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CONTINUOUS AND ANALYTIC INVARIANTS FOR DEFORMATIONS OF FREDHOLM COMPLEXES

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1. INTRODUCTION

The aim of this paper is to prove the norm continuity of certain invariants that are attached to Fredholm complexes of Banach spaces and their endomorphisms. In particular, we prove the continuity of the Lefschetz number, which can naturally be defined in this context. The analytic and smooth dependence of these invariants will also be considered.

The usual stability of the index of a Fredholm operator under small perturbations can, of course, be regarded as a continuity statement. Starting from this simple remark, the second named author has proved in [14] the norm continuity of the Lefschetz number attached to a Fredholm operator and a pair of operators that intertwines it, as a function of three arguments. The first named author has then noticed that similar results can be proved for a larger class of invariants, that is derived from the characteristic polynomial. In this paper we shall extend these considerations to the case of Fredholm complexes of Banach spaces. Roughly speaking, for each pair consisting of a Fredholm complex and an endomorphism of it we define a rational function, which is called here the characteristic function (see Definition 1.2 below), and show that this assignment is norm continuous in a neighbourhood of the origin in the complex plane. In particular, the coefficients of the Taylor expansion of the characteristic function at the origin are norm continuous complex-valued functions. In order to state more accurately the main result (Theorem 1.3 below), let us introduce some notations and definitions.

Let X and Y be Banach spaces (over the complex field $\mathbf{C}$), and let us denote by $\mathscr{L}(X, Y)$ the space of all continuous linear operators from X into Y. The space $\mathscr{L}(X,X)$ will be simply denoted by $\mathscr{L}(X)$. For every $S \in \mathscr{L}(X, Y)$ we denote by $\mathrm{N}(S)$, $\mathrm{R}(S)$ and $\gamma(S)$ the *null-space*, the *range* and the *reduced minimum modulus* [7] of S, respectively.

A *complex of Banach spaces* is a sequence $(X, \alpha) = (X^p, \alpha^p)_{p\in\mathbf{Z}}$ of Banach spaces X^p and operators $\alpha^p \in \mathscr{L}(X^p, X^{p+1})$ such that $\mathrm{R}(\alpha^p) \subset \mathrm{N}(\alpha^{p+1})$ for all $p \in \mathbf{Z}$.

The associated *cohomology* will be denoted by $(H^p(X,\alpha))_{p\in\mathbf{Z}}$, where, as usually, $H^p(X,\alpha)$ stands for the quotient $\mathrm{N}(\alpha^p)/\mathrm{R}(\alpha^{p-1})$. If we fix a family of Banach spaces $X = (X^p)_{p\in\mathbf{Z}}$, then we identify the complex $(X,\alpha) = (X^p,\alpha^p)_{p\in\mathbf{Z}}$ with the family of operators $\alpha = (\alpha^p)_{p\in\mathbf{Z}}$, which is said to be a *complex* on X. We denote by $\partial(X)$ the set of all complexes on X.

An *endomorphism* of the complex (X,α) is a family of operators $\theta = (\theta^p)_{p\in\mathbf{Z}}$ such that $\theta^p \in \mathscr{L}(X^p)$ and $\alpha^p\theta^p = \theta^{p+1}\alpha^p$ for all $p\in\mathbf{Z}$. We denote by $\mathrm{End}(X,\alpha)$ the set of all endomorphisms of (X,α).

Let $\xi = (\xi^p)_{p\in\mathbf{Z}}$ and $\eta = (\eta^p)_{p\in\mathbf{Z}}$ be arbitrary families of operators such that $\xi^p, \eta^p \in \mathscr{L}(X^p, X^{p+m})$ for all p, where m is a given integer. Then we define the *pseudodistance*

$$\|\xi-\eta\| = \sup_{p\in\mathbf{Z}} \|\xi^p-\eta^p\|, \tag{1.1}$$

which does not necessarily have finite values.

A complex (X,α) is said to be *Fredholm* [12], [13] if
(i) the function $p \to \dim H^p(X,\alpha)$ is finite and has finite support, and
(ii) $\gamma(\alpha) := \inf\{\gamma(\alpha^p);\ p\in\mathbf{Z}\} > 0$
(see [10], [9] for similar concepts).

Let us note that if X is of *finite length* (i.e., $X^p = 0$ for all but a finite collection of indices), then the condition (ii) is a consequence of (i). The set of all Fredholm complexes on X will be denoted by $\Phi(X)$.

For every $\alpha\in\Phi(X)$ we define the *index* of α by the formula

$$\mathrm{ind}(X,\alpha) = \mathrm{ind}\,\alpha = \sum_{p\in\mathbf{Z}} (-1)^p \dim H^p(X,\alpha). \tag{1.2}$$

The number (1.2) is, in fact, the *Euler characteristic* of the complex (X,α); it is invariant under small perturbations in the following sense:

1.1. THEOREM. *For every $\alpha\in\Phi(X)$ there exists a positive number ε_α such that if $\tilde\alpha\in\partial(X)$ and $\|\tilde\alpha-\alpha\|<\varepsilon_\alpha$, then $\tilde\alpha\in\Phi(X)$, $\dim H^p(X,\tilde\alpha)\leqslant\dim H^p(X,\alpha)$ for all p and* $\mathrm{ind}\,\tilde\alpha = \mathrm{ind}\,\alpha$.

The proof of this theorem is essentially contained in [12], Theorem 2.12 (see also [15]).

Let us define the set

$$\mathscr{F}(X) = \{(\alpha,\theta);\quad \alpha\in\Phi(X),\quad \theta\in\mathrm{End}(X,\alpha)\}.$$

The set $\mathscr{F}(X)$ will be given the topology induced by the pseudodistance (1.1).

1.2. DEFINITION. Let $(\alpha,\theta)\in\mathscr{F}(X)$. The *characteristic function* of the pair (α,θ) is the rational function

$$\chi_{(\alpha,\theta)}(z) = z^{\mathrm{ind}\,\alpha} \prod_{k\in\mathbf{Z}} \frac{\mathrm{Det}_{H^{2k}(X,\alpha)}(z^{-1}-\theta^{2k})}{\mathrm{Det}_{H^{2k+1}(X,\alpha)}(z^{-1}-\theta^{2k+1})},$$

which is well-defined for z in a neighbourhood of zero in the complex plane. Here $\mathrm{Det}_{H^p(X,\alpha)}(w - \theta^p)$ is the determinant of the mapping induced by $w - \theta^p$ in the quotient $H^p(X, \alpha)$, i.e. the characteristic polynomial of the mapping induced by θ^p in $H^p(X, \alpha)$; if $H^p(X, \alpha) = 0$, then we define this determinant to be equal to one.

Let $\mathcal{O}_0$ be the algebra of germs of analytic complex-valued functions in neighbourhoods of the origin, endowed with its natural topology of inductive limit of Banach algebras. The main result of this paper is the following:

1.3. THEOREM. *The assignment*

$$\mathcal{F}(X) \ni (\alpha, \theta) \to \chi_{(\alpha,\theta)} \in \mathcal{O}_0$$

is a continuous function.

The proof of Theorem 1.3 will be given in the third section. We only note that

$$\chi_{(\alpha,\theta)}(z) = 1 - L_\alpha(\theta)z + \ldots,$$

iwhere

$$L_\alpha(\theta) = \sum_{p\in\mathbf{Z}} (-1)^p \mathrm{Tr}_{H^p(X,\alpha)}(\theta^p) \tag{1.3}$$

s the *Lefschetz number* of the endomorphism θ with respect to the Fredholm complex α [1], [4]. Here $\mathrm{Tr}_{H^p(X,\alpha)}(\theta^p)$ is the trace of the mapping induced by θ^p in $H^p(X, \alpha)$; it is assumed to be zero when $H^p(X, \alpha) = 0$. Let us notice that (1.2) can be obtained from (1.3) when θ is the identity. Theorem 1.3 implies, in particular, the continuity of the mapping

$$\mathcal{F}(X) \ni (\alpha, \theta) \to L_\alpha(\theta) \in \mathbf{C}$$

(see Corollary 3.1).

The statement of Theorem 1.3 can be improved in the sense that the analytic (or smooth) variation of the argument (α, θ) is inherited by the characteristic function. Such results will be presented in the fourth section. The fifth (and the last) section contains some final comments and related examples.

2. THE AUXILIARY MACHINERY

In this section we present a sequence of lemmas that are needed for the proof of Theorem 1.3.

Let X be a Banach space and let $A \in \mathcal{L}(X)$. If M and N are closed subspaces of X that are invariant under A such that $M \subset N$ and $\dim N/M < \infty$, then we denote by $\mathrm{Det}_{N/M}(A)$ (resp. $\mathrm{Tr}_{N/M}(A)$) the *determinant* (resp. the *trace*) of the operator induced by A in N/M. If $M = N$, then we define $\mathrm{Det}_{N/M}(A) = 1$ and $\mathrm{Tr}_{N/M}(A) = 0$.

The next four lemmas, which are of classical style, have a more or less obvious proof that will be omitted.

2.1. LEMMA. *Let $A \in \mathscr{L}(X)$, and let $M \subset X$ be a closed subspace that is invariant under A, with* $\dim X/M < \infty$. *Let N be a complement of M in X, and let P denote the projection of X onto N along M. Then we have the equality*

$$\mathrm{Det}_N(PA \mid N) = \mathrm{Det}_{X/M}(A).$$

2.2. LEMMA. *Let $A \in \mathscr{L}(X)$, and let M and N be closed subspaces of X that are invariant under A, such that $M \subset N$,* $\dim X/N < \infty$ *and* $\dim N/M < \infty$. *Then we have the equality*

$$\mathrm{Det}_{X/M}(A) = \mathrm{Det}_{X/N}(A) \cdot \mathrm{Det}_{N/M}(A).$$

2.3. LEMMA. *Let X be finite dimensional, and let M and N be subspaces of X such that $X = M + N$ and $M \cap N = 0$. If P is the projection of X onto N along M, $A \in \mathscr{L}(X)$ and M is invariant under A, then we have the equality*

$$\mathrm{Det}_X(A) = \mathrm{Det}_M(A) \cdot \mathrm{Det}_N(PA \mid N).$$

2.4. LEMMA. *Let X be finite dimensional. Then for all A and B in $\mathscr{L}(X)$ we have the estimate*

$$|\mathrm{Det}_X(A) - \mathrm{Det}_X(B)| \leqslant n!n(\max\{\|A\|, \|B\|\})^{n-1}\|A - B\|,$$

where $n = \dim X$.

A technical result that is useful in the sequel is the following:

2.5. LEMMA. *Let $S \in \mathscr{L}(X, Y)$ and $T \in \mathscr{L}(Y, Z)$ be such that* $\mathrm{R}(S) = \mathrm{N}(T)$ *and* $\mathrm{R}(T)$ *is closed. Let also $A \in \mathscr{L}(X, Y)$ and $B \in \mathscr{L}(Y, Z)$ be with the property that* $\mathrm{R}(\tilde{S}) \subset \mathrm{N}(\tilde{T})$, *where $\tilde{S} = S + A$ and $\tilde{T} = T + B$. If $\varepsilon_A \geqslant \|A\|$, $\varepsilon_B \geqslant \|B\|$, $r_S \geqslant \gamma(S)^{-1}$, $r_T \geqslant \gamma(T)^{-1}$ and $(1 + \varepsilon_A r_S)(1 + \varepsilon_B r_T) < 2$, then* $\mathrm{R}(\tilde{S}) = \mathrm{N}(\tilde{T})$ *and*

$$\gamma(\tilde{S})^{-1} \leqslant \frac{r_S(1 + \varepsilon_B r_T)}{2 - (1 + \varepsilon_A r_S)(1 + \varepsilon_B r_T)}. \tag{2.1}$$

This result, which expresses the stability of the exactness under small perturbations, can be found in [12], Lemma 2.1 and Corollary 2.2.

2.2. LEMMA. *With the conditions of Lemma 2.5, for all $y \in \mathrm{N}(T)$ and $\tilde{y} \in \mathrm{N}(\tilde{T})$ one can find the elements x and $\tilde{x}$ in X with $Sx = y$, $\tilde{S}\tilde{x} = \tilde{y}$, and such that $\|\tilde{x} - x\|$ is as small as one wants if $\|\tilde{y} - y\|$ and $\|\tilde{S} - S\|$ are sufficiently small. Moreover, the choice of x can be made independently of that of $\tilde{x}$.*

Proof. We consider the numbers $r_S > \gamma(S)^{-1}$ and $r_{\tilde{S}} > \gamma(\tilde{S})^{-1}$. We can choose $x \in X$ such that $Sx = y$ and $\|x\| \leqslant r_S\|y\|$. Let us observe that $y_1 = \tilde{y} - \tilde{S}x \in \mathrm{N}(\tilde{T})$. Therefore we can find $x_1 \in X$ such that $\tilde{S}x_1 = \tilde{y} - \tilde{S}x$ and $\|x_1\| \leqslant r_{\tilde{S}}\|\tilde{y} - \tilde{S}x\|$. Let us set $\tilde{x} = x_1 + x$. Then we can write that

$$(2.2)\qquad \begin{aligned} &\|\tilde{x} - x\| \leqslant r_{\tilde{S}}\|\tilde{y} - \tilde{S}x\| \leqslant \\ &\leqslant r_{\tilde{S}}(\|\tilde{y} - y\| + r_S\|\tilde{S} - S\|\,\|y\|). \end{aligned}$$

Since the function $\tilde{S} \to r_{\tilde{S}}$ can be chosen to be bounded by (2.1), the number $\|\tilde{x} - x\|$ can be made as small as we desire when $\|\tilde{y} - y\|$ and $\|\tilde{S} - S\|$ are sufficiently small. Plainly, the choice of x is independent of that of $\tilde{x}$.

If X and Y are arbitrary Banach spaces, then we denote by $X \oplus Y$ their direct sum, endowed with the norm $\|x \oplus y\|^2 = \|x\|^2 + \|y\|^2$ for all $x \in X$ and $y \in Y$. We shall identify the subspace $X \oplus 0$ of $X \oplus Y$ with X and the subspace $0 \oplus Y$ with Y.

2.7. LEMMA. *Let $S \in \mathscr{L}(X, Y)$, $T \in \mathscr{L}(Y, Z)$ and $C \in \mathscr{L}(M, Y)$ be such that* $\mathrm{R}(S) + \mathrm{R}(C) = \mathrm{N}(T)$, *where M is a finite dimensional space and* $\mathrm{R}(T)$ *is closed Let us consider the mappings*

$$\sigma\colon \mathscr{L}(M, X) \oplus \mathscr{L}(M) \to \mathscr{L}(M, Y) \quad \textit{and} \quad \tau\colon \mathscr{L}(M, Y) \to \mathscr{L}(M, Z)$$

given by the formulas $\sigma(E \oplus F) = SE + CF$ for $E \in \mathscr{L}(M, X)$, $F \in \mathscr{L}(M)$, and $\tau(G) = TG$ for $G \in \mathscr{L}(M, Y)$. Then we have that $\mathrm{R}(\sigma) = \mathrm{N}(\tau)$ *and that* $\mathrm{R}(\tau)$ *is closed.*

Proof. Let us note that the space $\mathscr{L}(M, X)$ is isomorphic to the direct sum $X^{(m)}$ of m copies of X, by the mapping

$$\mathscr{L}(M, X) \ni E \to Ev_1 \oplus \ldots \oplus Ev_m \in X^{(m)},$$

where $\{v_1, \ldots, v_m\}$ is a fixed basis of M. Analogously, $\mathscr{L}(M)$, $\mathscr{L}(M, Y)$ and $\mathscr{L}(M, Z)$, can be identified with the spaces $M^{(m)}$, $Y^{(m)}$ and $Z^{(m)}$ respectively. Then the complex

$$\mathscr{L}(M, X) \oplus \mathscr{L}(M) \xrightarrow{\sigma} \mathscr{L}(M, Y) \xrightarrow{\tau} \mathscr{L}(M, Z)$$

is isomorphic to the complex

$$X^{(m)} \oplus M^{(m)} \xrightarrow{\hat{\sigma}} Y^{(m)} \xrightarrow{\hat{\tau}} Z^{(m)},$$

where

$$\hat{\sigma}(x_1 \oplus \ldots \oplus x_m \oplus w_1 \oplus \ldots \oplus w_m) = (Sx_1 + Cw_1) \oplus \ldots \oplus (Sx_m + Cw_m)$$

and

$$\hat{\tau}(y_1 \oplus \ldots \oplus y_m) = Ty_1 \oplus \ldots \oplus Ty_m$$

for all $x_j \in X$, $w_j \in M$, $y_j \in Y$ and $j = 1, \ldots, m$. The condition $R(S) + R(C) = N(T)$ insures the equality $R(\hat{\sigma}) = N(\hat{\tau})$, while $R(T) = \overline{R(T)}$ implies that $R(\hat{\tau})$ is closed. Therefore $R(\sigma) = N(\tau)$ and $R(\tau)$ is closed, by the mentioned isomorphism.

2.8. LEMMA. *Let S, T and C be as in Lemma 2.7. Let also $A \in \mathscr{L}(X)$ and $B \in \mathscr{L}(Y)$ be such that $SA = BS$ and $BN(T) \subset N(T)$. We define the operator $S_1 \in \mathscr{L}(X \oplus M, Y)$ by the equality $S_1(x \oplus v) = Sx + Cv$ for all $x \in X$ and $v \in M$. Then there exists an operator $A_1 \in \mathscr{L}(X \oplus M)$ such that $A_1 | X = A$ and $S_1A_1 = BS_1$.*

Proof. Every operator $A_1 \in \mathscr{L}(X \oplus M)$ such that $A_1 | X = A$ has necessarily the form $A_1(x \oplus v) = (Ax + Ev) \oplus Fv$ for all $x \in X$ and $v \in M$, where $E \in \mathscr{L}(M, X)$ and $F \in \mathscr{L}(M)$. The condition $S_1A_1 = BS_1$ is equivalent to the equality $SE + CF = BC$. Since $BC \in N(\tau)$ (where τ is defined as in Lemma 2.6), the existence of a pair (E, F) with the required property follows by Lemma 2.7.

2.9. LEMMA. *With the conditions of Lemma 2.8, if $A_1 \in \mathscr{L}(X \oplus M)$ is any operator with the property that $A_1 | X = A$ and $S_1A_1 = BS_1$, then we have the equality*

$$\text{Det}_{N(S_1)/N(S)}(w - A_1) \cdot \text{Det}_{R(S_1)/R(S)}(w - B) = \text{Det}_M(w - P_M A_1 | M), \tag{2.3}$$

where P_M is the canonical projection of $X \oplus M$ onto M and w is an arbitrary complex number.

Proof. Let N_0 be the space $R(S) \cap R(C)$, and let us denote by M_0 the space $C^{-1}(N_0)$. If M_1 is a complement of M_0 in M, then it is easily seen that $C\colon M_1 \to N_1$ is an isomorphism, where $N_1 = C(M_1)$, and that N_1 is a complement of $R(S)$ in $N(T)$ Let P_1 be the projection of $N(T)$ onto N_1 along $R(S)$, let and Q_0 be the projection of M onto M_0 along M_1. We set $Q_1 = 1_M - Q_0$, and let us denote by $\hat{Q}_0, \hat{Q}_1$ the projections $0 \oplus Q_0$, $0 \oplus Q_1$, acting in $X \oplus M$, respectively. We note the equality

$$\text{Det}_M(w - P_M A_1 | M) = \text{Det}_{M_0}(w - \hat{Q}_0 A_1 | M_0) \cdot \text{Det}_{M_1}(w - \hat{Q}_1 A_1 | M_1), \tag{2.4}$$

which follows by Lemma 2.3. Indeed, let us show that the space M_0 is invariant under $P_M A_1 | M$. We notice first that the operator A_1 has the form $A_1(x \oplus v) = (Ax + Ev) \oplus Fv$ (see the proof of Lemma 2.8). From the equality $S_1A_1 = BS_1$, we infer that $A_1 N(S_1) \subset N(S_1)$. Let us observe that

$$N(S_1) = \{x \oplus v_0;\ x \in X,\ v_0 \in M_0,\ Sx + Cv_0 = 0\}.$$

The invariance of $N(S_1)$ under A_1 implies the invariance of the subspace M_0 under F. Since $P_M A_1 \mid M_0 = \hat{Q}_0 A_1 \mid M_0 = F \mid M_0$, we obtain that (2.4) holds, by Lemma 2.3.

Next we prove the equality

$$\text{(2.5)} \qquad \mathrm{Det}_{M_0}(w - \hat{Q}_0 A_1 \mid M_0) = \mathrm{Det}_{N(S_1)/N(S)}(w - A_1).$$

Indeed, let us define the operator $U\colon N(S_1)/N(S) \to M_0$ by the equality $U(x \oplus \oplus v_0 + N(S)) = v_0$, where $x \oplus v_0 \in N(S_1)$. It is easily seen that U is an isomorphism of $N(S_1)/N(S)$ onto M_0. We also note that $U\hat{A}_1 = \hat{Q}_0(A_1 \mid M)\, U$, where $\hat{A}_1$ is the operator induced by A_1 in $N(S_1)/N(S)$; this equality follows from the above mentioned relation $FM_0 \subset M_0$. Therefore the operators $w - A_1$ and $w - \hat{Q}_0 A_1 \mid M_0$ are similar, so that the equality (2.5) holds true.

Now we show the relation

$$\text{(2.6)} \qquad \mathrm{Det}_{M_1}(w - \hat{Q}_1 A_1 \mid M_1) = \mathrm{Det}_{R(S_1)/R(S)}(w - B).$$

Indeed, from the equality $S_1 A_1 = B S_1$, we infer that $(C|M_1)\hat{Q}_1 A_1|M_1 = P_1 B(C\,|M_1)$, since $S_1|M_1 = C|M_1$ and $P_1 S_1 \hat{Q}_0 = 0$. Hence $w - \hat{Q}_1 A_1|M_1$ and $w - P_1 B|\, N_1$ are similar. On the other hand, by Lemma 2.1 we obtain that

$$\mathrm{Det}_{N_1}(w - P_1 B \mid N_1) = \mathrm{Det}_{R(S_1)/R(S)}(w - B),$$

since $R(S_1)$ $(= N(T))$ and $R(S)$ are invariant under B.

The equality (2.3) now follows from (2.4), (2.5) and (2.6).

For every pair of closed subspaces M and N of the Banach space X we set

$$\delta(M, N) = \sup_{\substack{x \in M \\ \|x\| \leqslant 1}} \mathrm{dist}(x, N).$$

We note that if S and $\tilde{S}$ are elements of $\mathscr{L}(X, Y)$, then we have the estimate

$$\text{(2.7)} \qquad \delta(N(S), N(\tilde{S})) \leqslant \gamma(\tilde{S})^{-1}\|\tilde{S} - S\|,$$

provided that $R(\tilde{S})$ is closed (see [12], Lemma 2.6).

2.10. Lemma. 1) *Let S, T, A, B and C be as in Lemma 2.8. Let also $\tilde{S}$, $\tilde{T}$, $\tilde{A}$, and $\tilde{B}$ have similar properties to those of S, T, A and B, respectively. If $\|\tilde{S} - S\|$, $\|\tilde{T} - T\|$ are sufficiently small and the function $\tilde{T} \to \gamma(\tilde{T})^{-1}$ is bounded for $\tilde{T}$ in a neighbourhood of T, then we can find $\tilde{C} \in \mathscr{L}(M, Y)$ such that* $R(\tilde{\sigma}) = N(\tilde{\tau})$, *where $\tilde{\sigma}$ and τ are defined as in Lemma 2.7, with $\tilde{S}$, $\tilde{C}$ and $\tilde{T}$ instead of S, C and T, respectively.*

2) *Let* S_1, $\tilde{S}_1$, A_1 *and* $\tilde{A}_1$ *be the extensions of* S, $\tilde{S}$, A *and* $\tilde{A}$ *to* $X \oplus M$, *in the sense of Lemma 2.8, respectively. If* $\|\tilde{S} - S\|$, $\|\tilde{T} - T\|$, $\|\tilde{A} - A\|$, $\|\tilde{B} - B\|$ *are sufficiently small and the function* $\tilde{T} \to \gamma(\tilde{T})^{-1}$ *is bounded for* $\tilde{T}$ *in a neighbourhood of* T, *then the operators* A_1 *and* $\tilde{A}_1$ *can be chosen such that* $\|\tilde{A}_1 - A_1\|$ *is as small as we desire.*

Proof. 1) Let $\{v_1, \ldots, v_m\}$ be a basis of the space M, that is chosen according to Auerbach's lemma [2]. In particular, $\|v_j\| = 1$, and for an arbitrary element $v = \lambda_1 v_1 + \ldots + \lambda_m v_m \in M$ we have $|\lambda_j| \leqslant \|v\|$ for all j. If $\delta > \delta(\mathrm{N}(T), \mathrm{N}(\tilde{T}))$, then we can find a system of vectors $\{\tilde{y}_1, \ldots, \tilde{y}_m\} \subset \mathrm{N}(\tilde{T})$ such that $\|Cv_j - \tilde{y}_j\| \leqslant \leqslant \delta\|C\|$ for all j. Let us define the operator

$$\tilde{C}\left(\sum_{j=1}^{m} \lambda_j v_j\right) = \sum_{j=1}^{m} \lambda_j \tilde{y}_j, \quad v = \sum_{j=1}^{m} \lambda_j v_j \in M.$$

It is easily seen that $\|\tilde{C} - C\| \leqslant m\delta\|C\|$. We also have $\|\tilde{\tau} - \tau\| = \|\tilde{T} - T\|$, and

$$\|\tilde{\sigma} - \sigma\| \leqslant (\|\tilde{S} - S\|^2 + \|\tilde{C} - C\|^2)^{1/2}.$$

If $\|\tilde{S} - S\|$, $\|\tilde{T} - T\|$ are sufficiently small and the function $\tilde{T} \to \gamma(\tilde{T})^{-1}$ is bounded in a neighbourhood of T, then $\|\tilde{C} - C\|$ can be made as small as we want, by the estimate (2.7). Therefore $\mathrm{R}(\tilde{\sigma}) = \mathrm{N}(\tilde{\tau})$, by Lemmas 2.5 and 2.7.

2) The operators A_1 and $\tilde{A}_1$ have the form $A_1(x \oplus v) = (Ax + Ev) \oplus Fv$ and $\tilde{A}_1(x \oplus v) = (\tilde{A}x + \tilde{E}v) \oplus \tilde{F}v$ for all $x \in X$ and $v \in M$, which follows from the proof of Lemma 2.8. Moreover, $SE + CF = BC$ and $\tilde{S}\tilde{E} + \tilde{C}\tilde{F} = \tilde{B}\tilde{C}$. By the first part of this proof and the estimate (2.2), we can choose (E, F) and $(\tilde{E}, \tilde{F})$ such that

$$\|E \oplus F - \tilde{E} \oplus \tilde{F}\| \leqslant r_{\tilde{\sigma}}(\|\tilde{B}\tilde{C} - BC\| + r_\sigma\|\tilde{\sigma} - \sigma\| \|BC\|),$$

where $r_{\tilde{\sigma}} > \gamma(\tilde{\sigma})^{-1}$ and $r_\sigma > \gamma(\sigma)^{-1}$. The operators A_1 and $\tilde{A}_1$ satisfy the following estimate:

$$\|(\tilde{A}_1 - A_1)(x \oplus v)\| = \|((\tilde{A} - A)x + (\tilde{E} - E)v) \oplus (\tilde{F} - F)v\| \leqslant$$

$$\leqslant (\|\tilde{A} - A\|^2 + \|\tilde{E} \oplus \tilde{F} - E \oplus F\|^2)^{1/2} \|x \oplus v\|.$$

If $\|\tilde{S} - S\|$, $\|\tilde{T} - T\|$, $\|\tilde{A} - A\|$, $\|\tilde{B} - B\|$ are sufficiently small and the function $\tilde{T} \to \gamma(\tilde{T})^{-1}$ is bounded for $\tilde{T}$ in a neighbourhood of T, then the function $\tilde{\sigma} \to r_{\tilde{\sigma}}$ can be chosen to be bounded for $\tilde{\sigma}$ in a neighbourhood of σ (by (2.1)), and therefore $\|\tilde{A}_1 - A_1\|$ can be made as small as we desire, by the previous estimate.

3. PROOF OF THE MAIN RESULT

The first part of this section is dedicated to the proof of Theorem 1.3. Then we present some consequences of this result.

Let $X = (X^p)_{p\in\mathbf{Z}}$ be a sequence of Banach spaces, and let $(\alpha, \theta) \in \mathscr{F}(X)$. For the sake of simplicity, let us denote by $\chi^p_\alpha(\theta, w)$ the polynomial $\mathrm{Det}_{H^p(X,\alpha)}(w - \theta^p)$. If $\lambda_{1,p}, \ldots, \lambda_{m_p,p}$ are the eigenvalues of θ^p in $H^p(X, \alpha)$, repeated according to their multiplicity, and hence $m_p = \dim H^p(X, \alpha)$, then we have

$$\chi^p_\alpha(\theta, z^{-1}) = z^{-m_p}(1 - z\lambda_{1,p}) \ldots (1 - z\lambda_{m_p,p}).$$

Therefore, with the assumption that $\lambda_{1,p} = \ldots = \lambda_{m_p,p} = 0$ if $m_p = 0$, we can write the equality

$$\chi_{(\alpha,\theta)}(z) = \prod_{k\in\mathbf{Z}} \frac{(1 - z\lambda_{1,2k}) \ldots (1 - z\lambda_{m_{2k},2k})}{(1 - z\lambda_{1,2k+1}) \ldots (1 - z\lambda_{m_{2k+1},2k+1})}. \tag{3.1}$$

Consequently the function $\chi_{(\alpha,\theta)}$ is analytic in a neigbourhood of zero and $\chi_{(\alpha,\theta)}(0) = 1$.

Let us deal with the continuity of the mapping

$$\mathscr{F}(X) \ni (\tilde{\alpha}, \tilde{\theta}) \to \chi_{(\tilde{\alpha},\tilde{\theta})} \in \mathcal{O}_0$$

at a certain point (α, θ). There is no loss of generality in assuming that $H^p(X, \alpha) = 0$ if $p < 0$. Let us define the number

$$n(X, \alpha) := \min\{n \geqslant 0;\ H^p(X, \alpha) = 0,\ p \geqslant n\}.$$

We shall prove our assertion by induction with respect to $n(X, \alpha)$.

If $n(X, \alpha) = 0$, then $\chi_{(\tilde{\alpha},\tilde{\theta})} = 1$ for every pair $(\tilde{\alpha}, \tilde{\theta})$ such that $\|\tilde{\alpha} - \alpha\| < \varepsilon_\alpha$, where ε_α is given by Theorem 1.1, and therefore the assertion holds true.

Now we assume that the assertion is true if $n(X, \alpha) = n \geqslant 0$, and let us consider a complex $\alpha \in \Phi(X)$ such that $n(X, \alpha) = n + 1$. We shall make an auxiliary construction for which we can apply the induction hypothesis. Namely, let M^{n-1} be a complement of $\mathrm{R}(\alpha^{n-1})$ in $\mathrm{N}(\alpha^n)$. Then we define $Y^{n-1} = X^{n-1} \oplus M^{n-1}$. Let β^{n-1} be the extension of α^{n-1} to Y^{n-1} in the sense of Lemma 2.8 (with α^{n-1} for S and the identity for C). Then we have $\mathrm{R}(\beta^{n-1}) = \mathrm{N}(\alpha^n)$ and $\mathrm{N}(\beta^{n-1}) = \mathrm{N}(\alpha^{n-1})$. Therefore, if $Y^p = X^p$ and $\beta^p = \alpha^p$ for $p \neq n - 1$, then the complex $(Y, \beta) = (Y^p, \beta^p)_{p\in\mathbf{Z}}$ has the property that $n(Y, \beta) = n$. Consequently we may apply the induction hypothesis to the complex (Y, β).

Let $\tilde{\alpha} \in \Phi(X)$ be a complex such that $\|\tilde{\alpha} - \alpha\| < \varepsilon_\alpha$, where $\varepsilon_\alpha > 0$ is provided by Theorem 1.1. Let $\tilde{\beta}^{n-1}$ be the extension of $\tilde{\alpha}^{n-1}$ to Y^{n-1} in the sense of Lemma 2.10 (with $\tilde{\alpha}^{n-1}$ for $\tilde{S}$; let us notice that Lemma 2.10 applies if $\varepsilon_\alpha > 0$ is sufficiently small). Let θ_1^{n-1} and $\tilde{\theta}_1^{n-1}$ be the extensions of θ^{n-1} and $\tilde{\theta}^{n-1}$ to Y^{n-1} in the sense of Lemma 2.10, respectively (with θ^{n-1} for A and $\tilde{\theta}^{n-1}$ for $\tilde{A}$). We also define $\theta_1^p = \theta^p$, $\tilde{\theta}_1^p = \tilde{\theta}^p$, $\tilde{\beta}^p = \tilde{\alpha}^p$ for $p \neq n-1$, and let us set $\theta_1 = (\theta_1^p)_{p\in \mathbf{Z}}$, $\tilde{\theta}_1 = (\tilde{\theta}_1^p)_{p\in\mathbf{Z}}$ and $\tilde{\beta} = (\tilde{\beta}^p)_{p\in\mathbf{Z}}$. By the induction hypothesis, the function $\chi_{(\tilde{\beta},\tilde{\theta}_1)}$ tends to the function $\chi_{(\beta,\theta_1)}$ in the topology of $\mathcal{O}_0$ as $(\tilde{\beta}, \tilde{\theta}_1)$ tends to (β, θ_1) in the topology of $\mathscr{F}(Y)$.

Let us assume momentarily that n is even. Then we have the equalities:

$$\chi_{(\tilde{\beta},\tilde{\theta}_1)}(z) = z^{\operatorname{ind}\tilde{\beta}} \prod_{k\leq \frac{n}{2}-2} \frac{\chi_{\tilde{\alpha}}^{2k}(\tilde{\theta}, z^{-1})}{\chi_{\tilde{\alpha}}^{2k+1}(\tilde{\theta}, z^{-1})} \times$$

$$\times \frac{\chi_{\tilde{\alpha}}^{n-2}(\tilde{\theta}, z^{-1})}{\chi_{\tilde{\beta}}^{n-1}(\tilde{\theta}_1, z^{-1})} \cdot \frac{\chi_{\tilde{\beta}}^{n}(\tilde{\theta}_1, z^{-1})}{\chi_{\tilde{\alpha}}^{n+1}(\tilde{\theta}, z^{-1})} \prod_{k\geq \frac{n}{2}+1} \frac{\chi_{\tilde{\alpha}}^{2k}(\tilde{\theta}, z^{-1})}{\chi_{\tilde{\alpha}}^{2k+1}(\tilde{\theta}, z^{-1})} =$$

$$= z^{\operatorname{ind}\tilde{\beta} - \operatorname{ind}\tilde{\alpha}} \chi_{(\tilde{\alpha},\tilde{\theta})}(z) \, (\operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\tilde{\theta}_1^{n-1} \mid M^{n-1}))^{-1},$$

where P_{n-1} is the canonical projecton of Y^{n-1} onto M^{n-1}. We have used here the equalities

$$\operatorname{Det}_{N(\tilde{\beta}^{n-1}), N(\tilde{\alpha}^{n-1})}(z^{-1} - \tilde{\theta}_1^{n-1}) \cdot \operatorname{Det}_{R(\tilde{\beta}^{n-1})/R(\tilde{\alpha}^{n-1})}(z^{-1} - \tilde{\theta}^n) =$$

$$= \operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\tilde{\theta}_1^{n-1} \mid M^{n-1}),$$

which is the relation (2.3) applied to this case, and

$$\chi_{\tilde{\beta}}^{n-1}(\tilde{\theta}_1, z^{-1}) = \operatorname{Det}_{N(\tilde{\beta}^{n-1})/N(\tilde{\alpha}^{n-1})}(z^{-1} - \tilde{\theta}_1^{n-1}) \chi_{\tilde{\alpha}}^{n-1}(\tilde{\theta}, z^{-1}),$$

(3.2)

$$\chi_{\tilde{\alpha}}^{n}(\tilde{\theta}, z^{-1}) = \operatorname{Det}_{R(\tilde{\beta}^{n-1})/R(\tilde{\alpha}^{n-1})}(z^{-1} - \tilde{\theta}^n) \chi_{\tilde{\beta}}^{n}(\tilde{\theta}, z^{-1}),$$

which are obtained by Lemma 2.2. (Of course, the second equality from (3.2) is trivial when we have $R(\tilde{\beta}^{n-1}) = N(\tilde{\alpha}^n)$, which happens if ε_α is small enough.)

When n is odd the calculation is similar and we deduce, in general, that the following equality holds:

$$\chi_{(\tilde{\beta},\tilde{\theta}_1)}(z) =$$

(3.3)

$$= z^{\operatorname{ind}\tilde{\beta} - \operatorname{ind}\tilde{\alpha}} \chi_{(\tilde{\alpha},\tilde{\theta})}(z) (\operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\tilde{\theta}_1^{n-1} \mid M^{n-1}))^{(-1)^{n-1}}.$$

A similar relation holds if $\tilde{\alpha}, \tilde{\beta}$, $\tilde{\theta}$ and $\tilde{\theta}_1$ are replaced by α, β, θ and θ_1. From Theorem 1.1 we have that $\operatorname{ind}\tilde{\beta} = \operatorname{ind}\beta$ and $\operatorname{ind}\tilde{\alpha} = \operatorname{ind}\alpha$ if $\varepsilon_\alpha > 0$ is sufficiently small. Consequently, on account of the equality (3.3), we can write that

$$\frac{\chi_{(\tilde{\alpha},\tilde{\theta})}(z)}{\chi_{(\alpha,\theta)}(z)} = \frac{\chi_{(\tilde{\beta},\tilde{\theta}_1)}(z)}{\chi_{(\beta,\theta_1)}(z)} \left(\frac{\varphi(\tilde{\theta}_1^{n-1}, z^{-1})}{\varphi(\theta_1^{n-1}, z^{-1})}\right)^{(-1)^n}, \tag{3.4}$$

where $\varphi(\theta_1^{n-1}, z^{-1}) = \operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\theta_1^{n-1} \mid M^{n-1})$.

Now we apply the induction hypothesis. Since in the algebra $\mathcal{O}_0$ the set of all invertible elements $\mathcal{O}_0^{-1}$ is open and the map $f \to f^{-1}$ is continuous on $\mathcal{O}_0^{-1}$ [16], there exists a neighbourhood V_0 of zero such that

$$\left|\frac{\chi_{(\tilde{\beta},\tilde{\theta}_1)}(z)}{\chi_{(\beta,\theta_1)}(z)} - 1\right| < \eta_0(\|\tilde{\beta} - \beta\| + \|\tilde{\theta}_1 - \theta_1\|), \quad z \in V_0,$$

where η_0 is a positive function such that $\eta_0(r) \to 0$ as $r \to 0$.

We also notice that the polynomial $z^{m_{n-1}}\varphi(\tilde{\theta}_1^{n-1}, z^{-1})$ is convergent to the polynomial $z^{m_{n-1}}\varphi(\theta_1^{n-1}, z^{-1})$ uniformly on compact subsets of the complex plane as $\|\tilde{\theta}_1 - \theta_1\| \to 0$, by Lemma 2.4, where $m_{n-1} = \dim M^{n-1}$. Since the value at zero of these polynomials is one, the above argument also applies to this case, and we can find a neighbourhood V_1 of zero such that

$$\left|\frac{\varphi(\tilde{\theta}_1^{n-1}, z^{-1})}{\varphi(\theta_1^{n-1}, z^{-1})} - 1\right| < \eta_1(\|\tilde{\theta}_1 - \theta_1\|), \quad z \in V_1,$$

where η_1 is a positive function such that $\eta_1(r) \to 0$ as $r \to 0$. We only note that $\|\tilde{\beta} - \beta\|$ and $\|\tilde{\theta}_1 - \theta_1\|$ tend to zero as $\|\tilde{\alpha} - \alpha\|$ and $\|\tilde{\theta} - \theta\|$ tend to zero, by Lemma 2.10. Hence there exists a neighbourhood V of zero and a positive function η such that $\eta(r) \to 0$ as $r \to 0$, and

$$|\chi_{(\tilde{\alpha},\tilde{\theta})}(z) - \chi_{(\alpha,\theta)}(z)| < \eta(\|\tilde{\alpha} - \alpha\| + \|\tilde{\theta} - \theta\|), \quad z \in V,$$

which is the desired continuity. The proof of Theorem 1.3 is complete.

3.1. Corollary. *The mapping*

$$\mathcal{F}(X) \ni (\alpha, \theta) \to L_\alpha(\theta) \in \mathbf{C}$$

is continuous.

Proof. From (3.1), we deduce, by a simple calculation, that

$$(\partial\chi_{(\alpha,\theta)}/\partial z)(0) = -L_\alpha(\theta).$$

Therefore the continuity of the mapping $(\alpha, \theta) \to L_\alpha(\theta)$ follows from Theorem 1.3, via Cauchy's inequalities.

3.2. REMARKS. 1°. The previous argument shows that all coefficients in the Taylor expansion of $\chi_{(\alpha,\theta)}$ at zero are continuous functions on the space $\mathscr{F}(X)$.

2°. The characteristic function (and the Lefschetz number as well) is still continuous at each point (α, θ) if we only ask that $\|\tilde{\alpha} - \alpha\| \to 0$ and $\|\tilde{\theta}^p - \theta^p\| \to 0$ for every $p \in \mathbf{Z}$. This condition is sufficient since $\chi^p_{\tilde{\alpha}}(\tilde{\theta}, w) = 1$ when $H^p(X, \alpha) = 0$ and $\|\tilde{\alpha} - \alpha\|$ is small enough, regardless the size of $\|\tilde{\theta}^p - \theta^p\|$.

3.3. PROPOSITION. *Let us define the set*

$$\mathscr{F}_1(X) = \{(\alpha, \theta) \in \mathscr{F}(X);\ \prod_{k \in \mathbf{Z}} \mathrm{Det}_{H^{2k+1}(X,\alpha)}(\theta^{2k+1}) \neq 0\}.$$

Then $\mathscr{F}_1(X)$ is open in $\mathscr{F}(X)$ and the mapping

$$\mathscr{F}_1(X) \ni (\alpha, \theta) \to \prod_{k \in \mathbf{Z}} \frac{\mathrm{Det}_{H^{2k}(X,\alpha)}(\theta^{2k})}{\mathrm{Det}_{H^{2k+1}(X,\alpha)}(\theta^{2k+1})} \tag{3.5}$$

is continuous.

Proof. Let us consider the function

$$M_\alpha(\theta, z) = (-z)^{-\mathrm{ind}\,\alpha}\chi_{(\alpha,\theta)}(z),$$

which is well-defined outside a disc with center at zero and a sufficiently large radius (this assertion follows easily by examining the equality (3.1)). Moreover, if $(\alpha, \theta) \in \mathscr{F}_1(X)$, then $M_\alpha(\theta, z)$ is analytic at infinity and $M_\alpha(\theta, \infty)$ is just the mapping (3.5). Hence it will be sufficient to prove the continuity of the mapping

$$\mathscr{F}_1(X) \ni (\alpha, \theta) \to M_\alpha(\theta, \cdot) \in \mathscr{O}_\infty, \tag{3.6}$$

where $\mathscr{O}_\infty$ is the algebra of germs of analytic functions in neighbourhoods of infinity, endowed with its natural topology of inductive limit of Banach algebras.

The proof of this assertion is similar to that of Theorem 1.3, so that we only sketch it.

We show first that the set $\mathscr{F}_1(X)$ is open in $\mathscr{F}(X)$. We use the notation and the inductive argument from the proof of Theorem 1.3.

If n is even, then from (3.3) we infer that

$$(-1)^{m_{n-1}}M_\alpha(\theta, z) = M_\beta(\theta, z)\cdot \mathrm{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\theta_1^{n-1} \mid M^{n-1}),$$

since $\operatorname{ind}\beta - \operatorname{ind}\alpha = -\dim M^{n-1} = -m_{n-1}$, by Proposition 2.9 from [12]. Thus $M_\alpha(\theta, \infty) \neq 0$ implies that $M_\beta(\theta_1, \infty) \neq 0$. By the induction hypothesis, we have that $M_{\tilde{\beta}}(\tilde{\theta}_1, \infty) \neq 0$ if $(\tilde{\beta}, \tilde{\theta}_1)$ is in a neighbourhood of (β, θ_1). Therefore $M_{\tilde{\alpha}}(\tilde{\theta}, \infty) \neq 0$ if $(\tilde{\alpha}, \tilde{\theta})$ is in a neighbourhood of (α, θ).

If n is odd, then (3.3) implies that

$$(-1)^{m_{n-1}} M_\beta(\theta_1, z) = M_\alpha(\theta, z) \cdot \operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\theta_1^{n-1} \mid M^{n-1}).$$

But in this case we can write that

$$\operatorname{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\theta_1^{n-1} \mid M^{n-1}) = \operatorname{Det}_{N(\alpha^n)/R(\alpha^{n-1})}(z^{-1} - \theta^n)$$

by (2.3), since $N(\alpha^n) = R(\beta^{n-1})$ and $N(\beta^{n-1}) = N(\alpha^{n-1})$. Moreover, $\chi_\alpha^n(\theta, 0) \neq 0$ by the hypothesis. Hence $M_\beta(\theta_1, \infty) \neq 0$ and we can argue as above. Consequently the set $\mathscr{F}_1(X)$ is open in $\mathscr{F}(X)$.

The continuity of the mapping (3.6) at the point (α, θ) (from $\mathscr{F}_1(X)$) follows by the formula

$$\frac{M_{\tilde{\alpha}}(\tilde{\theta}, z)}{M_\alpha(\theta, z)} = \frac{M_{\tilde{\beta}}(\tilde{\theta}_1, z)}{M_\beta(\theta_1, z)} \left(\frac{\varphi(\tilde{\theta}_1^{n-1}, z^{-1})}{\varphi(\theta_1^{n-1}, z^{-1})} \right)^{(-1)^n},$$

which is obtained from (3.4), as in the proof of Theorem 1.3.

We end this section with a version of Theorem 1.3, that is valid for complexes of unbounded operators.

Let $X = (X^p)_{p \in \mathbf{Z}}$ be a sequence of Banach spaces and let $D = (D^p)_{p \in \mathbf{Z}}$ be a sequence of linear spaces such that $D^p \subset X^p$ for all $p \in \mathbf{Z}$. We denote by $\partial(X; D)$ the set of all families of maps $\alpha = (\alpha^p)_{p \in \mathbf{Z}}$ such that $\alpha^p : D^p \to X^{p+1}$ is a closed linear operator and $R(\alpha^p) \subset N(\alpha^{p+1})$ for each p. An element $\alpha \in \partial(X; D)$ will still be called a *complex*. Let $\Phi(X; D)$ be the subset of those elements of $\partial(X; D)$ that are *Fredholm* (the definition is the same). An *endomorphism* of the complex $\alpha \in \partial(X; D)$ is a family $\theta = (\theta^p)_{p \in \mathbf{Z}}$ such that $\theta^p \in \mathscr{L}(X^p)$, $R(\theta^p | D^p) \subset D^p$ and $\alpha^p \theta^p = \theta^{p+1}\alpha^p$ for all p. We denote by $\operatorname{End}(X, \alpha)$ the family of all endomorphisms of the complex $\alpha \in \partial(X; D)$. Let us consider the set

$$\mathscr{F}(X; D) = \{(\alpha, \theta);\ \alpha \in \partial(X; D),\ \theta \in \operatorname{End}(X, \alpha)\},$$

which will be given the topology induced by the pseudodistance (1.1). It is plain that the characteristic function can also be defined in this context.

3.3. PROPOSITION. *The mapping*

$$\mathscr{F}(X; D) \ni (\alpha, \theta) \to \chi_{(\alpha, \theta)} \in \mathcal{O}_0$$

is continuous.

Proof. We reduce the statement to the bounded case, by a well-known procedure. Namely, we consider the following norm on the space D^p:

$$\|y\|_\alpha = \|y\| + \|\alpha^p y\|, \quad y \in D^p.$$

Endowed with this norm, the space D^p becomes a Banach space, since α^p is a closed operator. We note that $\|\alpha^p\|_\alpha \leqslant 1$ and that $\|\theta^p\|_\alpha \leqslant \max\{\|\theta^p\|, \|\theta^{p+1}\|\}$ for all p.

Now let $(\tilde{\alpha}, \tilde{\theta}) \in \mathscr{F}(X; D)$ be such that $\|\tilde{\alpha} - \alpha\| < \infty$ and $\|\tilde{\theta} - \theta\| < \infty$. We show that the operators induced by $\tilde{\alpha}^p$ and $\tilde{\theta}^p$ on D^p are bounded. Indeed, it is easily seen that

$$\|\tilde{\alpha}^p y\|_\alpha \leqslant (1 + \|\tilde{\alpha}^{p+1} - \alpha^{p+1}\|) \max\{1, \|\tilde{\alpha}^p - \alpha^p\|\} \|y\|_\alpha$$

and

$$\|\tilde{\theta}^p y\|_\alpha \leqslant \max\{\|\tilde{\theta}^p\|, \|\tilde{\alpha}^p - \alpha^p\| \|\tilde{\theta}^p\|, \|\tilde{\alpha}^p - \alpha^p\| \|\tilde{\theta}^{p+1}\|, \|\tilde{\theta}^{p+1}\|\} \|y\|_\alpha$$

for all $y \in D^p$ and $p \in \mathbf{Z}$. Furthermore, we also have that

$$\|(\tilde{\alpha}^p - \alpha^p) y\|_\alpha \leqslant \max\{\|\tilde{\alpha}^p - \alpha^p\|, \|\tilde{\alpha}^{p+1} - \alpha^{p+1}\|, \|\tilde{\alpha}^p - \alpha^p\| \|\tilde{\alpha}^{p+1} - \alpha^{p+1}\|\} \|y\|_\alpha$$

and

$$\|(\tilde{\theta}^p - \theta^p) y\|_\alpha \leqslant \max\{\|\tilde{\theta}^p - \theta^p\|, \|\tilde{\alpha}^p - \alpha^p\| \|\tilde{\theta}^p\|, \|\tilde{\alpha}^p - \alpha^p\| \|\tilde{\theta}^{p+1}\|, \|\tilde{\theta}^{p+1} - \theta^{p+1}\|\} \|y\|_\alpha$$

for all $y \in D^p$ and $p \in \mathbf{Z}$. These estimates show that $\|\tilde{\alpha} - \alpha\|_\alpha$ tends to zero as $\|\tilde{\alpha} - \alpha\|$ tends to zero, and that $\|\tilde{\theta}^p - \theta^p\|_\alpha$ tends to zero for each p as $\|\tilde{\theta} - \theta\|$ tends to zero. The desired assertion then follows from Theorem 1.3, on account of Remark 3.2.2°.

4. ANALYTIC AND SMOOTH DEPENDENCE OF PARAMETERS

In this section we study the variation of the characteristic function when the arguments vary analytically or smoothly. We show that, in general, the type of variation is preserved by the characteristic function. In the first part of this section we deal with the analytic dependence of parameters.

4.1. LEMMA. *Let X, Y and Z be Banach spaces, let $\Omega \subset \mathbf{C}^m$ be open, and let $S: \Omega \to \mathscr{L}(X, Y)$ and $T: \Omega \to \mathscr{L}(Y, Z)$ be analytic such that $T(\lambda)S(\lambda) = 0$ for all $\lambda \in \Omega$. Assume that $\mathrm{R}(S(\lambda_0)) = \mathrm{N}(T(\lambda_0))$ for a certain $\lambda_0 \in \Omega$. If $g: \Omega \to Y$ is analytic and $T(\lambda)g(\lambda) = 0$ in Ω, then we can find an open neighbourhood V_0 of λ_0 and an analytic function $f: V_0 \to X$ such that $S(\lambda)f(\lambda) = g(\lambda)$ in V_0. Moreover, if $g(\lambda_0) = 0$, then $f(\lambda_0)$ can be given an arbitrary value in $\mathrm{N}(S(\lambda_0))$.*

The proof of this lemma, which extends a result from [6], is sketched in [11] (see also [15]).

4.2. LEMMA. *Let X, Y, Z and W be Banach spaces, let $\Omega \subset \mathbf{C}^m$ be open, and let $S: \Omega \to \mathscr{L}(X, Y)$, $T: \Omega \to \mathscr{L}(Y, Z)$ and $U: \Omega \to \mathscr{L}(Z, W)$ be analytic such that $T(\lambda)S(\lambda) = 0$ and $U(\lambda)T(\lambda) = 0$ in Ω. Assume that $\mathrm{R}(T(\lambda_0)) = \mathrm{N}(U(\lambda_0))$ and that there exists a finite dimensional complement M of $\mathrm{R}(S(\lambda_0))$ in $\mathrm{N}(T(\lambda_0))$ for a certain $\lambda_0 \in \Omega$. Then there exist a neighbourhood V_0 of λ_0 and an analytic function $S_1: V_0 \to \mathscr{L}(X \oplus M, Y)$ such that $T(\lambda)S_1(\lambda) = 0$. $S_1(\lambda) \mid X = S(\lambda)$ and $\mathrm{R}(S_1(\lambda)) = \mathrm{N}(T(\lambda))$ for all $\lambda \in V_0$.*

Proof. Let us consider the operators $\tau(\lambda)$ from $\mathscr{L}(M, Y)$ into $\mathscr{L}(M, Z)$ given by $\tau(\lambda)(G) = T(\lambda)G$ for $G \in \mathscr{L}(M, Y)$, and $\nu(\lambda)$ from $\mathscr{L}(M, Z)$ into $\mathscr{L}(M, W)$ defined by $\nu(\lambda)(H) = U(\lambda)H$ for $H \in \mathscr{L}(M, Z)$. Since $\mathrm{R}(T(\lambda_0)) = \mathrm{N}(U(\lambda_0))$, Lemma 2.7 implies that $\mathrm{R}(\tau(\lambda_0)) = \mathrm{N}(\nu(\lambda_0))$. Then the previous lemma assures us of the existence of a $\mathscr{L}(M, Y)$-valued function C, that is analytic in a neighbourhood of λ_0, such that $\tau(\lambda)C(\lambda) = 0$ and $C(\lambda_0) = 1_M$. If we define $S_1(\lambda)(x \oplus v) = S(\lambda)x + C(\lambda)v$ for all $x \in X$ and $v \in M$, then we have $T(\lambda)S_1(\lambda) = 0$ and $S_1(\lambda) \mid X = S(\lambda)$. Since $\mathrm{R}(S_1(\lambda_0)) = \mathrm{N}(T(\lambda_0))$, the property $\mathrm{R}(S_1(\lambda)) = \mathrm{N}(T(\lambda))$ also holds in a neighbourhood of λ_0, by Lemma 2.5.

4.3. LEMMA. *Let the conditions of Lemma 4.2 be fulfilled. Let also $A: \Omega \to \mathscr{L}(X)$ and $B: \Omega \to \mathscr{L}(Y)$ be analytic, such that $S(\lambda)A(\lambda) = B(\lambda)S(\lambda)$ and $B(\lambda)\mathrm{N}(T(\lambda)) \subset \mathrm{N}(T(\lambda))$ for all $\lambda \in \Omega$. Then there exist an open neighbourhood $V_1 \subset V_0$ of λ_0 and an analytic function $A_1: V_1 \to \mathscr{L}(X \oplus M)$ such that $S_1(\lambda)A_1(\lambda) = B(\lambda)S_1(\lambda)$ and $A_1(\lambda) \mid X = A(\lambda)$ for all $\lambda \in V_1$.*

Proof. As in Lemma 2.7, we define the operators

$$\sigma(\lambda): \mathscr{L}(M, X) \oplus \mathscr{L}(M) \to \mathscr{L}(M, Y) \quad \text{and} \quad \tau(\lambda): \mathscr{L}(M, Y) \to \mathscr{L}(M, Z)$$

by the formulas $\sigma(\lambda)(E \oplus F) = S(\lambda)E + C(\lambda)F$ and $\tau(\lambda)(G) = T(\lambda)G$, where $C(\lambda)$ has been obtained in the proof of Lemma 4.2. We note that $\mathrm{R}(\sigma(\lambda_0)) = \mathrm{N}(\tau(\lambda_0))$, by Lemma 2.7. Therefore, from Lemma 4.1, we infer that there are a neighbourhood V_1 of λ_0 and two analytic functions $E(\lambda)$ and $F(\lambda)$ in V_1 such that

$$S(\lambda)E(\lambda) + C(\lambda)F(\lambda) = B(\lambda)C(\lambda).$$

Then the function $A_1(\lambda)$, given by

$$A_1(\lambda)(x \oplus v) = (A(\lambda)x + E(\lambda)v) \oplus F(\lambda)v, \quad x \in X, \ v \in M,$$

is analytic and has the required properties (see the proof of Lemma 2.8).

Let $X = (X^p)_{p \in \mathbf{Z}}$ be a sequence of Banach spaces, and let $\Omega \subset \mathbf{C}^m$ be an open set. A mapping

$$\Omega \ni \lambda \to (\alpha(\lambda), \theta(\lambda)) \in \mathscr{F}(X)$$

is said to be *analytic* if $\alpha^p(\lambda)$ and $\theta^p(\lambda)$ are analytic for each $p \in \mathbf{Z}$, and the mapping $\lambda \to \alpha(\lambda)$ is continuous on Ω in the topology of $\Phi(X)$ (induced by the pseudodistance (1.1)). Of course, the condition of continuity is superfluous if X is of finite length.

4.4. THEOREM. *Let $X = (X^p)_{p\in\mathbf{Z}}$ be a sequence of Banach spaces, and let $\Omega \subset \mathbf{C}^m$ be open. Let us consider an analytic mapping*

$$\Omega \ni \lambda \to (\alpha(\lambda),\ \theta(\lambda)) \in \mathscr{F}(X).$$

Then the mapping

$$\Omega \ni \lambda \to \chi_{(\alpha(\lambda),\theta(\lambda))} \in \mathcal{O}_0$$

is also analytic.

Proof. It is enough to prove the analyticity of the mapping $\chi_{(\alpha(\lambda),\theta(\lambda))}$ in a neighbourhood of an arbitrary point $\lambda_0 \in \Omega$. With no loss of generality, we may assume that $H^p(X, \alpha(\lambda_0)) = 0$ if $p < 0$. We use the inductive argument from the proof of Theorem 1.3. With obvious modifications of the notation in the proof of Theorem 1.3, from (3.4) we obtain the equality

$$\frac{\chi_{(\alpha(\lambda),\theta(\lambda))}(z)}{\chi_{(\alpha(\lambda_0),\theta(\lambda_0))}(z)} = \frac{\chi_{(\beta(\lambda),\theta_1(\lambda))}(z)}{\chi_{(\beta(\lambda_0),\theta_1(\lambda_0))}(z)} \left(\frac{\varphi(\theta_1^{n-1}(\lambda),\ z^{-1})}{\varphi(\theta_1^{n-1}(\lambda_0),\ z^{-1})}\right)^{(-1)^n} \tag{4.1}$$

where the extensions $\beta^{n-1}(\lambda)$ and $\theta_1^{n-1}(\lambda)$ can be chosen to depend analytically on λ in a neighbourhood of λ_0, by Lemmas 4.2 and 4.3. The mapping $\chi_{(\beta(\lambda),\theta_1(\lambda))}$ depends analytically on λ by the induction hypothesis, and the mapping

$$z^{m_{n-1}}\varphi(\theta_1^{n-1}(\lambda), z^{-1}) = z^{m_{n-1}}\mathrm{Det}_{M^{n-1}}(z^{-1} - P_{n-1}\theta_1^{n-1}(\lambda) \mid M^{n-1}) \tag{4.2}$$

depends analytically on λ by obvious reasons. From the properties of the algebra $\mathcal{O}_0$, it follows that the function (4.2) is invertible and that its inverse is also analytic. Consequently, from (4.1) we can derive the analyticity of the function $\chi_{(\alpha(\lambda),\theta(\lambda))}$, and the proof of the theorem is complete.

4.5. COROLLARY. *With the conditions of the preceding theorem, the mapping*

$$\Omega \ni \lambda \to L_{\alpha(\lambda)}(\theta(\lambda)) \in \mathbf{C}$$

is analytic.

4.6. COROLLARY. *Let us consider the analytic mapping*

$$\Omega \ni \lambda \to (\alpha(\lambda),\ \theta(\lambda)) \in \mathscr{F}_1(X).$$

Then the mapping

$$\Omega \ni \lambda \to M_{\alpha(\lambda)}(\theta(\lambda),\ \cdot) \in \mathcal{O}_\infty$$

is analytic.

4.7. REMARK. Since the proof of Lemma 4.1 can be easily adapted for real analytic functions, a version of Theorem 4.4 in such a context can also be given.

We are not aware of any version of Lemma 4.1 which would be valid for smooth functions and general Banach spaces. For this reason, in order to give a version of Theorem 4.4 in the case of smooth functions, we shall work, from now on, only with complexes of Hilbert spaces. In regard to terminology, we say that a function (generally vector-valued) is a *$\mathscr{C}^r$-function*, where r is a non-negative integer or infinity, if it is continuously differentiable up to order r on its domain of definition (which is usually an open subset of $\mathbf{R}^n$).

4.8. LEMMA. *Let X, Y and Z be Hilbert spaces, let $\Omega \subset \mathbf{R}^m$ be open, and let $S: \Omega \to \mathscr{L}(X, Y)$ and $T: \Omega \to \mathscr{L}(Y, Z)$ be $\mathscr{C}^r$-functions, such that $T(\lambda)S(\lambda) = 0$ for all $\lambda \in \Omega$. If $\mathrm{R}(S(\lambda_0)) = \mathrm{N}(T(\lambda_0))$ and $\mathrm{R}(T(\lambda_0))$ is closed for a certain $\lambda_0 \in \Omega$, then the orthogonal projection $P(\lambda)$ of X onto $\mathrm{N}(S(\lambda))$ is a $\mathscr{C}^r$-function in an open neighbourhood of λ_0.*

Proof. We note first that if $A: \Omega \to \mathscr{L}(X, Y)$ is a $\mathscr{C}^r$-function and $A(\lambda)$ is surjective for each λ, then the orthogonal projection of X onto $\mathrm{N}(A(\lambda))$ is a $\mathscr{C}^r$-function in Ω. Indeed, it is easily seen that the operator $A(\lambda)A(\lambda)^*$ is invertible on Y for each $\lambda \in \Omega$. Then the function

$$1_X - A(\lambda)^*(A(\lambda)A(\lambda)^*)^{-1}A(\lambda),$$

which is a $\mathscr{C}^r$-function on Ω, defines the orthogonal projection of X onto $\mathrm{N}(A(\lambda))$ at each point $\lambda \in \Omega$. (This argument has been suggested to us by C. Apostol.)

Next we consider the function $A: \Omega \to \mathscr{L}(X \oplus Z, Y)$, given by the formula

$$A(\lambda) = S(\lambda)P_X + T(\lambda)^*P_Z, \quad \lambda \in \Omega, \tag{4.3}$$

where P_X and P_Z are the natural projections of $X \oplus Z$ onto X and Z, respectively. Since $\mathrm{R}(S(\lambda_0)) = \mathrm{N}(T(\lambda_0))$ and $\mathrm{R}(T(\lambda_0))$ is closed, the operator $A(\lambda_0)$ is surjective. Therefore the operator $A(\lambda)$ is surjective for λ in a neighbourhood V of λ_0. By the first part of the proof, the orthogonal projection $Q(\lambda)$ of $X \oplus Z$ onto $\mathrm{N}(A(\lambda))$ is a $\mathscr{C}^r$-function in V. We also notice that

$$\mathrm{N}(A(\lambda)) = \mathrm{N}(S(\lambda)) \oplus \mathrm{N}(T(\lambda)^*), \quad \lambda \in \Omega.$$

Then $P(\lambda) = P_X Q(\lambda) \mid X$ is the projection of X onto $\mathrm{N}(S(\lambda))$, and it is a $\mathscr{C}^r$-function in V.

4.9. LEMMA. *With the conditions of the previous lemma, if $g: V \to Y$ is a $\mathscr{C}^r$-function such that $T(\lambda)g(\lambda) = 0$ in V, then we can find a $\mathscr{C}^r$-function $f: V \to X$ with the property that $S(\lambda)f(\lambda) = g(\lambda)$ in V.*

Proof. By the preceding lemma, the operator

$$A(\lambda)(1_{X\oplus Z} - Q(\lambda)) \tag{4.4}$$

is injective and surjective, and therefore invertible, for all $\lambda \in V$, where $A(\lambda)$ is given by (4.3) and $Q(\lambda)$ is the orthogonal projection of $X \oplus Z$ onto $\mathrm{N}(A(\lambda))$. Moreover, the inverse of (4.4) is a $\mathscr{C}^r$-function. Then the expression

$$f(\lambda) = (P_X - P_X Q(\lambda))\,(A(\lambda)\,(1_{X\oplus Z} - Q(\lambda)))^{-1} g(\lambda), \quad \lambda \in V,$$

defines a $\mathscr{C}^r$-function that satisfies $S(\lambda)f(\lambda) = g(\lambda)$ in V. The last assertion follows from the identity

$$(S(\lambda)P_X + T(\lambda)^* P_Z)(1_{X\oplus Z} - Q(\lambda))\,(A(\lambda)\,(1_{X\oplus Z} - Q(\lambda)))^{-1} g(\lambda) = g(\lambda),$$

and from the fact that $\mathrm{R}(S(\lambda))$ and $\mathrm{R}(T(\lambda)^*)$ are orthogonal.

4.10. LEMMA. *Let X, Y, Z and W be Hilbert spaces, let $\Omega \subset \mathbf{R}^m$ be open, and let $S\colon \Omega \to \mathscr{L}(X, Y)$, $T\colon \Omega \to \mathscr{L}(Y, Z)$ and $U\colon \Omega \to \mathscr{L}(Z, W)$ be $\mathscr{C}^r$-functions such that $T(\lambda)S(\lambda) = 0$ and $U(\lambda)T(\lambda) = 0$ in Ω. Assume that $\mathrm{R}(T(\lambda_0)) = \mathrm{N}(U(\lambda_0))$, that $\mathrm{R}(U(\lambda_0))$ is closed and that there exists a finite dimensional complement M of $\mathrm{R}(S(\lambda_0))$ in $\mathrm{N}(T(\lambda_0))$ for a certain $\lambda_0 \in \Omega$. Then there exist a neighbourhood V_0 of λ_0 and a $\mathscr{C}^r$-function $S_1\colon V_0 \to \mathscr{L}(X \oplus M, Y)$ such that $T(\lambda)S_1(\lambda) = 0$, $S_1(\lambda) \mid X = S(\lambda)$ and $\mathrm{R}(S_1(\lambda)) = \mathrm{N}(T(\lambda))$ for all $\lambda \in V_0$.*

Proof. Let $P(\lambda)$ be the orthogonal projection of Y onto $\mathrm{N}(T(\lambda))$, which is a $\mathscr{C}^r$-function in a neighbourhood of λ_0, by Lemma 4.8. Then $C(\lambda) = P(\lambda) \mid M$ is a $\mathscr{C}^r$-function, and we set $S_1(\lambda)\,(x \oplus v) = S(\lambda)x + C(\lambda)v$ for all $x \in X$ and $v \in M$. The function $S_1(\lambda)$ has the required properties in a neighbourhood of λ_0, as in the proof of Lemma 4.2.

4.11. LEMMA. *Let the conditions of Lemma 4.10 be fulfilled. Let also $A\colon \Omega \to \mathscr{L}(X)$ and $B\colon \Omega \to \mathscr{L}(Y)$ be $\mathscr{C}^r$-functions such that $S(\lambda)A(\lambda) = B(\lambda)\,S(\lambda)$ and $B(\lambda)\mathrm{N}(T(\lambda)) \subset \mathrm{N}(T(\lambda))$ for all $\lambda \in \Omega$. Then there exist an open neighbourhood $V_1 \subset V_0$ and a $\mathscr{C}^r$-function $A_1\colon V_1 \to \mathscr{L}(X \oplus M)$ such that $S_1(\lambda)A_1(\lambda) = B(\lambda)S_1(\lambda)$ and $A_1(\lambda) \mid X = A(\lambda)$ for all $\lambda \in V_1$.*

The proof of Lemma 4.11 is similar to that of Lemma 4.3; the only difference is the use of Lemma 4.9 instead of Lemma 4.1.

Let $X = (X^p)_{p\in\mathbf{Z}}$ be a sequence of Banach spaces, and let $\Omega \subset \mathbf{R}^m$ be an open set. A mapping

$$\Omega \ni \lambda \to (\alpha(\lambda), \theta(\lambda)) \in \mathscr{F}(X)$$

is said to be a *$\mathscr{C}^r$-function* if $\alpha^p(\lambda)$ and $\theta^p(\lambda)$ are $\mathscr{C}^r$-functions in Ω for all p, and the mapping $\lambda \to \alpha(\lambda)$ is continuous on Ω in the topology of $\Phi(X)$.

4.12. THEOREM. *Let $X = (X^p)_{p\in\mathbf{Z}}$ be a sequence of Hilbert spaces and let $\Omega\subset\mathbf{R}^m$ be open. Let us consider a $\mathscr{C}^r$-function*

$$\Omega \ni \lambda \to (\alpha(\lambda), \theta(\lambda)) \in \mathscr{F}(X).$$

Then the mapping

$$\Omega \ni \lambda \to \chi_{(\alpha(\lambda),\theta(\lambda))} \in \mathcal{O}_0$$

is also a $\mathscr{C}^r$-function.

The proof of this theorem is similar to that of Theorem 4.4, so that it will be omitted. We only mention that one uses Lemmas 4.10 and 4.11 instead of Lemmas 4.2 and 4.3.

4.13. COROLLARY. *With the conditions of the previous theorem, the mapping*

$$\Omega \ni \lambda \to L_{\alpha(\lambda)}(\theta(\lambda)) \in \mathbf{C}$$

is a $\mathscr{C}^r$-function.

4.14. COROLLARY. *Let us consider the $\mathscr{C}^r$-function*

$$\Omega \ni \lambda \to (\alpha(\lambda), \theta(\lambda)) \in \mathscr{F}_1(X).$$

Then the mapping

$$\Omega \ni \lambda \to M_{\alpha(\lambda)}(\theta(\lambda), \cdot) \in \mathcal{O}_\infty$$

is a $\mathscr{C}^r$-function.

5. FINAL COMMENTS AND EXAMPLES

The implications of Theorem 1.3 and of its versions and corollaries have not yet been completely worked out. In this section we shall present some immediate consequences, open problems and connections with similar mathematical objects.

1°. We begin this discussion with a consequence of our results for operator theory. Let X be a Banach space and let $S \in \mathscr{L}(X)$ be a Fredholm operator. Let also $\mathrm{C}(S)$ denote the *commutant* of S in $\mathscr{L}(X)$. If $A \in \mathrm{C}(S)$, then we can define the function

$$\chi_{(S,A)}(z) = \frac{\mathrm{Det}_{\mathrm{N}(S)}(1 - zA)}{\mathrm{Det}_{X/\mathrm{R}(S)}(1 - zA)},$$

which will be called the *characteristic function* of the pair (S, A).

5.1. PROPOSITION. *Let us consider the Taylor expansion of the characteristic function of the pair (S, A) at zero, namely*

$$\chi_{(S,A)}(z) = \sum_{k=0}^{\infty} z^k L_S^{(k)}(A).$$

If $L_S^{(m)}(A) \neq 0$ for a certain $m \geq 1$, then there exists a positive number $\varepsilon > 0$ such that if $\tilde{S} \in \mathscr{L}(X)$ is a Fredholm operator, $\tilde{A} \in C(\tilde{S})$, $\|\tilde{S} - S\| < \varepsilon$ and $\|\tilde{A} - A\| < \varepsilon$, then either $\tilde{A}$ or its adjoint has an eigenvalue.

Proof. Let $m \geq 1$ be minimal with the property that $L_S^{(m)}(A) \neq 0$. Then we have

$$\chi_{(S,A)}(z) = 1 + z^m(L_S^{(m)}(A) + \ldots).$$

Therefore $\chi_{(S,A)} \neq 1$. By Theorem 1.3, we also have $\chi_{(\tilde{S},\tilde{A})} \neq 1$ if $\|\tilde{S} - S\|$ and $\|\tilde{A} - A\|$ are sufficiently small. In this case either $N(\tilde{S}) \neq 0$ or $R(\tilde{S}) \neq X$. Accordingly, either $\tilde{A}$ has an eigenvalue or its adjoint has one.

Let us mention that the case $L_S^{(1)}(A) \neq 0$ has been treated in [14] (where $L_S(A) = -L_S^{(1)}(A)$ is called the *Lefschetz number* of A with respect to S).

2°. Most of the above considerations make sense in the case of a Fredholm system of operators [12] and an operator in its commutant. However, the interpretation of the results is not as easy as in the previous case. Any contribution in this respect would be of interest.

3°. Corollary 3.1 is not a precise extension of Theorem 1.1. Indeed, the semicontinuity of the functions $\alpha \to \dim H^p(X, \alpha)$ is not reflected by Corollary 3.1. As a matter of fact, nothing of this sort can, in general, be expected, as shown by the following:

5.2. EXAMPLE. Let X be equal to $\mathbf{C}^2$, and let $S(\lambda)$ and $A(\lambda)$ be operators on X given by

$$S(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}, \quad A(\lambda) = \begin{pmatrix} \lambda_0 & 0 \\ 0 & g(\lambda) \end{pmatrix},$$

where $\lambda \in \mathbf{C}$ is a parameter, g is a continuous function on $\mathbf{C}$ and $\lambda_0 \in \mathbf{C}$ is fixed. Then we have that

$$\mathrm{Tr}_{N(S(\lambda))} A(\lambda)) = \begin{cases} g(\lambda) & \text{if } \lambda \neq 0 \\ \lambda_0 + g(0) & \text{if } \lambda = 0. \end{cases}$$

In spite of this example, is there any (non-trivial) class of endomorphisms in $\mathrm{End}(X, \alpha)$ such that the function

$$(\alpha, \theta) \to \mathrm{Tr}_{H^p(X,\alpha)}(\theta^p)$$

restricted to that class is semi-continuous for each p?

4°. The characteristic function is invariant under similarities in the following sense: Let (X, α) and (Y, β) be complexes of Banach spaces, and let $\xi: (X, \alpha) \to (Y, \beta)$ be an isomorphism, i.e., $\xi = (\xi^p)_{p \in \mathbf{Z}}$ is such that each ξ^p is an isomorphism and $\xi^{p+1}\alpha^p = \beta^{p+1}\xi^p$ for all $p \in \mathbf{Z}$. Let $\theta \in \operatorname{End}(X, \alpha)$. If $\tau \in \operatorname{End}(Y, \beta)$ is given by $\tau^p := \xi^p\theta^p(\xi^p)^{-1}$, then we have that $\chi_{(\alpha,\theta)} = \chi_{(\beta,\tau)}$, by the invariance of the determinant under similarities.

5°. We shall use the terminology from [4]. Let K be a polyhedron (or more generally, a compact ANR), and let $f\colon K \to K$ be a continuous map. Let $H^*(K; \mathbf{C}) := \{H^p(K; \mathbf{C})\}_{p\in\mathbf{Z}}$ denote the singular $\mathbf{C}$-cohomology of K. Then the map f naturally induces a morphism $f^*\colon H^*(K; \mathbf{C}) \to H^*(K; \mathbf{C})$, $f^* = (f^p)_{p\in\mathbf{Z}}$. Let us define the function

$$\chi_f(z) = \prod_{k\in\mathbf{Z}} \frac{\operatorname{Det}_{H^{2k}(K;\,\mathbf{C})}(1 - zf^{2k})}{\operatorname{Det}_{H^{2k+1}(K;\,\mathbf{C})}(1 - zf^{2k+1})},$$

which is analytic in a neighbourhood of zero. By analogy with Definition 1.2, the mapping $\chi_f(z)$ may be called the *characteristic function* of f. It has a property of continuity which is similar to that given by Theorem 1.3.

Let $\operatorname{Hom}(K)$ denote the set of all continuous functions $f\colon K \to K$. The set $\operatorname{Hom}(K)$ can be given a metric space structure induced by the relation

$$\delta(f, g) = \sup_{\omega\in K} d(f(\omega), g(\omega)), \quad f, g \in \operatorname{Hom}(K),$$

where d is the metric of K. Then the mapping

$$\operatorname{Hom}(K) \ni f \to \chi_f \in \mathcal{O}_0$$

is continuous. Indeed, if $f \in \operatorname{Hom}(K)$ is given, then there exists a $\delta > 0$ such that every $g \in \operatorname{Hom}(K)$ with the property that $\delta(f, g) < \delta$ is homotopic to f (see [4, III.A]). Therefore the maps induced by f and g in $H^*(K; \mathbf{C})$ are equal [5], and the function χ_g is actually equal to χ_f for all such g's.

Now let us consider the Taylor expansion of $\chi_f(z)$ at zero, which can be written as

$$\chi_f(z) = \sum_{m=0}^{\infty} z^m L^m(f; \mathbf{C}).$$

We note that $L^{(0)}(f; \mathbf{C}) = 1$ and that $-L^{(1)}(f; \mathbf{C}) = L(f; \mathbf{C})$ is just the Lefschetz number of the map f.

It would be interesting to give the numbers $L^{(m)}(f; \mathbf{C})$ a topological interpretation ($m \geqslant 2$). As it is known, the number $-L^{(1)}(f; \mathbf{C})$ (i.e., the Lefschetz number) coincides with the fixed point index of the map f, by the theorem of Lefschetz-Hopf [5].

6°. Let us recall some facts from [1]. Let M be a smooth compact manifold, and let $E_0, E_1, \ldots, E_n$ be smooth vector bundles over M. Let $d_p\colon \Gamma(E_p) \to \Gamma(E_{p+1})$

be differential operators such that $d_{p+1} \cdot d_p = 0$ for all p, i.e., $(\Gamma(E), d) = (\Gamma(E_p), d_p)_{p=0}^n$ is a complex, where $\Gamma(E_p)$ is the space of smooth sections of E_p. Assume that the complex $(\Gamma(E), d)$ is elliptic. Then its homology $(H^p(\Gamma(E), d))_{p=0}^n$ is finite dimensional.

Let $f: M \to M$ be a smooth map, and let $\varphi_p: f^*E_p \to E_p$ be smooth bundle homomorphisms. Let T_p be defined by $(T_p s)(\omega) = \varphi_p s(f(\omega))$, if $s \in \Gamma(E_p)$. Assume, in addition, that $d_p T_p = T_{p+1} d_p$ for all p. Then $T = (T_p)_{p=0}^n$ is an endomorphism of the complex $(\Gamma(E), d)$, whose Lefschetz number is defined by the relation

$$L(T) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr}_{H^p(\Gamma(E), d)}(T_p).$$

A fixed point $\omega \in M$ of the map f is said to be *simple* if $\operatorname{Det}(1 - df_\omega) \neq 0$, where df_ω is the induced map on the tangent space at ω. Assume that $f: M \to M$ has only simple fixed points in M. Then there are only a finite number of such points, by the compactness of M.

The Atiyah-Bott theorem asserts that, with these conditions, one has the formula

$$L(T) = \sum_{f(\omega)=\omega} \frac{\sum_{p=0}^{n} (-1)^p \operatorname{Tr} \varphi_{p,\omega}}{|\operatorname{Det}(1 - df_\omega)|},$$

where $\operatorname{Tr}\varphi_{p,\omega}$ makes sense since $\varphi_{p,\omega}$ is an endomorphism of the vector space $(E_p)_\omega$, for every fixed point $\omega \in M$ of the map f.

In particular, this formula can be used to show that the $\mathscr{C}^\infty$-variation of all parameters (i.e. the differential operators from the elliptic complex, the bundle homomorphisms and the map f) implies that the Lefschetz number is also a $\mathscr{C}^\infty$-function.

A similar result can also be obtained as an application of Corollary 4.13. Indeed, let us denote by m_p the order of the operator d_p. Let $H^\sigma(E_p)$ be the Sobolev space of order σ associated with the sections of the vector bundle E_p. Then we have the following Fredholm complex of Hilbert spaces:

$$(H, D): \quad \begin{gathered} 0 \to H^0(E_0) \xrightarrow{D_0} H^{-m_0}(E_1) \xrightarrow{D_1} H^{-(m_0+m_1)}(E_2) \longrightarrow \cdots \\ \cdots \xrightarrow{D_{n-1}} H^{-(m_0+m_1+\ldots+m_n)}(E_n) \to 0, \end{gathered}$$

where D_p are the bounded operators induced by the operators d_p, $0 \leqslant p \leqslant n$. The operators φ_p are differential operators of order zero, so that if we assume that f is a diffeomorphism, then each operator T_p extends to a continuous operator $\theta_p \in \mathscr{L}(H^\sigma(E_p))$. Moreover, $\theta = (\theta_p)_{p=0}^n$ is a morphism of the complex (H, D).

By a theorem of Hodge [17], we can identify the space $H^p(H, D)$ with the space $H^p(\Gamma(E), d)$ (via harmonic forms), and we obtain that

$$\mathrm{Tr}_{H^p(H,\,D)}(\theta_p) = \mathrm{Tr}_{H^p(\Gamma(E),\,d)}(T_p).$$

Therefore, with our notation, $L(T) = L_D(\theta)$. By noticing that the Hilbert space operators D_p and θ_p vary smoothly when d_p, φ_p and f depend smoothly on a parameter (preserving the order of d_p), then $L_D(\theta)$ varies smoothly, by Corollary 4.13. Let us remark that this procedure applies even in case when f has multiple fixed points.

7° The analytic variation of the characteristic function can also be obtained in the frame of the deformation theory of analytic spaces [8], [3].

Let X and Y be analytic spaces, let $f\colon X \to Y$ be a proper morphism of analytic spaces, and let $\mathscr{P} = (\mathscr{P}^p)_{p\in \mathbf{Z}}$ be a complex of $\mathcal{O}_X$-sheaves which are flat on Y, with coherent cohomology and exact, except for a finite number of terms. Let $\Theta : \mathscr{P}^\bullet \to \mathscr{P}^\bullet$ be a morphism of complexes of $\mathcal{O}_X$-modules. For each $y \in Y$ we can define the *characteristic function*

$$\chi_{(\mathscr{P}^\bullet_y,\,\Theta_y)}(z) = \prod_{k\in \mathbf{Z}} \frac{\mathrm{Det}_{H^{2k}(\Gamma(X_y,\,\mathscr{P}^\bullet_y))}(1 - z\Theta_y^{2k})}{\mathrm{Det}_{H^{2k+1}(\Gamma(X_y,\,\mathscr{P}^\bullet_y))}(1 - z\,\Theta_y^{2k+1})}$$

where we have denoted by X_y, $\mathscr{P}^\bullet_y$ and Θ_y the analytic fibres of the corresponding entities (see [3] for details). With standard algebraic arguments (the construction of a universal complex of finite type free modules on Y [3, Chapter 3]), one can prove that the assignment

$$Y \ni y \mapsto \chi_{(\mathscr{P}_y,\,\Theta_y)} \in \mathcal{O}_0$$

is analytic.

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