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ALMOST UNIFORMLY CONTINUOUS AUTOMORPHISM GROUPS OF OPERATOR ALGEBRAS

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INTRODUCTION

Consider the following property of a C^*-dynamical system (A, G, α), where G is an abelian locally compact group and α is a strongly continuous homomorphism $G \to \mathrm{Aut}(A)$: α leaves each primitive ideal of A invariant, the homomorphism from G into the automorphism group of each primitive quotient of A induced by α has compact spectrum (equivalently, is uniformly continuous), and the "covering size" of this compact spectrum is bounded by a number which is independent of the primitive quotient. By the "covering size" of a compact subset of $\hat{G}$ we mean the smallest number of translates of a fixed compact neighbourhood of 0 in $\hat{G}$ needed to cover it. (Note that any two compact neighbourhoods of 0 in $\hat{G}$ give essentially the same notions of covering size — each bounded by a multiple of the other.)

In this paper, this property of a C^*-dynamical system, together with the analogous property of a W^*-dynamical system, is shown to arise naturally in connection with the following question, concerning the case $G = \mathbf{R}$.

Let X be a complex Banach space, endowed with an appropriate weak topology τ, and let α be a τ-continuous one-parameter group of τ-continuous linear isometries on X. In [3] a certain operator associated with α, called the *analytic generator* of α, and denoted by α_{-i}, is shown to display the following dichotomy:

$$\text{either } \sigma(\alpha_{-i}) \subset [0, +\infty), \text{ or } \sigma(\alpha_{-i}) = \mathbf{C}.$$

The case $\sigma(\alpha_{-i}) \subset [0, +\infty)$ occurs for example if X is the L^p-space with $1 < p < \infty$ associated to a semifinite normal trace on a semifinite W^*-algebra M, and α is induced by a weak*-continuous one-parameter automorphism group of M ([26], Theorem 4.2). In [25] it is shown that if (X, α) is a one-parameter C^*- or W^*-dynamical system with X commutative, then the case $\sigma(\alpha_{-i}) \subset [0, +\infty)$ occurs only if α acts identically. (That the case $\sigma(\alpha_{-i}) = \mathbf{C}$ can occur was first pointed out in [23].)

The purpose of this paper is to give a criterion for deciding the above dichotomy for an arbitrary one-parameter C^*- or W^*-dynamical system. Namely, the case $\sigma(\alpha_{-i}) \subset [0, +\infty)$ occurs precisely when α is "almost uniformly continuous" in the sense described above in the first paragraph. This definition is formulated for a C^*-dynamical system, but if instead of the set of primitive ideals one considers just some set of closed two-sided ideals with zero intersection, possibly depending on α, then one obtains an equivalent formulation in the C^*-case (see Theorem 3.5), and the definition is also suitable in the W^*-case (see Theorem 4.4).

1. GENERATORS AND SPECTRA

In this paragraph of preliminary character we begin by recalling some facts concerning the analytic generator of a one-parameter group. In order to include the cases both of C^*-dynamical systems and W^*-dynamical systems, we place ourselves within the framework of "dual pairs of Banach spaces".

We call *dual pair of Banach spaces* any pair $(X, \mathscr{F})$ of complex Banach spaces, together with a bilinear functional $X \times \mathscr{F} \ni (x, \varphi) \mapsto \langle x, \varphi \rangle \in \mathbf{C}$, such that

(i) $\|x\| = \sup_{\|\varphi\| \leqslant 1} |\langle x, \varphi \rangle|$ for any $x \in X$;

(ii) $\|\varphi\| = \sup_{\|x\| \leqslant 1} |\langle x, \varphi \rangle|$ for any $\varphi \in \mathscr{F}$;

(iii) the convex hull of any relatively $\sigma(X, \mathscr{F})$-compact subset of X is relatively $\sigma(X, \mathscr{F})$-compact;

(iv) the convex hull of any relatively $\sigma(\mathscr{F}, X)$-compact subset of $\mathscr{F}$ is relatively $\sigma(\mathscr{F}, X)$-compact.

If $(X, \mathscr{F})$ is a dual pair of Banach spaces then $(\mathscr{F}, X)$, endowed with the same bilinear pairing, is also a dual pair of Banach spaces.

If X is a complex Banach space then, by the Hahn-Banach theorem, the Kreĭn theorem on the relative weak compactness of the convex hull of a weakly compact set, the uniform boundedness principle, and the Alaoglu theorem, the pair (X, X^*) formed by X and its dual space X^*, with the natural bilinear pairing, is a dual pair of Banach spaces. Thus also (X^*, X) is a dual pair of Banach spaces.

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces. Then the uniform boundedness principle holds in X with respect to $\sigma(X, \mathscr{F})$ and in $\mathscr{F}$ with respect to $\sigma(\mathscr{F}, X)$ ([15], Theorem 2.8.6). In particular, the analyticity of X-valued or $\mathscr{F}$-valued maps with complex variable does not depend on the topology considered on X or on $\mathscr{F}$ ([15], Theorem 3.10.1). On the other hand, quite general X-valued or $\mathscr{F}$-valued maps, defined on a locally compact space endowed with a Radon measure, are weakly integrable ([1], Proposition 1.2; [3], Proposition 1.4).

If $(X, \mathscr{F})$ is a dual pair of Banach spaces and T a $\sigma(X, \mathscr{F})$-densely defined linear operator in X then one can define the adjoint $T^{\mathscr{F}}$ of T in $\mathscr{F}$ by

$$(\varphi, \psi) \in \operatorname{graph}(T^{\mathscr{F}}) \Leftrightarrow \langle x, \psi \rangle = \langle T(x), \varphi \rangle \quad \text{for all } \varphi \in \mathscr{D}_T .$$

$T^{\mathscr{F}}$ is always $\sigma(\mathscr{F}, X)$-closed. If T is $\sigma(X, \mathscr{F})$-densely defined and $\sigma(X, \mathscr{F})$-closed, then $T^{\mathscr{F}}$ will be $\sigma(\mathscr{F}, X)$-densely defined and $\sigma(\mathscr{F}, X)$-closed, and

$$(T^{\mathscr{F}})^X = T$$

([22], IV.7.1). With $B_{\mathscr{F}}(X)$ denoting the Banach algebra of all $\sigma(X, \mathscr{F})$-continuous linear operators on X, if $T \in B_{\mathscr{F}}(X)$, then

$$T^{\mathscr{F}}(\varphi) = \varphi \circ T, \quad \varphi \in \mathscr{F},$$

and $T^{\mathscr{F}} \in B_X(\mathscr{F})$.

If $(X, \mathscr{F})$ is a dual pair of Banach spaces and T a $\sigma(X, \mathscr{F})$-closed linear operator in X then the *resolvent set* of T is

$$\rho(T) = \{\lambda \in \mathbf{C} \,;\, \lambda - T \text{ is injective and } (\lambda - T)^{-1} \in B_{\mathscr{F}}(X)\},$$

and the *spectrum* of T is

$$\sigma(T) = \mathbf{C} \setminus \rho(T).$$

The standard power series argument shows that $\rho(T)$ is open in $\mathbf{C}$, so $\sigma(T)$ is closed. If T is also $\sigma(X, \mathscr{F})$-densely defined then

$$\sigma(T) = \sigma(T^{\mathscr{F}}).$$

We note that if $\mathscr{F} = X^*$ or $X = \mathscr{F}^*$ then, by the closed graph theorem, the Banach-Šmulian theorem on the weak* continuity of linear functionals, and the Alaoglu theorem,

$$\rho(T) = \{\lambda \in \mathbf{C} \,;\, \lambda - T \text{ is bijective}\}.$$

Let $(X, \mathscr{F})$ be a dual pair of Banach spaces. A *one-parameter group* U in $B_{\mathscr{F}}(X)$ is a mapping $U\colon \mathbf{R} \to B_{\mathscr{F}}(X)$ such that

$$U_0 = \text{identity map of } X,$$

$$U_{t_1+t_2} = U_{t_1} U_{t_2}, \quad t_1, t_2 \in \mathbf{R}.$$

U is called *$\sigma(X, \mathscr{F})$-continuous* if for each $x \in X$ the mapping

$$\mathbf{R} \ni t \mapsto U_t(x) \in X$$

is $\sigma(X, \mathscr{F})$-continuous. In this case one can define the *dual group* $U^{\mathscr{F}}$ in $B_X(\mathscr{F})$ by

$$U_t^{\mathscr{F}} = (U_t)^{\mathscr{F}}, \quad t \in \mathbf{R}$$

and $U^{\mathscr{F}}$ is $\sigma(\mathscr{F}, X)$-continuous. We note that if $\mathscr{F} = X^*$ then a $\sigma(X, \mathscr{F})$-continuous one-parameter group in $B(X) = B_{\mathscr{F}}(X)$ is always strongly continuous.

Since we have C^*- and W^*-dynamical systems in view, we shall denote in the sequel by U a $\sigma(X,\mathscr{F})$-continuous one-parameter group of *isometries* in $B_{\mathscr{F}}(X)$, where $(X,\mathscr{F})$ is a dual pair of Banach spaces. Then $U^{\mathscr{F}}$ will be a $\sigma(\mathscr{F},X)$-continuous one-parameter group of isometries in $B_X(\mathscr{F})$.

The infinitesimal generator D_U of U is the linear operator in X defined by

$$(x,\ y)\in \mathrm{graph}(D_U) \Leftrightarrow \begin{cases} \text{for each } \varphi\in\mathscr{F} \text{ the function} \\ \mathbf{R}\ni t\mapsto \langle U_t(x),\varphi\rangle\in\mathbf{C} \\ \text{is differentiable at } 0 \text{ and} \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\langle U_t(x),\varphi\rangle|_{t=0}=\langle y,\varphi\rangle. \end{cases}$$

Standard arguments from the theory of operator semigroups (see for example [15]) show that D_U is a $\sigma(X,\mathscr{F})$-densely defined $\sigma(X,\mathscr{F})$-closed linear operator in X, D_U determines U uniquely, and

$$(D_U)^{\mathscr{F}}=D_{U^{\mathscr{F}}}\ .$$

Moreover,

$$\sigma(D_U)\subset \mathrm{i}\mathbf{R},$$

so

$$\sigma(-\mathrm{i}D_U)\subset\mathbf{R}.$$

For each $f\in L^1(\mathbf{R})$ one can define

$$\int_{-\infty}^{+\infty} f(t)U_t\mathrm{d}t\in B_{\mathscr{F}}(X)$$

by the equalities

$$\left\langle\left(\int_{-\infty}^{+\infty} f(t)U_t\mathrm{d}t\right)(x),\varphi\right\rangle=\int_{-\infty}^{+\infty} f(t)\langle U_t(x),\varphi\rangle\mathrm{d}t,\qquad x\in X,\ \varphi\in\mathscr{F}.$$

Then

$$L^1(\mathbf{R})\ni f\mapsto U_f=\int_{-\infty}^{+\infty} f(t)U_t\mathrm{d}t\in B_{\mathscr{F}}(X)$$

is a homomorphism of the convolution algebra $L^1(\mathbf{R})$ into $B_{\mathscr{F}}(X)$. Denote by $\hat{f}$ the inverse Fourier transform of $f\in L^1(\mathbf{R})$, defined by

$$\hat{f}(s):=\int_{-\infty}^{+\infty} f(t)\,\mathrm{e}^{\mathrm{i}st}\mathrm{d}t,\qquad s\in\mathbf{R},$$

and by $A(\mathbf{R})$ the function algebra $\{\hat{f}\,;\, f\in L^1(\mathbf{R})\}$; then the map

$$A(\mathbf{R}) \ni \hat{f} \mapsto U_f \in B_{\mathscr{F}}(X)$$

is also a homomorphism, but now with the pointwise multiplication on $A(\mathbf{R})$. The support $\sigma(U)$ of the above homomorphism, defined by

$$\sigma(U) = \left\{\lambda\in\mathbf{R}\ ;\ \begin{matrix}\text{for each neighbourhood } N \text{ of } \lambda \text{ there is}\\ f_N\in L^1(\mathbf{R}) \text{ with } \operatorname{supp}(\hat{f}_N)\subset N \text{ and } U_{f_N}\neq 0\end{matrix}\right\},$$

is called the *spectrum* of U. It is the smallest closed subset F of $\mathbf{R}$ with the property

$$f\in L^1(\mathbf{R}),\ \ F\cap \operatorname{supp}(\hat{f}) = \emptyset \Rightarrow U_f = 0$$

([1]). It turns out that

$$\sigma(-\mathrm{i}D_U) = \sigma(U)$$

([12]), so

$$\sigma(D_U) = \mathrm{i}\sigma(U).$$

We note that the above equality has been familiar for a long time to people dealing with generalized scalar operators (for an introduction to their theory we refer to [6]).

On the other hand, the *analytic generator* $U_{-\mathrm{i}}$ of U is the linear operator in X defined by

$$(x,y)\in \operatorname{graph}(U_{-\mathrm{i}}) \Leftrightarrow \begin{cases}\mathbf{R}\ni t\mapsto U_t(x)\in X \text{ has a } \sigma(X,\mathscr{F})\text{-}\\ \text{-continuous extension on the strip}\\ \{\zeta\in\mathbf{C}\ ;\ -1\leqslant \operatorname{Im}\zeta\leqslant 0\}, \text{ which is}\\ \text{analytic on the interior and whose}\\ \text{value at } -\mathrm{i} \text{ is } y.\end{cases}$$

The linear operator $U_{-\mathrm{i}}$ is $\sigma(X,\mathscr{F})$-densely defined and $\sigma(X,\mathscr{F})$-closed and it determines U uniquely ([3]). We note also that

$$(U_{-\mathrm{i}})^{\mathscr{F}} = (U^{\mathscr{F}})_{-\mathrm{i}}$$

([24], Theorem 1.1).

If $x\in \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(U_{-\mathrm{i}})^k}$, then $\mathbf{R}\ni t\mapsto U_t(x)\in X$ has an entire extension, whose power series expansion yields

$$U_{-\mathrm{i}}(x) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!}(D_U)^n(x).$$

Since $\bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(U_{-i})^k}$ is an essential domain for U_{-i} ([3]), one might expect that by a hypothetical spectral mapping theorem,

$$\sigma(U_{-i}) = \exp(-i\sigma(D_U)) = \exp\sigma(U).$$

Indeed, this happens for example in the case of strongly continuous one-parameter groups of unitaries on a complex Hilbert space.

In [3], Lemma 3.1 and Theorem 3.6, it was shown that the point spectrum of U_{-i} is always contained in $[0, +\infty)$, and

$$\sigma(U_{-i}) \begin{cases} \text{either} = \mathbf{C}, \\ \text{or} \quad \subset [0, +\infty). \end{cases}$$

The above dichotomy was refined in [25], Theorem 4.3, as follows:

$$\sigma(U_{-i}) \begin{cases} \text{either} = \mathbf{C}, \\ \text{or} \quad = \exp\sigma(U) \subset [0, +\infty). \end{cases}$$

But the case $\sigma(U_{-i}) = \mathbf{C}$ can really occur; a first example was given in [23] and others in [25]. The spectrum problem for U consists in deciding whether $\sigma(U_{-i}) = \mathbf{C}$ or not.

In order to get a general criterion for solving the spectrum problem for U we recall some definitions.

For each $z \in \mathbf{C}$ the *analytic extension* of U at z, U_z, is defined by

$$(x, y) \in \operatorname{graph}(U_z) \Leftrightarrow \begin{cases} \mathbf{R} \ni t \mapsto U_t(x) \in X \text{ has a } \sigma(X, \mathscr{F})\text{-} \\ \text{-continuous extension on the strip} \\ \{\zeta \in \mathbf{C}\ ;\ (\operatorname{Im}\zeta)(\operatorname{Im}z) \geqslant 0,\ |\operatorname{Im}\zeta| \leqslant |\operatorname{Im}z|\}, \\ \text{which is analytic on the interior and} \\ \text{whose value at } z \text{ is } y. \end{cases}$$

Thus the analytic generator of U is the analytic extension of U at $-i$. The group property of U is preserved for the analytic extension:

$$U_{z_1+z_2} = U_{z_1}U_{z_2}, \quad (\operatorname{Im}z_1)(\operatorname{Im}z_2) \geqslant 0,$$

$$U_{-z} = (U_z)^{-1}, \quad \text{all } z \in \mathbf{C}$$

([3]).

On the other hand, for each closed set $F \subset \mathbf{R}$ one can define the spectral subspace

$$X^U(F) = \{x \in X\ ;\ U_f(x) = 0 \text{ for each } f \in L^1(\mathbf{R}) \text{ with } F \cap \operatorname{supp}(\hat{f}) = \emptyset\}$$

([1]). We assume familiarity in handling spectral subspaces, for whose theory we refer to [18] and [7].

For each $\lambda \in \mathbf{R}$ we have

$$X^U((-\infty, \lambda]) = \left\{x \in \bigcap_{k=0}^{\infty} \mathscr{D}_{U_{-ki}} ;\ \varlimsup_{k\to\infty} \| U_{-ki}(x)\|^{1/k} \leqslant e^{\lambda}\right\} =$$

$$= \left\{x \in \bigcap_{\operatorname{Im} z \leqslant 0} \mathscr{D}_{U_z} ;\ \| U_z(x)\| \leqslant e^{\lambda|\operatorname{Im} z|}\|x\| \text{ for } \operatorname{Im} z \leqslant 0\right\},$$

$$X^U([\lambda, +\infty)) = \left\{x \in \bigcap_{k=0}^{\infty} \mathscr{D}_{U_{ki}} ;\ \varlimsup_{k\to\infty} \| U_{ki}(x)\|^{1/k} \leqslant e^{-\lambda}\right\} =$$

$$= \left\{x \in \bigcap_{\operatorname{Im} z \geqslant 0} \mathscr{D}_{U_z} ;\ \| U_z(x)\| \leqslant e^{-\lambda|\operatorname{Im} z|}\|x\| \text{ for } \operatorname{Im} z \geqslant 0\right\}$$

([3] and [24]). Roughly speaking, the analytic extension of U constitutes in a certain sense the Fourier-Laplace transform of the spectral-subspace-valued "quasimeasure" $X^U(\cdot)$, and the above equalities correspond to the Paley-Wiener theorem (for more details see [4] and [5]).

It follows that for each $\lambda \in \mathbf{R}$,

$$X^U((-\infty, \lambda]) \subset \mathscr{D}_{U_{-i}},$$

$$U_{-i}X^U((-\infty, \lambda]) \subset X^U((-\infty, \lambda]),$$

$$\| U_{-i} \mid X^U((-\infty, \lambda])\| \leqslant e^{\lambda},$$

and

$$X^U([\lambda, +\infty)) \subset \mathscr{D}_{U_i},$$

$$U_i X^U([\lambda, +\infty)) \subset X^U([\lambda, +\infty)),$$

$$\| U_i \mid X^U([\lambda, +\infty))\| \leqslant e^{-\lambda}.$$

The following criterion makes explicit ideas used in the proofs of [25], Theorem 5.1 and Corollary 4.4, and it completes [25], Theorem 2.3.

THEOREM 1.1. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and U a $\sigma(X, \mathscr{F})$--continuous one-parameter group of isometries in $B_{\mathscr{F}}(X)$. Then the following conditions are equivalent:*

(i) $\sigma(U_{-i}) \neq \mathbf{C}$;

(ii) *for each $\varepsilon > 0$ there is $P_\varepsilon \in B_{\mathscr{F}}(X)$ such that $P_\varepsilon|X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$ is a linear projection from $X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$ onto $X^U((-\infty, -\varepsilon])$ with kernel $X^U([\varepsilon, +\infty))$ and such that*

$$\|P_\varepsilon\| \leqslant (1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^2 \|(1 + U_{-i})^{-1}\| + 40\left(1 + \frac{16}{\varepsilon}\right)^5;$$

(iii) *there are* $-\infty < \mu_1 < \lambda_1 < +\infty$ *with*

$$X = X^U((-\infty, \lambda_1]) + X^U([\mu_1, +\infty))$$

and $-\infty < \mu_2 < \lambda_2 < +\infty$ *with*

$$\mathscr{F} = \mathscr{F}^{U^{\mathscr{F}}}((-\infty, \lambda_2]) + \mathscr{F}^{U^{\mathscr{F}}}([\mu_2, +\infty)).$$

Proof. Let us assume that (ii) holds and choose some $\varepsilon > 0$. Let further $f \in L^1(\mathbf{R})$ be such that

$$\hat{f}(s) = 1 \quad \text{for } s \in [-\varepsilon, \varepsilon],$$

$$\operatorname{supp}(\hat{f}) \subset [-2\varepsilon, 2\varepsilon].$$

Then for any $x \in X$,

$$U_f(x) \in X^U([-2\varepsilon, 2\varepsilon]),$$

$$x - U_f(x) \in X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty)),$$

so

$$x = (U_f(x) + P_\varepsilon(x - U_f(x))) + (1 - P_\varepsilon)(x - U_f(x)) \in X^U((-\infty, 2\varepsilon]) + X^U([\varepsilon, +\infty)).$$

Thus the first equality from (iii) is satisfied with

$$\lambda_1 = 2\varepsilon, \quad \mu_1 = \varepsilon.$$

On the other hand, for each $\varphi \in \mathscr{F}$ we have

$$\langle x, (1 - P_\varepsilon)^{\mathscr{F}}(\varphi)\rangle = \langle (1 - P_\varepsilon)(x), \varphi\rangle = 0, \quad x \in X^U((-\infty, -\varepsilon]),$$

so by [18], or [7], we get

$$(1 - P_\varepsilon)^{\mathscr{F}}(\varphi) \in \sigma(\mathscr{F}, X)\text{-closure of } \bigcup_{\nu > -\varepsilon} \mathscr{F}^{U^{\mathscr{F}}}([\nu, +\infty)) \subset \mathscr{F}^{U^{\mathscr{F}}}([-\varepsilon, +\infty)).$$

Similarly, for each $\varphi \in \mathscr{F}$,

$$\langle x, (P_\varepsilon)^{\mathscr{F}}(\varphi)\rangle = \langle P_\varepsilon(x), \varphi\rangle = 0, \quad x \in X^U([\varepsilon, +\infty)),$$

so

$$(P_\varepsilon)^{\mathscr{F}}(\varphi) \in \sigma(\mathscr{F}, X)\text{-closure of } \bigcup_{\nu < \varepsilon} \mathscr{F}^{U^{\mathscr{F}}}((-\infty, \nu]) \subset \mathscr{F}^{U^{\mathscr{F}}}((-\infty, \varepsilon]).$$

Hence

$$\varphi = (P_\varepsilon)^{\mathscr{F}}(\varphi) + (1 - P_\varepsilon)^{\mathscr{F}}(\varphi) \in \mathscr{F}^{U^{\mathscr{F}}}((-\infty, \varepsilon]) + \mathscr{F}^{U^{\mathscr{F}}}([-\varepsilon, +\infty)), \quad \varphi \in \mathscr{F},$$

and we conclude that also the second equality from (iii) is satisfied, with

$$\lambda_2 = \varepsilon, \quad \mu_2 = -\varepsilon.$$

Next, if (iii) holds then

$$X = \mathscr{D}_{U_{-\mathrm{i}}} + \mathscr{D}_{U_{\mathrm{i}}},$$

$$\mathscr{F} = \mathscr{D}_{U^{\mathscr{F}}_{-\mathrm{i}}} + \mathscr{D}_{U^{\mathscr{F}}_{\mathrm{i}}},$$

and [25], Theorem 2.3 yields (i).

Finally, let us prove the implication (i) ⇒ (ii). Assume that (i) holds. Then by [3], Theorem 3.6, $\sigma(U_{-\mathrm{i}}) \subset [0, +\infty)$. Let $\varepsilon > 0$ be arbitrary. Take a function $f_\varepsilon \in L^1(\mathbf{R})$ such that

$$\hat{f}_\varepsilon \in C^2(\mathbf{R}),$$

$$0 \leqslant \hat{f}_\varepsilon \leqslant 1,$$

$$f_\varepsilon(s) = 1 \quad \text{for } s \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right],$$

$$\operatorname{supp}(\hat{f}_\varepsilon) \subset \left[-\frac{3\varepsilon}{4}, \frac{3\varepsilon}{4}\right]$$

(choose $\hat{f}_\varepsilon$ first). Since

$$\sigma(U_{-\mathrm{i}}) \subset [0, +\infty),$$

we may consider the operators

$$(\lambda - U_{-\mathrm{i}})^{-1} \in B_{\mathscr{F}}(X), \quad |\lambda| = 1, \ \lambda \neq 1.$$

Let us estimate

$$\|(\lambda - U_{-\mathrm{i}})^{-1}(1 - U_{f_\varepsilon})\|$$

for $|\lambda| = 1, \ \lambda \neq 1$.

Let $|\lambda| = 1, \ \lambda \neq 1$. Then by [3], Corollary 3.3 we have

$$\lambda(\lambda - U_{-\mathrm{i}})^{-1} - (1 + U_{-\mathrm{i}})^{-1} = \int_{-\infty}^{+\infty} g_\lambda(t) U_t \, \mathrm{d}t,$$

where $g_\lambda \colon \{z \in \mathbf{C} \, ; \, |\mathrm{Im} z| < 1\} \to \mathbf{C}$ is the analytic function defined by

$$g_\lambda(z) = \frac{1}{2} \frac{1 - (-\lambda)^{-\mathrm{i}z}}{\sin \pi \mathrm{i} z}, \qquad |\mathrm{Im} z| < 1, \ z \neq 0.$$

Applying the above equality to the group of multiplications by $\mathrm{e}^{\mathrm{i}st}$, we have in particular

$$\frac{\lambda}{\lambda - \mathrm{e}^s} - \frac{1}{1 + \mathrm{e}^s} = \int_{-\infty}^{+\infty} g_\lambda(t) \, \mathrm{e}^{\mathrm{i}st} \mathrm{d}t, \qquad s \in \mathbf{R},$$

that is,

$$\hat{g}_\lambda(s) = \frac{(1+\lambda)e^s}{(\lambda - e^s)(1+e^s)}, \qquad s \in \mathbf{R}.$$

Therefore, setting

$$h_{\lambda,\varepsilon} = g_\lambda - g_\lambda * f_\varepsilon ,$$

we have

$$\lambda(\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon}) - (1 + U_{-i})^{-1}(1 - U_{f_\varepsilon}) = \int_{-\infty}^{+\infty} h_{\lambda,\varepsilon}(t) U_t \, dt,$$

where

$$(*) \qquad \hat{h}_{\lambda,\varepsilon}(s) = \frac{(1+\lambda)\, e^s(1 - \hat{f}_\varepsilon(s))}{(\lambda - e^s)(1+e^s)}, \qquad s \in \mathbf{R}.$$

Thus,

$$\|\lambda(\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon})\| \leqslant \|(1 + U_{-i})^{-1}\| \|1 - U_{f_\varepsilon}\| + \left\| \int_{-\infty}^{+\infty} h_{\lambda,\varepsilon}(t) U_t \, dt \right\| \leqslant$$

$$\leqslant (1 + \|f_\varepsilon\|_1) \|(1 + U_{-i})^{-1}\| + \|h_{\lambda,\varepsilon}\|_1 .$$

But by the inversion formula,

$$h_{\lambda,\varepsilon}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{h}_{\lambda,\varepsilon}(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

$$(it)^2 h_{\lambda,\varepsilon}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{h}''_{\lambda,\varepsilon}(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

so we get successively,

$$(1 + t^2) h_{\lambda,\varepsilon}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\hat{h}_{\lambda,\varepsilon}(s) - \hat{h}''_{\lambda,\varepsilon}(s))\, e^{-its} ds, \qquad t \in \mathbf{R},$$

$$\|h_{\lambda,\varepsilon}\|_1 \leqslant \frac{1}{2} \|\hat{h}_{\lambda,\varepsilon} - \hat{h}''_{\lambda,\varepsilon}\|_1 .$$

An elementary computation, using the formula $(*)$, shows that

$$\|\hat{h}_{\lambda,\varepsilon} - \hat{h}''_{\lambda,\varepsilon}\|_1 = \int_{|s| \geqslant \frac{\varepsilon}{2}} |\hat{h}_{\lambda,\varepsilon}(s) - \hat{h}''_{\lambda,\varepsilon}(s)| \, ds \leqslant$$

$$\leqslant \frac{40|1+\lambda|}{(1 - e^{-\varepsilon/2})^3} \max\{1, \|\hat{f}'_\varepsilon\|_\infty, \|\hat{f}''_\varepsilon\|_\infty\}.$$

Again by the inversion formula, we have also

$$(1 + t^2)f_\varepsilon(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\hat{f}_\varepsilon(s) - \hat{f}_\varepsilon''(s))\, e^{-its} ds, \qquad t \in \mathbf{R},$$

$$\|f_\varepsilon\|_1 \leqslant \frac{1}{2}\|\hat{f}_\varepsilon - \hat{f}_\varepsilon''\|_1 \leqslant 2\varepsilon \max\{1, \|\hat{f}_\varepsilon''\|_\infty\},$$

so we conclude that

$$\|(\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon})\| \leqslant$$

$$\leqslant \left[(1 + 2\varepsilon)\|(1 + U_{-i})^{-1}\| + \frac{40}{(1 - e^{-\varepsilon/2})^3}\right]\max\{1, \|\hat{f}_\varepsilon'\|, \|\hat{f}_\varepsilon''\|\}.$$

On the contour

$$C = \{\lambda \in \mathbf{C} \ ; \ |\lambda| = 1\},$$

the mapping

$$C \setminus \{1\} \ni \lambda \mapsto (\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon}) \in B_{\mathscr{F}}(X)$$

is norm continuous and, by the above estimation, bounded, so we can define

$$P_\varepsilon = \frac{1}{2\pi i}\int_C (\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon}) d\lambda \in B_{\mathscr{F}}(X).$$

Then

$$\|P_\varepsilon\| \leqslant \frac{1}{2\pi}\int_C \|(\lambda - U_{-i})^{-1}(1 - U_{f_\varepsilon})\| \, d|\lambda| \leqslant$$

$$\leqslant \left[(1 + 2\varepsilon)\|(1 + U_{-i})^{-1}\| + \frac{40}{(1 - e^{-\varepsilon/2})^3}\right]\max\{1, \|\hat{f}_\varepsilon'\|_\infty, \|\hat{f}_\varepsilon''\|_\infty\}.$$

If $x \in X^U((-\infty, -\varepsilon])$ then

$$P_\varepsilon(x) = \frac{1}{2\pi i}\int_C (\lambda - U_{-i})^{-1}(x) d\lambda,$$

and since

$$\|U_{-i} \mid X^U((-\infty, -\varepsilon])\| \leqslant e^{-\varepsilon},$$

standard analytic functional calculus arguments (see [10], VII.3.9—10) show that

$$P_\varepsilon(x) = x.$$

Similarly, if $x \in X^U([\varepsilon, +\infty))$ then

$$P_\varepsilon(x) = 0.$$

Since

$$X^U((-\infty, -\varepsilon]) + X^U([\varepsilon, +\infty))$$

is $(X, \mathscr{F})$-dense in

$$X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$$

and $P_\varepsilon \in B_{\mathscr{F}}(X)$, it follows that

$$P_\varepsilon \mid X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$$

is a linear projection from $X^U((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$ onto $X^U((-\infty, -\varepsilon])$ with kernel $X^U([\varepsilon, +\infty))$.

We complete the proof by showing that with an appropriate choice of f_ε the above estimation of $\|P_\varepsilon\|$ yields that in (ii).

Let us define the continuous function $k: \mathbf{R} \to \mathbf{R}$ by

$$k(s) = \begin{cases} 0 & \text{for } s \in \left[0, \frac{1}{2}\right] \\ -128 & \text{for } s = \frac{9}{16} \\ 128 & \text{for } s = \frac{11}{16} \\ 0 & \text{for } s \in \left[\frac{3}{4}, +\infty\right), \end{cases}$$

k is linear on each of the intervals $\left[\frac{1}{2}, \frac{9}{16}\right]$, $\left[\frac{9}{16}, \frac{11}{16}\right]$, $\left[\frac{11}{16}, \frac{3}{4}\right]$,

$$k(s) = k(-s) \quad \text{for all } s \in \mathbf{R}.$$

Then the function $l: \mathbf{R} \to \mathbf{R}$ defined by

$$l(s) = \int_{-\infty}^{s} \left(\int_{-\infty}^{u} k(v)\mathrm{d}v \right) \mathrm{d}u, \qquad s \in \mathbf{R},$$

belongs to $C^2(\mathbf{R})$ and satisfies

$$0 \leqslant l \leqslant 1,$$

$$l(s) = 1 \quad \text{for } s \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$\operatorname{supp}(l) \subset \left[-\frac{3}{4}, \frac{3}{4}\right],$$

$$\|l'\|_\infty = 8,$$

$$\|l''\|_\infty = \|k\|_\infty = 128.$$

Denote by f the Fourier transform of l:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} l(s)\, e^{-its} ds, \qquad t \in \mathbf{R}.$$

Then $f \in L^1(\mathbf{R})$ and $\hat{f} = l$, so for $f_\varepsilon \in L^1(\mathbf{R})$ defined by

$$f_\varepsilon(t) = \varepsilon f(\varepsilon t), \qquad t \in \mathbf{R},$$

we have

$$\hat{f}_\varepsilon(s) = \hat{f}\left(\frac{s}{\varepsilon}\right) = l\left(\frac{s}{\varepsilon}\right), \qquad s \in \mathbf{R}.$$

Thus, f_ε satisfies all conditions we required for the construction of P_ε, and, moreover,

$$\|\hat{f}_\varepsilon'\|_\infty = \frac{8}{\varepsilon},$$

$$\|\hat{f}_\varepsilon''\|_\infty = \frac{128}{\varepsilon^2}.$$

Hence we have the following estimation of the norm of the corresponding P_ε:

$$\|P_\varepsilon\| \leqslant \left[(1 + 2\varepsilon)\|(1 + U_{-i})^{-1}\| + \frac{40}{(1 + e^{-\varepsilon/2})^3}\right]\max\left\{1, \frac{8}{\varepsilon}, \frac{128}{\varepsilon^2}\right\} \leqslant$$

$$\leqslant \left[(1 + 2\varepsilon)\|(1 + U_{-i})^{-1}\| + 40\left(1 + \frac{2}{\varepsilon}\right)^3\right]\left(1 + \frac{16}{\varepsilon}\right)^2 \leqslant$$

$$\leqslant (1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^2\|(1 + U_{-i})^{-1}\| + 40\left(1 + \frac{16}{\varepsilon}\right)^5. \qquad ▨$$

The next statement is a quantitative completion of [3], Theorem 5.2.

THEOREM 1.2. *Let* $(X, \mathscr{F})$ *be a dual pair of Banach spaces and* U *a* $\sigma(X, \mathscr{F})$*-continuous one-parameter group of isometries in* $B_{\mathscr{F}}(X)$*. Then for each*

$$-\infty < \mu < \lambda < +\infty$$

we have

$$X^U([\mu, \lambda]) \subset \mathscr{D}_{(1+U_{-i})^{-1}},$$

$$(1 + U_{-i})^{-1} X^U([\mu, \lambda]) \subset X^U([\mu, \lambda]),$$

$$\|(1 + U_{-i})^{-1} \mid X^U([\mu, \lambda])\| \leqslant 3(\lambda - \mu)\left(1 + \frac{2}{\lambda - \mu} + \frac{16}{(\lambda - \mu)^2}\right).$$

Proof. By [3], Theorem 3.2,

$$X^U([\mu, \lambda]) \subset \mathscr{D}_{U_{-i}} \subset \mathscr{D}_{(1+U_{-i})^{-1}}.$$

Now let $x \in X^U([\mu, \lambda])$ and $y = (1 + U_{-i})^{-1}(x)$. By [3], Corollary 2.5 we have for all $f \in L^1(\mathbf{R})$ with $[\mu, \lambda] \cap \operatorname{supp}(\hat{f}) = \emptyset$,

$$(1 + U_{-i})U_f(y) = U_f(1 + U_{-i})(y) = U_f(x) = 0,$$

so by the injectivity of $1 + U_{-i}$,

$$U_f(y) = 0.$$

Thus,

$$y \in X^U([\mu, \lambda]).$$

Finally, in order to get the desired estimation of the norm of the restriction of $(1 + U_{-i})^{-1}$ to $X^U([\mu, \lambda])$, we let $x \in X^U([\mu, \lambda])$ and $y = (1 + U_{-i})^{-1}(x)$ and look for an estimation of $\|y\|$ in terms of $\|x\|$.

In the proof of Theorem 1.1 we have seen that there exists a function $f \in L^1(\mathbf{R})$ with

$$\hat{f} \in C^2(\mathbf{R}),$$
$$0 \leqslant \hat{f} \leqslant 1,$$
$$\hat{f}(s) = 1 \quad \text{for } s \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$
$$\operatorname{supp}(\hat{f}) \subset \left[-\frac{3}{4}, \frac{3}{4}\right],$$
$$\|\hat{f}'\|_\infty = 8,$$
$$\|\hat{f}''\|_\infty = 128.$$

For each $\delta > 1$ we define $f_\delta \in L^1(\mathbf{R})$ by

$$f_\delta(t) = e^{-i\frac{\lambda+\mu}{2}t}\,\delta(\lambda - \mu) f(\delta(\lambda - \mu)t), \quad t \in \mathbf{R}.$$

Then

$$\hat{f}_\delta(s) = \hat{f}\left(\frac{s - \frac{\lambda + \mu}{2}}{\delta(\lambda - \mu)}\right), \quad s \in \mathbf{R},$$

so

$$\hat{f}_\delta \in C^2(\mathbf{R}),$$
$$0 \leqslant \hat{f}_\delta \leqslant 1,$$
$$\hat{f}_\delta(s) = 1 \quad \text{for } s \in \left[\mu - \frac{\delta - 1}{2}(\lambda - \mu),\ \lambda + \frac{\delta - 1}{2}(\lambda - \mu)\right],$$
$$\operatorname{supp}(\hat{f}_\delta) \subset \left[\mu - \frac{3\delta - 2}{4}(\lambda - \mu),\ \lambda + \frac{3\delta - 2}{4}(\lambda - \mu)\right],$$
$$\|\hat{f}_\delta'\|_\infty = \frac{8}{\delta(\lambda - \mu)},$$
$$\|\hat{f}_\delta''\|_\infty = \frac{128}{\delta^2(\lambda - \mu)^2}.$$

Since $y \in X^U([\mu, \lambda])$ and $\hat{f}_3$ is equal to 1 on a neighbourhood of $[\mu, \lambda]$, we have

$$y = \int_{-\infty}^{+\infty} f_3(t) U_t(y) \mathrm{d}t.$$

But f_3 is an entire function and

$$\mathbf{C} \ni z \mapsto \int_{-\infty}^{+\infty} f_3(t - z) U_t(y) \mathrm{d}t \in X$$

is an entire extension of

$$\mathbf{R} \ni r \mapsto U_r(y) = \int_{-\infty}^{+\infty} f_3(t - r) U_t(y) \mathrm{d}t \in X,$$

so

$$U_{-\mathrm{i}}(y) = \int_{-\infty}^{+\infty} f_3(t + \mathrm{i}) U_t(y) \mathrm{d}t.$$

Hence

(*) $$x = (1 + U_{-\mathrm{i}})(y) = U_{f_3 + f_3(\cdot\, + \mathrm{i})}(y).$$

Now the inverse Fourier transform $(f_3 + f_3(\cdot\, + \mathrm{i}))^\wedge$ of $f_3 + f_3(\cdot\, + \mathrm{i})$ is

$$\mathbf{R} \ni s \mapsto (1 + \mathrm{e}^s)\hat{f}_3(s).$$

Since

$$s \in [\mu - (\lambda - \mu),\ \lambda + (\lambda - \mu)] \Rightarrow (f_3 + f_3(\cdot\, + \mathrm{i}))^\wedge(s) = 1 + \mathrm{e}^s$$

and

$$\mathrm{supp}(\hat{f}_2) \subset [\mu - (\lambda - \mu),\ \lambda + (\lambda - \mu)],$$

the quotient

$$\frac{\hat{f}_2}{(f_3 + f_3(\cdot\, + \mathrm{i}))^\wedge}$$

is a well defined function in $C^2(\mathbf{R})$ with compact support, hence it is the inverse Fourier transform of some $g \in L^1(\mathbf{R})$. Thus we have

$$\hat{g} \in C^2(\mathbf{R}),$$

$$\mathrm{supp}(\hat{g}) \subset [\mu - (\lambda - \mu),\ \lambda + (\lambda - \mu)],$$

$$\hat{f}_2(s) = \hat{g}(s)(f_3 + f_3(\cdot\, + \mathrm{i}))^\wedge(s) = \hat{g}(s)(1 + \mathrm{e}^s)\hat{f}_3(s), \quad s \in \mathbf{R}.$$

Since $y \in X^U([\mu, \lambda])$ and $\hat{f}_2$ is equal to 1 on a neighbourhood of $[\mu, \lambda]$, we have

$$y = U_{f_2}(y) = U_g U_{f_3 + f_3(\cdot\, + \mathrm{i})}(y).$$

Using (*) we get

$$y = U_g(x),$$

$$\|y\| \leqslant \|g\|_1 \|x\|.$$

Hence we need only an appropriate estimation of $\|g\|_1$.

By the inversion formula,

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

$$(it)^2 g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}''(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

so

$$g(t) = \frac{1}{2\pi} \frac{1}{1+t^2} \int_{-\infty}^{+\infty} (\hat{g}(s) - \hat{g}''(s))\, e^{-its} ds, \qquad t \in \mathbf{R}.$$

Hence

$$\|g\|_1 \leqslant \frac{1}{2} \|\hat{g} - \hat{g}''\|_1 .$$

Using the formula

$$\hat{g}(s) = \frac{1}{1+e^s} \frac{\hat{f}_2(s)}{\hat{f}_3(s)}, \qquad s \in \mathbf{R},$$

an elementary computation shows us that

$$\|\hat{g} - \hat{g}''\|_1 = \int_{\mu-(\lambda-\mu)}^{\lambda+(\lambda-\mu)} |\hat{g}(s) - \hat{g}''(s)| ds \leqslant$$

$$\leqslant 3(\lambda-\mu)\|\hat{g} - \hat{g}''\|_\infty \leqslant$$

$$\leqslant 3(\lambda-\mu)\left(\left\|\frac{\hat{f}_2}{\hat{f}_3}\right\|_\infty + \left\|\left(\frac{\hat{f}_2}{\hat{f}_3}\right)'\right\|_\infty + \left\|\left(\frac{\hat{f}_2}{\hat{f}_3}\right)''\right\|_\infty\right).$$

But

$$\operatorname{supp}(\hat{f}_2) \subset [\mu - (\lambda-\mu),\ \lambda + (\lambda-\mu)] \subset \{s \in \mathbf{R}\,;\, \hat{f}_3(s) = 1\},$$

so

$$\left\|\frac{\hat{f}_2}{\hat{f}_3}\right\|_\infty = \|\hat{f}_2\|_\infty = 1,$$

$$\left\|\left(\frac{\hat{f}_2}{\hat{f}_3}\right)'\right\|_\infty = \|\hat{f}_2'\|_\infty = \frac{4}{\lambda-\mu},$$

$$\left\|\left(\frac{\hat{f}_2}{\hat{f}_3}\right)''\right\|_\infty = \|\hat{f}_2''\|_\infty = \frac{32}{(\lambda-\mu)^2}.$$

We conclude that

$$\|\hat{g} - \hat{g}''\|_1 \leqslant 3(\lambda - \mu)\left(1 + \frac{4}{\lambda - \mu} + \frac{32}{(\lambda - \mu)^2}\right),$$

$$\|g_1\| \leqslant \frac{1}{2}\|\hat{g} - \hat{g}''\|_1 \leqslant 3(\lambda - \mu)\left(1 + \frac{2}{\lambda - \mu} + \frac{16}{(\lambda - \mu)^2}\right).$$ ▨

It is well known that U is *uniformly continuous*, i.e.,

$$\mathbf{R} \ni t \mapsto U_t \in B_{\mathscr{F}}(X)$$

is continuous with the norm topology on $B_{\mathscr{F}}(X)$, if and only if $\sigma(U)$ is compact, and if and only if $\sigma(U_{-i})$ is a compact subset of $(0, +\infty)$ (see [3], Theorem 5.2 and Corollary 5.7). We end the section by giving in this case an estimate for $\|(1+U_{-i})^{-1}\|$.

For a closed subset K of $\mathbf{R}$ we shall call the number of the elements of the set

$$\{k \in \mathbf{Z}\,;\ K \cap [k, k+1] \neq \emptyset\}$$

the *covering size* of K. We shall denote the covering size of K by

$$|K|_{\mathrm{cover}}\,.$$

COROLLARY 1.3. *Let $(X, \mathscr{F})$ be a dual pair of Banach spaces and U a uniformly continuous group of isometries in $B_{\mathscr{F}}(X)$. Then*

$$\|(1 + U_{-i})^{-1}\| \leqslant 4680|\sigma(U)|^2_{\mathrm{cover}}\,.$$

Proof. With

$$n = |\sigma(U)|_{\mathrm{cover}}\,,$$

there are integers

$$k_1 < k_2 < \ldots < k_n$$

such that

$$\sigma(U) \subset \bigcup_{j=1}^{n} [k_j, k_j + 1]\,.$$

With

$$C_1, \ldots, C_m$$

the connected components of $\bigcup_{j=1}^{n} [k_j, k_j + 1]$, each C_p is the union of a certain number of intervals $[k_j, k_j + 1]$, $\sum_{p=1}^{m} \mathrm{length}(C_p) = n$, and the sets

$$\left\{s \in \mathbf{R}\,;\ \mathrm{dist}(s, C_p) < \frac{1}{2}\right\}, \quad p = 1, \ldots, m$$

are mutually disjoint.

Keep for the moment $1 \leqslant p \leqslant m$ fixed. Then C_p is of the form

$$[a_p, b_p], \quad a_p, b_p \in \mathbf{Z}, \quad a_p < b_p\,.$$

Defining the continuous function $g_p\colon \mathbf{R} \to \mathbf{R}$ by

$$g_p(s) = \begin{cases} 0 & \text{for } s \in \left(-\infty, a_p - \frac{3}{8}\right] \\ 128 & \text{for } s = a_p - \frac{5}{16} \\ -128 & \text{for } s = a_p - \frac{3}{16} \\ 0 & \text{for } s \in \left[a_p - \frac{1}{8}, b_p + \frac{1}{8}\right] \\ -128 & \text{for } s = b_p + \frac{3}{16} \\ 128 & \text{for } s = b_p + \frac{5}{16} \\ 0 & \text{for } s \in \left[b_p + \frac{3}{8}, +\infty\right), \end{cases}$$

g_p is linear on each of the intervals

$$\left[a_p - \frac{3}{8}, a_p - \frac{5}{16}\right], \quad \left[a_p - \frac{5}{16}, a_p - \frac{3}{16}\right], \quad \left[a_p - \frac{3}{16}, a_p - \frac{1}{8}\right],$$

$$\left[b_p + \frac{1}{8}, b_p + \frac{3}{16}\right], \quad \left[b_p + \frac{3}{16}, b_p + \frac{5}{16}\right], \quad \left[b_p + \frac{5}{16}, b_p + \frac{3}{8}\right],$$

and then $h_p\colon \mathbf{R} \to \mathbf{R}$ by

$$h_p(s) = \int_{-\infty}^{s} \left(\int_{-\infty}^{u} g_p(v)\mathrm{d}v \right) \mathrm{d}u, \qquad s \in \mathbf{R},$$

we have

$$h_p \in C^2(\mathbf{R}),$$

$$0 \leqslant h_p \leqslant 1,$$

$$h_p(s) = 1 \quad \text{for } s \in \left[a_p - \frac{1}{8}, b_p + \frac{1}{8}\right],$$

$$\operatorname{supp}(h_p) \subset \left[a_p - \frac{3}{8}, b_p + \frac{3}{8}\right],$$

$$\|h_p''\|_\infty = 128.$$

Denoting by f_p the Fourier transform of h_p,

$$f_p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h_p(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

we have

$$f_p \in L^1(\mathbf{R}), \quad \hat{f}_p = h_p .$$

Moreover, since

$$(it)^2 f_p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h_p''(s)\, e^{-its} ds, \qquad t \in \mathbf{R},$$

we get successively

$$f_p(t) = \frac{1}{2\pi} \frac{1}{1+t^2} \int_{-\infty}^{+\infty} (h_p(s) - h_p''(s))\, e^{-its} ds, \qquad t \in \mathbf{R},$$

$$\|f_p\|_1 \leqslant \frac{1}{2} \|h_p - h_p''\|_1 \leqslant$$

$$\leqslant \frac{1}{2} (b_p - a_p + 1)(\|h_p\|_\infty + \|h_p''\|_\infty) \leqslant$$

$$\leqslant 65(b_p - a_p + 1) = 65(\text{length}(C_p) + 1).$$

Now let $x \in X$ be arbitrary. Since

$$\sum_{p=1}^{m} \hat{f}_p(s) = 1 \quad \text{for } s \in \bigcup_{p=1}^{m} \left(a_p - \frac{1}{8}, b_p + \frac{1}{8}\right) \supset \sigma(U),$$

we have

$$x = U_{\sum_{p=1}^{m} f_p}(x) = \sum_{p=1}^{m} U_{f_p}(x),$$

so

$$(1 + U_{-i})^{-1}(x) = \sum_{p=1}^{m} (1 + U_{-i})^{-1} U_{f_p}(x).$$

But for each $1 \leqslant p \leqslant m$,

$$U_{f_p}(x) \in X^U(\text{supp}(\hat{f}_p)) \subset X^U\left(\left[a_p - \frac{1}{2}, b_p + \frac{1}{2}\right]\right),$$

whence by Theorem 1.2 we obtain

$$\|(1 + U_{-i})^{-1} U_{f_p}(x)\| \leqslant 18(b_p - a_p + 1)\|U_{f_p}(x)\| \leqslant$$

$$\leqslant 18(\text{length}(C_p) + 1)\|f_p\|_1 \|x\| \leqslant$$

$$\leqslant 1170(\text{length}(C_p) + 1)^2 \|x\| .$$

Thus, we have

$$\begin{aligned}\|(1+U_{-i})^{-1}(x)\| &\leqslant \sum_{p=1}^{m} \|(1+U_{-i})^{-1}U_{f_p}(x)\| \leqslant \\ &\leqslant 1170 \sum_{p=1}^{m} (\mathrm{length}(C_p)+1)^2\|x\| \leqslant \\ &\leqslant 1170 \left(\sum_{p=1}^{m} \mathrm{length}(C_p)+m\right)^2 \|x\| \leqslant \\ &\leqslant 4680\, n^2\|x\|.\end{aligned}$$

We conclude that

$$\|(1+U_{-i})^{-1}\| \leqslant 4680\, n^2 = 4680|\sigma(U)|^2_{\mathrm{cover}}\,. \qquad ▨$$

2. COMPUTATIONS FOR C^*- AND W^*-DYNAMICAL SYSTEMS

We assume familiarity in handling C^*- and W^*-algebras. For their theory we refer to [21] and [19].

We recall that a *one-parameter C^*-dynamical system* is a pair (A, α) formed by a C^*-algebra A and a $\sigma(A, A^*)$-continuous, hence strongly continuous, one-parameter group α of $*$-automorphisms of A; a *one-parameter W^*-dynamical system* is a pair (M, α) consisting of a W^*-algebra M and a $\sigma(M, M_*)$-continuous, hence s^*-continuous, one-parameter group α of (automatically $\sigma(M, M_*)$-continuous) $*$-automorphisms of M (see [19], Section 7.4).

First we reduce the spectrum problem for one-parameter W^*-dynamical systems to that for one-parameter C^*-dynamical systems.

PROPOSITION 2.1. *Let (M, α) be a one-parameter W^*-dynamical system and consider the set*

$$A = \{x \in M\ ;\ \mathbf{R} \ni t \mapsto \alpha_t(x) \in M \text{ is norm continuous}\}.$$

Then A is a $\sigma(M, M_*)$-dense α-invariant C^*-subalgebra of M and, denoting by $\alpha|A$ the strongly continuous one-parameter group of $*$-automorphisms of A defined by

$$(\alpha|A)_t(a) = \alpha_t(a), \quad t \in \mathbf{R},\ a \in A,$$

we have

$$\sigma(\alpha|A) = \sigma(\alpha),$$

$$\sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i})\,.$$

Moreover, for each $\lambda \notin \sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i})$ we have

$$(\lambda - (\alpha|A)_{-i})^{-1} = (\lambda - \alpha_{-i})^{-1}|A,$$

$$\|(\lambda - (\alpha|A)_{-i})^{-1}\| = \|(\lambda - \alpha_{-i})^{-1}\|.$$

Proof. That A is a $\sigma(M, M^*)$-dense α-invariant C^*-subalgebra of M follows by [19], Lemma 7.5.1.

For each $f \in L^1(\mathbf{R})$ the operator α_f is $\sigma(M, M_*)$-continuous and

$$\alpha_f | A = (\alpha | A)_f ,$$

so by the $\sigma(M, M_*)$-density of A in M,

$$(\alpha | A)_f = 0 \Leftrightarrow \alpha_f = 0.$$

It follows that

$$(*) \qquad \sigma(\alpha | A) = \sigma(\alpha).$$

Next we prove that

$$(**) \qquad \sigma((\alpha | A)_{-i}) = \mathbf{C} \Leftrightarrow \sigma(\alpha_{-i}) = \mathbf{C}.$$

By [3], Theorem 3.6, $(**)$ is equivalent to

$$-1 \notin \sigma((\alpha | A)_{-i}) \Leftrightarrow -1 \notin \sigma(\alpha_{-i}).$$

Let $-1 \notin \sigma((\alpha | A)_{-i})$ and put

$$c = \|(1 + (\alpha | A)_{-i})^{-1}\|.$$

Take an arbitrary $x \in M$. By the Kaplansky density theorem there exists a net $(a_\iota)_{\iota \in \mathfrak{J}}$ in A such that

$$\|a_\iota\| \leqslant \|x\|, \quad \iota \in \mathfrak{J},$$

$$a_\iota \to x \quad \text{in} \quad \sigma(M, M_*).$$

Writing

$$b_\iota = (1 + (\alpha | A)_{-i})^{-1}(a_\iota), \quad \iota \in \mathfrak{J},$$

we have

$$\|b_\iota\| \leqslant c\|a_\iota\| \leqslant c\|x\|, \quad \iota \in \mathfrak{J},$$

$$a_\iota = b_\iota + \alpha_{-i}(b_\iota), \quad \iota \in \mathfrak{J}.$$

But by the Alaoglu theorem there exists a subnet $(b_{\iota_\varkappa})_{\varkappa \in \mathfrak{K}}$ of $(b_\iota)_{\iota \in \mathfrak{J}}$ which converges in the $\sigma(M, M_*)$-topology to some $y \in M$. Then, after passage to this subnet,

$$\alpha_{-i}(b_\iota) = a_\iota - b_\iota \to x - y \quad \text{in } \sigma(M, M_*),$$

and since the operator α_{-i} is $\sigma(M, M_*)$-closed, it follows that $y \in \mathscr{D}_{\alpha_{-i}}$ and

$$\alpha_{-i}(y) = x - y,$$

$$x = (1 + \alpha_{-i})(y).$$

Since x was arbitrary, we conclude that $1 + \alpha_{-i}$ is surjective. But by [3], Lemma 3.1 it is also injective, so $1 \notin \sigma(\alpha_{-i})$.

Conversely, let us assume that $-1 \notin \sigma(\alpha_{-i})$. Since the analyticity of M-valued mappings with a complex variable is the same in the $\sigma(M, M_*)$- and in the norm topology, we have

$$\bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(\alpha|A)_{ki}} = \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{\alpha_{ki}} .$$

By [3], Lemma 3.1, $1 + (\alpha|A)_{-i}$ is injective and by [3], Theorem 3.2,

$$\bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(\alpha|A)_{ki}} \subset \mathscr{D}_{(1+(\alpha|A)_{-i})^{-1}} ,$$

so

$$(1 + (\alpha|A)_{-i})^{-1}| \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(\alpha|A)_{ki}} = (1 + \alpha_{-i})^{-1}| \bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{\alpha_{ki}}$$

is bounded. But $(1 + (\alpha|A)_{-i})^{-1}$ is closed and $\bigcap_{k=-\infty}^{+\infty} \mathscr{D}_{(\alpha|A)_{ki}}$ by [3], Lemma 2.2 is norm dense in A; hence

$$\mathscr{D}_{(1+(\alpha|A)_{-i})^{-1}} = A .$$

Therefore $-1 \notin \sigma((\alpha|A)_{-i})$.

We conclude that (**) holds. Using (*), (**), and [25], Theorem 4.3 we obtain

$$\sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i}) .$$

Finally, let $\lambda \notin \sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i})$. Then the operators

$$(\lambda - (\alpha|A)_{-i})^{-1}, \quad (\lambda - \alpha_{-i})^{-1}$$

coincide on

$$\bigcap_{k=-\infty}^{\infty} \mathscr{D}_{(\alpha|A)_{ki}} = \bigcap_{k=-\infty}^{\infty} \mathscr{D}_{\alpha_{ki}} .$$

But this linear subspace of A is norm dense in A, so it follows that

$$(\lambda - (\alpha|A)_{-i})^{-1} = (\lambda - \alpha_{-i})^{-1}|A.$$

By the Kaplansky density theorem and the $\sigma(M, M_*)$-continuity of $(\lambda - \alpha_{-i})^{-1}$ we get also

$$\|(\lambda - (\alpha|A)_{-i})^{-1}\| = \|(\lambda - \alpha_{-i})^{-1}\|.$$ ▨

Next we reduce the spectrum problem for one-parameter C^*-dynamical systems to that for one-parameter W^*-dynamical systems:

PROPOSITION 2.2. *Let (A, α) be an one-parameter C^*-dynamical system. Then*

$$\sigma(\alpha_{-i}) \neq \mathbf{C}$$

if and only if the formula

$$(\alpha^{**})_t(x) = (\alpha_t)^{**}(x), \quad t \in \mathbf{R},\ x \in A^{**}$$

defines a $\sigma(A^{**}, A^*)$*-continuous one-parameter group* α^{**} *of* $*$*-automorphisms of the* W^**-algebra* A^{**} *and*

$$\sigma((\alpha^{**})_{-i}) \neq \mathbf{C}.$$

Moreover, in this case

$$\sigma(\alpha^{**}) = \sigma(\alpha),$$

$$\sigma((\alpha^{**})_{-i}) = \sigma(\alpha_{-i}),$$

and for each $\lambda \notin \sigma((\alpha^{**})_{-i}) = \sigma(\alpha_{-i})$ *we have*

$$(\lambda - (\alpha^{**})_{-i})^{-1}|A = (\lambda - \alpha_{-i})^{-1},$$

$$\|(\lambda - (\alpha^{**})_{-i})^{-1}\| = \|(\lambda - \alpha_{-i})^{-1}\|.$$

Proof. Let us first assume that $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then by Theorem 1.1 there are $-\infty < \mu < \lambda < +\infty$ such that, with α^{A^*} denoted simply by α^*,

$$A^* = (A^*)^{\alpha^*}((-\infty, \lambda]) + (A^*)^{\alpha^*}([\mu, +\infty)).$$

But by [24], Theorem 4.1 (which is a slight modification of [1], Theorem 5.3), for each

$$\varphi \in (A^*)^{\alpha^*}((-\infty, \lambda]) \quad \text{or} \quad \varphi \in (A^*)^{\alpha^*}([\mu, +\infty))$$

the mapping

$$\mathbf{R} \ni t \mapsto \alpha_t^*(\varphi) \in A^*$$

is norm continuous. Therefore α^* is a strongly continuous one-parameter group, so one can consider its dual group α^{**} with respect to the natural duality between A^* and A^{**}. Since

$$\sigma(\alpha^{**}) = \sigma(\alpha^*) = \sigma(\alpha),$$

$$\sigma((\alpha^{**})_{-i}) = \sigma((\alpha^*)_{-i}) = \sigma(\alpha_{-i}),$$

we have in particular $\sigma((\alpha^{**})_{-i}) \neq \mathbf{C}$.

Conversely, let us assume that α^{**} is $\sigma(A^{**}, A^*)$-continuous and $\sigma((\alpha^{**})_{-i}) \neq \mathbf{C}$. Put

$$A_n = \{x \in A^{**}\ ;\ \mathbf{R} \ni t \mapsto (\alpha^{**})_t(x) \in A^{**} \text{ is norm continuous}\};$$

by Proposition 2.1 A_n is an α^{**}-invariant C^*-subalgebra of A^{**} and $(A_n, \alpha^{**}|A_n)$ is a one-parameter C^*-dynamical system with

$$\sigma((\alpha^{**}|A_n)_{-i}) = \sigma((\alpha^{**})_{-i}) \neq \mathbf{C}.$$

But A is an α^{**}-invariant C^*-subalgebra of A_n and $\alpha = \alpha^{**}|A$, so [25], Lemma 4.1 yields

$$\sigma(\alpha_{-i}) \subset \sigma((\alpha^{**}|A_n)_{-i}) \neq \mathbf{C},$$

$$\sigma(\alpha_{-i}) \neq \mathbf{C}.$$

Finally, if $\lambda \notin \sigma((\alpha^{**})_{-i}) = \sigma(\alpha_{-i})$ then

$$(\lambda - (\alpha^{**})_{-i})^{-1}|A = (\lambda - \alpha_{-i})^{-1},$$

and the last equality of the statement of Proposition 2.2 follows. ▨

Using Propositions 2.1 and 2.2 we will be able to reduce the spectrum problem for one-parameter C^*- and W^*-dynamical systems to the treatment of certain particular cases. The next criterion, whose proof is based on an idea already used in the proof of [25], Theorem 5.1, enables us to settle the spectral problem in these particular cases.

THEOREM 2.3. *Let (M, α) be a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then for each $\varepsilon > 0$ and each $x \in M^\alpha((-\infty, -\varepsilon])$ with $\|x\| < 1$ we have*

$$\|\ln(1 - x)\| \leqslant \pi(1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^2 \|(1 + \alpha_{-i})^{-1}\| + 40\pi\left(1 + \frac{16}{\varepsilon}\right)^5,$$

where $\ln\colon \mathbf{C}\setminus(-\infty, 0] \to \mathbf{C}$ *is the analytic function defined by*

$$\ln(r\,e^{i\theta}) = \ln r + i\theta, \quad r > 0,\ |\theta| < \pi.$$

Proof. Let $\varepsilon > 0$. By Theorem 1.1 there is a bounded linear projection Q_ε of $M^\alpha((-\infty, -\varepsilon] \cup [\varepsilon, +\infty))$ onto $M^\alpha((-\infty, -\varepsilon])$ with kernel $M^\alpha([\varepsilon, +\infty))$ and such that

$$\|Q_\varepsilon\| \leqslant (1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^2 \|(1 + \alpha_{-i})^{-1}\| + 40\left(1 + \frac{16}{\varepsilon}\right)^5.$$

Now let $x \in M^\alpha((-\infty, -\varepsilon])$ with $0 \neq \|x\| < 1$. Then the formulas

$$f_+(z) = \frac{\pi}{2} + i\ln(1 - \|x\|z),$$

$$f_-(z) = \frac{\pi}{2} - i\ln(1 - \|x\|z)$$

define analytic functions f_+, f_- on

$$\left\{z \in \mathbf{C}\,;\ |z| < \frac{1}{\|x\|}\right\} \supset \{z \in \mathbf{C}\,;\ |z| \leqslant 1\}.$$

Since

$$\operatorname{Re} f_{\pm}(z) \geqslant 0 \quad \text{for all } |z| < \frac{1}{\|x\|},$$

applying the von Neumann inequality ([20], Section 153, Theorem B or Appendix, §4) we get successively

$$\operatorname{Re} f_{\pm}\left(\frac{1}{\|x\|}\, x\right) \geqslant 0,$$

$$\frac{\pi}{2} \pm \operatorname{Im}\ln(1 - x) \geqslant 0,$$

$$\|\operatorname{Im}\ln(1 - x)\| \leqslant \frac{\pi}{2}.$$

But by known properties of the spectral subspaces of α we have

$$\ln(1 - x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k \in M^{\alpha}((-\infty, -\varepsilon]),$$

so

$$[\ln(1 - x)]^* \in M^{\alpha}([\varepsilon, +\infty)).$$

It follows that

$$\ln(1 - x) = Q_{\varepsilon}(\ln(1 - x) - [\ln(1 - x)]^*) =$$

$$= 2\mathrm{i} Q_{\varepsilon}(\operatorname{Im}\ln(1 - x)),$$

$$\|\ln(1 - x)\| \leqslant 2\|Q_{\varepsilon}\|\,\|\operatorname{Im}\ln(1 - x)\| \leqslant$$

$$\leqslant \pi\|Q_{\varepsilon}\| \leqslant$$

$$\leqslant \pi(1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^2 \|(1 + \alpha_{-\mathrm{i}})^{-1}\| + 40\pi\left(1 + \frac{16}{\varepsilon}\right)^5. \qquad ▨$$

The next statement is an immediate consequence of [25], Theorem 5.1, but for completeness we give it here with proof:

COROLLARY 2.4. *Let (M, α) be a one-parameter W^*-dynamical system with $\sigma(\alpha_{-\mathrm{i}}) \neq \mathbf{C}$ and Z the centre of M. Then*

$$\alpha_t(z) = z, \quad z \in Z,\ t \in \mathbf{R}.$$

Proof. Each α_t leaves Z invariant, so we may define a W^*-dynamical system $(Z, \alpha|Z)$ by

$$(\alpha|Z)_t(z) = \alpha_t(z), \quad t \in \mathbf{R},\ z \in Z.$$

By [25], Lemma 4.1 we have

$$\sigma((\alpha|Z)_{-i}) \neq \mathbf{C}.$$

Let us assume that $\alpha|Z$ does not act identically. Then $\sigma(\alpha|Z) \neq \{0\}$, and since $\sigma(\alpha|Z) = -\sigma(\alpha|Z)$, we have

$$\sigma(\alpha|Z) \cap (-\infty, 0) \neq \emptyset.$$

Hence there exists $\varepsilon > 0$ such that

$$Z^{\alpha|Z}((-\infty, -\varepsilon]) \neq \{0\}.$$

By Theorem 2.3,

$$\|\ln(1 - z)\| \leqslant c_\varepsilon = \pi(1 + 2\varepsilon)\left(1 + \frac{16}{\varepsilon}\right)^3 \|(1 + (\alpha|Z)_{-i})^{-1}\| + 40\pi\left(1 + \frac{16}{\varepsilon}\right)^5$$

for all $z \in Z^{\alpha|Z}((-\infty, -\varepsilon])$ with $\|z\| < 1$.

Let $z_0 \in Z^{\alpha|Z}((-\infty, -\varepsilon])$ with $\|z_0\| = 1$. By the Gelfand representation theory there exists a nonzero complex homomorphism χ_0 of Z with $|\chi_0(z_0)| = 1$. Replacing z_0 with $\overline{\chi_0(z_0)}z_0$ we may assume without loss of generality that

$$\chi_0(z_0) = 1.$$

Now for each $0 < \delta < 1$ we have

$$\delta z_0 \in Z^{\alpha|Z}((-\infty, -\varepsilon]), \quad \|\delta z_0\| < 1,$$

so

$$\ln\frac{1}{1-\delta} = |\ln(1 - \delta)| =$$

$$= |\ln(1 - \delta\chi_0(z_0))| =$$

$$= |\chi_0(\ln(1 - \delta z_0))| \leqslant$$

$$\leqslant \|\ln(1 - \delta z_0)\| \leqslant c_\varepsilon,$$

which is not possible. Thus our assumption that $\alpha|Z$ does not act identically is false. ▨

For C^*-dynamical systems we shall need the following consequence of the above corollary.

COROLLARY 2.5. *Let* (A, α) *be a one-parameter* C^**-dynamical system with* $\sigma(\alpha_{-i}) \neq \mathbf{C}$. *Then for each closed two-sided ideal* I *in* A *we have*

$$\alpha_t(I) = I, \quad t \in \mathbf{R},$$

and the formula

$$(\alpha^I)_t(x/I) = \alpha_t(x)/I, \quad t \in \mathbf{R},\ x/I \in A/I,$$

defines a strongly continuous one-parameter group α^I *of* $*$*-automorphisms of the quotient* C^**-algebra* A/I, *satisfying*

$$\sigma((\alpha^I)_{-i}) \subset \sigma(\alpha_{-i}).$$

Proof. Let α^{**} be as in Proposition 2.2. Then by Proposition 2.2,

$$\sigma((\alpha^{**})_{-i}) = \sigma(\alpha_{-i}) \neq \mathbf{C},$$

so by Corollary 2.4 we have

$$(\alpha^{**})_t(z) = z, \quad t \in \mathbf{R},$$

for all central elements z of A^{**}.

But the $\sigma(A^{**}, A^*)$-closure of I in A^{**} is a two-sided ideal in A^{**}, so by [21], Proposition 1.10.5 it is of the form $A^{**}p$, where p is a central projection of A^{**}. It follows that for all $t \in \mathbf{R}$,

$$(\alpha^{**})_t(A^{**}p) = A^{**}p,$$

so

$$\alpha_t(I) = \alpha_t(A \cap (A^{**}p)) = A \cap (A^{**}p) = I.$$

Obviously, the formula in the statement defines a strongly continuous one-parameter group α^I of $*$-automorphisms of A/I, and using [25], Lemma 4.1, it is easy to see that

$$\sigma((\alpha^I)_{-i}) \subset \sigma(\alpha_{-i}).$$ ▨

Next we apply Theorem 2.3 to inner W^*-dynamical systems.

PROPOSITION 2.6. *Let* (M, α) *be a one-parameter* W^**-dynamical system with* $\sigma(\alpha_{-i}) \neq \mathbf{C}$ *and such that for some* s*-continuous one-parameter group*

$$u : \mathbf{R} \to \text{unitaries in } M$$

we have

$$\alpha_t(x) = u_t x u_t^*, \quad t \in \mathbf{R} \ \ x \in M.$$

Then there exists a family $(p_\iota)_{\iota\in\mathfrak{J}}$ of mutually orthogonal central projections in M with $\sum_{\iota\in\mathfrak{J}} p_\iota = 1$ such that for each $\iota\in\mathfrak{J}$ the formula

$$(\alpha|Mp_\iota)_t(x) = \alpha_t(x), \quad t\in\mathbf{R},\ x\in Mp_\iota$$

defines a uniformly continuous one-parameter group $\alpha|Mp_\iota$ of $*$-automorphisms of Mp_ι and

$$|\sigma(\alpha|Mp_\iota)|_{\text{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Proof. First we note that by Corollary 2.4 we have for each central projection p in M,

$$\alpha_t(Mp) = Mp, \quad t\in\mathbf{R},$$

so the formula

$$(\alpha|Mp)_t(x) = \alpha_t(x), \quad t\in\mathbf{R},\ x\in Mp,$$

defines a W^*-dynamical system $(Mp, \alpha|Mp)$. Therefore, by Zorn's lemma, it is enough to prove that for any nonzero central projection p_0 in M there is a nonzero central projection $p\leqslant p_0$ in M such that $\sigma(\alpha|Mp)$ is compact and satisfies the inequality

$$|\sigma(\alpha|Mp)|_{\text{cover}} \leqslant 18\exp(2c),$$

where

$$c = 3\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 2\cdot 10^8.$$

To do this, let us assume that there is a central projection $p_0\neq 0$ in M such that for every central projection $0\neq p\leqslant p_0$ in M we have

$$|\sigma(\alpha|Mp)|_{\text{cover}} > 18\exp(2c),$$

and look for a contradiction.

We may consider M as a von Neumann algebra in a complex Hilbert space H, that is, a weak-operator-closed self-adjoint subalgebra of $B(H)$ containing the identity operator on H ([21], Theorem 1.16.7). By the Stone representation theorem ([20], Section 137) there exists a self-adjoint operator a in H such that

$$u_t = \exp(ita), \quad t\in\mathbf{R}.$$

Moreover, since the u_t's belong to M, a is affiliated with M. Now by [7], 6.19 (iii),

$$\sigma(\alpha|Mp) \subset \sigma(ap) - \sigma(ap), \quad 0\neq p\leqslant p_0 \text{ a central projection in } M;$$

hence by a combinatorial computation,

$$|\sigma(ap)|_{\text{cover}} > 3\exp c,\ 0\neq p\leqslant p_0 \text{ a central projection in } M.$$

Denote by χ_k the characteristic function of $[k, k+1]$ and by e_k the corresponding spectral projection of a:

$$e_k = \chi_k(a) \in M.$$

The above condition means that

$$\operatorname{card}\{k \in \mathbf{Z} \ ; \ e_k p \neq 0\} > 3 \exp c, \ 0 \neq p \leqslant p_0 \text{ a central projection in } M.$$

We claim that, with n denoting the integral part of $\exp c$, there exist $k_1, \ldots$ $\ldots, k_n \in \mathbf{Z}$, with

$$|k_j - k_m| \geqslant 2 \quad \text{for } 1 \leqslant j, m \leqslant n \text{ and } j \neq m,$$

and nonzero equivalent projections

$$f_1, \ldots, f_n \in M,$$

such that

$$f_j \leqslant e_{k_j}, \quad 1 \leqslant j \leqslant n.$$

Indeed, there is $k_1 \in \mathbf{Z}$ with

$$0 \neq f_{1,1} = e_{k_1} p_0.$$

If $n \geqslant 2$ then, with

$$0 \neq p_1 = \text{central support of } f_{1,1} \text{ in } M \leqslant p_0,$$

using the inequality

$$\operatorname{card}\{k \in \mathbf{Z} \ ; \ e_k p_1 \neq 0\} > 3n \geqslant 6,$$

we get $k_2 \in \mathbf{Z}$ with

$$|k_2 - k_1| \geqslant 2,$$

$$e_{k_2} p_1 \neq 0.$$

Since

$$\text{central support of } e_{k_2} p_1 \text{ in } M \leqslant p_1 =$$

$$= \text{central support of } f_{1,1} \text{ in } M,$$

by the comparability theorem ([21], Theorem 2.1.3) there exist nonzero equivalent projections

$$f_{2,1}, f_{2,2} \in M,$$

$$f_{2,1} \leqslant f_{1,1}, \quad f_{2,2} \leqslant e_{k_2} p_1.$$

Next, if $n \geqslant 3$ then, with

$$0 \neq p_2 = \text{common central support of } f_{2,1} \sim f_{2,2} \text{ in } M \leqslant p_1,$$

using the inequality

$$\operatorname{card}\{k \in \mathbf{Z}\ ;\ e_k p_2 \neq 0\} > 3n \geqslant 9,$$

we get $k_3 \in \mathbf{Z}$ with

$$|k_3 - k_1| \geqslant 2, \quad |k_3 - k_2| \geqslant 2,$$
$$e_{k_3} p_2 \neq 0.$$

Since

$$\text{central support of } e_{k_3} p_2 \text{ in } M \leqslant p_2 =$$
$$= \text{common central support of } f_{2,1} \sim f_{2,2} \text{ in } M,$$

by the comparability theorem there exist nonzero equivalent projections

$$f_{3,1},\, f_{3,2},\, f_{3,3} \in M,$$

$$f_{3,1} \leqslant f_{2,1}, \quad f_{3,2} \leqslant f_{2,2}, \quad f_{3,3} \leqslant e_{k_3} p_2 .$$

By induction we get $k_1, \ldots, k_n \in \mathbf{Z}$ with

$$|k_j - k_m| \geqslant 2 \quad \text{for } 1 \leqslant j, m \leqslant n \text{ and } j \neq m,$$

and projections in M as follows:

$$\begin{array}{ccccc}
e_{k_1} & & & & \\
\geqslant & & & & \\
f_{1,1} & & e_{k_2} & & \\
\geqslant & & \geqslant & & \\
f_{2,1} & \sim & f_{2,2} & & \\
\geqslant & & \geqslant & & \\
\vdots & & \vdots & & e_{k_n} \\
\geqslant & & \geqslant & & \geqslant \\
0 \neq f_{n,1} & \sim & f_{n,2} & \sim \ \ldots \ \sim & f_{n,n} .
\end{array}$$

Thus the claimed statement holds with

$$f_j = f_{n,j}, \quad 1 \leqslant j \leqslant n.$$

We may assume, reordering the k_j's if necessary, that

$$k_1 < k_2 < \ldots < k_n ,$$

so

$$k_{j+1} - k_j \geqslant 2 \quad \text{for } j = 1, \ldots, n-1.$$

Let, for any $1 \leqslant j \leqslant n-1$, $u_j \in M$ be a partial isometry with

$$u_j^* u_j = f_{j+1}, \quad u_j u_j^* = f_j,$$

and put

$$u = \sum_{j=1}^{n-1} u_j \in M.$$

For each $1 \leqslant j \leqslant n-1$ and $m \geqslant 1$,

$$\exp(ma)\, u_j \exp(-ma) = \exp(ma) e_{k_j} u_j e_{k_{j+1}} \exp(-ma)$$

is defined on the domain of $\exp(-ma)$ and satisfies

$$\|\exp(ma) u_j \exp(-ma)\| \leqslant \|\exp(ma) e_{k_j}\| \, \|\exp(-ma) e_{k_{j+1}}\| \leqslant$$

$$\leqslant e^{m(k_j+1)} e^{-mk_{j+1}} =$$

$$= e^{-m(k_{j+1}-k_j-1)} \leqslant$$

$$\leqslant e^{-m},$$

so by [3], Theorem 6.2,

$$u_j \in \mathscr{D}_{\alpha_{-mi}} \quad \text{and} \quad \|\alpha_{-mi}(u_j)\| \leqslant e^{-m}.$$

Hence for any $1 \leqslant j \leqslant n-1$,

$$u_j \in \bigcap_{m=0}^{\infty} \mathscr{D}_{\alpha_{-mi}} \quad \text{and} \quad \overline{\lim_{m\to\infty}} \|\alpha_{-mi}(u_j)\|^{1/m} \leqslant e^{-1},$$

that is,

$$u_j \in M^{\alpha}((-\infty, -1]).$$

Thus

$$u \in M^{\alpha}((-\infty, -1]).$$

Using Theorem 2.3 we obtain for every $0 < \delta < 1$,

$$\|\ln(1-\delta u)\| \leqslant 867\pi \|(1+\alpha_{-i})^{-1}\| + 56{,}794{,}280\pi.$$

But $u^n = 0$, so

$$\ln(1-\delta u) = \delta u + \frac{\delta^2}{2} u^2 + \ldots + \frac{\delta^{n-1}}{n-1} u^{n-1}, \quad 0 < \delta < 1,$$

and letting δ increase to 1 in the above inequality, we get

$$\left\| u + \frac{1}{2} u^2 + \ldots + \frac{1}{n-1} u^{n-1} \right\| \leqslant 867\pi \|(1+\alpha_{-i})^{-1}\| + 56{,}794{,}280\pi.$$

On the other hand, choosing

$$\xi_n \in f_n H \quad \text{with } \|\xi_n\| = 1$$

and defining

$$\xi_j = u_j u_{j+1} \dots u_{n-1}(\xi_n), \quad 1 \leqslant j \leqslant n-1,$$

we have

$$\xi_j \in f_j H \quad \text{and} \quad \|\xi_j\| = 1,\ 1 \leqslant j \leqslant n,$$

so the norm of

$$\xi = \frac{1}{\sqrt{n}}(\xi_1 + \dots + \xi_n)$$

is equal to 1. Hence

$$\left\| u\xi + \frac{1}{2}u^2\xi + \dots + \frac{1}{n-1}u^{n-1}\xi \right\| \leqslant 867\pi \|(1+\alpha_{-1})^{-1}\| + 56{,}794{,}280\pi.$$

But

$$u\xi = \frac{1}{\sqrt{n}}(\xi_1 + \dots + \xi_{n-2} + \xi_{n-1})$$

$$u^2\xi = \frac{1}{\sqrt{n}}(\xi_1 + \dots + \xi_{n-2}),$$

$$\dots\dots\dots\dots\dots\dots$$

$$u^{n-1}\xi = \frac{1}{\sqrt{n}}\xi_1,$$

so we get, successively,

$$\left\| u\xi + \frac{1}{2}u^2\xi + \dots + \frac{1}{n-1}u^{n-1}\xi \right\|^2 =$$

$$= \frac{1}{n}\left\|\left(1 + \frac{1}{2} + \dots + \frac{1}{n-1}\right)\xi_1 + \dots + \left(1 + \frac{1}{2}\right)\xi_{n-2} + \xi_{n-1}\right\|^2 =$$

$$= \frac{1}{n}\left[\left(1 + \frac{1}{2} + \dots + \frac{1}{n-1}\right)^2 + \dots + \left(1 + \frac{1}{2}\right)^2 + 1\right] \geqslant$$

$$\geqslant \frac{1}{n}\cdot[(\ln n)^2 + \dots + (\ln 2)^2 + (\ln 1)^2] \geqslant$$

$$\geqslant \frac{1}{n^2}\left(\sum_{j=1}^{n} \ln j\right)^2 \geqslant \left(\ln\frac{n}{e}\right)^2,$$

$$\left\| u\xi + \frac{1}{2}\cdot u^2\xi + \dots + \frac{1}{n-1}u^{n-1}\xi \right\| \geqslant \ln\frac{n}{e} = (\ln n) - 1.$$

It follows that

$$\ln n \leqslant 867\pi \|(1 + \alpha_{-i})^{-1}\| + 56{,}794{,}280\pi + 1 \leqslant c - \ln 2,$$

so

$$2n \leqslant \exp c,$$

which is not possible because n is the integral part of $\exp c \geqslant 1$. ▨

Now we are able to solve the spectrum problem for W^*-dynamical systems in type I factors with separable predual.

COROLLARY 2.7. *Let H be a separable complex Hilbert space and $(B(H), \alpha)$ a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is uniformly continuous and*

$$|\sigma(\alpha)|_{\text{cover}} \leqslant 18 \exp(6 \cdot 10^3 \|(1 + \alpha_{-i})^{-1}\| + 4 \cdot 10^8).$$

Proof. By a well known result of V. Bargmann there exists a strongly continuous one-parameter group of unitaries on H such that

$$\alpha_t(x) = u_t x u_t^*, \quad t \in \mathbf{R}, \ x \in B(H)$$

(see for example [16], Remark (4.14) or [17], Section 4.1). Hence our statement follows immediately from Proposition 2.6. ▨

The corresponding statement for C^*-dynamical systems is

COROLLARY 2.8. *Let A be a separable prime C^*-algebra and (A, α) a one--parameter C^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is uniformly continuous and*

$$|\sigma(\alpha)|_{\text{cover}} \leqslant 18 \exp(6 \cdot 10^3 \|(1 + \alpha_{-i})^{-1}\| + 4 \cdot 10^8).$$

Proof. By a theorem of Dixmier (see for example [19], Proposition 4.3.6) A is primitive, so it has a faithful irreducible $*$-representation $\pi : A \to B(H)$. Since A is separable, H will be separable.

On the other hand, by Proposition 2.2 α can be extended to a $\sigma(A^{**}, A^*)$--continuous one-parameter group α^{**} of $*$-automorphisms of A^{**} with

$$\sigma((\alpha^{**})_{-i}) = \sigma(\alpha_{-i}) \neq \mathbf{C}$$

and

$$\|(1 + (\alpha^{**})_{-i})^{-1}\| = \|(1 + \alpha_{-i})^{-1}\|.$$

But π can be extended to a normal $*$-representation $\tilde{\pi} : A^{**} \to B(H)$, and there is a central projection p of A^{**} such that

$$\ker(\tilde{\pi}) = A^{**}(1 - p),$$

$$\tilde{\pi} | A^{**}p : A^{**}p \to B(H) \quad \text{is a } *\text{-isomorphism}$$

(see the proof of [21], Proposition 1.16.2). By Corollary 2.4,

$$(\alpha^{**})_t(p) = p, \quad t \in \mathbf{R},$$

so there exists a one-parameter W^*-dynamical system $(B(H), \beta)$ with

$$\tilde{\pi} \circ (\alpha^{**})_t = \beta_t \circ \tilde{\pi}, \quad t \in \mathbf{R}.$$

Using [25], Lemma 4.1, it is easy to see that

$$\sigma(\beta_{-i}) \subset \sigma((\alpha^{**})_{-i}),$$

so

$$\sigma(\beta_{-i}) \neq \mathbf{C},$$

and

$$\|(1 + \beta_{-i})^{-1}\| \leqslant \|(1 + (\alpha^{**})_{-i})^{-1}\| = \|(1 + \alpha_{-i})^{-1}\| .$$

Hence by Corollary 2.7, β is uniformly continuous and

$$|\sigma(\beta)|_{\text{cover}} \leqslant 18 \exp(6 \cdot 10^3 \|(1 + \alpha_{-i})^{-1}\| + 4 \cdot 10^8).$$

Finally, since π is injective and

$$\pi \circ \alpha_t = \beta_t \circ \pi, \quad t \in \mathbf{R},$$

we have $\sigma(\alpha) \subset \sigma(\beta)$, so the above inequality yields the required one. ∎

3. THE SPECTRUM PROBLEM FOR C^*-DYNAMICAL SYSTEMS

In this paragraph we show how Corollaries 2.5 and 2.8 lead to the description of the one-parameter C^*-dynamical systems whose analytic generator has positive spectrum.

First we extend Corollary 2.8 to arbitrary prime C^*-algebras, eliminating the separability assumption. To do this, we need the prime C^*-algebra version of a result of Blackadar concerning simple C^*-algebras ([2], Proposition 2.2):

PROPOSITION 3.1. *Let A be a prime C^*-algebra and let B_0 be a separable C^*-subalgebra of A. Then there exists a separable prime C^*-subalgebra B of A containing B_0.*

Proof. Since B_0 is separable, its primitive spectrum $\mathrm{Prim}(B_0)$ is second countable (see [9], Proposition 3.3.4 or [19], Corollary 4.3.4). Choosing a countable basis for the topology of $\mathrm{Prim}(B_0)$, we enumerate the disjoint pairs of nonempty open sets in this basis as

$$p_1^0, p_2^0, \ldots .$$

Let us denote by I_k^0, J_k^0 the orthogonal nonzero closed two-sided ideals of B_0 whose spectra are the two open sets in p_k^0 (see [9], 3.2 or [19], 4.1).

Since I_1^0, J_1^0 are nonzero and A is prime, we have

$$I_1^0 A J_1^0 \neq \{0\}.$$

Choose a separable C^*-subalgebra B_1 of A such that $B_0 \subset B_1$ and

$$I_1^0 B_1 J_1^0 \neq \{0\}.$$

Next, choose a countable basis for the topology of $\mathrm{Prim}(B_1)$ and enumerate the disjoint pairs of nonempty open sets in this basis as

$$p_1^1, p_2^1, \ldots .$$

Let I_k^1, J_k^1 be the orthogonal nonzero closed two-sided ideals of B_1 with spectra the two open sets in p_k^1.

Again, since I_2^0, J_2^0, respectively, I_1^1, J_1^1 are nonzero and A is prime, we have

$$I_2^0 A J_2^0 \neq \{0\}, \quad I_1^1 A J_1^1 \neq \{0\}.$$

Choose a separable C^*-subalgebra B_2 of A such that $B_1 \subset B_2$ and

$$I_2^0 B_2 J_2^0 \neq \{0\}, \quad I_1^1 B_2 J_1^1 \neq \{0\}.$$

Continue in this way

1) to construct an increasing sequence

$$B_0 \subset B_1 \subset B_2 \subset \ldots$$

of separable C^*-subalgebras of A,

2) to choose a countable basis for the topology of each $\mathrm{Prim}(B_n)$, and

3) to enumerate the disjoint pairs of nonempty open sets in the chosen basis of $\mathrm{Prim}(B_n)$ as

$$p_1^n, p_2^n, \ldots$$

so that, denoting by I_k^n, J_k^n the orthogonal nonzero closed two-sided ideals of B_n with spectra the two open sets in p_k^n, we have

$$I_k^j B_n J_k^j \neq \{0\}, \quad n \geqslant 1,\ 0 \leqslant j \leqslant n-1,\ 1 \leqslant k \leqslant n-j.$$

Set the closure of the union $\bigcup_{n \geqslant 1} B_n$ equal to B. Then B is a separable C^*-subalgebra of A containing B_0, and

$$(*) \qquad I_k^j B J_k^j \neq \{0\}, \quad j \geqslant 0,\ k \geqslant 1.$$

We complete the proof by showing that B is prime.

Let us assume that B is not prime. Then there are two nonzero closed two-sided ideals I and J of B with

$$IJ = \{0\}.$$

Choose $j \geq 0$ with

$$I \cap B_j \neq \{0\}, \quad J \cap B_j \neq \{0\}.$$

Since $I \cap B_j$ and $J \cap B_j$ are orthogonal nonzero closed two-sided ideals of B_j, their spectra are disjoint nonempty open subsets of $\mathrm{Prim}(B_j)$. Therefore each one of these spectra contains one of the open sets in some p_k^j, $k \geq 1$. In other words, there exists $k \geq 1$ such that each one of the ideals $I \cap B_j$, $J \cap B_j$ contains one of I_k^j, J_k^j, say

$$I_k^j \subset I \cap B_j, \quad J_k^j \subset J \cap B_j.$$

But then

$$I_k^j B J_k^j \subset IBJ \subset IJ = \{0\},$$

in contradiction with $(*)$. ▨

Using Proposition 3.1 one gets easily

COROLLARY 3.2. *Let A be a prime C^*-algebra, (A, α) a one-parameter C^*-dynamical system and $S \subset A$ a countable set. Then there exists an α-invariant separable prime C^*-subalgebra B of A containing S.*

Proof. Let us denote by A_1 the C^*-subalgebra of A generated by

$$\{\alpha_r(x) \; ; \; x \in S, \; r \text{ rational}\}.$$

Then A_1 is separable and α-invariant.

Next, by Proposition 3.1 there exists a separable prime C^*-subalgebra B_1 of A containing A_1. The C^*-subalgebra A_2 generated by

$$\{\alpha_r(x) \; ; \; x \in B_1, \; r \text{ rational}\}$$

is separable and α-invariant.

Continuing in this way we get a sequence

$$A_1 \subset B_1 \subset A_2 \subset B_2 \subset \ldots$$

of separable C^*-subalgebras of A such that

$$S \subset A_1,$$

$$A_k \text{ is } \alpha\text{-invariant}, \quad k \geq 1,$$

$$B_k \text{ is prime}, \quad k \geq 1.$$

Then the closure B of

$$\bigcup_{k\geqslant 1} A_k = \bigcup_{k\geqslant 1} B_k$$

is a separable C^*-subalgebra of A containing S, which is simultaneously α-invariant and prime. ▨

Now we are able to extend Corollary 2.8 to the inseparable case.

COROLLARY 3.3. *Let A be a prime C^*-algebra and (A, α) a one-parameter C^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is uniformly continuous and*

$$|\sigma(\alpha)|_{\text{cover}} \leqslant 18 \exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Proof. Let us assume that

$$|\sigma(\alpha)|_{\text{cover}} > 18 \exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8)$$

and look for a contradiction.

Let $\lambda_1, \ldots, \lambda_n \in \sigma(\alpha)$ be such that

$$|\{\lambda_1, \ldots, \lambda_n\}|_{\text{cover}} > 18 \exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

By the definition of $\sigma(\alpha)$, for any $1\leqslant j \leqslant n$ and for any $k \geqslant 1$ there are

$$f_{j,k} \in L^1(\mathbf{R}) \quad \text{with } \operatorname{supp}(\hat{f}_{j,k}) \subset \left(\lambda_j - \frac{1}{k}, \lambda_j + \frac{1}{k}\right)$$

and

$$x_{j,k} \in A \quad \text{with } \alpha_{f_{j,k}}(x_{j,k}) \neq 0.$$

Since the set $\{x_{j,k}\,;\, 1\leqslant j\leqslant n,\, k\geqslant 1\}$ is countable, by Corollary 3.2 there exists an α-invariant separable prime C^*-subalgebra B of A containing it. Defining the strongly continuous one-parameter group $\alpha|B$ of $*$-automorphisms of B by

$$(\alpha|B)_t(b) = \alpha_t(b), \quad t \in \mathbf{R},\, b\in B,$$

we have

$$\lambda_j \in \sigma(\alpha|B) \quad \text{for } 1 \leqslant j \leqslant n.$$

Thus

$$|\sigma(\alpha|B)|_{\text{cover}} \geqslant |\{\lambda_1, \ldots, \lambda_n\}|_{\text{cover}} >$$

$$> 18 \exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

On the other hand, by [25], Lemma 4.1 we have

$$\sigma((\alpha|B)_{-i}) \neq \mathbf{C},$$

$$(1 + (\alpha|B)_{-i})^{-1} = (1+\alpha_{-i})^{-1}|B,$$

so

$$\|(1+(\alpha|B)_{-i})^{-1}\| \leqslant \|(1+\alpha_{-i})^{-1}\|.$$

Consequently

$$|\sigma(\alpha|B)|_{\text{cover}} > 18\exp(6\cdot 10^3\|(1+(\alpha|B)_{-i})^{-1}\| + 4\cdot 10^8),$$

which contradicts Corollary 2.8. ▨

The corresponding statement for W^*-dynamical systems is

Corollary 3.4. *Let M be a factor and (M, α) a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is uniformly continuous and*

$$|\sigma(\alpha)|_{\text{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Proof. Consider the C^*-subalgebra

$$A := \{x \in M;\, \mathbf{R} \ni t \mapsto \alpha_t(x) \in M \ \text{ is norm continuous}\}.$$

By Proposition 2.1 A is a $\sigma(M, M_*)$-dense α-invariant C^*-subalgebra of M, the restriction $\alpha|A$ of α to A is a strongly continuous one-parameter group of $*$-automorphisms of A, and

$$\sigma(\alpha|A) = \sigma(\alpha),$$

$$\sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i}) \neq \mathbf{C},$$

$$\|(1+(\alpha|A)_{-i})^{-1}\| = \|(1+\alpha_{-i})^{-1}\|.$$

But A is prime: if I and J are norm-closed two-sided ideals of A with $IJ = \{0\}$, then their (M, M_*)-closures are by [21], Proposition 1.10.5 of the form Mp, Mq, where p and q are orthogonal central projections in M; hence, M being a factor, it follows successively that

$$\text{either } p = 0 \text{ or } q = 0,$$
$$\text{either } I = \{0\} \text{ or } J = \{0\}.$$

Thus, by Corollary 3.3 we have

$$|\sigma(\alpha|A)|_{\text{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+(\alpha|A)_{-i})^{-1}\| + 4\cdot 10^8),$$

and by the above relationships between α and $\alpha|A$, our statement follows. ▨

It is interesting to note that the statement of Corollary 3.4 was established first for W^*-dynamical systems on type I factors with separable predual (Corollary 2.7), from which case we deduced it for C^*-dynamical systems on separable prime C^*-algebras (Corollary 2.8), and then on arbitrary prime C^*-algebras (Corollary 3.3); only using this last case were we able to prove the statement for W^*-dynamical systems on arbitrary factors.

Now we solve the spectrum problem for arbitrary one-parameter C^*-dynamical systems:

THEOREM 3.5. *Let* (A, α) *be a one-parameter* C^**-dynamical system. Then the following statements are equivalent:*

(i) $\sigma(\alpha_{-i}) \neq \mathbf{C}$;

(ii) *there exists a family* $(I_\iota)_{\iota\in\mathfrak{I}}$ *of closed two-sided ideals of* A *such that*

$$\bigcap_{\iota\in\mathfrak{I}} I_\iota = \{0\},$$

for each $\iota \in \mathfrak{I}$, I_ι *is* α*-invariant and* α *induces a uniformly continuous one-parameter group* α^{I_ι} *of* $*$*-automorphisms of the quotient* C^**-algebra* A/I_ι, *and*

$$\sup_{\iota\in\mathfrak{I}} |\sigma(\alpha^{I_\iota})|_{\mathrm{cover}} < +\infty;$$

(iii) *every prime closed two-sided ideal* I *of* A *is* α*-invariant and* α *induces a uniformly continuous one-parameter group* α^I *of* $*$*-automorphisms of the quotient* C^**-algebra* A/I, *and*

$$\sup_{I\ \mathrm{prime}} |\sigma(\alpha^I)|_{\mathrm{cover}} < +\infty.$$

Moreover, if the above equivalent statements hold then

$$\|(1+\alpha_{-i})^{-1}\| \leqslant 4680 \sup_{\iota\in\mathfrak{I}} |\sigma(\alpha^{I_\iota})|_{\mathrm{cover}},$$

$$\sup_{I\ \mathrm{prime}} |\sigma(\alpha^I)|_{\mathrm{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Proof. Using Corollaries 2.5 and 3.3, it is easy to see that (i) $\Rightarrow$ (iii) and

$$\sup_{I\ \mathrm{prime}} |\sigma(\alpha^I)|_{\mathrm{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Since the primitive two-sided ideals of A are prime and their intersection is $\{0\}$, we have also (iii) $\Rightarrow$ (ii).

Finally, let us assume that (ii) holds.

Since $\bigcap_{\iota\in\mathfrak{I}} I_\iota = \{0\}$, the $*$-homomorphism

$$A \ni x \mapsto (x/I_\iota)_{\iota\in\mathfrak{I}} \in \prod_{\iota\in\mathfrak{I}} (A/I_\iota)$$

is injective, hence isometric. Therefore

$$\|x\| = \sup_{\iota\in\mathfrak{I}} \|x/I_\iota\|, \quad x \in A.$$

Let $x \in \mathscr{D}_{\alpha_{-i}}$. Then x belongs to the domain of $(1+\alpha_{-i})^{-1}$ ([3], Corollary 3.3) and it is easy to see that

$$(1+\alpha_{-i})^{-1}(x)/I_\iota = (1+(\alpha^{I_\iota})_{-i})^{-1}(x/I_\iota), \quad \iota \in \mathfrak{I};$$

hence

$$\|(1+\alpha_{-i})^{-1}(x)\| = \sup_{\iota\in\mathfrak{I}}\|(1+(\alpha^{I_\iota})_{-i})^{-1}(x/I_\iota)\| \leqslant$$
$$\leqslant(\sup_{\iota\in\mathfrak{I}}\|(1+(\alpha^{I_\iota})_{-i})^{-1}\|)\|x\|.$$

But by Corollary 1.3,

$$\|(1+(\alpha^{I_\iota})_{-i})^{-1}\| \leqslant 4680|\sigma(\alpha^{I_\iota})|_{\mathrm{cover}}, \quad \iota\in\mathfrak{I},$$

so

$$\|(1+\alpha_{-i})^{-1}(x)\| \leqslant 4680\sup_{\iota\in\mathfrak{I}}|\sigma(\alpha^{I_\iota})|_{\mathrm{cover}}\|x\|.$$

We conclude that

$$\|(1+\alpha_{-i})^{-1}|\mathscr{D}_{\alpha_{-i}}\| \leqslant 4680\sup_{\iota\in\mathfrak{I}}|\sigma(\alpha^{I_\iota})|_{\mathrm{cover}}.$$

Since $(1+\alpha_{-i})^{-1}$ is a closed operator and $\mathscr{D}_{\alpha_{-i}}$ is dense in A ([3], Theorem 2.4), it follows that $(1+\alpha_{-i})^{-1}$ is everywhere defined and

$$\|(1+\alpha_{-i})^{-1}\| \leqslant 4680\sup_{\iota\in\mathfrak{I}}|\sigma(\alpha^{I_\iota})|_{\mathrm{cover}}.$$

In particular, $\sigma(\alpha_{-i}) \neq \mathbf{C}$. ∎

4. THE SPECTRUM PROBLEM FOR W^*-DYNAMICAL SYSTEMS

In this paragraph we extend Corollary 3.4 to global W^*-algebras by proving a W^*-algebra counterpart to Theorem 3.5.

First we consider W^*-dynamical systems on W^*-algebras with separable predual:

LEMMA 4.1. *Let M be a W^*-algebra with separable predual and (M,α) a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is inner; that is, there exists an* s-*continuous one-parameter group*

$$u\colon \mathbf{R} \to \textit{unitaries in } M$$

with

$$\alpha_t(x) = u_t x u_t^*, \quad t\in\mathbf{R},\ x\in M.$$

Proof. By the classical reduction theory for W^*-algebras with separable predual (see e.g. [21], Section 3.2) we can consider M to be a W^*-subalgebra of some $L^\infty(\Omega,\mu,B(H))$ with

- Ω a compact metric space,
- μ a positive Radon measure on Ω,
- H an infinite-dimensional separable complex Hilbert space,
- $B(H)$ endowed with the s-topology,

in such a way that the centre of M is $L^\infty(\Omega, \mu, \mathbf{C}) \subset L^\infty(\Omega, \mu, B(H))$; if $(a_n)_{n\geq 1}$ is a sequence in M which generates M, then for μ-almost every $\omega \in \Omega$ the sequence $(a_n(\omega))_{n\geq 1}$ generates a factor $M(\omega)$ in $B(H)$, and for each $a \in L^\infty(\Omega, \mu, B(H))$,

$$a \in M \Leftrightarrow a(\omega) \in M(\omega) \quad \mu\text{-almost everywhere};$$

in particular, the factors $M(\omega)$ are uniquely defined modulo μ-negligible subsets of Ω.

Since M is countably generated, by Proposition 2.1 there exists an s-dense α-invariant separable C^*-subalgebra A of M, such that the restriction $\alpha|A$ of α to A is a strongly continuous one-parameter group of $*$-automorphisms of A; taking into account [25], Lemma 4.1 we get

$$\sigma((\alpha|A)_{-i}) \neq \mathbf{C}.$$

Let B be a norm-dense countable self-adjoint $\mathbf{Q}(\mathrm{i})$-subalgebra of A (where $\mathbf{Q}(\mathrm{i}) := \mathbf{Q} + \mathrm{i}\mathbf{Q} \subset \mathbf{C}$) such that

$$\alpha_r(B) = B, \quad r \in \mathbf{Q}.$$

Choose a Borel set $\Omega_0 \subset \Omega$, $\Omega \setminus \Omega_0$ μ-negligible, such that the functions

$$\Omega_0 \ni \omega \mapsto \|b(\omega)\|, \quad b \in B$$

are Borel,

$$\|b(\omega)\| \leq \|b\|, \quad b \in B, \omega \in \Omega_0,$$

and for each $\omega \in \Omega_0$ the factor $M(\omega)$ is generated by $\{b(\omega)\,;\, b \in B\}$.

Let $\omega \in \Omega_0$ be fixed. Then

$$B \ni b \mapsto b(\omega) \in M(\omega)$$

is a contractive, $\mathbf{Q}(\mathrm{i})$-linear, multiplicative, $*$-preserving map, so it can be extended to a $*$-homomorphism

$$\pi_\omega : A \to M(\omega)$$

with s-dense image. By Corollary 2.5 the formula

$$(\alpha_A^\omega)_t \pi_\omega(a) = \pi_\omega(\alpha_t(a)), \quad t \in \mathbf{R}, \ a \in A$$

defines a strongly continuous one-parameter group α_A^ω of $*$-automorphisms of the C^*-algebra $\pi_\omega(A)$, such that

$$\sigma((\alpha_A^\omega)_{-i}) \neq \mathbf{C}.$$

Furthermore, since $\pi_\omega(A)$ is s-dense in $M(\omega)$, using Proposition 2.2 and Corollary 2.4, and arguing as in the proof of Corollary 2.8 it is easy to see that α_A^ω can be extended to an s-continuous one-parameter group α^ω of $*$-automorphisms of $M(\omega)$, such that

$$\sigma((\alpha^{\omega})_{-i}) \neq \mathbf{C}.$$

Finally, by Corollary 3.4 α^{ω} is uniformly continuous so by well known results (see e.g. [19], 8.5.5, 8.6.5) there exists a unique self-adjoint $h_{\omega} \in M(\omega)$ with

$$(\alpha^{\omega})_t(x) = \exp(ith_{\omega})x\exp(-ith_{\omega}), \quad t \in \mathbf{R},\ x \in M(\omega),$$

$$\|h_{\omega}\| = \frac{1}{2}\|\delta^{\omega}\|,$$

where the bounded derivation $\delta^{\omega} : M(\omega) \to M(\omega)$ is defined by

$$\delta^{\omega}(x) = \text{norm-}\lim_{t\to 0}\frac{1}{t}((\alpha^{\omega})_t(x) - x), \quad x \in M(\omega).$$

For each $b \in B$ and $r \neq 0$ rational,

$$\frac{1}{r}(\alpha_r(b) - b) \in B,$$

so the function

$$\Omega_0 \ni \omega \mapsto \left\|\frac{1}{r}((\alpha^{\omega})_r(b(\omega)) - b(\omega))\right\| = \left\|\frac{1}{r}(\alpha_r(b) - b)(\omega)\right\|$$

is Borel. It follows that for each $b \in B$ the function

$$\Omega_0 \ni \omega \mapsto \|\delta^{\omega}(b(\omega))\| = \lim_{\mathbf{Q}\ni r\to 0}\frac{1}{r}\|(\alpha^{\omega})_r(b(\omega)) - b(\omega)\|$$

is also Borel. Since $\{b(\omega)\ ;\ b \in B,\ \|b\| \leqslant 1\}$ is s-dense in the closed unit ball of $M(\omega)$, we deduce that the function

$$\Omega_0 \ni \omega \mapsto \|\delta^{\omega}\| = \sup_{\substack{b\in B\\ \|b\|\leqslant 1}}\|\delta^{\omega}(b(\omega))\|$$

is Borel, and hence also the function

$$\Omega_0 \ni \omega \mapsto \|h_{\omega}\| = \frac{1}{2}\|\delta^{\omega}\|.$$

Let us set

$$\Omega_k = \{\omega \in \Omega_0\ ;\ k - 1 \leqslant \|h_{\omega}\| < k\}, \quad k \geqslant 1,$$

$$B(H)_{\mathrm{sa},k} = \{x \in B(H)\ ;\ x = x^*,\quad \|x\| \leqslant k\}.$$

Then $\Omega_1, \Omega_2, \ldots$ form a partition of Ω_0 into Borel sets, and $B(H)_{sa,k}$, endowed with the s-topology, is separable and metrizable with a complete metric. The characteristic function p_k of Ω_k is a central projection in M and we have

$$\sum_{k=1}^{\infty} p_k = 1.$$

By Corollary 2.4 each p_k is fixed by α, so α leaves invariant the reduced W^*-algebras Mp_k and it is enough to prove the innerness of α on each Mp_k.

Let $k \geqslant 1$ be fixed. The product Borel space

$$\Omega_k \times B(H)_{sa,k}$$

is standard, and

$$\Gamma_k = \{(\omega, h) \in \Omega_k \times B(H)_{sa,k}\,;\ \alpha_r(b)(\omega) = \exp(irh)b(\omega)\exp(-irh) \text{ for all } r \in \mathbf{Q} \text{ and } b \in B,\ \|h\| \leqslant \|h_\omega\|\}$$

is a Borel subset of $\Omega_k \times B(H)_{sa,k}$. It follows that Γ_k itself is a standard Borel space. By the definition of the h_ω's

$$\Gamma_k \ni (\omega, h) \mapsto \omega \in \Omega_k$$

is a bijective Borel mapping, hence its inverse

$$\Omega_k \ni \omega \mapsto (\omega, h_\omega) \in \Gamma_k$$

is also Borel (for the properties of standard Borel spaces used we refer to [9], B20, B21). Consequently,

$$\Omega_k \ni \omega \mapsto h_\omega \in B(H)_{sa,k}$$

is a bounded Borel mapping $a_k \in L^\infty(\Omega_k, \mu, B(H))$. Since

$$h_\omega \in M(\omega), \quad \omega \in \Omega_k,$$

a_k belongs to Mp_k. Defining the s-continuous one-parameter group u^k of unitaries in Mp_k by

$$(u^k)_t = p_k + \frac{1}{1!} ita_k + \frac{1}{2!}(ita_k)^2 + \ldots, \quad t \in \mathbf{R},$$

we get, successively,

$$\alpha_r(b)\, p_k = (u^k)_r b (u^k)_r^*, \quad r \in \mathbf{Q},\ b \in B,$$

$$\alpha_t(x)\, p_k = (u^k)_t x (u^k)_t^*, \quad t \in \mathbf{R},\ x \in M.$$ ▨

Next we consider W^*-dynamical systems on countably generated W^*-algebras:

LEMMA 4.2. *Let M be a countably generated W^*-algebra and (M, α) a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then there exists a family $(p_\iota)_{\iota \in \mathfrak{J}}$ of mutually orthogonal central projections in M with $\sum_{\iota \in \mathfrak{J}} p_\iota = 1$, such that α leaves invariant each reduced W^*-algebra Mp_ι and each restriction $\alpha | Mp_\iota$ is uniformly continuous.*

Proof. Let P be a maximal set of mutually orthogonal central projections in M, such that each Mp, $p \in P$, has separable predual. Write

$$q_0 = 1 - \sum_{p \in P} p.$$

Let us assume that $q_0 \neq 0$. We consider Mq_0 imbedded in some $B(H)$ as a von Neumann algebra. Since $q_0 \neq 0$, there exists $\xi_0 \in H$ with $\|\xi_0\| = 1$. The orthogonal projection e' onto $\overline{Mq_0\xi_0}$ belongs to the commutant $(Mq_0)'$ of Mq_0. Denoting by $z(e')$ the central support of e', we have

$$0 \neq z(e') \in Mq_0$$

and $Mz(e')$ is $*$-isomorphic to the induced von Neumann algebra $(Mq_0)_{e'}$. But M is countably generated, so $e'H = \overline{Mq_0\xi_0}$ is a separable Hilbert space. Consequently $(Mq_0)_{e'}$ has separable predual and thus $Mz(e')$ too. This contradicts the maximality of P.

Hence we have

$$\sum_{p \in P} p = 1.$$

By Corollary 2.4 and Lemma 4.1 it follows that α is inner. Now the statement of Lemma 4.2 is a direct consequence of Proposition 2.6. ∎

Now we are able to prove the innerness of a general one-parameter W^*-dynamical system whose analytic generator has positive spectrum:

LEMMA 4.3. *Let (M, α) be a one-parameter W^*-dynamical system with $\sigma(\alpha_{-i}) \neq \mathbf{C}$. Then α is inner.*

Proof. Let P be a maximal set of mutually orthogonal central projections in M, such that the restriction of α to each (by Corollary 2.4 automatically α-invariant) Mp, $p \in P$, is inner. With

$$p_0 = \sum_{p \in P} p, \quad q_0 = 1 - p_0,$$

the restriction of α to Mp_0 is inner and for no nonzero central projection $q \leqslant q_0$ is $\alpha | Mq$ inner.

Let us assume that there exists a nonzero projection $e \leqslant q_0$ in $M^\alpha = \{x \in M;\ \alpha_t(x) = x \text{ for all } t \in \mathbf{R}\}$, such that $\alpha | eMe$ is inner. Arguing as in the proof of [8], Lemma 1.5.2, it is easy to see that in this case also $\alpha | Mz(e)$, where $z(e)$ is the central support of e, is inner. This contradicts the maximality of P.

Consequently, for no nonzero projection $e \leqslant q_0$ in M^α is $\alpha|eMe$ inner. In particular, for no nonzero projection $e \leqslant q_0$ in M^α is $\alpha\,|\,eMe$ uniformly continuous. We shall prove that if $q_0 \neq 0$, then there exists a projection $0 \neq e \leqslant q_0$ in M^α with $\alpha|eMe$ uniformly continuous, in contradiction to the above statement. Thus it will follow that $q_0 = 0$, that is, $p_0 = 1$.

Hence let us assume that $q_0 \neq 0$.

Let $n \geqslant 1$ be an arbitrary integer. Denoting for each $x = x^* \in M$ the projection

$$\bigvee_{t\in\mathbf{R}} \mathrm{supp}(\alpha_t(x)) = \bigvee_{t\in\mathbf{R}} \alpha_t(\mathrm{supp}(x))$$

by $\mathrm{supp}_\alpha(x)$ and calling it the α-support of x, let X_n be a maximal set of self-adjoint elements of the closed unit ball of $(Mq_0)^\alpha((-\infty, -n] \cup [n, +\infty))$ with mutually orthogonal α-supports. Put

$$x_n = \sum_{x\in X_n} x \in (Mq_0)^\alpha((-\infty, n] \cup [n, +\infty)).$$

Now let N be the α-invariant W^*-subalgebra of Mq_0 generated by q_0 and the sequence $(x_n)_{n\geqslant 1}$. Then N is countably generated and its unit element is $q_0 \neq 0$, so by Lemma 4.2 there exists a central projection $e \neq 0$ of N such that α fixes e and the restriction $\alpha|Ne$ is uniformly continuous. Thus there exists an integer $n_e \geqslant 1$ with

$$Ne = (Ne)^\alpha([-n_e, n_e]).$$

Therefore we have, simultaneously,

$$x_{n_e+1}e \in Ne \subset (Mq_0)^\alpha([-n_e, n_e]),$$

$$x_{n_e+1}\, e \in (Mq_0)^\alpha((-\infty, -n_e - 1] \cup [n_e + 1, +\infty,)),$$

which is possible only for

$$x_{n_e+1}e = 0.$$

Let $x \in X_{n_e+1}$ be arbitrary. By the above equality we get, successively,

$$xe = \mathrm{supp}_\alpha(x)x_{n_e+1}e = 0,$$

$$\mathrm{supp}(x)e = 0,$$

$$\alpha_t(\mathrm{supp}(x))e = \alpha_t(\mathrm{supp}(x)e) = 0, \quad t \in \mathbf{R}$$

$$\mathrm{supp}_\alpha(x)e = (\bigvee_{t\in\mathbf{R}} \alpha_t(\mathrm{supp}(x)))e = 0.$$

We conclude that e is orthogonal to the α-support of every $x \in X_{n_e+1}$. By the maximality of X_{n_e+1} it follows successively that

$$(eMe)^{\alpha}((-\infty, -n_e - 1] \cup [n_e + 1, +\infty)) = \{0\},$$

$$eMe = (eMe)^{\alpha}([-n_e - 2, n_e + 2]).$$

But this implies that $\alpha|eMe$ is uniformly continuous. ▨

Finally we prove the promised W^*-algebra counterpart to Theorem 3.5. We recall that if M is a W^*-algebra, Z its centre and Ω the maximal ideal space of Z, then following [13], one can consider for each $\omega \in \Omega$ the norm-closed (automatically two-sided) ideal $[\omega]$ of M.

THEOREM 4.4. *Let (M, α) be a one-parameter W^*-dynamical system, Z the centre of M, and Ω the maximal ideal space of Z. Then the following statements are equivalent:*

(j) $\sigma(\alpha_{-i}) \neq \mathbf{C}$;

(jj) *there exists a family $(I_\iota)_{\iota \in \mathfrak{I}}$ of norm-closed two-sided ideals of M such that*

$$\bigcap_{\iota \in \mathfrak{I}} I_\iota = \{0\},$$

for each $\iota \in \mathfrak{I}$, I_ι is α-invariant and α induces a uniformly continuous one-parameter group α^{I_ι} of $*$-automorphisms of the quotient C^*-algebra M/I_ι, and

$$\sup_{\iota \in \mathfrak{I}} |\sigma(\alpha^{I_\iota})|_{\text{cover}} < +\infty;$$

(jjj) *there exists a dense open subset $\mathfrak{D}$ of Ω such that, for each $\omega \in \mathfrak{D}$, $[\omega]$ is α-invariant and α induces a uniformly continuous one-parameter group α^{ω} of $*$-automorphisms of the quotient C^*-algebra $M/[\omega]$, and*

$$\sup_{\omega \in \mathfrak{D}} |\sigma(\alpha^{\omega})|_{\text{cover}} < +\infty;$$

(jjjj) *there exists a family $(p_\iota)_{\iota \in \mathfrak{I}}$ of mutually orthogonal central projections in M such that*

$$\sum_{\iota \in \mathfrak{I}} p_\iota = 1,$$

for each $\iota \in \mathfrak{I}$, p_ι is fixed by α and α induces a uniformly continuous one-parameter group $\alpha|Mp_\iota$ of $*$-automorphisms of Mp_ι, and

$$\sup_{\iota \in \mathfrak{I}} |\sigma(\alpha|Mp_\iota)|_{\text{cover}} < +\infty.$$

Moreover, if the above equivalent statements hold then

$$\|(1+\alpha_{-i})^{-1}\| \leqslant 4680 \sup_{\iota\in\mathfrak{J}} |\sigma(\alpha^{I_\iota})|_{\text{cover}},$$

$$\sup_{\omega\in\mathfrak{D}} |\sigma(\alpha^{\omega})|_{\text{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Proof. Using Lemma 4.3 and Proposition 2.6, we get immediately the implication (j) $\Rightarrow$ (jjjj), with the estimation

$$\sup_{\iota\in\mathfrak{J}} |\sigma(\alpha|Mp_\iota)|_{\text{cover}} \leqslant 18\exp(6\cdot 10^3\|(1+\alpha_{-i})^{-1}\| + 4\cdot 10^8).$$

Let us next assume that (jjjj) holds. Each p_ι corresponds to the characteristic function of some closed and open subset K_ι of Ω and the open set

$$\mathfrak{D} = \bigcup_{\iota\in\mathfrak{J}} K_\iota$$

is dense in Ω. Let $\omega \in \mathfrak{D}$. Then ω belongs to some $K_{\iota(\omega)}$ and since $\alpha|Mp_{\iota(\omega)}$ is uniformly continuous, hence inner, and

$$[\omega] = [\omega] \cap Mp_{\iota(\omega)} + M(1 - p_{\iota(\omega)}),$$

α leaves $[\omega]$ invariant and induces a uniformly continuous one-parameter group α^{ω} of $*$-automorphisms of

$$M/[\omega] = Mp_{\iota(\omega)}/([\omega] \cap Mp_{\iota(\omega)}).$$

Obviously,

$$\sigma(\alpha^{\omega}) \subset \sigma(\alpha|Mp_{\iota(\omega)}),$$

so we have

$$|\sigma(\alpha^{\omega})|_{\text{cover}} \leqslant |\sigma(\alpha|Mp_{\iota(\omega)})|_{\text{cover}} \leqslant \sup_{\iota\in\mathfrak{J}} |\sigma(\alpha|Mp_\iota)|_{\text{cover}}.$$

Hence (jjj) holds.

By the formula from p. 232 of [13] for the norm on M in terms of the norms on the quotient C^*-algebras $M/[\omega]$, and by [13], Lemma 10, we have

$$\bigcap_{\omega\in\mathfrak{D}} [\omega] = \{0\}.$$

Therefore, (jjj) $\Rightarrow$ (jj).

Finally, let us assume that (jj) holds. Let

$$A = \{x \in M\,;\ \mathbf{R} \ni t \mapsto \alpha_t(x) \in M \text{ is norm continuous}\}.$$

By Proposition 2.1, A is an α-invariant C^*-subalgebra of M, $\alpha|A$ is a strongly continuous one-parameter group of $*$-automorphisms of A, and we have

$$\sigma((\alpha|A)_{-i}) = \sigma(\alpha_{-i}),$$

$$\|(\lambda - (\alpha|A)_{-i})^{-1}\| = \|(\lambda - \alpha_{-i})^{-1}\| \quad \text{for } \lambda \notin \sigma(\alpha_{-i}).$$

Now $(I_\iota \cap A)_{\iota \in \mathfrak{J}}$ is a family of closed two-sided ideals of A with intersection zero, and for each $\iota \in \mathfrak{J}$, $\mathfrak{J}_\iota \cap A$ is α-invariant and α induces a uniformly continuous one-parameter group $\alpha^{I_\iota \cap A}$ of $*$-automorphisms of

$$A/(I_\iota \cap A) \subset M/I_\iota,$$

namely,

$$\alpha^{I_\iota \cap A} = \alpha^{I_\iota}|A/(I_\iota \cap A).$$

Then we have, successively,

$$\sigma(\alpha^{I_\iota \cap A}) \subset \sigma(\alpha^{I_\iota}), \quad \iota \in \mathfrak{J},$$

$$\sup_{\iota \in \mathfrak{J}} |\sigma(\alpha^{I_\iota \cap A})|_{\text{cover}} \leqslant \sup_{\iota \in \mathfrak{J}} |\sigma(\alpha^{I_\iota})|_{\text{cover}} < +\infty.$$

By Theorem 3.5,

$$\sigma((\alpha|A)_{-i}) \neq \mathbf{C},$$

$$\|(1 + (\alpha|A)_{-i})^{-1}\| \leqslant 4680 \sup_{\iota \in \mathfrak{J}} |\sigma(\alpha^{I_\iota \cap A})|_{\text{cover}},$$

and we conclude that

$$\sigma(\alpha_{-i}) \neq \mathbf{C},$$

$$\|(1 + \alpha_{-i})^{-1}\| \leqslant 4680 \sup_{\iota \in \mathfrak{J}} |\sigma(\alpha^{I_\iota})|_{\text{cover}}.$$

In particular, (j) holds. ▨

Proposition 2.2 and Theorem 4.4 yield immediately the following completion to Theorem 3.5:

COROLLARY 4.5. *Let (A, α) be a one-parameter C^*-dynamical system. Then the following statements are equivalent:*

(i) $\sigma(\alpha_{-i}) \neq \mathbf{C}$;

(iiii) *there exists a family $(p_\iota)_{\iota \in \mathfrak{J}}$ of mutually orthogonal central projections in A^{**} such that*

$$\sum_{\iota \in \mathfrak{J}} p_\iota = 1,$$

for each $\iota \in \mathfrak{J}$, p_ι *is fixed by all* $(\alpha_t)^{**}$ *with* $t \in \mathbf{R}$ *and*

$$\alpha^{**}|A^{**}p_\iota : \mathbf{R} \ni t \mapsto (\alpha_t)^{**}|A^{**}p_\iota$$

is a uniformly continuous one-parameter group of $*$*-automorphisms of* $A^{**}p_\iota$, *and*

$$\sup_{\iota \in \mathfrak{J}} |\sigma(\alpha^{**}|A^{**}p_\iota)|_{\mathrm{cover}} < +\infty .$$

5. FINAL REMARKS

Let us consider for a one-parameter C^*- or W^*-dynamical system (A, α) the following condition:

(N) there exist constants $\varepsilon > 0$ and $c_\varepsilon > 0$ such that

$$x \in A^\alpha((-\infty, -\varepsilon]),\ \|x\| < 1 \Rightarrow \|\ln(1 - x)\| \leqslant c_\varepsilon .$$

By Proposition 2.2 and Theorem 2.3 we have

$$\sigma(\alpha_{-i}) \neq \mathbf{C} \Rightarrow \alpha \text{ satisfies (N)}.$$

On the other hand, a careful examination of the proof of Proposition 2.6 shows that, assuming α inner, we have also

$$\alpha \text{ satisfies (N)} \Rightarrow \sigma(\alpha_{-i}) \neq \mathbf{C}.$$

Thus we can raise the following

PROBLEM. Is it true for an arbitrary C^*- or W^*-dynamical system (A, α) that

$$\sigma(\alpha_{-i}) \neq \mathbf{C} \Leftrightarrow \alpha \text{ satisfies (N)?}$$

In the proof of Lemma 4.2 we have proved that every countably generated W^*-algebra is a direct product of W^*-algebras with separable predual. One also has the following

PROPOSITION. *A W^*-algebra M is countably generated over its centre Z if and only if it can be imbedded as W^*-subalgebra in a type $I_{\aleph_0}$ W^*-algebra N, in such a way that Z becomes the centre of N.*

Proof. If M allows an imbedding as in the statement, then M is countably generated over its centre by [11], Lemma 4.

Conversely, let us assume that M is countably generated over its centre Z. One can consider M to be a von Neumann algebra in some complex Hilbert space H such that for some involutive antilinear isometry $J: H \to H$ we have

$$M' = JMJ$$

(for the whole topic of "standard" representations we refer to [14]). In this spatial representation M' will also be countably generated over Z. It follows that the von Neumann algebra generated by M and M'

$$R = (M \cup M')'' = Z'$$

is countably generated over Z. R is of type I, M is a W^*-subalgebra of R, and the centre of R is Z.

Let P be a maximal set of mutually orthogonal projections in Z, such that for each $p \in P$, Rp is countably decomposable.

Assuming that

$$q_0 = 1 - \sum_{p \in P} p \neq 0,$$

one has $\xi_0 \in q_0 H$ with $\|\xi_0\| = 1$, and the orthogonal projection $0 \neq q \leqslant q_0$ onto $\overline{R\xi_0}$ belongs to $R' = Z$; if R is generated by Z and by the countable subring B of R, then $\{b\xi_0 ; b \in B\}$ is a countable separating set for Rq in $\overline{R\xi_0}$:

$$x \in Rq,\ xb\xi_0 = 0 \quad \text{for all } b \in B \Rightarrow$$
$$\Rightarrow x(zb)\xi_0 = z(xb\xi_0) = 0 \quad \text{for all } z \in Z \text{ and } b \in B \Rightarrow$$
$$\Rightarrow x|\overline{R\xi_0} = 0 \Rightarrow$$
$$\Rightarrow x = 0.$$

Hence Rq is countably decomposable, in contradiction to the maximality of P.

Thus $q_0 = 0$, that is

$$\sum_{p \in P} p = 1.$$

Since a countably decomposable homogeneous type I W^*-algebra is always of type I_n with $n \leqslant \aleph_0$, it follows that each homogeneous component of R is of type I_n for some $n \leqslant \aleph_0$. Therefore the tensor product of R with a factor of type $I_{\aleph_0}$ satisfies the conditions for N. ▨

By the preceding proposition the classical reduction theory of von Neumann as described in [9], Chapter II is available for von Neumann algebras which are countably generated over their centres (and precisely for these). Note that the extension of von Neumann's measurable choice principle due to R. J. Aumann (see [11]) applies in this situation.

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