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PRESERVATION OF ESSENTIAL MATRIX RANGES BY COMPACT PERTURBATIONS

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1. INTRODUCTION

In [12], it was shown that if T is a bounded operator on a separable Hilbert space, then for any positive integer n, there exists a compact operator K such that the first n matrix ranges of $T+K$ are the same as the first n essential matrix ranges of T. They then asked if for every T there exists a compact operator K such that all the matrix ranges of $T+K$ are the same as all the essential matrix ranges of T. In this paper we obtain several partial results concerning this question.

We prove that to answer the above question it is sufficient to assume that T is block-diagonal (Corollary 3.5). For a block-diagonal operator we give a complete characterization of whether of not such a compact perturbation exists in terms of a distance formula (Theorem 3.13). These results allow us to give a somewhat different proof of the result of [12] referred to above (Theorem 3.12).

For a general operator we attempt to show that the existence of such a compact operator is completely determined by the essential matrix ranges of the operator. We find a formula involving the essential matrix ranges which implies the existence of the desired compact (Theorem 3.15). In the converse direction we only succeed in showing that if the essential matrix ranges of some operator fail to satisfy this formula, then there exists an operator for which no such compact perturbation can be found (Theorem 3.16).

Finally, we show that the original question can be re-formulated strictly in terms of sets of finite matrices (Corollary 3.19).

2. BLOCK DIAGONALIZATIONS

Let $\mathscr{H}$ denote a separable, complex Hilbert space, $\mathscr{L}(\mathscr{H})$ the bounded linear operators on $\mathscr{H}$, $\mathscr{K}(\mathscr{H})$ the ideal of compact operators, and $Q(\mathscr{H})$ the quotient (Calkin) algebra. For T in $\mathscr{L}(\mathscr{H})$, we shall use $\dot{T}$ to denote its image in $Q(\mathscr{H})$. We shall use M_n to denote the C^*-algebra of $n \times n$ matrices.

If $\mathscr{A}$ and $\mathscr{B}$ are C^*-algebras, then a linear map $\varphi: \mathscr{A} \to \mathscr{B}$ is called *completely positive* provided the maps $\varphi \otimes 1_n: \mathscr{A} \otimes M_n \to \mathscr{B} \otimes M_n$ are positive for every integer n.

We shall call a map $\delta: \mathscr{A} \to \mathscr{L}(\mathscr{H})$ *block diagonal*, if there is a sequence $\{\mathscr{H}_n\}$ of finite dimensional subspaces of $\mathscr{H}$ with

$$\mathscr{H} = \sum_{n=1}^{\infty} \oplus \mathscr{H}_n$$

and maps

$$\delta_n: \mathscr{A} \to \mathscr{L}(\mathscr{H}_n)$$

such that

$$\delta = \sum_{n=1}^{\infty} \oplus \delta_n .$$

In [4] it was shown that every operator is almost a compression of a block diagonal operator. The following is a summary of those results.

THEOREM 2.1. [4, Proposition 3] *Let $\mathscr{A}$ be a separable, unital C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ containing $\mathscr{K}(\mathscr{H})$, let $T \in \mathscr{A}$ and let $\varepsilon > 0$ be given. Then there is a separable, Hilbert space $\mathscr{H}'$, a pair of unital, completely positive maps,*

$$\delta: \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H}'), \quad \lambda: \mathscr{L}(\mathscr{H}') \to \mathscr{L}(\mathscr{H})$$

and an isometry

$$V: \mathscr{H} \to \mathscr{H}'$$

such that:

1. $\lambda \circ \delta(A) - A$ *is compact, for every* $A \in \mathscr{A}$,
2. $\|\lambda \circ \delta(T) - T\| < \varepsilon$,
3. δ *is a block diagonal map,*
4. $\lambda(X) = V^*XV$ *for every* $X \in \mathscr{L}(\mathscr{H}')$,
5. VV^* *essentially reduces* $\delta(\mathscr{A})$.

We shall call a pair of maps (δ, λ) satisfying 1 — 3 of the above Theorem a *localizing pair*. We shall need localizing pairs with an additional property. A map

$$\delta: \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H}')$$

will be called *compact preserving*, if

$$\delta(\mathscr{K}(\mathscr{H})) \subseteq \mathscr{K}(\mathscr{H}').$$

Clearly, the compact preserving maps are precisely those maps for which the map

$$\dot{\delta}: Q(\mathscr{H}) \to Q(\mathscr{H}')$$

given by

$$\dot{\delta}(\dot{T}) = \delta(\dot{T})$$

is well defined. A localizing pair (δ, λ) will be called compact preserving provided both maps are compact preserving.

THEOREM 2.2. *The maps δ and λ of Theorem 2.1 can be constructed to be compact preserving.*

Proof. Since isometric compressions of compact operators are compact, λ is necessarily compact preserving by [4].

The key ingredient to the proof of Theorem 2.1 is the construction of the map δ from a countable, quasi-central approximate unit [4, p. 329] for $\mathscr{K}(\mathscr{H})$ in $\mathscr{A}$ consisting of finite rank operators. It will suffice to show that by a perspicuous choice of the quasi-central approximate unit, the construction of Theorem 2.1 will yield a map δ which is compact preserving. In the construction of δ [4, p. 334], given an appropriate increasing, countable, quasi-central approximate unit, $\{F_n\}$ of finite rank operators, one sets

$$E_n = (F_n - F_{n-1})^{1/2},$$

defines δ_n to be the compression of $\mathscr{L}(\mathscr{H})$ to the space,

$$\mathscr{H}_n = E_n\mathscr{H}$$

and sets

$$\delta = \sum_{n=1}^{\infty} \oplus \delta_n .$$

We claim that if the sequence $\{F_n\}$ is chosen such that, in addition, the range projections of the E_n's converge to 0 in the strong operator topology, then δ will be compact preserving. Indeed, if this is so, then for any $K \in \mathscr{K}(\mathscr{H})$, we have that $\|\delta_n(K)\|$ will tend to 0. Thus, since $\delta_n(K)$ is finite rank for all n, we have

$$\delta(K) = \sum_{n=1}^{\infty} \oplus \delta_n(K)$$

is compact.

To construct $\{F_n\}$ with this additional property, we return to the proof of the existence of a quasi-central approximate unit in [4, Theorem 1]. We let $\{e_n\}$ be an orthogonal basis for $\mathscr{H}$, let $\{A_n\}$ be dense in $\mathscr{A}$, let $\{K_n\}$ be dense in $\mathscr{K}(\mathscr{H})$, and let Q_j be the projection onto the span of $\{e_1, \ldots, e_j\}$. We have that $\{Q_j\}$ is an approximate unit for $\mathscr{K}(\mathscr{H})$ and that its convex hull, Λ, with the usual ordering on positive operators is a convex approximate unit for $\mathscr{K}(\mathscr{H})$ [4, p. 330].

Hence, there exists $F_1 \in A_1$ such that

$$\|F_1K_1 - K_1\| \leqslant 1$$

and

$$\|F_1A_1 - A_1F_1\| \leqslant 1/2,$$

[4, Lemma 1]. Inductively,

$$A_n = \{F \in A \mid F \geqslant F_n,\ F \geqslant Q_n\}$$

is a convex approximate unit for $\mathscr{K}(\mathscr{H})$, since it is a cofinal subset of A, and hence [4, p. 331] there exists $F_{n+1} \in A_n$ such that

$$\|F_{n+1}K_i - K_i\| \leqslant 1/n + 1$$

and

$$\|F_{n+1}A_i - A_iF_{n+1}\| \leqslant 1/n + 1 \quad \text{for } 1 \leqslant i \leqslant n + 1.$$

Note that since $F_{n+1} \geqslant Q_n$ the sequence $\{F_n\}$ and any subsequence of it, will have the desired property that the range projections of $E_n = (F_n - F_{n-1})^{1/2}$ tend strongly to 0.

A careful reading of [4, p. 333] shows that if this quasi-central approximate unit is used in the proof of [4, Theorem 2] then the block diagonal map of [4, Proposition 3] will be compact preserving. ▨

REMARK 2.3. We note that in the above construction, if one sets

$$A_n = \{F \in A \mid F \geqslant P_n,\ F \geqslant Q_n\}$$

where P_n is the range projection of F_n, then $E_n \cdot E_k = 0$ for $|n - k| \geqslant 2$. This guarantees that $\mathscr{H}_n$ is orthogonal to $\mathscr{H}_k$ for $|n - k| \geqslant 2$, which means that the map λ of Theorem 2.1 is tri-block diagonal. Hence, the composition $\lambda \circ \delta$ will map $\mathscr{A}$ into a tri-block diagonal subspace. Such maps appear in [12], [13] and [14]. Other variations are also possible by a suitable choice of the A_n's.

We recall that, if $\mathscr{A}$ is a unital C^*-algebra and $T \in \mathscr{A}$, then the *n-th matrix range* of T is the set

$$W^n(T) = \{L \in M_n \,.\, L = \varphi(T) \text{ for some unital, completely positive map, } \varphi : \mathscr{A} \to M_n\}.$$

Since completely positive maps into M_n can always be extended to larger C^*-algebras [2, Theorem 1.2.3] the definition of $W^n(T)$ is independent of the particular C^*-algebra that T belongs to. If $T \in \mathscr{L}(\mathscr{H})$, then the *n-th essential matrix range* of T is defined to be $W^n(\dot{T})$. The n-th matrix range of T is a compact, convex set bounded by $\|T\|$. These sets enjoy an additional property referred to variously as matricial convexity [6, p. 753] or C^*-convexity [9]. A subset $\mathscr{S} \subseteq M_n$ is *C^*-convex*, if for $A_1, \ldots, A_m$

in M_n with

$$\sum_{j=1}^{m} A_j^* A_j = 1_n$$

and $X_1, \ldots, X_m$ in $\mathcal{S}$ we have that

$$\sum_{j=1}^{m} A_j^* X_j A_j$$

belongs to $\mathcal{S}$.

The following will allow us to reduce many questions concerning matrix ranges to the case of a block diagonal operator.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a separable C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ containing $\mathcal{K}(\mathcal{H})$ and let (δ, λ) be a compact preserving, localizing pair for $\mathcal{A}$. Then for any $A \in \mathcal{A}$,*

$$W^n(\dot{\delta}(\dot{A})) = W^n(\dot{A})$$

and

$$W^n(\delta(A)) \subseteq W^n(A).$$

Proof. This follows from the fact that compositions of completely positive maps are completely positive, and the identities, $\dot{\lambda} \circ \dot{\delta}(\dot{A}) = \dot{A}$, $\delta \circ \lambda(\delta(\dot{A})) = \delta(\dot{A})$. ▨

3. THE MATRIX RANGE PRESERVATION PROPERTY

In this section we study the problem of finding compact perturbations of an operator whose matrix ranges are equal to the essential matrix ranges of the operator. By using the block diagonalization techniques of the previous section, we are able to characterize this problem in terms of some distance formulas. We are also able to give a new proof of one of the results of [12].

DEFINITION 3.1. Let $T \in \mathcal{L}(\mathcal{H})$ and let $j \geqslant 1$ be a positive integer. We shall say that T has the *j-th matrix range preservation property* (*j-MRPP*), provided that there exists a compact operator K such that

$$W^n(T + K) = W^n(\dot{T}),$$

for all $n \leqslant j$. If there exists a compact operator K such that,

$$W^n(T + K) = W^n(\dot{T}),$$

for all n, then we say that T has the *matrix range preservation property* (*MRPP*).

We note that if

$$W^j(T+K) = W^j(\dot{T}),$$

then

$$W^n(T+K) = W^n(\dot{T})$$

for any $n \leqslant j$, since any completely positive map into M_n will factor through M_j.

In [12, § 5] it was shown that every operator has the j-MRPP and they asked whether every operator has the MRPP. It was also noted there that the MRPP is equivalent to the existence of a certain type of lifting of T, [13, § 1].

Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras, $\mathscr{S} \subseteq \mathscr{A}$ a self-adjoint subspace containing the identity. A unital linear map $\varphi: \mathscr{S} \to \mathscr{B}$ is *completely isometric* if

$$\varphi \otimes 1_n: \mathscr{S} \otimes M_n \to \mathscr{B} \otimes M_n$$

is isometric for all n. Completely isometric maps are necessarily completely positive [2, Proposition 1.2.8]. By [3, Theorem 2.4.2], if $\mathscr{S}$ is the span of 1, S, and S^*, then φ is completely isometric if and only if

$$W^n(\varphi(S)) = W^n(S)$$

for all $S \in \mathscr{S}$. Thus, if $\mathscr{S}$ is the span of $\dot{1}$, $\dot{T}$, and $\dot{T}^*$ in $Q(\mathscr{H})$, then T has the MRPP if and only if $\mathscr{S}$ has a unital, completely isometric lifting of the quotient map. Also, it is not difficult to show that if $\mathscr{S}$ is a 3-dimensional, self-adjoint subspace of $Q(\mathscr{H})$ containing the unit, then $\mathscr{S}$ is the span of $\dot{1}$, $\dot{T}$, and $\dot{T}^*$ for some T. Thus, every operator has the MRPP if and only if every 3-dimensional, self-adjoint subspace of $Q(\mathscr{H})$ containing the unit has a unital completely isometric lifting of the quotient map.

We shall now show that there exists a 5-dimensional, self-adjoint subspace of $Q(\mathscr{H})$ containing the unit, which has no unital completely isometric lifting of the quotient map. In spite of this result, based on some other results of this section, we conjecture that all operators have the MRPP.

The following is perhaps well known.

LEMMA 3.2. *Let $\mathscr{A}$ be a unital C^*-algebra, and let $\varphi: \mathscr{A} \to \mathscr{A}$ be a unital, completely positive map. Then $\{a \mid \varphi(a) = a,\ \varphi(a^*a) = a^*a,\ \varphi(aa^*) = aa^*\}$ is a C^*-subalgebra of $\mathscr{A}$.*

Proof. The above set is clearly norm-closed and $*$-closed. Since

$$\varphi(a^*a) = \varphi(a^*)\,\varphi(a)$$

if and only if

$$\varphi(ba) = \varphi(b)\,\varphi(a)$$

for all $b \in \mathscr{A}$ [7], we can easily see that the set is also closed under products. ▨

THEOREM 3.3. *There exists a 5-dimensional self-adjoint subspace of $Q(\mathscr{H})$, containing the unit, which does not have a unital, completely isometric lifting of the quotient map.*

Proof. In [1], an example is given of a unital, C^*-subalgebra $\mathscr{A}$ of $Q(\mathscr{H})$, which does not have a unital, completely positive lifting of the quotient map. Furthermore, $\mathscr{A}$ is generated by two unitaries and a projection, and so by [10, Theorem 1], $\mathscr{A} \otimes M_2$ is singly generated. Let a be this generator.

We claim that the span, $\mathscr{S}$, of 1, a, a^*, a^*a, and aa^* does not have a completely isometric lifting of the quotient map from $\mathscr{L}(\mathscr{H}) \otimes M_2$ to $Q(\mathscr{H}) \otimes M_2$. Assuming the contrary, by [2, Theorem 1.2.3] this completely isometric lifting could be extended to a completely positive map φ from $Q(\mathscr{H}) \otimes M_2$ to $\mathscr{L}(\mathscr{H}) \otimes M_2$. Since the composition of φ with the quotient map is completely positive and the identity on $\mathscr{S}$, by Lemma 3.1, this composition must be the identity on all of $\mathscr{A} \otimes M_2$, i.e., φ is a completely positive lifting of the quotient map on $\mathscr{A} \otimes M_2$. But the map which sends a to the compression of $\varphi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ to the $(1, 1)$-entry, yields a completely positive lifting of the quotient map on $\mathscr{A}$. This contradiction completes the proof of the theorem. ▨

The following result essentially allows us to reduce the study of the MRPP and j-MRPP to block diagonal operators.

PROPOSITION 3.4. *Let $\mathscr{A}$ be a unital, separable C^*-subalgebra of $\mathscr{L}(\mathscr{H})$ containing $\mathscr{K}(\mathscr{H})$, let (δ, λ) be a compact preserving, localizing pair for $\mathscr{A}$, and let $T \in \mathscr{A}$. Then T has the j-*MRPP *(respectively,* MRPP) *if and only if $\delta(T)$ has the j-*MRPP *(respectively,* MRPP).

Proof. If there exists a compact K such that

$$W^n(\dot{T}) = W^n(T + K), \quad \text{for } 1 \leqslant n \leqslant j,$$

then

$$W^n(\delta(\dot{T})) \subseteq W^n(\delta(T + K)) \subseteq W^n(T + K) = W(\dot{T}) = W^n(\delta(\dot{T}))$$

by Proposition 2.4, and so $\delta(K)$ is a compact which works for $\delta(T)$.

The converse and the proof for the MRPP follows similarly. ▨

COROLLARY 3.5. *Every operator has the j-*MRPP *(respectively,* MRPP) *if and only if every block diagonal operator has the j-*MRPP *(respectively,* MRPP).

In the remainder of this section we shall give some partial characterizations of these preservation properties in terms of some asymptotic distance formulas.

For a metric space (X, d), $x \in X$, and $Y \subseteq X$, we define

$$\mathrm{d}(x, Y) = \inf\{\mathrm{d}(x, y) \mid y \in Y\}.$$

If $W \subseteq X$, then we set

$$d(W, Y) = \sup\{d(w, Y) \mid w \in W\}.$$

We now define some new objects, which in some sense measure, for an operator T, how much the set $W^n(T)$ is determined by $W^j(T)$.

DEFINITION 3.6. For T an element of a C^*-algebra $\mathscr{A}$, and strictly positive integers n and j, we set,

$$\mathscr{S}_{n,j}(T) = \{A \in M_n \mid W^j(A) \subseteq W^j(T)\}.$$

The following Proposition summarizes some of the structure of these sets.

LEMMA 3.7. *Let*

$$B = \sum_{k=1}^{\infty} \oplus A_k$$

acting on

$$\mathscr{H} = \sum_{k=1}^{\infty} \oplus \mathscr{H}_k .$$

If for some T and some j, $W^j(A_k) \subseteq W^j(T)$ for all k, then $W^j(B) \subseteq W^j(T)$.

Proof. Let A_k' be the direct sum of j copies of A_k acting on the direct sum of j copies of $\mathscr{H}_k$, $\mathscr{H}_k'$, and let

$$B' = \sum_{k=1}^{\infty} \oplus A_k' .$$

Since the C^*-algebras generated by B' and B are $*$-isomorphic, $W^j(B) = W^j(B')$ and it will suffice to show that $W^j(B') \subseteq W^j(T)$. Let

$$V : \mathbf{C}^j \to \mathscr{H}' = \sum_{k=1}^{\infty} \oplus \mathscr{H}_k'$$

be an isometry and let P_k be the projection from $\mathscr{H}'$ to $\mathscr{H}_k'$. Since $\dim \mathscr{H}_k' \geqslant j$, we may use polar decomposition to write $P_k V = V_k R_k$, where $R_k \in M_j$ is positive and

$$V_k : \mathbf{C}^j \to \mathscr{H}_k'$$

is an isometry, and

$$\sum_{k=1}^{\infty} R_k^2 = 1.$$

We have that,

$$V_k^* A_k' V_k \in W^j(A_k') = W^j(A_k) \subseteq W^j(T).$$

Set

$$L_k = V_k^* A'_k V_k,$$

then

$$V^* B' V = \sum_{k=1}^{\infty} R_k L_k R_k$$

which belongs to $W^j(T)$ by [9, Remark 10]. Since

$$V^* B' V \in W^j(T)$$

for all isometries V, an application of [6, Theorem 3.5] yields the result. ▨

PROPOSITION 3.8. *For all strictly positive integers n and j, the sets* $\mathcal{S}_{n,j}(T)$ *are compact,* C^**-convex and contained in the closed balls of radius* $2\|T\|$ *for* $j = 1$ *and* $\|T\|$ *for* $j \geqslant 2$. *Furthermore,*

$$W^n(T) \subseteq \mathcal{S}_{n,j}(T)$$

with equality when $j \geqslant n$.

Proof. It is easy to see that the above sets are all closed. By [15, Theorem 2.2] for any operator A, and any $j \geqslant 2$,

$$\sup\{\|L\| \mid L \in W^j(A)\} = \|A\|.$$

Hence, if $j \geqslant 2$ and $A \in \mathcal{S}_{n,j}(T)$, then

$$\|A\| \leqslant \sup\{\|L\| \mid L \in W^j(T)\} = \|T\|.$$

The corresponding statement for $j = 1$, follows similarly by using the inequalities [8, p. 114],

$$\frac{1}{2}\|A\| \leqslant \sup\{|\lambda| \mid \lambda \in W^1(A)\} \leqslant \|A\|.$$

This also establishes compactness.

If $A \in W^n(T)$, then necessarily, $W^j(A) \subseteq W^j(T)$ since compositions of completely positive maps are completely positive, and so $W^n(T) \subseteq \mathcal{S}_{n,j}(T)$. Equality is clear if $n = j$, so let $j > n$ and set $k = j - n$. If $A \in \mathcal{S}_{n,j}(T)$ and L is any element of $W^k(A)$, then

$$(A \oplus L) \in W^j(A) \subseteq W^j(T).$$

Since compressions are completely positive maps, and compositions of completely positive maps are completely positive, $A \in W^n(T)$.

Finally, to see that $\mathcal{S}_{n,j}(T)$ is C^*-convex, let $A, B \in \mathcal{S}_{n,j}(T)$ and let $X, Y \in M_n$ with $X^*X + Y^*Y = 1$. By Lemma 3.7,

$$W^j(A \oplus B) \subseteq W^j(T)$$

and since $X^*AX + Y^*BY$ is the image of $A \oplus B$ under a unital, completely positive map,

$$W^j(X^*AX + Y^*BY) \subseteq W^j(A \oplus B) \subseteq W^j(T)$$

and hence,

$$X^*AX + Y^*BY \in \mathscr{S}_{n,j}(T).$$

By [9, Theorem 15] this is sufficient to insure that $\mathscr{S}_{n,j}(T)$ is C^*-convex. ▨

We are now in a position to give a somewhat different proof of the result of [12] which says that every operator has the j-MRPP.

LEMMA 3.9. *Let*

$$B = \sum_{k=1}^{\infty} \oplus A_k, \quad A_k \in M_{n_k}$$

be a block diagonal operator acting on

$$\mathscr{H} = \sum_{k=1}^{\infty} \oplus \mathscr{H}_k,$$

and let j be a fixed positive integer. Then B has the j-MRPP if and only if

$$\lim_{k\to+\infty} \mathrm{d}(A_k, \mathscr{S}_{n_k,j}(\dot{B})) = 0.$$

Proof. Let K be a compact operator on $\mathscr{H}$, such that

$$W^n(B + K) = W^n(\dot{B}) \quad \text{for } 1 \leqslant n \leqslant j.$$

Let P_k be the projection from $\mathscr{H}$ to $\mathscr{H}_k$ and let

$$\delta: \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$$

be defined by

$$\delta(X) = \sum_{k=1}^{\infty} P_k X P_k.$$

Then δ is compact preserving, completely positive, and $\delta(B) = B$. Setting

$$T = \delta(K) = \sum_{k=1}^{\infty} \oplus T_k$$

with $T_k \in M_{n_k}$ we obtain a block diagonal compact operator and so $\|T_k\| \to 0$ as $k \to +\infty$. Since

$$W^j(A_k + T_k) \subseteq W^j(B + T) = W^j(\delta(B + K)) \subseteq W^j(B + K) = W^j(\dot{B}),$$

we have that

$$A_k + T_k \in \mathcal{S}_{n_k, j}(\dot{B}).$$

Hence,

$$\mathrm{d}(A_k, \mathcal{S}_{n_k,j}(\dot{B})) \leqslant \|T_k\|$$

from which it follows that the limit is 0.

Conversely, if

$$\mathrm{d}(A_k, \mathcal{S}_{n_k,j}(\dot{B})) \to 0 \quad \text{as } k \to +\infty$$

then there exists $T_k \in M_{n_k}$ such that $\|T_k\| \to 0$ as $k \to +\infty$ and

$$A_k + T_k \in \mathcal{S}_{n_k,j}(\dot{B}).$$

Setting

$$K = \sum_{k=1}^{\infty} \oplus T_k$$

yields a compact operator and since

$$W^j(A_k + T_k) \subseteq W^j(\dot{B})$$

we have that

$$W^j(B + K) \subseteq W^j(\dot{B})$$

from which equality follows. Finally, as remarked at the beginning of the section equality of the j-th matrix ranges implies equality of the n-th matrix ranges for any $n \leqslant j$. ▨

The following lemma is essentially contained in [12, Theorem 4.6] and [15, Theorem 4.2].

LEMMA 3.10. *Let $W^1(T)$ contain the closed unit disc, then $W^n(T)$ contains the closed unit ball.*

Proof. Clearly, $W^n(T)$ contains all scalar matrices of modulus less than or equal to 1. Now let $E_1, \ldots, E_n$ be pairwise orthogonal, rank 1 projections and let $\lambda_1, \ldots, \lambda_n$ be scalars of modulus 1. By C^*-convexity,

$$\sum_{j=1}^{n} E_j(\lambda_j 1_n)E_j = \sum_{j=1}^{n} \lambda_j E_j$$

belongs to $W^n(T)$, and hence $W^n(T)$ contains all unitary matrices. But the closed convex hull of the unitaries is the unit ball of M_n [11, Proposition 1.1.12] and $W^n(T)$ is convex, hence the unit ball must be contained in $W^n(T)$. ▨

LEMMA 3.11. *Let*

$$B = \sum_{k=1}^{\infty} \oplus A_k,$$

then for any j,

$$\lim_{k\to+\infty} \mathrm{d}(W^j(A_k),\ W^j(\dot{B})) = 0.$$

Proof. Suppose not, then for some $\varepsilon > 0$, there exists a sequence

$$L_m \in W^j(A_{k_m}),$$

with

$$\mathrm{d}(L_m, W^j(\dot{B})) \geqslant \varepsilon.$$

Let L be a limit point of this sequence, then

$$\mathrm{d}(L, W^j(\dot{B})) \geqslant \varepsilon,$$

but by [6, Theorem 3.1], $L \in W^j(\dot{B})$. This contradiction proves the lemma. ▨

THEOREM 3.12. [12, §5] *For any positive integer j, every operator has the j-*MRPP.

Proof. By Corollary 3.5, it is sufficient to prove it for the case of a block diagonal operator. So let

$$B = \sum_{k=1}^{\infty} \oplus A_k, \quad A_k \in M_{n_k}$$

be block diagonal. As in [12] we must consider two cases.

First, we assume that $W^1(\dot{B})$ has no interior, so that it must be contained in a line segment. Translating B by a scalar and multiplying by a non-zero scalar changes neither the hypotheses nor the conclusions. Thus we may and do assume that $W^1(\dot{B})$ is the interval [0, 1]. In this case $\dot{B}$ is self-adjoint, the spectrum of $\dot{B}$ is contained in [0, 1], by and [3, Proposition 2.4.1], $W^j(\dot{B}) = \{P : 0 \leqslant P \leqslant 1\}$. Defining,

$$f(t) = \begin{cases} 0 & \text{for } t \leqslant 0 \\ t & \text{for } 0 \leqslant t \leqslant 1 \\ 1 & \text{for } 1 \leqslant t \end{cases},$$

we have

$$f(\mathrm{Re}(\dot{B})) = \dot{B}.$$

Hence, if we set

$$T_k = f(\mathrm{Re}(A_k)),$$

then $\|T_k - A_k\| \to 0$ as $k \to +\infty$, and

$$T = \sum_{k=1}^{\infty} \oplus\, T_k$$

will be a compact perturbation of B. Since each T_k is self-adjoint with spectrum contained in $[0, 1]$, by [9, Corollary 20],

$$W^j(T_k) \subseteq \{P \mid 0 \leqslant P \leqslant 1\} = W^j(\dot{B}).$$

Thus, by Lemma 3.7,

$$W^j(T) \subset W^j(\dot{B}),$$

which implies

$$W^n(T) = W^n(\dot{B})$$

for all $n \leqslant j$.

Thus, we may assume that $W^1(\dot{B})$ contains the closed unit disc. By Lemma 3.10, $W^j(\dot{B})$ contains the closed unit ball.

Given $\varepsilon > 0$, by Lemma 3.11, we may choose k_0 such that for $k \geqslant k_0$, we have

$$\mathrm{d}(W^j(A_k), W^j(\dot{B})) < \varepsilon.$$

Let

$$\varphi \colon M_{n_k} \to M_j$$

be any unital, completely positive map and let

$$L \in W^j(B),$$

such that

$$\|L - \varphi(A_k)\| < \varepsilon .$$

Since $W^j(\dot{B})$ is convex and contains the unit ball, a little geometry shows that

$$\frac{\varphi(A_k)}{(1 + \varepsilon)}$$

belongs to $W^j(\dot{B})$. Indeed, setting

$$t = (\|\varphi(A_k) - L\| + 1)^{-1},$$

we have that

$$tL + (1 - t)\, \frac{\varphi(A_k) - L}{\|\varphi(A_k) - L\|}$$

is in $W^j(\dot{B})$, but this last quantity is equal to $t\varphi(A_k)$ and $t \geqslant (1+\varepsilon)^{-1}$. Finally, since φ was arbitrary, we have that

$$W^j\left(\frac{A_k}{(1+\varepsilon)}\right) \subseteq W^j(\dot{B}),$$

so that

$$\mathrm{d}(A_k, \mathscr{S}_{n_k, j}(B)) \leqslant \frac{\varepsilon}{1+\varepsilon} \|A_k\|,$$

and hence

$$\lim_{k\to\infty} \mathrm{d}(A_k, \mathscr{S}_{n_k, j}(\dot{B})) = 0.$$

The Theorem now follows by applying Lemma 3.9. ▨

We note that,

$$\|\dot{T}\| = \sup\{\|L\| \mid L \in W^2(\dot{T})\}.$$

Thus, if K is the compact operator obtained in the above construction with $j = 2$, then $\|T+K\| = \|\dot{T}\|$. A careful reading of Theorem 2.2 combined with the above construction yields the explicit form of K.

We now turn our attention to the MRPP. We begin with the case of a block diagonal operator.

THEOREM 3.13. *Let*

$$B = \sum_{k=1}^{\infty} \oplus A_k, \quad A_k \in M_{n_k}$$

be a block diagonal operator, acting on

$$\mathscr{H} = \sum_{k=1}^{\infty} \oplus \mathscr{H}_k$$

where $\dim \mathscr{H}_k = n_k$. *Then* B *has the* MRPP *if and only if*

$$\lim_{k\to+\infty} \mathrm{d}(A_k, W^{n_k}(\dot{B})) = 0.$$

Proof. Suppose that B has the MRPP, then there exists a compact operator K such that

$$W^n(B+K) = W^n(\dot{B}).$$

Let P_k be the projection from $\mathscr{H}$ onto $\mathscr{H}_k$, and define

$$\delta\colon \mathscr{L}(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$$

by

$$\delta(T) = \sum_{k=1}^{\infty} P_k T P_k .$$

Notice that δ is completely positive, compact preserving, and $\delta(B) = B$. Hence,

$$W^n(\dot{B}) \subseteq W^n(B + \delta(K)) = W^n(\delta(B + K)) \subseteq W^n(B + K) = W^n(\dot{B}),$$

for all n. Writing

$$\delta(K) = \sum_{k=1}^{\infty} \oplus K_k \quad \text{with } K_k \in M_{n_k},$$

we have that

$$\lim_{k\to\infty} \|K_k\| = 0,$$

since $\delta(K)$ is compact. Furthermore, since

$$A_k + K_k \in W^{n_k}(B + \delta(K)) = W^{n_k}(\dot{B}),$$

we have

$$\mathrm{d}(A_k,\ W^{n_k}(\dot{B})) \leqslant \|K_k\|$$

and so,

$$\lim_{k\to+\infty} \mathrm{d}(A_k, W^{n_k}(\dot{B})) = 0.$$

Conversely, assuming that this limit is 0, then we may choose a sequence $K_k \in M_{n_k}$, such that

$$A_k + K_k \in W^{n_k}(\dot{B})$$

and

$$\lim_{k\to+\infty} \|K_k\| = 0.$$

Setting

$$K = \sum_{k=1}^{\infty} \oplus K_k ,$$

we claim yields the desired compact operator. Indeed, since

$$A_k + K_k \in W^{n_k}(\dot{B}) \quad \text{for all } k,$$

there exists completely positive maps,

$$\varphi_k : Q(\mathscr{H}) \to \mathscr{L}(\mathscr{H}_k) \quad \text{with } \varphi_k(\dot{B}) = A_k + K_k .$$

Defining

$$\varphi : Q(\mathscr{H}) \to \mathscr{L}(\mathscr{H}) \quad \text{by } \varphi = \sum_{k=1}^{\infty} \oplus \varphi_k$$

yields a completely positive map with $\varphi(\dot{B}) = B + K$, and hence for any n,

$$W^n(\dot{B}) \subseteq W^n(B + K) = W^n(\varphi(\dot{B})) \subseteq W^n(\dot{B}),$$

from which the equality,

$$W^n(\dot{B}) = W^n(B + K)$$

follows. ▨

We can now give a partial characterization of the MRPP in terms of the matrix ranges for an arbitrary operator.

PROPOSITION 3.14. *Let $T \in \mathscr{A}$ for some C^*-algebra $\mathscr{A}$, then the collection of numbers,*

$$\{\mathrm{d}(\mathscr{S}_{n,j}(T), W^n(T))\}_{n,j=1}^{+\infty}$$

is bounded above, monotone increasing in n, and monotone decreasing in j. Hence,

$$\lim_{j\to\infty} \sup_n \mathrm{d}(\mathscr{S}_{n,j}(T), W^n(T))$$

exists for any operator T.

Proof. By Proposition 3.8, since $\mathscr{S}_{n,j}(T)$ and $W^n(T)$ are always contained in the ball of radius $2\|T\|$, we see that the collection of numbers is bounded.

For j fixed, let $m > n$, let $A \in \mathscr{S}_{m,j}(T)$, let $B \in W^m(T)$, and let $V\colon \mathbf{C}^n \to \mathbf{C}^m$ be an isometry, then $V^*AV \in \mathscr{S}_{n,j}(T)$ and $V^*BV \in W^n(T)$. Hence,

$$\|A - B\| \geqslant \|V^*(A - B)V\| \geqslant \mathrm{d}(V^*AV, W^n(T)),$$

so that

$$\mathrm{d}(\mathscr{S}_{m,j}(T), W^m(T)) \geqslant \sup\{\mathrm{d}(V^*AV, W^n(T)) \mid A \in \mathscr{S}_{m,j}(T)\}.$$

But we claim that

$$\{V^*AV \mid A \in \mathscr{S}_{m,j}(T)\} = \mathscr{S}_{n,j}(T),$$

from which it follows that the numbers are monotone increasing in n. To establish this last claim, let $U\colon \mathbf{C}^{m-n} \to \mathbf{C}^m$ be an isometry whose range is orthogonal to the range of V. If $C \in \mathscr{S}_{n,j}(T)$ and $D \in W^{m-n}(T)$, then it is easily checked that

$$(VCV^* + UDU^*) \in \mathscr{S}_{m,j}(T).$$

Furthermore,

$$V^*(VCV^* + UDU^*)V = C$$

from which the claim follows.

If $k > j$ and $A \in M_n$ with $W^k(A) \subseteq W^k(T)$, then $W^j(A) \subseteq W^j(T)$. Hence,

$$\mathscr{S}_{n,k}(T) \subseteq \mathscr{S}_{n,j}(T)$$

from which it follows that the distances are monotone decreasing in j. ▨

THEOREM 3.15. *Let* $T \in \mathscr{L}(\mathscr{H})$, *if*

$$\limsup_{j\to\infty}\sup_n d(\mathscr{S}_{n,j}(\dot{T}),\ W^n(\dot{T})) = 0,$$

then T *has the* MRPP.

Proof. Let $\mathscr{A}$ be a separable, unital C^*-algebra containing T and $\mathscr{K}(\mathscr{H})$, and let (δ, λ) be a compact preserving, localizing pair for $\mathscr{A}$. Let

$$\delta(T) = B = \sum_{k=1}^{\infty} \oplus A_k, \quad \text{with } A_k \in M_{n_k}.$$

By Lemma 3.9 and Theorem 3.12, we have that,

$$\lim_{k\to+\infty} d(A_k, \mathscr{S}_{n_k,j}(\dot{B})) = 0.$$

Since, by Proposition 2.4, $W^n(\dot{B}) = W^n(\dot{T})$, it follows that $\mathscr{S}_{n,j}(\dot{B}) = \mathscr{S}_{n,j}(\dot{T})$. From the inequality,

$$d(A_k, W^{n_k}(\dot{B})) \leqslant d(A_k, \mathscr{S}_{n_k,j}(\dot{B})) + d(\mathscr{S}_{n_k,j}(\dot{B}),\ W^{n_k}(\dot{B})),$$

one sees that

$$\lim_{k\to+\infty} d(A_k,\ W^{n_k}(\dot{B}) = 0.$$

Hence, by Theorem 3.13, B has the MRPP, and so, by Proposition 3.4, T has the MRPP. ▨

We do not know if the converse of the above theorem is true. However, we do have the following partial converse.

THEOREM 3.16. *If for some* $T \in \mathscr{L}(\mathscr{H})$,

$$\limsup_{j}\sup_n d(\mathscr{S}_{n,j}(\dot{T}), W^n(\dot{T})) \neq 0,$$

then there is a bounded, block diagonal operator which does not have the MRPP.

Proof. Since, by Proposition 3.14, the terms in the above limit are monotone decreasing in j, for some $\varepsilon > 0$, we may choose for each j an integer n_j, such that

$$d(\mathscr{S}_{n_j,j}(\dot{T}),\ W^{n_j}(\dot{T})) \geqslant \varepsilon.$$

Let $A_j \in \mathscr{S}_{n_j,j}(T)$ be chosen such that

$$d(A_j, W^{n_j}(\dot{T})) \geqslant \varepsilon,$$

and set

$$B = \sum_{j=1}^{\infty} \oplus A_j.$$

Note that B is bounded by Proposition 3.8. Since $W^j(A_j) \subseteq W^j(\dot{T})$, for any $l \leqslant j$, we will have $W^l(A_j) \subseteq W^l(\dot{T})$. By Lemma 3.7, if

$$B_l = \sum_{j=l}^{\infty} \oplus A_j ,$$

then $W^l(B_l) \subseteq W^l(\dot{T})$ and hence $W^l(\dot{B}_l) \subseteq W^l(\dot{T})$. But by [6, Theorem 3.1], $W^l(\dot{B}_l) = W^l(\dot{B})$, so that $W^l(\dot{B}) \subseteq W^l(\dot{T})$. But now

$$\mathrm{d}(A_j, W^{n_j}(\dot{B})) \geqslant \mathrm{d}(A_j, W^{n_j}(\dot{T})) \geqslant \varepsilon$$

and so by Theorem 3.13, B does not have the MRPP. ▨

COROLLARY 3.17. *Every bounded operator has the* MRPP *if and only if*

$$\lim_j \sup_n \mathrm{d}(\mathscr{S}_{n,j}(\dot{T}), W^n(\dot{T})) = 0$$

for every $T \in \mathscr{L}(\mathscr{H})$.

Corollary 3.17 shows that the question of whether or not every operator has the MRPP is essentially a question about sequences of compact, C^*-convex sets of matrices, which satisfy a coherence condition. We make this more precise in what follows.

Recall [9, Proposition 31], that a set $W^n \subseteq M_n$ is compact and C^*-convex if and only if there exists a bounded operator T with $W^n(T) = W^n$. For a set $W^j \subseteq M_j$, we define

$$\mathscr{S}_n(W^j) = \{A \in M_n \mid W^j(A) \subseteq W^j\}.$$

We shall call a sequence $\{W^n\}$ of compact, C^*-convex sets, with $W^n \subseteq M_n$, *coherent*, provided that $W^n \subseteq \mathscr{S}_n(W^j)$ for all n and j (see also [3, p. 301]).

PROPOSITION 3.18. *Let* $\{W^n\}$ *be a sequence of compact,* C^**-convex sets, with* $W^n \subseteq M_n$. *There exists a bounded operator* B, *with* $W^n(B) = W^n$ *if and only if the sequence is coherent.*

Proof. Assume that $\{W^n\}$ is a coherent sequence. For each n, choose a countable dense subset of W^n and let $\{A_k\}$ be an enumeration of the union of these countable sets. It is easy to verify that for

$$B = \sum_{k=1}^{\infty} \oplus A_k ,$$

one has $W^n(B) = W^n$. ▨

The converse is trivial.

COROLLARY 3.19. *Every bounded operator has the* MRPP *if and only if*

$$\lim_j \sup_n \mathrm{d}(\mathscr{S}_n(W^j), W^n) = 0$$

for all coherent sequences $\{W^n\}$ *of compact,* C^**-convex sets.*

We close with four questions. We note that an affirmative answer to any of these questions implies an affirmative answer to the latter questions.

1. Does there exist constants $\{c_{n,j}\}$ with $\lim_j \sup_n c_{n,j} = 0$, such that if $\{W^n\}$ is a coherent sequence of compact, C^*-convex sets, each of which is contained in the unit ball, then

$$d(\mathcal{S}_n(W^j), W^n) \leqslant c_{n,j} ?$$

2. Does every operator have the MRPP ?

3. Is the converse of Theorem 3.15 true ?

4. If $W^n(\dot{T}) = W^n(\dot{S})$ for all n, then does S have the MRPP if and only if T has the MRPP?

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