which establishes Proposition 3.15.

LEMMA 3.16. *If K is a totally disconnected compact subset of* $\mathbf{C}$ *then* $R_{n,c}(K) = C_{n,c}(K)$.

Proof. Notice first that if X is a totally disconnected compact Hausdorff space then the Stone-Weierstrass Theorem implies that $\{f \in C(X) : f(X) \text{ is finite}\}$ is dense in $C(X)$. Thus we see (using Lemma 3.12) that Lemma 3.16 will be proved if we show for every $\varepsilon > 0$ and $\varphi \in C(K)$ with $\varphi(K)$ finite there is a rational function f with poles off K s.t.

$$\max_{z \in K} |\varphi - f| < \varepsilon$$

(3.17) and

$$\max_{z \in K} |f^{(j)}| < \varepsilon \quad 1 \leqslant j \leqslant n-1 .$$

Let $\varphi(K) = \{z_i | 1 \leqslant i \leqslant N\}$ and let $K_i = \varphi^{-1}(\{z_i\})$ for $1 \leqslant i \leqslant N$. Clearly, K_i is both open and closed in K so that there exist disjoint compact sets E_i with $K_i \subseteq E_i^0$ (E_i^0 is the interior of E_i in $\mathbf{C}$). Set $E = \bigcup_{i=1}^{N} E_i$ and define $\chi \in C(E)$ by

$$\chi(z) = z_i \quad 1 \leqslant i \leqslant N,\ z \in E_i .$$

By Bishop's localization Theorem, $\chi \in R(E)$. Thus there is a sequence of rational functions, $\{f_k\}$ with poles off E (and hence off K) with f_k converging uniformly to χ on E (and hence on K). Since χ is constant on each component of E^0, if $j \geqslant 1$ then $f_k^{(j)}$ converges uniformly to 0 on each compact subset of E^0 (and hence on K). Thus (3.17) holds for k sufficiently large and the proof of Lemma 3.16 is complete.

THEOREM 3.18. *If $S \in \operatorname{sub}\mathscr{J}_c^n(\mathscr{H})$ and $\sigma(S)$ is totally disconnected then $S \in \mathscr{J}_c^n(\mathscr{H})$.*

Proof. By Proposition 3.14, $\sigma(S)$ is an (n, c)-spectral set for S and by Lemma 3.16, $R_{n,c}(\sigma(S)) = C_{n,c}(\sigma(S))$. Thus by Proposition 3.15, $S \in \mathscr{J}_c^n(\mathscr{H})$.

As a corollary to Theorem 3.18 one obtains Theorem C and Theorem D of the introduction. Theorem C is immediate. To prove Theorem D observe that by Theorem 5.26 of [1], $T \in \operatorname{sub}\mathscr{J}_c^2(\mathscr{H})$ (for an appropriate c). Hence by Theorem 3.18 and the assumption on $\sigma(T)$, $T \in \mathscr{J}_c^2(\mathscr{H})$.

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JIM AGLER
Department of Mathematics,
University of Virginia,
Charlottesville, VA 22903,
U.S.A.

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