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A TRACE FORMULA FOR WIENER-HOPF OPERATORS

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INTRODUCTION

For a function $\sigma \in L_\infty(\mathbf{R})$ the Wiener-Hopf operator $W(\sigma)$ on $L_2(\mathbf{R}^+)$ is defined by

$$W(\sigma)\varphi = P(\sigma\hat{\varphi})^{\vee}$$

where

$$\hat{\varphi}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)dx, \quad \check{\Phi}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\Phi(\xi)d\xi$$

and P is the projection from $L_2(\mathbf{R})$ to $L_2(\mathbf{R}^+)$.

We consider here the following questions: For which functions f and σ is the operator

$$f(W(\sigma)) - W(f\circ\sigma) \tag{1}$$

of trace class? If it is trace class, what is its trace?

First, it is not hard to show that if σ is real-valued then a necessary condition that

$$W(\sigma)^2 - W(\sigma^2)$$

be trace class is that the distributional inverse Fourier transform $\check{\sigma}$ be equal on $\mathbf{R}\setminus\{0\}$ to a function satisfying

$$\int_{-\infty}^{\infty} |x|\,|\check{\sigma}(x)|^2 dx < \infty \tag{2}$$

(this integral and analogous ones are understood to be taken over $\mathbf{R}\setminus\{0\}$) and that if σ_1 and σ_2 (real-valued or not) satisfy (2) then

$$W(\sigma_1)W(\sigma_2) - W(\sigma_1\sigma_2)$$

is trace class. (The Toeplitz analogue of this is very well known and quite trivial. It is less straightforward here since $\check{\sigma}$ is a distribution rather than a function. Proofs will be given in the next section.) It follows that if (2) holds then

$$\mathrm{W}(\sigma)^n - \mathrm{W}(\sigma^n)$$

is trace class for all n and so (1) is trace class for all polynomials f. We shall assume about σ only that (2) holds and, of course, that $\sigma \in L_\infty$.

To state the main result of the paper we introduce two pieces of notation. First, $\mathrm{d}\omega$ denotes the measure on $\mathbf{R}^2$ given by

$$\mathrm{d}\omega(\xi_1, \xi_2) = \frac{1}{8\pi^2}|\xi_1 - \xi_2|^{-2}\mathrm{d}\xi_1\mathrm{d}\xi_2 .$$

The usefulness of this measure lies in the identity

$$\iint[\sigma_1(\xi_1) - \sigma_1(\xi_2)][\sigma_2(\xi_1) - \sigma_2(\xi_2)]\mathrm{d}\omega(\xi_1, \xi_2) = \frac{1}{2}\int|x|\check{\sigma}_1(x)\check{\sigma}_2(-x)\mathrm{d}x, \tag{3}$$

satisfied by all σ_1, σ_2 for which (2) holds, which will also be derived in the next section. Second, for complex numbers α and β and a function f belonging to C^1 on the line segment joining α and β we write

$$U(\alpha, \beta, f) = \int_0^1 \frac{f((1-\theta)\alpha + \theta\beta) - [(1-\theta)f(\alpha) + \theta f(\beta)]}{\theta(1-\theta)}\,\mathrm{d}\theta.$$

THEOREM 1. *Assume* (2) *holds and that either of the following conditions is satisfied:*

(a) *f is analytic on a neighborhood of the closed convex hull Σ of the essential range of σ;*

(b) *σ is real valued, $f \in L_1(\mathbf{R})$ and $t^2\hat{f}(t) \in L_1(\mathbf{R})$.*

Then the operator (1) *is trace class and its trace equals*

$$\iint U(\sigma(\xi_1), \sigma(\xi_2), f)\mathrm{d}\omega(\xi_1, \xi_2).$$

Observe that since the spectrum of $\mathrm{W}(\sigma)$ is always contained in Σ the condition (a) on f guarantees (and is needed to guarantee) that $f(\mathrm{W}(\sigma))$ may be defined by the analytic functional calculus. As for the more interesting condition (b) when σ is real valued, the operator $f(\mathrm{W}(\sigma))$ is defined by the spectral theorem for any f which is bounded on Σ. This operator depends only on the restriction of f to Σ. Thus condition (b) need not be satisfied by f itself but only by a function equal to it on Σ. What we really have, therefore, is a local condition on $f''(x)$.

This condition is probably not overwhelmingly more than is necessary. For if f does satisfy condition (b) then the supremum of

$$\mathrm{tr}[f(W(\sigma)) - W(f \circ \sigma)]$$

taken over all real-valued σ satisfying

$$\|\sigma\|_\infty \leqslant 2A, \quad \int |x|\,|\check{\sigma}(x)|^2 dx \leqslant 1$$

is at least a constant times

$$\max\{|f''(x)| : -A \leqslant x \leqslant A\}.$$

This can be seen by using the formula

$$(3') \quad U(\alpha, \beta, f) = (\beta - \alpha)^2 \int_0^1 f''((1-\theta)\alpha + \theta\beta)[\theta \log\theta + (1-\theta)\log(1-\theta)]d\theta$$

to deduce that if for some constant x we have $\lim_{n\to\infty}\sigma_n(\xi) = x$ uniformly in ξ then

$$U(\sigma_n(\xi_1), \sigma_n(\xi_2), f) = \left[-\frac{1}{2} f''(x) + o(1)\right][\sigma_n(\xi_1) - \sigma_n(\xi_2)]^2$$

uniformly in ξ_1, ξ_2 and then setting

$$\sigma_n(\xi) = x + n^{-1/2}\xi^{-1}[e^{i(n+1)\xi} - e^{in\xi}]$$

and applying the theorem and identity (3) with $\sigma_1 = \sigma_2 = \sigma_n$. This suggests that a necessary, and perhaps also sufficient, condition that (1) be trace class for all σ satisfying (2) is that f'' be locally bounded.

Two special cases of the theorem, corresponding to $f(z)$ equal to z^2 and $\log z$, are especially simple. Elementary computation show that

$$U(\alpha, \beta, z^2) = -(\alpha - \beta)^2$$

$$U(\alpha, \beta, \log z) = \frac{1}{2}(\log\alpha - \log\beta)^2$$

and so the theorem and identity (3) give

$$\mathrm{tr}[W(\sigma)^2 - W(\sigma^2)] = -\frac{1}{2}\int |x|\check{\sigma}(x)\check{\sigma}(-x)dx = -\int_0^\infty x\check{\sigma}(x)\check{\sigma}(-x)dx$$

and similarly

$$\mathrm{tr}[\log(W(\sigma) - W(\log\sigma)] = \frac{1}{2}\int_0^\infty x(\log\sigma)^\vee(x)(\log\sigma)^\vee(-x)\mathrm{d}x.$$

These formulas, with suitable conditions imposed on σ, have been known for some time. The similarity of the right sides is striking and is due directly to the fact that for these two special f, $U(\alpha, \beta, f)$ is the form $[F(\beta) - F(\alpha)]^2$ for some function F. Modulo translation and addition of a linear function these are the only two smooth functions with this property.

Following the proof of Theorem 1 we shall consider the finite Wiener-Hopf operator $W_\alpha(\sigma)$ on $L_2(0, \alpha)$, defined by

$$W_\alpha(\sigma)\varphi = P_\alpha(\sigma\hat{\varphi})^\vee$$

where P_α is the projection from $L_2(\mathbf{R})$ to $L_2(0, \alpha)$, and prove the following.

THEOREM 2. *Under either of the conditions of Theorem* 1 *we have*

$$\lim_{\alpha\to\infty}\mathrm{tr}[f(W_\alpha(\sigma)) - W_\alpha(f\circ\sigma)] = 2\iint U(\sigma(\xi_1), \sigma(\xi_2), f)\mathrm{d}\omega(\xi_1, \xi_2).$$

A discrete analogue of this, for self-adjoint Toeplitz matrices, was proved by L. M. Libkind [3]. His conditions were different from ours, as well as his expression for the limit. The methods of the present paper apply equally well (and more easily) to the discrete case.

In view of the second assertion of Theorem 1 the content of Theorem 1 is really that

$$\lim_{\alpha\to\infty}\mathrm{tr}[f(W_\alpha(\sigma)) - W_\alpha(f\circ\sigma)] = 2\mathrm{tr}[f(W(\sigma)) - W(f\circ\sigma)].$$

That this holds, at least with stronger conditions on f and σ, is already known [6]. The proof under the present conditions will be only slightly more involved than the proof of the first (and simpler) assertion of Theorem 1.

If in Theorem 2 we take $f(z) = \log z$ we obtain information on the asymptotics of determinants of finite Wiener-Hopf operators. Recall that an operator is of determinant class if it differs from I by an operator of trace class, and its determinant is then defined [1, Chapter IV].

THEOREM 3. *If there is a determination of* $\log\sigma$ *which is bounded and satisfies* (2) *then*

$$W_\alpha(\sigma)\,e^{-W_\alpha(\log\sigma)}$$

is of determinant class and

(4) $$\lim_{\alpha\to\infty} \det W_\alpha(\sigma) e^{-W_\alpha(\log\sigma)} = \exp\left\{\int_0^\infty x(\log\sigma)^\vee(x)(\log\sigma)^\vee(-x)dx\right\}.$$

If in addition $\log\sigma \in L_1(\mathbf{R})$ *then* $W_\alpha(\sigma)$ *is of determinant class and*

$$\lim_{\alpha\to\infty} \frac{\det W_\alpha(\sigma)}{\exp\left\{\frac{\alpha}{2\pi}\int \log\sigma(\xi)d\xi\right\}} = \exp\left\{\int_0^\infty x(\log\sigma)^\vee(x)(\log\sigma)^\vee(-x)dx\right\}.$$

This is of course the continuous analogue of the strong Szegö limit theorem for Toeplitz determinants, long known to hold under suitable conditions on σ [2].

We end this introduction with a conjecture of which Theorem 2 is a very special case. Recall that the pseudodifferential operator on $\mathbf{R}^n$ with symbol $\sigma(x, \xi)$ is the operator A given by the formula

$$Af(x) = (2\pi)^{-n}\int e^{i\xi\cdot x}\sigma(x, \xi)\hat{f}(\xi)d\xi.$$

If σ satisfies appropriate conditions then A is trace class and

$$\operatorname{tr}A = (2\pi)^{-n}\iint \sigma(x, \xi)dxd\xi.$$

More generally if Ω is a domain in $\mathbf{R}^n$ and P denotes projection onto $L_2(\Omega)$ then

(5) $$\operatorname{tr}PAP = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_\Omega \sigma(x, \xi)dxd\xi.$$

The question is, given a suitable function f and operator A, what is the trace of $f(PAP)$? It is of course hopeless to expect an exact expression for this in general, but not at all hopeless if f or A depends on a parameter in a suitable way and one is looking for, say, an asymptotic formula.

A reasonable first approximation to $f(PAP)$ is $Pf(A)P$ and a reasonable first approximation to $f(A)$ is the operator with symbol $f(\sigma)$. This leads us to the formula, for a first approximation,

(6) $$\operatorname{tr}f(PAP) \approx (2\pi)^{-n}\int_{\mathbf{R}^n}\int_\Omega f(\sigma(x, \xi))dxd\xi.$$

Examples of this abound. To give just one, probably the most famous, if $\sigma(x, \xi) = |\xi|^2$ and

$$f(x) = \begin{cases} 0 & x > \lambda \\ 1 & x < \lambda \end{cases}$$

then (6) is Weyl's formula on the asymptotic distribution of the eigenvalues of the Laplace operator on Ω with appropriate boundary conditions. Our conjecture is that a certain formula is in the same (imprecise) way a universal second approximation. We need some notation. Denote by $\mathscr{X}$ the set of all pairs $X = (x, \eta)$ where $x \in \partial\Omega$ and $\eta \in T_x$ (the tangent hyperplane to $\partial\Omega$ at x). Thus $\mathscr{X}$ seems to be the tangent bundle of $\partial\Omega$ although it is better to think of it as the cotangent bundle. There is a natural measure $dX = d\eta dx$ on $\mathscr{X}$, where dx denotes surface measure on $\partial\Omega$ and $d\eta$ Lebesgue measure on T_x. Given a symbol $\sigma(x, \xi)$ we define for each $X = (x, \eta) \in \mathscr{X}$ the function σ_X of the real variable ξ by

$$\sigma_X(\xi) = \sigma(x, \eta + \xi \nu_x)$$

where ν_x is the unit inner normal to $\partial\Omega$ at x. The proposed formula for a second approximation reads

$$\begin{aligned} \operatorname{tr} f(PAP) \approx (2\pi)^{-n} \int_{\mathbf{R}^n}\int_{\Omega} \left[f(\sigma) - \frac{1}{2} f''(\sigma) \sum_{k=1}^{n} \frac{\partial\sigma}{\partial x_k}\frac{\partial\sigma}{\partial \xi_n} \right] dx d\xi \, + \\ + (2\pi)^{-n+1} \int_{\mathscr{X}}\int_{\mathbf{R}^2} U(\sigma_X(\xi_1), \sigma_X(\xi_2), f) d\omega(\xi_1, \xi_2) dX. \end{aligned} \tag{7}$$

In case the symbol σ is $\sigma(\xi/\alpha)$ and Ω is the interval $[0, 1]$ in $\mathbf{R}^1$ then PAP is unitarily equivalent to $W_\alpha(\sigma)$. And (7) is just Theorem 2 if $f(\sigma) \in L_1(\mathbf{R})$ since σ is independent of x and $\partial[0, 1]$ consists of two points making the same contribution.

Although to a first approximation the operator $f(A)$ has symbol $f(\sigma)$, to a second approximation its symbol is the integrand in the first integral above. (See, for example, [5, § 2].) In view of this, formula (5), and Theorem 1 an alternative formula is

$$\operatorname{tr}[f(PAP) - Pf(A)P] \approx (2\pi)^{-n+1} \int_{\mathscr{X}} \operatorname{tr}[f(W(\sigma_X)) - W(f \circ \sigma_X)] dX.$$

A precise version of this was proved in [6] for a class of symbols of the form $\sigma(\xi/\alpha)$ in $\mathbf{R}^n$ and f satisfying condition (a).

Curiously enough if $\sigma(x, \xi) = -|\xi|^2$, so that A is the Laplace operator, and $f(x) = e^{tx}$ then (7) is a correct second approximation as $t \to 0+$ if PAP is interpreted as the Laplace operator with Dirichlet boundary condition. This is seen by comparing what (7) gives (it is easy to compute the integrals exactly) with the known asymptotic expansion for the heat operator with this boundary condition [4].

PRELIMINARIES

We denote by $\mathcal{K}$ the set of all functions $\sigma \in L_\infty(\mathbf{R})$ for which (2) holds.

PROPOSITION 1. *Suppose* $\sigma \in L_\infty(\mathbf{R})$. *Then* $\sigma \in \mathcal{K}$ *if and only if* $\sigma(\xi_1) - \sigma(\xi_2) \in L_2(\mathrm{d}\omega)$ *and then*

$$\int |\sigma(\xi_1) - \sigma(\xi_2)|^2 \mathrm{d}\omega(\xi_1, \xi_2) = 1/2 \int |x|\, |\check{\sigma}(x)|^2 \mathrm{d}x.$$

Proof. Assume $\sigma \in \mathcal{K}$ and fix η. The inverse Fourier transform of $\sigma(\xi + \eta) - \sigma(\xi)$ is

$$(\mathrm{e}^{-\mathrm{i}\eta x} - 1)\check{\sigma}(x)$$

which is equal, in $\mathbf{R}\setminus\{0\}$, to an L_2 function. Thus $\sigma(\xi + \eta) - \sigma(\xi)$ equals an L_2 function $\tau_\eta(\xi)$ plus the Fourier transform of a distribution supported at $\{0\}$, i.e., a polynomial. Now the integral of $\sigma(\xi + \eta) - \sigma(\xi)$ over an interval J is bounded by a constant independent of J, and the integral of τ_η over J is bounded by a constant times $|J|^{1/2}$. It follows that the polynomial in question equals 0. Thus

$$\sigma(\xi + \eta) - \sigma(\xi) = \tau_\eta(\xi)$$

and Parseval's identity gives

$$\int |\sigma(\xi + \eta) - \sigma(\eta)|^2 \mathrm{d}\xi = 2\pi \int |\mathrm{e}^{-\mathrm{i}\eta x} - 1|^2 |\check{\sigma}(x)|^2 \mathrm{d}x.$$

Dividing by η^2, integrating with respect to η, and changing variables yield the asserted relation. The converse is proved similarly.

Formula (3) follows easily from the proposition by a standard argument.

We shall denote by $|||\sigma|||$ the norm of $\sigma(\xi_1) - \sigma(\xi_2)$ in $L_2(\mathrm{d}\omega)$. Thus $\sigma \in \mathcal{K}$ if and only if

$$\|\sigma\|_\infty + |||\sigma||| < \infty.$$

A trivial but extremely useful fact is that if $\sigma \in \mathcal{K}$ and f satisfies a Lipschitz condition with Lipschitz constant A on the essential range of σ then $f(\sigma) \in \mathcal{K}$ and

$$|||f(\sigma)||| \leqslant A|||\sigma|||. \tag{8}$$

Proposition 2. *Given* $\sigma \in \mathcal{K}$ *there exist* $\sigma_n \in \mathcal{K}$ *such that*

(i) $|||\sigma_n - \sigma||| \to 0$ *as* $n \to \infty$,

(ii) $\sigma_n \to \sigma$ *almost everywhere as* $n \to \infty$,

(iii) $\|\sigma_n\|_\infty = \mathrm{O}(1)$ *as* $n \to \infty$,

(iv) *each* $\check{\sigma}_n$ *is of the form* $\mathrm{c}_n\delta + s_n$ *where* c_n *is a constant and* s_n *is an* L_2 *function supported in a compact subset of* $\mathbf{R}\setminus\{0\}$.

Proof. Assume first that $\check{\sigma}$ has compact support, and let $\varphi(x)$ be a smooth function with compact support equal to 1 on a neighborhood of $x = 0$. Set

$$\tau_n(\xi) = [(1 - \varphi(nx))\check{\sigma}(x)]^{\wedge}.$$

Since the inverse Fourier transform of the distribution $\sigma'(\xi)$ is the L_2 function

$$-ix\check{\sigma}(x)$$

(the proof of Proposition 1 showed that $(e^{-i\eta x} - 1)\check{\sigma}(x)$ is always an L_2 function) use of Parseval's identity shows that

$$\tau_n'(\xi) \to \sigma'(\xi)$$

in L_2. Integrating shows that

$$\sigma_n(\xi) = \tau_n(\xi) - \tau_n(0) + \sigma(0) \to \sigma(\xi)$$

locally uniformly. And the other properties (i), (iii), and (iv) are easily checked. If $\check{\sigma}$ does not have compact support we first replace σ by

$$[\varphi(x/m)\check{\sigma}(x)]^{\wedge},$$

find a sequence $\sigma_{m,n}$ for each of these, and then choose an appropriate sequence $\{\sigma_{m,n_m}\}$. The details are left to the reader.

We now prove the two assertions made early in the introduction.

PROPOSITION 3. *If σ is a real-valued function in L_∞ and if $W(\sigma^2) - W(\sigma)^2$ is trace class then $\sigma \in \mathscr{K}$. If $\sigma_1, \sigma_2 \in \mathscr{K}$ then the operator*

$$H(\sigma_1, \sigma_2) = W(\sigma_1\sigma_2) - W(\sigma_1)W(\sigma_2) \tag{9}$$

is trace class and

$$\|H(\sigma_1, \sigma_2)\| \leqslant |||\sigma_1||| \, |||\sigma_2|||, \tag{10}$$

$$\operatorname{tr} H(\sigma_1, \sigma_2) = \int_0^\infty x\check{\sigma}_1(x)\check{\sigma}_2(-x)dx. \tag{11}$$

Proof. If we define the operator $A(\sigma)$ by

$$A(\sigma)\varphi = (\sigma\hat{\varphi})^{\vee}$$

then

$$H(\sigma_1, \sigma_2) = PA(\sigma_1)(I - P)A(\sigma_2)P.$$

To prove the first assertion observe that this identity gives for real-valued σ

$$W(\sigma^2) - W(\sigma)^2 = [PA(\sigma)(I - P)][PA(\sigma)(I - P)]^*.$$

Since the operator on the left is trace class, $PA(\sigma)(I-P)$ is Hilbert-Schmidt. Now if σ belonged to L_1 then $A(\sigma)$ would be the integral operator with continuous kernel $\check{\sigma}(x-y)$ and we would have

$$\|PA(\sigma)(I-P)\|_2^2 = \int_0^\infty \int_{-\infty}^0 |\check{\sigma}(x-y)|^2 \mathrm{d}y\mathrm{d}x = \int_0^\infty x|\check{\sigma}(x)|^2\mathrm{d}x.$$

If σ is real-valued this equals

$$1/2 \int_{-\infty}^\infty |x|\,|\check{\sigma}(x)|^2\mathrm{d}x = |||\sigma|||^2$$

since $|\sigma(-x)| = |\sigma(x)|$. For general σ we introduce the functions

$$\sigma_\varepsilon^\pm(\xi) = (\varepsilon^{-1}\mathrm{e}^{-|x|/\varepsilon}\chi_{\mathbf{R}^\pm}(x))^\vee(\xi) = \frac{1}{1 \pm \mathrm{i}\varepsilon\xi}.$$

Since

$$(I-P)A(\sigma_\varepsilon^+)P = 0, \quad PA(\sigma_\varepsilon^-)(I-P) = 0$$

we have

$$PA(\sigma_\varepsilon^+)PA(\sigma)(I-P)A(\sigma_\varepsilon^-)(I-P) = PA(\sigma_\varepsilon^+)A(\sigma)A(\sigma_\varepsilon^-)(I-P).$$

The operator on the left has Hilbert-Schmidt norm at most

$$\|PA(\sigma)(I-P)\|_2$$

since $\|A(\sigma_\varepsilon^\pm)\| = \|\sigma_\varepsilon^\pm\|_\infty = 1$. The operator on the right is

$$PA(\sigma_\varepsilon)(I-P)$$

where

$$\sigma_\varepsilon(\xi) = \frac{1}{1+\varepsilon^2\xi^2}\sigma(\xi).$$

Since this is an L_1 function we deduce that $\sigma_\varepsilon \in \mathscr{K}$ for all ε and

$$|||\sigma_\varepsilon||| \leqslant \|PA(\sigma)(I-P)\|_2.$$

If we let $\varepsilon \to 0$ Fatou's lemma tells us that

$$|||\sigma||| \leqslant \|PA(\sigma)(I-P)\|_2$$

and so $\sigma \in \mathscr{K}$.

To prove the second assertion assume first that $\sigma_i = c_i\delta + s_i$ where c_i are constants and s_i are L_2 functions supported in compact subsets of $\mathbf{R}\setminus\{0\}$. Then $H(\sigma_1, \sigma_2)$ is the integral operator with continuous kernel

$$\int_0^\infty s_1(x+z)s_2(-z-y)dz.$$

Its trace norm is at most $|||\sigma_1|||\,|||\sigma_2|||$ and its trace equals

$$\int_0^\infty\int_0^\infty s_1(x+z)s_2(-z-x)dzdx = \int_0^\infty x\check{\sigma}_1(x)\check{\sigma}_2(-x)dx.$$

For arbitrary $\sigma_1, \sigma_2 \in \mathscr{K}$ find sequences $\{\sigma_{1,n}\}$ and $\{\sigma_{2,n}\}$ as in Proposition 2. Then (ii) and (iii) imply that the operators $A(\sigma_{1,n})$ converge strongly to $A(\sigma_1)$. However since $\{\sigma_{1,n}\}$ is Cauchy in the norm $|||\ |||$ the operators $PA(\sigma_{1,n})(I-P)$ form a Cauchy sequence in the Hilbert-Schmidt norm. It follows that

$$PA(\sigma_{1,n})(I-P) \to PA(\sigma_1)(I-P)$$

in the Hilbert-Schmidt norm, and similarly

$$(I-P)A(\sigma_{2,n})P \to (I-P)A(\sigma_2)P.$$

Hence the relations (10) and (11) for σ_1 and σ_2 follow from the corresponding relation for $\sigma_{1,n}$ and $\sigma_{2,n}$.

THE TRACE FORMULA

The proof of Theorem 1 consists of three parts. We show first that under either condition (a) or (b) the operator (1) is trace class. We then show that, because of convenient density and continuity properties, to prove in general that the trace is given by the proposed formula it suffices to prove it in the special case where $\check{\sigma} = c\delta + s$ with s supported in a compact subset of $\mathbf{R}\setminus\{0\}$ and $f(z) = (z-\lambda)^{-1}$ with λ sufficiently large. Finally this is done with the aid of a Wiener-Hopf factorization.

We begin the first part of the proof by assuming that condition (a) is satisfied for $f(z) = z^{-1}$. Thus we assume $0 \notin \Sigma$. By (8) this implies that $\sigma^{-1} \in \mathscr{K}$ and

$$|||\sigma^{-1}||| \leqslant \|\sigma^{-1}\|_\infty^2|||\sigma|||.$$

Hence by (10)

$$\|I - W(\sigma)W(\sigma^{-1})\|_1 \leqslant \|\sigma^{-1}\|_\infty^2|||\sigma|||^2.$$

Since $\|W(\sigma)^{-1}\|$ is at most the reciprocal of the distance from 0 to Σ, which in turn is at most $\|\sigma^{-1}\|_\infty$, we obtain

$$\|W(\sigma)^{-1} - W(\sigma^{-1})\|_1 \leqslant \|\sigma^{-1}\|_\infty^3 |||\sigma|||^2 .$$

Now let f be any function for which condition (a) holds. Let Γ be a rectifiable path enclosing Σ, say at a distance δ from it, such that f is analytic inside and on Γ. Then if $\lambda \in \Gamma$ we have, since $|||\sigma - \lambda||| = |||\sigma|||$,

$$\|(\lambda - W(\sigma))^{-1} - W((\lambda - \sigma)^{-1})\|_1 \leqslant \delta^{-3}|||\sigma|||^2$$

and multiplying by $f(\lambda)/2\pi i$ and integrating give

$$\|f(W(\sigma)) - W(f \circ \sigma)\|_1 \leqslant (2\pi)^{-1}\delta^{-3}|||\sigma|||^2 \int_\Gamma |f(\lambda)|\,|d\lambda| . \tag{12}$$

We now pass to conditions (b) and consider first the case $f(x) = e^{itx}$ where t is a real parameter. We have

$$\frac{d}{dt} W(e^{it\sigma})\, e^{-itW(\sigma)} = [W(e^{it\sigma}i\sigma) - W(e^{it\sigma})W(i\sigma)]\, e^{-itW(\sigma)} =$$

$$= - iH(e^{it\sigma}, \sigma)\, e^{-itW(\sigma)} .$$

Replacing t by τ, integrating with respect to τ from 0 to t, and multiplying on the right by $e^{itW(\sigma)}$ give

$$W(e^{it\sigma}) = e^{itW(\sigma)} - i\int_0^t H(e^{i\tau\sigma}, \sigma)\, e^{i(t-\tau)W(\sigma)} d\tau. \tag{13}$$

It follows from (8) that

$$|||e^{i\tau\sigma}||| \leqslant |\tau|\, |||\sigma|||.$$

Since $e^{i(t-\tau)W(\sigma)}$ is unitary its operator norm is 1. Hence, from (10),

$$\|H(e^{i\tau\sigma}, \sigma)\, e^{i(t-\tau)W(\sigma)}\|_1 \leqslant \frac{1}{2} \cdot t^2 |||\sigma|||^2 .$$

Now for an arbitrary function f satisfying condition (b) we multiply by

$$(2\pi)^{-1}\, e^{itx}\hat{f}(t)$$

and integrate over t to deduce

$$\|f(W(\sigma)) - W(f \circ \sigma)\|_1 \leqslant (4\pi)^{-1}|||\sigma|||^2 \int t^2 |\hat{f}(t)|\, dt. \tag{14}$$

We are left with the proof of the identity

$$\text{tr}[f(\mathrm{W}(\sigma)) - \mathrm{W}(f\circ\sigma)] = \iint U(\sigma(\xi_1), \sigma(\xi_2), f)\mathrm{d}\omega(\xi_1, \xi_2) \tag{15}$$

under either condition. We show first that it suffices to prove this for entire functions f. So assume (15) holds for entire functions and that, first, condition (a) holds. Let Γ be a contour surrounding Σ, with f analytic inside and on Γ. Let $\{f_n\}$ be a sequence of polynomials converging uniformly to f inside and on Γ. Then (12) implies that

$$\text{tr}[f_n(\mathrm{W}(\sigma)) - \mathrm{W}(f_n\circ\sigma)] \to \text{tr}[f(\mathrm{W}(\sigma)) - f(\mathrm{W}(f\circ\sigma))] \tag{16}$$

as $n\to\infty$. On the other hand it follows easily from (3′) that

$$\left|\iint U(\sigma(\xi_1), \sigma(\xi_2), f)\mathrm{d}\omega(\xi_1, \xi_2)\right| \leqslant \frac{1}{2}\,|||\sigma|||^2 \max_{\Sigma}|f''| \tag{17}$$

and so

$$\iint U(\sigma(\xi_1), \sigma(\xi_2), f_n)\mathrm{d}\omega(\xi_1, \xi_2) \to \iint U(\sigma(\xi_1), \sigma(\xi_2), f)\mathrm{d}\omega(\xi_1, \xi_2). \tag{18}$$

Thus, since (15) holds for each f_n, it holds for f itself.

Suppose f satisfies condition (b) and replace f by

$$f_n = (\hat{f}\chi_{[-n,n]})^{\vee}$$

so that f_n is entire. Then (16) holds because of (14), and (18) holds because of (17) and the fact that

$$\|(f-f_n)''\|_\infty \leqslant (2\pi)^{-1}\int t^2|\hat{f}(t) - \hat{f}_n(t)|\mathrm{d}t \to 0.$$

Hence, since (15) holds for each f_n it holds for f.

We have shown that it suffices to prove (15) for f entire. Let σ_n be as in Proposition 2. We shall show that if (15) holds for each σ_n (and our given entire function f) then it holds for σ. We have the estimate

$$\|H(\sigma_n, (\sigma_n - \lambda)^{-1})\|_1 \leqslant |||\sigma_n|||\;|||(\sigma_n - \lambda)^{-1}||| \leqslant |||\sigma_n|||^2\|(\sigma_n - \lambda)^{-1}\|_\infty^2 .$$

It follows that if

$$|\lambda| > \sup_n(\|\sigma_n\|_\infty + |||\sigma_n|||) \tag{19}$$

then $I - H(\sigma_n, (\sigma_n - \lambda)^{-1})$ is invertible. Since

$$I - \mathrm{W}(\sigma_n - \lambda)\mathrm{W}((\sigma_n - \lambda)^{-1}) = H(\sigma_n, (\sigma_n - \lambda)^{-1})$$

and $W(\sigma_n - \lambda)^{-1}$ is invertible also we have

$$W(\sigma_n - \lambda)^{-1} = W((\sigma_n - \lambda)^{-1})[I - H(\sigma_n, (\sigma_n - \lambda)^{-1})]^{-1} =$$

$$= W((\sigma_n - \lambda)^{-1})\{I + H(\sigma_n, (\sigma_n - \lambda)^{-1})[I - H(\sigma_n, (\sigma_n - \lambda)^{-1})]^{-1}\}$$

and so

$$W(\sigma_n - \lambda)^{-1} - W((\sigma_n - \lambda)^{-1}) = W((\sigma_n - \lambda)^{-1})H(\sigma_n, (\sigma_n - \lambda)^{-1}) [I - H(\sigma_n, (\sigma_n - \lambda)^{-1})]^{-1}. \tag{20}$$

Now $|||\sigma_n - \sigma||| \to 0$ as $n \to \infty$. Alternatively,

$$\sigma_n(\xi_1) - \sigma_n(\xi_2) \to \sigma(\xi_1) - \sigma(\xi_2)$$

in $L_2(d\omega)$. We claim that also $|||(\sigma_n - \lambda)^{-1} - (\sigma - \lambda)^{-1}||| \to 0$. In fact

$$(\sigma_n(\xi_1) - \lambda)^{-1} - (\sigma_n(\xi_2) - \lambda)^{-1} = (\sigma_n(\xi_1) - \lambda)^{-1}(\sigma_n(\xi_2) - \lambda)^{-1}[\sigma_n(\xi_2) - \sigma_n(\xi_1)].$$

The first factor on the right converges boundedly and almost everywhere to

$$(\sigma(\xi_1) - \lambda)^{-1}(\sigma(\xi_2) - \lambda)^{-1}$$

while the second factor converges in $L_2(d\omega)$ to $\sigma(\xi_2) - \sigma(\xi_1)$. It follows that the product converges in $L_2(d\omega)$ to

$$(\sigma(\xi_1) - \lambda)^{-1}(\sigma(\xi_2) - \lambda)^{-1}[\sigma(\xi_2) - \sigma(\xi_1)] = (\sigma(\xi_1) - \lambda)^{-1} - (\sigma(\xi_2) - \lambda)^{-1},$$

which establishes the claim.

It follows from these two limit relations, and from (10), that

$$H(\sigma_n, (\sigma_n - \lambda)^{-1}) \to H(\sigma, (\sigma - \lambda)^{-1})$$

in trace norm. Moreover the fact that

$$(\sigma_n - \lambda)^{-1} \to (\sigma - \lambda)^{-1}$$

boundedly and almost everywhere assures that

$$W((\sigma_n - \lambda)^{-1}) \to W((\sigma - \lambda)^{-1})$$

strongly. Putting these things together and using (20) and its analogue with σ_n replaced by σ, we deduce that

$$W(\sigma_n - \lambda)^{-1} - W((\sigma_n - \lambda)^{-1}) \to W(\sigma - \lambda)^{-1} - W((\sigma - \lambda)^{-1})$$

in trace norm. Moreover this holds uniformly for any compact set of λ's satisfying (19). So if f is entire multiplying by $-f(\lambda)/2\pi i$ and integrating over a suitable contour give

$$f(W(\sigma_n)) - W(f\circ\sigma_n) \to f(W(\sigma)) - W(f\circ\sigma).$$

In particular

$$\text{(21)}\qquad \operatorname{tr}[f(W(\sigma_n)) - W(f\circ\sigma_n)] \to \operatorname{tr}[f(W(\sigma)) - W(f\circ\sigma)].$$

Next we show that

$$\text{(22)}\qquad \iint U(\sigma_n(\xi_1), \sigma_n(\xi_2), f)\,d\omega(\xi_1, \xi_2) \to \iint U(\sigma(\xi_1), \sigma(\xi_2), f)\,d\omega(\xi_1, \xi_2).$$

Formula (3′) gives

$$U(\sigma_n(\xi_1), \sigma_n(\xi_2), f) =$$

$$= (\sigma_n(\xi_1) - \sigma_n(\xi_2))^2 \int_0^1 f''((1-\theta)\sigma_n(\xi_1) + \theta\sigma_n(\xi_2))[\theta\log\theta + (1-\theta)\log(1-\theta)]\,d\theta,$$

with a similar identity if σ_n is replaced by σ. Now

$$(\sigma_n(\xi_1) - \sigma_n(\xi_2))^2 \to (\sigma(\xi_1) - \sigma(\xi_2))^2$$

in $L_1(d\omega)$ and it follows from properties (ii) and (iii) of the σ_n that the integral on the right side converges boundedly and almost everywhere to the corresponding integral with σ_n replaced by σ. It follows that

$$U(\sigma_n(\xi_1), \sigma_n(\xi_2), f) \to U(\sigma(\xi_1), \sigma(\xi_2), f)$$

in $L_1(d\omega)$ and (22) is established.

Combining this with (21) shows that, as claimed, to prove (15) for σ it suffices to prove it for each σ_n. So we shall drop the subscript and simply assume that $\check{\sigma} = c\delta + s$ where s is an L_2 function supported in a compact subset of $\mathbf{R}\setminus\{0\}$. In fact, it is convenient to assume less about s, namely that

$$\int |s(x)|\,dx < \infty, \quad \int |x|\,|s(x)|^2\,dx < \infty.$$

The corresponding functions σ form a Banach algebra $\mathscr{A}$ under pointwise multiplication with the norm

$$\|\sigma\| = |c| + \int |s(x)|\,dx + \left\{\iint |x|\,|s(x)|^2\,dx\right\}^{1/2}.$$

As before it suffices to prove (15) for the functions $f(z) = (z - \lambda)^{-1}$ (with λ sufficiently large), or with $f(z) = z^{-1}$ and σ replaced by $\sigma - \lambda$, or with $f(z) = z^{-1}$ and σ replaced by $(\sigma - \lambda)/(c - \lambda)$. Thus we may assume that $c = 1$ and (since λ may be arbitrarily large) that

(23) $$\|1 - \sigma\| < 1.$$

On the subalgebra $\mathscr{A}_0$ of $\mathscr{A}$ consisting of those σ for which the corresponding δ-summand of σ vanishes there are the continuous mappings $\sigma \to \sigma^{\pm}$ given by

$$\sigma^{\pm} = (\check{\sigma}\chi_{\mathbf{R}^{\pm}})^{\wedge}.$$

Since our σ has corresponding $c = 1$ and satisfies (23) there is a $\log\sigma \in \mathscr{A}_0$ and we have the Wiener-Hopf factorization $\sigma = \sigma_{-}\sigma_{+}$ where

$$\sigma_{\pm} = \exp\{(\log\sigma)^{\pm}\}.$$

Moreover using well known properties of these factors gives

$$W(\sigma)^{-1} - W(\sigma^{-1}) = W(\sigma_{+}^{-1})W(\sigma_{-}^{-1}) - W(\sigma_{+}^{-1}\sigma_{-}^{-1}) = -H(\sigma_{+}^{-1}, \sigma_{-}^{-1}).$$

The trace of the right hand side is given by (11). Since $(\sigma_{+}^{-1})^{\vee}$ vanishes for $x < 0$ we can write this as

(24) $$\operatorname{tr}[W(\sigma)^{-1} - W(\sigma^{-1})] = -\int_{-\infty}^{\infty} x(\sigma_{+}^{-1})^{\vee}(x)(\sigma_{-}^{-1})^{\vee}(-x)\mathrm{d}x.$$

We now consider the bilinear form on $\mathscr{A}$

$$(\varphi, \psi) = \int x\check{\varphi}(x)\check{\psi}(-x)\mathrm{d}x$$

and prove the identity

$$(\mathrm{e}^{\varphi}, \mathrm{e}^{\psi}) = \frac{1}{2}(\mathrm{e}^{\varphi+\psi}, \psi - \varphi).$$

We may suppose that $\check{\varphi}$ and $\check{\psi}$ have no δ-summand. If $x\check{\varphi}(x)$ and $x\check{\psi}(x)$ both belong to L_2 then $\varphi', \psi' \in L_2$ and

$$(x\check{\varphi})^{\wedge} = \mathrm{i}\varphi', \quad (x\check{\psi})^{\wedge} = \mathrm{i}\psi'.$$

Parseval's identity and integration by parts give therefore

$$(\varphi, \psi) = \frac{\mathrm{i}}{2\pi}\int \varphi'(\xi)\psi(\xi)\mathrm{d}\xi = -\frac{\mathrm{i}}{2\pi}\int \varphi(\xi)\psi'(\xi)\mathrm{d}\xi =$$

$$= \frac{\mathrm{i}}{\pi}\int [\varphi'(\xi)\psi(\xi) - \varphi(\xi)\psi'(\xi)]\mathrm{d}\xi.$$

Replacing φ resp. ψ by $e^{\varphi}-1$ resp. $e^{\psi}-1$ and using

$$\int \varphi' e^{\varphi} d\xi = \int \psi' e^{\psi} d\xi = 0$$

give

$$(e^{\varphi}, e^{\psi}) = \frac{i}{\pi} \int e^{\varphi(\xi)+\psi(\xi)}[\varphi'(\xi) - \psi'(\xi)] d\xi = \frac{1}{2}(e^{\varphi+\psi}, \psi - \varphi).$$

To remove the restriction that $x\check{\varphi}(x)$ and $x\check{\psi}(x)$ belong to L_2 we need only observe that such pairs φ, ψ are dense in $\mathscr{A}\times\mathscr{A}$ and both sides of the identity are continuous on $\mathscr{A}\times\mathscr{A}$.

Applying the identity with

$$\varphi = -(\log\sigma)^{+}, \quad \psi = -(\log\sigma)^{-}$$

shows that (24) may be written

$$\text{(25)} \qquad \operatorname{tr}[W(\sigma)^{-1} - W(\sigma^{-1})] = \frac{1}{2}(\sigma^{-1}, (\log\sigma)^{-} - (\log\sigma)^{+})$$

and since

$$[(\log\sigma)^{-} - (\log\sigma)^{+}]^{\vee} = -\operatorname{sgn} x(\log\sigma)^{\vee}(x)$$

we have

$$(\sigma^{-1}, (\log\sigma)^{-} - (\log\sigma)^{+}) = \int |x|(\sigma^{-1})^{\vee}(x)(\log\sigma)^{\vee}(-x)dx.$$

Hence, because of identity (3), relation (25) may be rewritten

$$\operatorname{tr}[W(\sigma)^{-1} - W(\sigma^{-1})] = \iint [\sigma(\xi_1)^{-1} - \sigma(\xi_2)^{-1}][\log\sigma(\xi_1) - \log\sigma(\xi_2)] d\omega(\xi_1, \xi_2).$$

However it is an elementary fact that

$$U(\alpha, \beta, z)^{-1} = (\alpha^{-1} - \beta^{-1})(\log\alpha - \log\beta)$$

and so we have established (15) in the required special case and Theorem 1 is proved.

THE LIMIT THEOREMS

Recall that Theorem 2 is equivalent to the assertion that

$$\text{(26)} \qquad \lim_{\alpha\to\infty} \operatorname{tr}[f(W_\alpha(\sigma)) - W_\alpha(f\circ\sigma)] = 2\operatorname{tr}[f(W(\sigma)) - W(f\circ\sigma)]$$

if either condition (a) or (b) holds.

To prove this under condition (a) we consider first the case $f(z) = z^{-1}$, assuming that $0 \notin \Sigma$. Then 0 does not belong to the convex hull of the essential range of σ^{-1} so both $W(\sigma)$ and $W(\sigma^{-1})$ are invertible. Since

$$I - W(\sigma)W(\sigma^{-1}) = H(\sigma, \sigma^{-1})$$

we conclude that $I - H(\sigma, \sigma^{-1})$ is invertible and

$$W(\sigma)^{-1} - W(\sigma^{-1}) = W(\sigma^{-1})H(\sigma, \sigma^{-1})[I - H(\sigma, \sigma^{-1})]^{-1}. \tag{27}$$

The analogue of (9) for finite Wiener-Hopf operators is

$$W_\alpha(\sigma_1\sigma_2) - W_\alpha(\sigma_1)W_\alpha(\sigma_2) = P_\alpha H(\sigma_1, \sigma_2)P_\alpha + Q_\alpha H(\sigma_2, \sigma_1)^t Q_\alpha \tag{28}$$

where P_α is the projection from $L_2(\mathbf{R}^+)$ to $L_2(0, \alpha)$, where Q_α is P_α followed by the unitary operator $\varphi(x) \to \varphi(\alpha - x)$, and where the "t" denotes transpose. This is easily checked. In particular we have

$$I - W_\alpha(\sigma)W_\alpha(\sigma^{-1}) = P_\alpha H(\sigma, \sigma^{-1})P_\alpha + Q_\alpha H(\sigma^{-1}, \sigma)^t Q_\alpha = A_\alpha + B_\alpha,$$

say. The analogue of (20) is now

$$W_\alpha(\sigma)^{-1} - W_\alpha(\sigma^{-1}) = W_\alpha(\sigma^{-1})(A_\alpha + B_\alpha)[I - (A_\alpha + B_\alpha)]^{-1}. \tag{29}$$

Since $Q_\alpha \to 0$ weakly as $\alpha \to \infty$ and $H(\sigma^{-1}, \sigma)$ is compact, both B_α and B_α^* converge to 0 strongly. And since $H(\sigma, \sigma^{-1})$ is trace class it is an easy consequence of this that

$$\|A_\alpha B_\alpha\|_1 \to 0, \quad \|B_\alpha A_\alpha\|_1 \to 0. \tag{30}$$

Now since $I - H(\sigma, \sigma^{-1})$ is an invertible operator on $L_2(\mathbf{R}^+)$ and

$$A_\alpha \to H(\sigma, \sigma^{-1})$$

in trace norm (and so in operator norm) the operators $I - A_\alpha$ are invertible with uniformly bounded inverses. Similarly for B_α. It follows from this and (30) that

$$\|(A_\alpha + B_\alpha)[I - (A_\alpha + B_\alpha)]^{-1} - A_\alpha(I - A_\alpha)^{-1} - B_\alpha(I - B_\alpha)^{-1}\|_1 \to 0.$$

Hence from (29)

$$\operatorname{tr}[W_\alpha(\sigma)^{-1} - W_\alpha(\sigma^{-1})] = \operatorname{tr} W_\alpha(\sigma^{-1})A_\alpha(I - A_\alpha)^{-1} + \operatorname{tr} W_\alpha(\sigma^{-1})B_\alpha(I - B_\alpha)^{-1} + o(1)$$

as $\alpha \to \infty$. Since as $\alpha \to \infty$

$$W_\alpha(\sigma^{-1})A_\alpha(I - A_\alpha)^{-1} \to W(\sigma^{-1})H(\sigma, \sigma^{-1})[I - H(\sigma, \sigma^{-1})]^{-1} = W(\sigma)^{-1} - W(\sigma^{-1})$$

in trace norm we have

$$\operatorname{tr}W_\alpha(\sigma^{-1})A_\alpha(I - A_\alpha)^{-1} \to \operatorname{tr}[W(\sigma)^{-1} - W(\sigma^{-1})].$$

As for the expression involving B_α note that

$$W_\alpha(\sigma^{-1})Q_\alpha = Q_\alpha W_\alpha(\sigma^{-1})^t,$$

that Q_α is P_α followed by a unitary operator, and that the trace is invariant under transpose. Using these facts we deduce that

$$\operatorname{tr}W_\alpha(\sigma^{-1})B_\alpha(I - B_\alpha)^{-1} \to \operatorname{tr}[I - H(\sigma^{-1}, \sigma)]^{-1}H(\sigma^{-1}, \sigma)W(\sigma^{-1}).$$

On the other hand the identity

$$I - W(\sigma^{-1})W(\sigma) = H(\sigma^{-1}, \sigma)$$

gives

$$W(\sigma)^{-1} - W(\sigma^{-1}) = [I - H(\sigma^{-1}, \sigma)]^{-1}H(\sigma^{-1}, \sigma)W(\sigma^{-1})$$

and so we have shown that also

$$\operatorname{tr}W_\alpha(\sigma^{-1})B_\alpha(I - B_\alpha)^{-1} \to \operatorname{tr}[W(\sigma)^{-1} - W(\sigma^{-1})].$$

This proves (26) in the special case $f(z) = z^{-1}$ if $0 \notin \Sigma$. It follows that (26) holds for $f(z) = (z - \lambda)^{-1}$ if $\lambda \notin \Sigma$ and it is easy to check that it holds uniformly for λ belonging to a compact subset of the complement of Σ. Cauchy's formula therefore gives (26) for any f satisfying condition (a).

For the proof under condition (b) we obtain as the analogue of (13) the identity

$$e^{itW_\alpha(\sigma)} - W_\alpha(e^{it\sigma}) = i\int_0^t P_\alpha H(e^{i\tau\sigma}, \sigma)P_\alpha e^{i(t-\tau)W_\alpha(\sigma)}d\tau +$$

$$+ i\int_0^t Q_\alpha H(\sigma, e^{i\tau\sigma})^t Q_\alpha e^{i(t-\tau)W_\alpha(\sigma)}d\tau.$$

As $\alpha \to \infty$

$$P_\alpha H(e^{i\tau\sigma}, \sigma)P_\alpha \to H(e^{i\tau\sigma}, \sigma)$$

in the trace norm and

$$e^{i(t-\tau)W_\alpha(\sigma)} \to e^{i(t-\tau)W(\sigma)}$$

strongly, uniformly for bounded τ. Hence the trace of the first term in the right has the limit

$$\operatorname{tr} i\int_0^t H(e^{i\tau\sigma}, \sigma)e^{i(t-\tau)W(\sigma)}d\tau = \operatorname{tr}[e^{itW(\sigma)} - W(e^{it\sigma})].$$

(See (13).) As before the trace of the second term has the same limit and so

$$\mathrm{tr}[e^{itW_\alpha(\sigma)} - W_\alpha(e^{it\sigma})] \to 2\mathrm{tr}[e^{itW(\sigma)} - W(e^{it\sigma})].$$

Since the trace on the left is $O(t^2)$ uniformly in α we can multiply by $\hat{f}(t)$ and integrate, taking the limit under the integral sign, and so deduce (26). This completes the proof of Theorem 2.

To prove Theorem 3, define

$$\sigma_\lambda = e^{\lambda \log\sigma}.$$

If λ is a sufficiently small complex number then the essential range of σ_λ lies in the right half-plane and so condition (a) is satisfied if $f(z) = \log z$, the principal value of the logarithm being taken. The conclusion of Theorem 2 may then be written

$$\lim_{\alpha\to\infty} \mathrm{tr}[\log W_\alpha(\sigma_\lambda) - W_\alpha(\log\sigma_\lambda)] = \int_0^\infty x(\log\sigma_\lambda)^\vee(x)(\log\sigma_\lambda)^\vee(-x)\,dx. \tag{30}$$

Now for any two operators A and B whose sum is trace class, the operator $e^A e^B$ is determinant class and

$$\det e^A e^B = \exp \mathrm{tr}(A + B).$$

If we apply this identity to

$$A = \log W_\alpha(\sigma_\lambda), \quad B = -W_\alpha(\log\sigma_\lambda)$$

and use (30) we see that (4) holds for σ_λ if λ is sufficiently small.

Next consider the identity

$$\frac{d}{d\lambda} W_\alpha(\sigma_\lambda)e^{-W_\alpha(\log\sigma_\lambda)} = [W_\alpha(\sigma_\lambda\log\sigma) - W_\alpha(\sigma_\lambda)W_\alpha(\log\sigma)]e^{-W_\alpha(\log\sigma_\lambda)}. \tag{31}$$

For λ belonging to any compact subset of the complex plane the second factor on the right is uniformly bounded in operator norm while the first factor is uniformly bounded in trace norm. (See (28).) It follows that the operators

$$I - W_\alpha(\sigma_\lambda)e^{-W_\alpha(\log\sigma_\lambda)} \tag{32}$$

are uniformly bounded in trace norm and so the functions

$$\det W_\alpha(\sigma_\lambda)e^{-W_\alpha(\log\sigma_\lambda)}$$

are uniformly bounded. Since they are entire functions of λ (this follows from the fact, a consequence of (31), that (32) are analytic trace class operator-valued functions of λ) the relation (4) for small λ implies the relation for all λ, and in particular for $\lambda = 1$.

Finally if $\log\sigma \in L_1(\mathbf{R})$ then $W_\alpha(\log\sigma)$ is trace class and so

$$e^{-W_\alpha(\log\sigma)}$$

is determinant class and

$$\det\left(e^{-W_\alpha(\log\sigma)}\right) = \exp\{-\operatorname{tr} W_\alpha(\log\sigma)\} = \exp\left\{-\frac{\alpha}{2\pi}\int \log\sigma(\xi)d\xi\right\}.$$

Thus the last assertion of the theorem is a consequence of the preceding one.

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