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A HYPONORMAL WEIGHTED SHIFT WHOSE SPECTRUM IS NOT A SPECTRAL SET

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Let $\mathscr{H}$ be a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbf{Z}\}$. Also, let $\mathscr{B}(\mathscr{H})$ denote the set of bounded linear operators on $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, let $\Delta(T)$ denote the spectrum of T in $\mathscr{B}(\mathscr{H})$ and let $\mathrm{r}(T)$ denote the spectral radius of T. A compact set $K \supset \Delta(T)$ is said to be a *spectral set* for T if

$$\|f(T)\| \leqslant \sup\{|f(z)| : z \in K\} \equiv \|f\|_K$$

for all rational functions f with poles off K.

An operator $T \in \mathscr{B}(\mathscr{H})$ is called a bilateral weighted shift if there exists $\{w_n : n \in \mathbf{Z}\} \subset \mathbf{C}$ such that $Te_n = w_n e_{n+1}$ for all integers n. In this paper we will consider the weighted shift T with $w_n = \frac{1}{2}$ for $n < 0$ and $w_n = 1$ for $n \geqslant 0$. The following facts are easy to establish (see Shields, [2]):

a) $\|T\| = 1$;

b) T is invertible and $\|T^{-1}\| = 2$;

c) $\Delta(T) = \left\{z \in \mathbf{C} : \frac{1}{2} \leqslant |z| \leqslant 1\right\}$;

d) $\mathrm{r}(T) = \|T\|$ and $\mathrm{r}(T^{-1}) = \|T^{-1}\|$;

e) T is hyponormal.

We will answer the second part of Question 7 of Shields [2] by showing that $\Delta(T)$ is not a spectral set for T. We let

$$f_m(z) = \frac{1}{2} z^{-1}\left(1 - \frac{1}{2} z^m\right)^{-1} = \frac{1}{2} z^{-1} + \sum_{n=1}^{\infty} 2^{-n-1} z^{mn-1}$$

for $m \geqslant 1$. We now have

$$\|f_m\|_{\Delta(T)} = \max\{\|f_m\|_{|z|=1}, \ \|f_m\|_{|z|=\frac{1}{2}}\}$$

by the maximum modulus theorem. From this equality, it is easy to see that:

i) $\|f_m\|_{\Delta(T)} = (1 - 2^{-m-1})^{-1}$,

ii) $\|f_m\|_{\Delta(T)} \to 1$ as $m \to \infty$.

However,

$$f_m(T) = \frac{1}{2} T^{-1} + \sum_{n=1}^{\infty} 2^{-n-1} T^{mn-1}$$

and

$$\|f_m(T)\| \geqslant \|f_m(T)e_0\|.$$

We then compute $f_m(T)e_0$ and get:

$$f_m(T)e_0 = e_{-1} + \sum_{n=1}^{\infty} 2^{-n-1} e_{mn-1}.$$

Hence

$$\|f_m(T)\| \geqslant \left(1 + \sum_{n=1}^{\infty} 4^{-n-1}\right)^{1/2} = (13/12)^{1/2}.$$

If $m \geqslant 4$, then

$$\|f_m(T)\| > \|f_m\|_{\Delta(T)}$$

and this inequality shows that $\Delta(T)$ is not a spectral set for T.

One may ask whether T provides an answer to the question of whether every operator whose spectrum is a c-spectral set is similar to an operator whose spectrum is a spectral set. It was pointed out to me that the weighted shift T is similar to a subnormal weighted shift S whose weight sequence is constructed as follows.

Let

$$\alpha = \frac{1}{2}(\delta_{1/2} + \delta_1),$$

$$\beta(n) = \left[\int_{1/2}^{1} r^{2n} \mathrm{d}\alpha(r)\right]^{1/2},$$

and

$$v_n = \beta(n+1)/\beta(n) \quad \text{for } n \in \mathbf{Z}.$$

Then the weighted shift S with weight sequence $\{v_n : n \in \mathbf{Z}\}$ is a subnormal weighted shift (see Shields, [2], Proposition 27). If we let $He_n = h_n e_n$ for $n \in \mathbf{Z}$, where

$$h_n = \begin{cases} \beta(n)^{-1} & \text{if } n > 0, \\ 1 & \text{if } n = 0, \\ 2^{-n}\beta(n)^{-1} & \text{if } n < 0, \end{cases}$$

then it is straightforward to check that H is an invertible operator and $T = HSH^{-1}$. Also, we see that $\Delta(T)$ is a c-spectral set for T with $c = \|H\| \cdot \|H^{-1}\| = \sqrt{2}$. From the preceding discussion, we see that the weighted shift T does not provide an answer to the question above.

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