J. OPERATOR THEORY
9(1983), 27–52

ON THE CARLEMAN RESOLVENT ESTIMATE OF SCHRÖDINGER TYPE OPERATOR WITH ARBITRARY POTENTIAL

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1. INTRODUCTION

Let $a(x) = \{a_{jk}(x)\}$, $x \in \mathbf{R}^m$, $1 \leqslant m \leqslant 3$, be a real, positive definite (for each $x \in \mathbf{R}^m$), $m \times m$-matrix with the elements $a_{jk}(x) \in C^\infty(\mathbf{R}^m)$. We denote by $\nu^2(x)(\mu^2(x))$ the lowest (respectively the largest) eigenvalue of $a(x)$ and we impose on $a(x)$ the restriction $\nu_0 \leqslant \nu(x) \leqslant \mu(x) \leqslant \mu_0$, $x \in \mathbf{R}^m$, with some constants $0 < \nu_0 \leqslant \mu_0$. Consider an arbitrary self-adjoint extension A in $L^2(\mathbf{R}^m)$ of the minimal operator corresponding to the elliptic differential expression

$$S = -\operatorname{div} a(x) \operatorname{grad} + V(x) \tag{1}$$

with the real potential $V(x) \in C^\infty(\mathbf{R}^m)$ having no restrictions at the infinity. Let $\mathrm{R}(x, y, \tau)$ be the kernel of the resolvent $(A - \mathrm{i}\tau)^{-1}$, $\tau \neq 0$, $\mathrm{Im}\tau = 0$. It is known (see e.g. [3, Chapter 6, Section 2]) that $(A - \mathrm{i}\tau)^{-1}$ is a Carleman operator, that is, the integral

$$I_\tau(x) = \int_{\mathbf{R}^m} |\mathrm{R}(x, y, \tau)|^2 \mathrm{d}y$$

is a locally bounded function of x. Certain problems of the spectral theory demand the information about the growth rate of $I_\tau(x)$ at the infinity. The well-known result of this kind is the uniform boundedness of $I_\tau(x)$ in $\mathbf{R}^m$ in the case of Schrödinger expression $-\Delta + V(x)$ with the potential bounded from below. It follows from Kovalenko-Semenov results [9, Theorem 1.4] that the same property holds if we add to the semibounded function $V(x)$ another real function $V_-(x)$ which is a relatively small perturbation of the Laplace operator.

Let $B_{y,\delta}$ be the ball $|x - y| < \delta$ in $\mathbf{R}^m$, $\chi_{y,\delta}(x)$ being its characteristic function; $\|\cdot\|_p$ stands for the norm in $L^p(\mathbf{R}^m)$. The aim of the present paper is to prove the following estimate of $I_\tau(x)$, for expressions S with arbitrary behaviour of potential at the infinity.

THEOREM 1. *Fix* $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. *There exists a function* $c_0(r)$, $r > 0$, *depending only on* m, p, μ_0, ν_0, $\lim_{r\to\infty} c_0(r) = 0$, *such that*

(2)
$$I_\tau(x) \leqslant \delta^{-m} c_0^2(|\tau|)[1 + \delta\|\chi_{x,\delta}V_-\|_p^{\frac{p}{2p-m}}]^m, \quad x \in \mathbf{R}^m,$$

$V_-(x) := -\min\{0, V(x)\}$, *for every* $\delta \in (0, 4^{-1}\pi\mu_0)$.

In particular, this result delivers useful information on the existence of wave operators corresponding to the pair of real Schrödinger operators $-\Delta + V(x)$, $-\Delta + V(x) + W(x)$. In the case $V(x) \equiv 0$ the wave operators exist and are complete if $W(x) \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$ [5, Chapter 10, Section 4]. Now take the potential $V(x) \in C^\infty(\mathbf{R}^m)$ with arbitrary behaviour at the infinity. Consider some self-adjoint extension A of the corresponding minimal operator $-\Delta + V(x)$. It is quite natural to ask under what conditions on $W(x)$ the complete wave operators for the pair A, $A + W$ exist. It is easy to prove the following consequence of Theorem 1.

THEOREM 2. *Fix* $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. *Let* $A_{ac} \neq 0$, A_{ac} *being the absolutely continuous part of* A. *For every real* $W(x)$ *satisfying the condition*

(3)
$$\mathcal{H}_{\delta,j} = \int_{\mathbf{R}^m} |W(x)|^j [1 + \delta\|\chi_{x,\delta}V_-\|_p^{\frac{p}{2p-m}}]^m \mathrm{d}x < \infty,$$

where $j = 1,2$, *with some* $\delta \in (0, 4^{-1}\pi\mu_0)$, *the wave operators corresponding to the pair* A, $A + W$ *exist and are complete. If*

$$\int_{B_{x,r}} |V_-(y)|^p \,\mathrm{d}y \leqslant c, \quad x \in \mathbf{R}^m,$$

for some constants r, c, *then the above mentioned restrictions on* $W(x)$ *are reduced to* $W(x) \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$.

Proof of Theorem 2. It follows from (2) that every $f \in \mathcal{D}(A) = \mathrm{R}((A - \mathrm{i}\tau)^{-1})$ is in $L^\infty_{\mathrm{loc}}(\mathbf{R}^m)$, its growth at the infinity being dominated by $[1 + \delta\|\chi_{x,\delta}V_-\|_p^{\frac{p}{2p-m}}]^{\frac{m}{2}}$, $\delta \in (0,4^{-1}\pi\mu_0)$. This implies $\mathcal{D}(A) \subseteq \mathcal{D}(W)$ which means boundedness of the operator $W(A - \mathrm{i}\tau)^{-1}$ for every $\tau \neq 0$, $\mathrm{Im}\,\tau = 0$. The kernel of $W(A - \mathrm{i}\tau)^{-1}$ is $W(x)\mathrm{R}(x, y, \tau)$.

The inequality (2) and the condition (3), where $j = 2$, imply that the Hilbert-Schmidt norm of $W(A-i\tau)^{-1}$ may be estimated from above by $\mathscr{H}_{\delta,2}^{\frac{1}{2}} \cdot \delta^{-\frac{m}{2}} c_0(|\tau|) \to 0$ $(\tau \to \infty)$, $\delta \in (0, 4^{-1}\pi\mu_0)$. Thus W is A-bounded with A-bound 0. In the same way (by using (3) with $j = 1$) we check that $|W|^{\frac{1}{2}}(A - i\tau)^{-1}$ is a Hilbert-Schmidt operator if $\tau \neq 0$, $\operatorname{Im}\tau = 0$. Theorem 2 follows from these facts by applying Theorem 4.9, Example 4.10 and Problem 4.14 from [5, Chapter 10].

In Theorem 1 we restrict ourselves for simplicity to C^∞-potentials. The result remains true also if

$$V(x) = V_+(x) - V_-(x),\ 0 \leqslant V_+(x) \in L^1_{\mathrm{loc}}(\mathbf{R}^m),\ 0 \leqslant V_-(x) \in L^p_{\mathrm{loc}}(\mathbf{R}^m),\ p \geqslant 1,\ p > \frac{m}{2}.$$

We do not discuss here how to construct self-adjoint realizations of S satisfying the above mentioned conditions. Certain tools used in the present paper play an important part in the author's approach to this problem sketched in [12], [13, § 4] (other approaches: Kato [6], Knowles [7]).

For dimensions $m \geqslant 4$ the method of proving Theorem 1 leads to the analogous estimates of the operators $(A - i\tau)^{-l}$, $l > \frac{m}{4}$. This result is briefly discussed in Section 8.

2. AUXILIARY LEMMAS

In the proof of Theorem 2 we use certain auxiliary statements in which the dimension m is arbitrary. Consider the real elliptic expression

$$T = -\operatorname{div} a^0(x) \operatorname{grad} \tag{4}$$

with the coefficients $a^0_{jk}(x) \in C^\infty(\mathbf{R}^m)$ stabilized in the neighbourhood of ∞, that is

$$a^0_{jk}(x) \equiv \text{const}, \quad |x| > r_0,$$

for some $r_0 > 0$. Let the eigenvalues of the positive definite matrix $a^0(x)$ be situated in the segment $[\nu_0^2, \mu_0^2]$, the constants ν_0, μ_0 being defined in Section 1. Denote by $F(x, y, t)$, $x, y \in \mathbf{R}^m$, $t > 0$, the fundamental solution of the parabolic equation $\frac{\partial u}{\partial t} + T[u] = 0$ and by $G_{0,a}(x - y, t)$ the fundamental solution of the equation $\frac{\partial u}{\partial t} - a^2 \Delta u = 0$, $a > 0$.

LEMMA 1. *There exist constants $a > 0$, $b > 0$ depending only on ν_0, μ_0 and m such that $0 \leqslant F(x, y, t) \leqslant bG_{0,a}(x - y, t)$, $x, y \in \mathbf{R}^m$, $t > 0$.*

This result follows from Aronson's Theorem [1] concerning the bounds for the fundamental solution of parabolic equation.

The kernel of the resolvent $(-\Delta + k)^{-1}$, $k > 0$, is $R_k(x - y)$ where $R_k(x) = (2\pi)^{-\frac{m}{2}}(k|x|^{-2})^{\frac{m-2}{4}} K_{\frac{m-2}{2}}(k^{\frac{1}{2}}|x|)$, $K_\nu(x)$ being the modified Bessel function of the third kind (see [3, Chapter 6, § 2], [2]).

LEMMA 2. *Let $m \geqslant 2$, $\beta \geqslant 1$. Then $R_k(x) \in L^\beta(\mathbf{R}^m)$ if and only if*

$$\|R_k(\cdot)\| = c_{m,\beta} k^{\frac{m}{2}(1-\beta^{-1})-1},$$

$c_{m,\beta}$ being the constants depending only on m, β. In the case $m = 1$ the same result is true for $1 \leqslant \beta \leqslant \infty$.

Lemma 2 follows from the asymptotic formulae

$$K_0(r) \sim c \cdot \ln r, \; r \downarrow 0;$$

$$K_\nu(r) \sim cr^{-|\nu|}, \; r \downarrow 0, \; \nu \neq 0;$$

$$K_\nu(r) \sim cr^{-\frac{1}{2}} e^{-r}, \; r \to \infty \quad [2].$$

Consider the minimal self-adjoint operator A_0^+ in $L^2(\mathbf{R}^m)$ associated with $S_0^+ = T + V_+(x)$, $0 \leqslant V_+(x) \in C_0^\infty(\mathbf{R}^m)$, and its resolvent $(A_0^+ + k)^{-1}$, $k > 0$ being an integral operator with the kernel $R_k^+(x, y)$.

LEMMA 3. *There exist constants $a > 0, b > 0$ depending only on m, μ_0, ν_0 such that*

$$0 \leqslant R_k^+(x, y) \leqslant ba^{-2} R_{ka^{-2}}(x - y) \tag{5}$$

for every $(x, y) \in \mathbf{R}^m \times \mathbf{R}^m$, $x \neq y$, and $k > 0$.

Proof. Let A' be the minimal self-adjoint operator corresponding to the expression T. The fundamental solution $F(x, y, t)$ is a kernel of $e^{-tA'}$ (respectively, $G_{0,a}(x - y, t)$ is a kernel of $e^{-t(-a^2\Delta)}$). By Lemma 1 there exist $a > 0$, $b > 0$ depending only on m, μ_0, ν_0 such that

$$|e^{-tA'} f(x)| \leqslant be^{-t(-a^2\Delta)} |f|(x),$$

$x \in \mathbf{R}^m$, $t > 0$, $f \in L^2(\mathbf{R}^m)$. Since $F(x, y, t) \geqslant 0$, $V_+(x) \geqslant 0$ it follows that

$$|e^{-\frac{t}{n}A'} e^{-\frac{t}{n}V_+} f(x)| \leqslant e^{-\frac{t}{n}A'} |f|(x), \quad x \in \mathbf{R}^m,$$

$$|(e^{-\frac{t}{n}A'} e^{-\frac{t}{n}V_+})^n f(x)| \leqslant e^{-tA'} |f|(x) \leqslant$$

$$\leqslant b e^{-t(-a^2\Delta)} |f|(x), \quad n = 1,2, \ldots .$$

Making $n \to \infty$ and using Trotter formula we get

$$|e^{-tA_0^+} f(x)| \leqslant b e^{-t(-a^2\Delta)} |f|(x), \tag{6}$$

$x \in \mathbf{R}^m$, $t > 0$, $f \in L^2(\mathbf{R}^m)$. The resolvent and the semigroup corresponding to the self-adjoint operator $A \geqslant 0$ are connected by $\int_0^\infty e^{-kt} e^{-tA} \, dt = (A + k)^{-1}$. Applying this formula to $A = A_0^+$, $A = -a^2\Delta$ along with (6) we see that

$$\begin{aligned} |(A_0 + k)^{-1} f(x)| &\leqslant b\,(-a^2\Delta + k)^{-1} |f|(x) \leqslant \\ &\leqslant b a^{-2} (-\Delta + k a^{-2})^{-1} |f|(x) \end{aligned} \tag{7}$$

holds in $\mathbf{R}^m$ for every $f \in L^2(\mathbf{R}^m)$. Since nonnegativeness of $F(x, y, t)$, $G_{0, ka^{-2}}(x-y, t)$ imply the same properties of $R_k^+(x, y)$, $R_k(x-y)$, the inequality (5) follows from (7). We exclude the diagonal $x = y$ because resolvent kernels may have singularities there (infinite differentiability of coefficients $a_{jk}^0(x)$, $V_+(x)$ provides the smoothness of these kernels when $x \neq y$).

The next preliminary result gives an estimate from below for $\inf A_0$, A_0 being the minimal self-adjoint operator associated with the expression $S_0 = T + V_0(x)$, $V_0(x) = V_+(x) - V_-(x)$, $0 \leqslant V_\pm(x) \in C_0^\infty(\mathbf{R}^m)$.

LEMMA 4. *Let $p \geqslant 1$, $p > \frac{m}{2}$ $(m \geqslant 1)$. There exists a constant α_1 depending only on m, v_0, p such that*

$$\inf A_0 \geqslant -k_1, \quad k_1 = (\alpha_1 \|V_-\|_p)^{\frac{2p}{2p-m}}. \tag{8}$$

Proof. We follow the ideas of Faris [4]. Estimating $(V_- f, f)$ by the Hölder inequality leads to

$$0 \leqslant (V_- f, f) \leqslant \|V_-\|_p \|f\|_{2s}^2, \quad f \in C_0^\infty(\mathbf{R}^m),\ p^{-1} + s^{-1} = 1.$$

Since $2s > 2$, the number l, $l^{-1} + (2s)^{-1} = 1$, satisfies $1 \leqslant l \leqslant 2$. By Haussdorf-Young inequality $\|f\|_{2s}^2 \leqslant \|\check{f}\|_l^2$, $\check{f}(x) = \int_{\mathbf{R}^m} f(y)\, e^{2\pi i \langle x, y\rangle} dy$ being the inverse Fourier transform of $f(y)$, $\langle x, y\rangle = \sum_{j=1}^{m} x_j y_j$. It follows that

$$(9) \qquad 0 \leqslant (V_- f, f) \leqslant \|V_-\|_p \|\check{f}\|_l^2, \quad f \in C_0^\infty(\mathbf{R}^m),\ p^{-1} + 1 = 2l^{-1}.$$

Write $\|\check{f}\|_l^2$ in the form $\|(1 + \sigma^2|x|^2)^{-\frac{1}{2}}(1 + \sigma^2|x|^2)^{\frac{1}{2}}\check{f}\|_l^2$ with arbitrary $\sigma > 0$ and apply Hölder inequality with indices $2l^{-1}$, r, $r^{-1} + 2^{-1} l = 1$. We infer

$$(10) \qquad \|\check{f}\|_l^2 \leqslant \|(1 + \sigma^2|x|^2)^{-\frac{1}{2}}\|_{lr}^2 \|(1 + \sigma^2|x|^2)^{\frac{1}{2}}\check{f}\|_2^2.$$

Relations connecting l and r, l and p imply $(lr)^{-1} + 2^{-1} = l^{-1} = (2p)^{-1} + 2^{-1}$, $lr = 2p$, so

$$(11) \qquad \|(1 + \sigma^2|x|^2)^{-\frac{1}{2}}\|_{lr}^2 = \|(1 + \sigma^2|x|^2)^{-1}\|_p = c_p \sigma^{-\frac{m}{p}},$$

$c_p = \|(1 + |x|^2)^{-1}\|_p$ being the constant depending only on m, p; $c_p < \infty$ since $p \geqslant 1$, $p > \frac{m}{2}$. Using the well-known properties of the Fourier transform one sees that

$$(12) \qquad \begin{aligned} \|(1 + \sigma^2|x|^2)^{\frac{1}{2}}\check{f}\|_2^2 &= (1 + \sigma^2|x|^2\check{f}, \check{f}) = \\ &= \|f\|_2^2 + \frac{\sigma^2}{4\pi^2}(-\Delta f, f). \end{aligned}$$

Substituting $\|\check{f}\|_l^2$ in (9) by the expression following from (10), (11), (12) and noting hat $(-\Delta f, f) \leqslant v_0^{-2}(S_0[f], f) \leqslant v_0^{-2}[(T[f], f) + (V_+ f, f)]$ we come to the inequality

$$0 \leqslant (V_- f, f) \leqslant c_p\|V_-\|_p[\sigma^{2-\frac{m}{p}} v_0^{-2}(T[f] + V_+ f, f) + \sigma^{-\frac{m}{p}}\|f\|_2^2],$$

$\in C_0^\infty(\mathbf{R}^m)$, $\sigma > 0$. In the case $\sigma = \sigma_0 = (v_0^2 c_p^{-1} \|V_-\|_p^{-1})^{\frac{p}{2p-m}}$ it follows that $\inf A_0$ is bounded from below by the constant (8), $\alpha_1 = c_p v_0^{-\frac{m}{p}}$ depending only on m, p, v_0.

We conclude this section with

LEMMA 5. *The property*

$$\|Bg\|_2 \leqslant c\|g\|_1, \quad g \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m) = \mathscr{L},$$

where B is a bounded self-adjoint operator in $L^2(\mathbf{R}^m)$, $m \geqslant 1$, and c is a fixed constant, implies

$$B\varphi \in L^\infty(\mathbf{R}^m), \quad \|B\varphi\|_\infty \leqslant c\|\varphi\|_2 \tag{13}$$

for every $\varphi \in L^2(\mathbf{R}^m)$.

Proof. The Cauchy-Buniakovski inequality applied to $(B\varphi, g) = (\varphi, Bg)$ yields

$$\left|\int_{\mathbf{R}^m} B\varphi(x)\overline{g(x)}\,\mathrm{d}x\right| \leqslant c\|\varphi\|_2 \int_{\mathbf{R}^m} |g(x)|\,\mathrm{d}x, \quad g \in \mathscr{L}.$$

In the case $g(x) = \chi_{y,r}(x)$ this estimate reduces to $\left|\int_{|x-y|<r} B\varphi(x)\,\mathrm{d}x\right| \leqslant$ $\leqslant c\|\varphi\|_2 \int_{|x-y|<r} \mathrm{d}x$ for every $y \in \mathbf{R}^m$, $r > 0$. Applying Lebesgue differentiation theorem we obtain (13).

3. ESTIMATE OF $\|(A_0 + k)^{-1}f\|_\infty$

As before, T is the expression (4) obeing the restrictions mentioned at the beginning of Section 2, $V(x) = V_+(x) - V_-(x)$, $0 \leqslant V_\pm(x) \in C_0^\infty(\mathbf{R}^m)$. Here we want to prove the estimate of $\|(A_0 + k)^{-1}f\|_\infty$, $f \in L^2(\mathbf{R}^m)$, $k > 0$ sufficiently large, A_0 being the minimal self-adjoint operator associated with $S_0 = T + V$.

PROPOSITION 1. *Fix $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. There exist constants $\alpha_0 > 0$ depending only on m, p, μ_0, ν_0, and $c > 0$ depending only on m, μ_0, ν_0 such that the resolvent $(A_0 + k)^{-1}$*

(a) *exists,*

(b) *satisfies*

$$\|(A_0 + k)^{-1}f\|_\infty \leqslant ck^{\frac{m}{4}-1}\left[1 - \left(\frac{k_0}{k}\right)^{\frac{2p-m}{2p}}\right]^{-1}\|f\|_2, \tag{14}$$

$f \in L^2(\mathbf{R}^m)$, for every $k > k_0$, $k_0 = (\alpha_0\|V_-\|_p)^{\frac{2p}{2p-m}}$.

The proof of Proposition 1 is based on three lemmas. Consider the integral operator

$$\mathscr{H}_k f(x) = \int_{\mathbf{R}^m} \Phi_k(x, y) f(y)\, \mathrm{d}y, \tag{15}$$

$k > 0$, $\Phi_k(x, y) = V_-(x)\, R_k^+(x, y)$, $R_k^+(x, y)$ being the kernel of the resolvent $(A_0^+ + k)^{-1}$. One may consider $\mathscr{H}_k$ as an operator in $L^p(\mathbf{R}^m)$ for different p's. In the case $p = 2, \mathscr{H}_k = V_-(A_0^+ + k)^{-1}$.

Lemma 6. *Fix* $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. *Then* $\mathscr{H}_k f \in L^1(\mathbf{R}^m)$ *for every* $f \in L^1(\mathbf{R}^m)$,

$$\|\mathscr{H}_k f\|_1 \leqslant \alpha_2 k^{\frac{m-2p}{2p}} \|V_-\|_p \|f\|_1, \tag{16}$$

α_2 *being a constant depending only on* m, p, μ_0, ν_0.

Proof. It follows from Lemma 3 and Hölder inequality that

$$\|\mathscr{H}_k f_1\| \leqslant b a^{-2} \int_{\mathbf{R}^m} V_-(x) \left[\int_{\mathbf{R}^m} R_{ka^{-2}}(x - y) |f(y)|\, \mathrm{d}y \right] \leqslant$$

$$\leqslant b a^{-2} \|V_-\|_p \|R_{ka^{-2}} * |f|\,\|_\beta, \quad p^{-1} + \beta^{-1} = 1,$$

with $\beta \in \left[1, \frac{m}{m-2}\right)$, $m = 2,3$, and $1 \leqslant \beta \leqslant \infty, m = 1$. By Lemma 2 and Young inequality

$$\|R_{ka^{-2}} * |f|\,\|_\beta \leqslant \|R_{ka^{-2}}\|_\beta \|f\|_1 \leqslant c_{m,\beta} k^{\frac{m}{2}(1-\beta^{-1})} \|f\|_1$$

this implying (16).

Now we turn to the integral operator

$$P_k f(x) = \int_{\mathbf{R}^m} R_k^+(x, y) f(y)\, \mathrm{d}y, \quad k > 0. \tag{17}$$

Lemma 7. *In the case* $1 \leqslant m \leqslant 3$, P_k *is a bounded operator from* $L^1(\mathbf{R}^m)$ *to* $L^2(\mathbf{R}^m)$ *its norm satisfying* $\|P_k\| \leqslant c k^{\frac{m}{4}-1}$; *c is a constant which depends only on* m, μ_0, ν_0.

Proof. Applying Lemma 3 we get for every $f \in L^1(\mathbf{R}^m)$:

$$\|P_k f\|_2 \leqslant ba^{-2}\|R_{ka^{-2}} * |f|\,\|_2. \tag{18}$$

By Lemma 2, $R_{ka^{-2}} \in L^2(\mathbf{R}^m)$ if and only if $1 \leqslant m \leqslant 3$. For these dimensions from Lemma 2 with $\beta = 2$, Young inequality, and (18) it follows $\|P_k f\|_2 \leqslant ck^{\frac{m}{4}-1}\|f\|_1$, $f \in L^1(\mathbf{R}^m)$, the constant c depending only on m, μ_0, ν_0.

It is seen from Lemma 6 and the condition $p > \dfrac{m}{2}$ that the norm $\|\mathscr{H}_k\|$ of the operator $\mathscr{H}_k$ acting in $L^1(\mathbf{R}^m)$ is less than 1 if $k > k_2 = (\alpha_2\|V_-\|_p)^{\frac{2p-m}{2p}}$, the constant α_2 taken from Lemma 6. For $k > k_2$ the operator $B_k = (1 - \mathscr{H}_k)^{-1} = \sum_{j=0}^{\infty} \mathscr{H}_k^j$ is bounded in $L^1(\mathbf{R}^m)$,

$$\|B_k\| \leqslant \left[1 - \left(\frac{k_2}{k}\right)^{\frac{2p-m}{2p}}\right]^{-1}, \tag{19}$$

and $P_k B_k$ is bounded from $L^1(\mathbf{R}^m)$ to $L^2(\mathbf{R}^m)$. The expression (8) for the constant k_1 is very much alike the one for k_2. Let

$$\alpha_0 = \max\{\alpha_1, \ \alpha_2\}, \quad k_0 = (\alpha_0\|V_-\|_p)^{\frac{2p}{2p-m}};$$

note that α_0 depends only on m, p, μ_0, ν_0. For every $k > k_0$ the resolvent $(A_0 + k)^{-1}$ and the operator $P_k B_k$ are correctly defined since $k_0 \geqslant k_1$, $k_0 \geqslant k_2$.

LEMMA 8. *The equality*

$$(A_0 + k)^{-1} f = P_k B_k f, \quad f \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m) = \mathscr{L}, \tag{20}$$

$1 \leqslant m \leqslant 3$, *is true for every* $k > k_0$.

Proof. By the definition of k_2 the series $B_k f = \sum_{j=0}^{\infty} \mathscr{H}_k^j f$ converges in $L^1(\mathbf{R}^m)$ for all $k > k_0 \geqslant k_2$ and $P_k B_k f = \sum_{j=0}^{\infty} P_k \mathscr{H}_k^j f$ holds, the series converging in $L^2(\mathbf{R}^m)$. Consider the obvious equality

$$(P_k B_k f, \ (A_0 + k)\varphi) = \sum_{i=0}^{\infty} (P_k \mathscr{H}_k^j f, \ (A_0 + k)\varphi), \tag{21}$$

$\varphi \in C_0^\infty(\mathbf{R}^m)$, $k > k_0$. Since $f \in \mathscr{L} \subset L^2(\mathbf{R}^m)$ the general term on the right may be written in the form

$$((A_0^+ + k)^{-1} [V_-(A_0^+ + k)^{-1}]^j f,\ (A_0 + k - V_-)\ \varphi) =$$

$$= ([V_-(A_0^+ + k)^{-1}]^j f, \varphi) - ([V_-(A_0^+ + k)^{-1}]^j f,\ (A_0^+ + k)^{-1} V_- \varphi) =$$

$$= ([V_-(A_0^+ + k)^{-1}]^j f,\ \varphi) - ([V_-(A_0^+ + k)^{-1}]^{j+1} f,\ \varphi).$$

It follows from $f \in \mathscr{L} \subset L^1(\mathbf{R}^m)$, $\varphi \in C_0^\infty(\mathbf{R}^m) \subset L^\infty(\mathbf{R}^m)$ that this expression is equal to $(\mathscr{H}_k^j f, \varphi) - (\mathscr{H}_k^{j+1} f, \varphi)$ which implies the convergence of the series (21) to (f, φ). We see that the formula (21) is equivalent to

$$(P_k B_k f,\ (A_0 + k)\varphi) = ((A_0 + k)^{-1} f,\ (A_0 + k)\varphi), \quad \varphi \in C_0^\infty(\mathbf{R}^m),\ k > k_0.$$

Using the density of $(A_0 + k)C_0^\infty(\mathbf{R}^m)$, $k > k_0 \geqslant k_1$, in $L^2(\mathbf{R}^m)$ one obtains (20).

Given $f \in \mathscr{L}$ the inequality

$$\|(A_0 + k)^{-1} f\|_2 \leqslant \|P_k\|\, \|B_k\|\, \|f\|_1 \leqslant$$

$$\leqslant ck^{\frac{m}{4}-1} \left[1 - \left(\frac{k_2}{k}\right)^{\frac{2p-m}{2p}}\right]^{-1} \|f\|_1 \leqslant ck^{\frac{m}{4}-1} \left[1 - \left(\frac{k_0}{k}\right)^{\frac{2p-m}{2p}}\right]^{-1} \|f\|_1,$$

$k > k_0$, holds by Lemmas 8 and 7, and (19). To conclude the proof of Proposition 1 one should use Lemma 5.

4. PRELIMINARY POINTWISE ESTIMATE OF $|\hat{\varphi}(tA_0^{1/2})f(x)|$

Throughout this section we fix some even function $\varphi(\tau) \in C_0^\infty(\mathbf{R})$, $\varphi(\tau) \geqslant 0$, $\operatorname{supp}\varphi(\tau) \subseteq [-1, 1]$,

$$\int_{-1}^{1} \varphi(\tau) d\tau = 2 \int_{0}^{1} \varphi(\tau)\, d\tau = 1. \tag{22}$$

Our aim is to deduce from Proposition 1 the analogous estimate for the operators $\hat{\varphi}(tA_0^{1/2})$, $t > 0$, where

$$\hat{\varphi}(\mu) = \int_{-1}^{1} \varphi(\tau) \cos \mu\tau\, d\tau = 2 \int_{0}^{1} \varphi(\tau) \cos \mu\tau\, d\tau, \tag{23}$$

where A_0 is the self-adjoint operator introduced in Section 3. To make the formulae more compact we use the function

$$\theta(\rho) = (1 + \rho)\rho^{-\left(1-\frac{m}{4}\right)}, \quad \rho > 0,\ 1 \leqslant m \leqslant 3,$$

having the only minimum at $\rho = \rho_0$

$$\rho_0 = \frac{4}{m} - 1. \tag{24}$$

As in Section 1 we denote by $\chi_{y,\rho}$ the characteristic function of the ball $|x - y| < \rho$ and the operator of multiplication by $\chi_{y,\rho}$.

PROPOSITION 2. *Fix* $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$, $t_0 > 0$. *Let* k_0 *be the constant from Proposition* 1,

$$K_0 := \max\{\rho_0, k_0 t_0^2\}.$$

Given $f \in L^2(\mathbf{R}^m)$ *at each moment* $0 < t \leqslant t_0$, *the following inequality holds a.e. in* $\mathbf{R}^m$:

$$|\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)| \leqslant \gamma_0 t^{-\frac{m}{2}}\theta(K_0)\cosh(t_0 k_0^{\frac{1}{2}})\|\chi_{x,\mu_0 t}f\|_2, \tag{25}$$

γ_0 *being a constant depending only on* $\varphi(\tau)$, m, μ_0, ν_0, p.

The first step in proving Proposition 2 will be the following

REMARK. The estimate (25) will be true for every $f \in L^2(\mathbf{R}^m)$ if it is proved for functions from $C_0^\infty(\mathbf{R}^m)$. It is sufficient to note that from the convergence $C_0^\infty(\mathbf{R}^m) \ni f_n \to f$ in $L^2(\mathbf{R}^m)$ it follows that $\|\chi_{x,\mu_0 t}f_n\|_2 \to \|\chi_{x,\mu_0 t}f\|_2$ at each point $x \in \mathbf{R}^m$ and the convergence $\hat{\varphi}(tA_0^{\frac{1}{2}})f_n \to \hat{\varphi}(tA_0^{\frac{1}{2}})f$ in $L^2(\mathbf{R}^m)$; this implies the convergence a.e. of certain subsequence $\hat{\varphi}(tA_0^{\frac{1}{2}})f_{n_j}(x)$. If the estimate (25) holds for functions from $C_0^\infty(\mathbf{R}^m)$ then substituting f by f_{n_j} in (25) and passing $j \to \infty$ one obtains the same inequality for every $f \in L^2(\mathbf{R}^m)$.

Next we prove

LEMMA 9. *Fix* $t_0 > 0$. *Let* $f \in C_0^\infty(\mathbf{R}^m)$, $\operatorname{supp} f \subset G$, $G \subset \mathbf{R}^m$ *being a bounded open set. For every* $t \in [0, t_0)$ *the supports of the functions* $\cos(tA_0^{\frac{1}{2}})f(x)$, $\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)$

lie in the open $t_0\mu_0$-neighbourhood $G^{t_0\mu_0}$ of the set G, the mentioned functions belonging to $C_0^\infty(\mathbf{R}^m)$.

Proof. Consider the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} + S_0[u] = 0, \quad u|_{t=0} = f, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \tag{26}$$

$S_0 = T + V$, T being the expression (4) satisfying the conditions mentioned at the beginning of Section 2. The upper bound for eigenvalues of matrix $\{a_{jk}^0(x)\}$ being μ_0 it follows from the finite speed of propagation property of the hyperbolic equation that (26) describes the propagation process with the speed $\leqslant \mu_0$. This fact together with the condition $\operatorname{supp} f \subset G$ imply that the solution $u(t, x)$ of (26) (which belongs to $C^\infty([0, \infty)\times\mathbf{R}^m)$ since the coefficients of S_0 and f are in $C^\infty(\mathbf{R}^m)$) posseses the property

$$\operatorname{supp} u(t, x) \subseteq G^{t\mu_0} \subset G^{t_0\mu_0}, \quad 0 \leqslant t < t_0. \tag{27}$$

The vector-function $U(t) = u(t, \cdot)$, $t \geqslant 0$, satisfies the Cauchy problem

$$U'' + A_0 U = 0, \quad U|_{t=0} = f, \quad U'|_{t=0} = 0$$

(U', U'' being strong derivatives in L^2-norm) from which it follows the representation

$$U(t) = u(t, \cdot) = \cos(tA_0^{\frac{1}{2}})f, \quad t \geqslant 0 \tag{28}$$

(see, e.g. [10, § 3]). Substituting here t by $t\tau$, then multiplying by $2\varphi(\tau)$, and finally integrating over $0 \leqslant \tau \leqslant 1$ one comes to

$$2\int_0^1 \varphi(\tau)u(t\tau, x)\,d\tau = 2\int_0^1 \varphi(\tau)\cos(t\tau A_0^{\frac{1}{2}})f\,d\tau = \varphi(tA_0^{\frac{1}{2}})f, \quad t \geqslant 0. \tag{29}$$

The statement of Lemma 9 is an immediate consequence of formulae (27), (28), (29).

The following consequence of Lemma 9 will be used in proving Proposition 2.

LEMMA 10. *Fix $x' \in \mathbf{R}^m$, $\varepsilon > 0$. Then*

$$\chi_{x',\varepsilon}\hat{\varphi}(tA_0^{\frac{1}{2}}) = \chi_{x',\varepsilon}\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon+t\mu_0} \tag{30}$$

for every $t > 0$.

Proof. Lemma 9 implies in the case $G = B_{x',\varepsilon}$, $f \in C_0^\infty(\mathbf{R}^m)$, $\operatorname{supp} f \subset G$, that

$$\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon}f = \hat{\varphi}(tA_0^{\frac{1}{2}})f = \chi_{x',\varepsilon+t_0\mu_0}\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon}f, \quad 0 \leqslant t < t_0,$$

for arbitrary $t_0 > 0$.

By continuity this equality spreads on elements $f = \chi_{x',\varepsilon}g$, $g \in L^2(\mathbf{R}^m)$. Passing to the limit as $t_0 \downarrow t$, $t \geqslant 0$ arbitrary, one receives:

$$\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon} = \chi_{x',\varepsilon+t\mu_0}\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon}, \quad t \geqslant 0.$$

Passing from the equality of operators to the equality of their adjoints one comes to (30).

We shall also use the next remark.

LEMMA 11. *Fix* $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. *Take a scalar function* $\Phi(\lambda) \in C(\mathbf{R})$ *which satisfies*

$$\sup_{\lambda \geqslant -k_0} |\Phi(\lambda)|(\lambda + k_0 + s) \leqslant a_0 = \text{const.} \tag{31}$$

Then $\Phi(A_0)f \in L^\infty(\mathbf{R}^m)$ *for every* $f \in L^2(\mathbf{R}^m)$ *and* $\|\Phi(A_0)f\|_\infty \leqslant a_0\alpha(s)\|f\|_2$,

$$\alpha(s) = c(k_0 + s)^{\frac{m}{4}-1}\left[1 - \left(\frac{k_0}{k_0 + s}\right)^{\frac{2p-m}{2p}}\right]^{-1}, \tag{32}$$

where k_0, c *are the constants from* (14) *depending only on* m, μ_0, ν_0, p.

Proof. Write $\Phi(A_0)f$ in the form $(A_0 + k_0 + s)^{-1}Bf$, $B = (A_0 + k_0 + s)\cdot \Phi(A_0)$. It follows from (31) and $\inf A_0 \geqslant -k_0$ that B is bounded in $L^2(\mathbf{R}^m)$ with its norm less than a_0. Using Proposition 1, we obtain:

$$\|\Phi(A_0)f\|_\infty = \|(A_0 + k_0 + s)^{-1}Bf\|_\infty \leqslant$$

$$\leqslant c(k_0 + s)^{\frac{m}{4}-1}\left[1 - \left(\frac{k_0}{k_0 + s}\right)^{\frac{2p-m}{2p}}\right]^{-1}\|Bf\|_2,$$

which leads to (32) since $\|Bf\|_2 \leqslant a_0\|f\|_2$.

Fix $t > 0$. In proving Proposition 2, we apply Lemma 11 to $\Phi(\lambda) = \hat{\varphi}(t\lambda^{\frac{1}{2}})$, $\hat{\varphi}(\mu)$ being the function (23). The condition (31) becomes

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})|(\lambda + k_0 + s) \leqslant \tag{33}$$

$$\leqslant \delta t^{-2}[1 + (k_0 + s)t^2] \cosh(tk_0^{\frac{1}{2}}),$$

$$\lambda \geqslant -k_0, \ \delta = 2\max\{\delta_0, \delta_2\}, \ \delta_l = \max_{\tau \in [-1, 1]} |\varphi^{(l)}(\tau)|, \quad l = 0, 1, 2, \ldots .$$

To verify this inequality note that

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})| \leqslant \begin{cases} \delta_0, & \lambda \geqslant 0; \\ \delta_0 \cosh(tk_0^{\frac{1}{2}}), & 0 > \lambda \geqslant -k_0. \end{cases} \tag{34}$$

The integration by parts in (23) implies also

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})| \leqslant \delta_l(t\lambda^{\frac{1}{2}})^{-l}, \quad \lambda > 0, \ l = 1,2, \ldots .$$

Comparing this estimate with (34), $\lambda > 0$, we see that

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})| \leqslant \begin{cases} \delta_0, & 0 < \lambda \leqslant \lambda_l; \\ \delta_l(t\lambda^{\frac{1}{2}})^{-l}, & \lambda > \lambda_l; \end{cases} \tag{35}$$

$$\lambda_l = (\delta_l \delta_0^{-1})^{\frac{2}{l}} t^{-2}, \quad l = 1, 2, \ldots .$$

Let $s > 0$. Multiplying (35) and (34), $0 > \lambda \geqslant -k_0$, by $(\lambda + k_0 + s)^{\frac{l}{2}}$ we obtain:

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})|(\lambda + k_0 + s)^{\frac{l}{2}} \leqslant \begin{cases} \delta_0(\lambda_l + k_0 + s)^{\frac{l}{2}}, & \lambda \geqslant 0; \\ \delta_0(k_0 + s)^{\frac{l}{2}} \cosh(tk_0^{\frac{1}{2}}), & 0 > \lambda \geqslant -k_0; \end{cases}$$

leading to

$$|\hat{\varphi}(t\lambda^{\frac{1}{2}})|(\lambda + k_0 + s)^{\frac{l}{2}} \leqslant \delta_0(\lambda_l + k_0 + s)^{\frac{l}{2}} \cosh(tk_0^{\frac{1}{2}}), \quad \lambda \geqslant -k_0.$$

If $l = 2$ the last estimate implies (33).

By Lemma 11 it follows that

$$\|\hat{\varphi}(tA_0^{\frac{1}{2}})f\|_\infty \leqslant a_0(t,s)\alpha(s)\|f\|_2, \quad f\in L^2(\mathbf{R}^m),$$

$$a_0(t,s) = \delta t^{-2}[1+(k_0+s)t^2]\cosh(tk_0^{\frac{1}{2}}),$$

for every $t>0$, $s>0$. We may write $a_0(t,s)\alpha(s)$ in the form $c\delta\cosh(tk_0^{\frac{1}{2}})\omega(s)\psi(t,s)$,

$$\omega(s) = (k_0+s)^{\frac{m}{4}}\left[1-\left(\frac{k_0}{k_0+s}\right)^{\frac{2p-m}{2p}}\right]^{-1},$$

$$\psi(t,s) = 1+(k_0+s)^{-1}t^{-2}.$$

Remind that c depends only on m_0, μ_0, ν_0 and δ depends only on $\varphi(\tau)$. Fix $\alpha = \text{const} > 0$, $r>0$, $0<t\leqslant[\alpha(k_0+r)^{-1}]^{\frac{1}{2}} < (\alpha k_0^{-1})^{\frac{1}{2}}$. Let $s=s_0$ be the positive solution of the equation $(k_0+s)\,t^2 = 2\alpha$; then

$$\omega(s_0) = \left[1-\left(\frac{k_0}{2\alpha}t^2\right)^{\frac{2p-m}{2p}}\right]^{-1}(2\alpha)^{\frac{m}{4}}t^{-\frac{m}{2}} \leqslant (2\alpha)^{\frac{m}{4}}\left[1-2^{-\frac{2p-m}{2p}}\right]^{-1}t^{-\frac{m}{2}},$$

$$\psi(t,s_0) = 1+(2\alpha)^{-1} \leqslant \alpha^{-1}(1+\alpha),$$

that is

$$\begin{gathered}\|\hat{\varphi}(tA_0^{\frac{1}{2}})f\|_\infty \leqslant \gamma_0 t^{-\frac{m}{2}}\theta(\alpha)\cosh(tk_0^{\frac{1}{2}})\|f\|_2, \quad f\in L^2(\mathbf{R}^m),\\ \gamma_0 = c\delta 2^{\frac{m}{4}}\left(1-2^{-\frac{2p-m}{2p}}\right)^{-1}, \quad \theta(\alpha) = \alpha^{-\left(1-\frac{m}{4}\right)}(1+\alpha),\end{gathered} \tag{36}$$

for every $t\in(0,[\alpha(k_0+r)^{-1}]^{\frac{1}{2}}]$.

Fix $t_0>0$, $r>0$. It follows from Lemma 10 and from (36), $\alpha = (k_0+r)t_0^2$, that

$$\begin{aligned}\|\chi_{x',\varepsilon}\hat{\varphi}(tA_0^{\frac{1}{2}})f\|_\infty &= \|\chi_{x',\varepsilon}\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon+t\mu_0}f\|_\infty \leqslant\\ &\leqslant \|\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x',\varepsilon+t\mu_0}f\|_\infty \leqslant\\ &\leqslant \gamma_0 t^{-\frac{m}{2}}\theta((k_0+r)t^2)\cosh(t_0k_0^{\frac{1}{2}})\|\chi_{x',\varepsilon+t\mu_0}f\|_2,\end{aligned} \tag{37}$$

$f \in L^2(\mathbf{R}^m)$, $x' \in \mathbf{R}^m$, $\varepsilon > 0$, $t \in (0, t_0]$. If $f \in C_0^\infty(\mathbf{R}^m)$, then the function $\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)$ is also in $C_0^\infty(\mathbf{R}^m)$ for every $t > 0$. The obtained estimate implies that $|\hat{\varphi}(tA_0^{\frac{1}{2}})f(x')|$ is bounded by the right side of (37) for every $x' \in \mathbf{R}^m$, $t \in (0, t_0]$. By passing $\varepsilon \to 0$ and changing x' for x one comes to

$$|\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)| \leqslant \gamma_0 t^{-\frac{m}{2}} \theta((k_0 + r)t_0^2) \cosh(t_0 k_0^{\frac{1}{2}}) \|\chi_{x,\, t\mu_0} f\|_2,$$

$x \in \mathbf{R}^m$, $t \in (0, t_0]$, $r > 0$ arbitrary. Since $\theta(\rho)$, $\rho > 0$, has a unique minimum in ρ_0 (see(20)) minimizing $\theta((k_0 + r)t_0^2)$ as a function of $r > 0$, it implies (25) for $f \in C_0^\infty(\mathbf{R}^m)$. It follows from the remark made at the beginning of our proof of Proposition 2 that the same estimate remains true for every $f \in L^2(\mathbf{R}^m)$.

5. FINAL POINTWISE ESTIMATE OF $|\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)|$

The aim of this section is to improve the estimate obtained in Section 4. First we consider the minimal self-adjoint operator the same as in Section 4, that is corresponding to the differential expression $S_0 = T + V_+(x) - V_-(x)$,

$$0 \leqslant V_\pm(x) \in C_0^\infty(\mathbf{R}^m), \tag{38}$$

the restrictions on T being the same as in Section 2. Let

$$k_{x,\,t} = t^2(\alpha_0 \|\chi_{x,\, t\mu_0} V_-\|_p)^{\frac{2p}{2p-m}}, \tag{39}$$

$$K_{x,\,t} = \max\{\rho_0, k_{x,\,t}\}. \tag{40}$$

The constant α_0 is taken from Proposition 1; it depends only on m, μ_0, ν_0, p. Here $\rho_0 > 0$ is the unique minimum point of the function $\theta(\rho) = \rho^{-\left(1-\frac{m}{4}\right)}(1 + \rho), \rho > 0$ (see (20)).

PROPOSITION 3. *Fix $p \geqslant 1$, $p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. There exists a constant γ_0 depending only on m, p, $\varphi(\tau)$, μ_0, ν_0 such that*

$$|\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)| \leqslant \gamma_0 t^{-\frac{m}{2}} \theta(K_{x,\,t}) \cosh(k_{x,\,t}^{\frac{1}{2}}) \|\chi_{x,\, t\mu_0} f\|_2 \tag{41}$$

holds a.e. in $\mathbf{R}^m$ for every $f \in L^2(\mathbf{R}^m)$ at any moment $t > 0$.

Fix $x_0 \in \mathbf{R}^m$, $r > 0$, $t_0 > 0$. Remind that $B_{x_0,\rho}$ stands for the ball $|x - x_0| < \rho$ in $\mathbf{R}^m$ and that $\chi_{x_0,\rho}$ is the characteristic function of $B_{x_0,\rho}$. Denote $B = B_{x_0, r+t_0\mu_0}$. Consider the auxiliary differential expression

$$S_1 = T + W_+(x) - W_-(x), \quad 0 \leqslant W_\pm(x) \in C_0^\infty(\mathbf{R}^m), \tag{42}$$
$$W_\pm(x) = V_\pm(x), \quad x \in B;$$

the corresponding minimal self-adjoint operator is denoted by A_1. We begin the proof of Proposition 3 by

LEMMA 12. *For every* $t \in [0, t_0]$

$$\chi_{x_0,r}\hat{\varphi}(tA_0^{\frac{1}{2}}) = \chi_{x_0,r}\hat{\varphi}(tA_1^{\frac{1}{2}}) \tag{43}$$

holds.

Proof. Compare the Cauchy problems ($j = 0, 1$)

$$\frac{\partial^2 u}{\partial t^2} + S_j[u] = 0, \quad \frac{\partial u}{\partial t}\Big|_{t=0} = 0,$$

$$u|_{t=0} = f \in C_0^\infty(\mathbf{R}^m), \quad \operatorname{supp} f \subset B_{x_0,r}.$$

It follows from Lemma 9 (see also (28)) that the supports of their solutions are in B for $0 \leqslant t < t_0$. Since the first equation ($j = 0$) coincides in B with the second ($j = 1$), by the uniqueness of the solution of the Cauchy problem and by (28) we have:

$$\cos(tA_0^{\frac{1}{2}})f = \cos(tA_1^{\frac{1}{2}})f, \tag{44}$$

$0 \leqslant t < t_0$. Passing $t \to t_0$ we receive the same property for $t = t_0$. By using (29) and the same representation for $\hat{\varphi}(tA_1^{\frac{1}{2}})f$ we see that the equality (44) implies

$$\hat{\varphi}(tA_0^{\frac{1}{2}})f = \hat{\varphi}(tA_1^{\frac{1}{2}})f, \quad 0 \leqslant t \leqslant t_0, \tag{45}$$

$f \in C_0^\infty(\mathbf{R}^m)$, $\operatorname{supp} f \subset B_{x_0,r}$. Such functions approximate every $g(x) = \chi_{x_0,r}(x)h(x)$, $h \in L^2(\mathbf{R}^m)$, so that it follows from (45) that

$$\hat{\varphi}(tA_0^{\frac{1}{2}})\chi_{x_0,r} = \hat{\varphi}(tA_1^{\frac{1}{2}})\chi_{x_0,r}, \quad 0 \leqslant t \leqslant t_0.$$

Passing to the adjoints one obtains (43).

Take $f \in C_0^\infty(\mathbf{R}^m)$, then $\hat{\varphi}(tA_0^{\frac{1}{2}})f \in C_0^\infty(\mathbf{R}^m)$ by Lemma 9. Lemma 19 implies

$$\hat{\varphi}(tA_0^{\frac{1}{2}})f(x) = \hat{\varphi}(tA_1^{\frac{1}{2}})f(x), \quad x \in B_{x_0, r}, \ 0 \leqslant t \leqslant t_0.$$

Proposition 2 delivers an estimate of the right hand side for every $x \in \mathbf{R}^m$, so that

$$(46) \qquad |\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)| \leqslant \gamma_0 t^{-\frac{m}{2}} \theta(K_1) \cosh(t_0 k_1^{\frac{1}{2}}) \|\chi_{x, t\mu_0} f\|_2,$$

$0 < t \leqslant t_0, x \in B_{x_0, r}$;

$$(47) \qquad k_1 = (\alpha_0 \|W_-\|_p)^{\frac{2p}{2p-m}}, \quad K_1 = \max\{\rho_0, k_1 t_0^2\},$$

where α_0, ρ_0, γ_0 are the same constants as in (39), (40), (41). Take the sequence $W_-^{(n)}(x) \in C_0^\infty(\mathbf{R}^m)$, $n = 1, 2, \ldots$, satisfying (42) and converging in $L^p(\mathbf{R}^m)$ to $\chi_B(x)V_-(x)$, where $\chi_B(x)$ is the characteristic function of the ball B. Denote the constants (57) for $W_- = W^{(n)}$ by $k_1^{(n)}$, $K_1^{(n)}$. The estimate (46) is true with $k_1 = k_1^{(n)}$, $K_1 = K_1^{(n)}$, $n = 1,2, \ldots$, so by passing to the limit when $n \to \infty$ we obtain:

$$(48) \qquad \begin{gathered} |\hat{\varphi}(tA_0^{\frac{1}{2}})f(x)| \leqslant \gamma_0 t^{-\frac{m}{2}} \theta(\tilde{K}_{x_0, t_0, r}) \cosh(t_0 \tilde{k}_{x_0, t_0, r}^{\frac{1}{2}}) \|\chi_{x, t\mu_0} f\|_2, \\ x \in B_{x_0, r}, \ f \in C_0^\infty(\mathbf{R}^m), \ 0 < t \leqslant t_0, \ \tilde{k}_{x_0, t_0, r} = (\alpha_0 \|\chi_B V_-\|_p)^{\frac{2p}{2p-m}}, \\ \tilde{K}_{x_0, t_0, r} = \max\{\rho_0, \ t_0^2 \tilde{k}_{x_0, t_0, r}\}. \end{gathered}$$

This estimate may be extended to all $f \in L^2(\mathbf{R}^m)$, since $C_0^\infty(\mathbf{R}^m)$ is dense in $L^2(\mathbf{R}^m)$ and the operators $\hat{\varphi}(tA_0^{\frac{1}{2}})$ are bounded (the inequality (48) holds a.e. in $\mathbf{R}^m$ if $f \in L^2(\mathbf{R}^m)$).

Now consider $\left| \int_{B_{x_0, r}} \hat{\varphi}(t_0 A_0^{\frac{1}{2}}) f(x) \, dx \right|$. Let $m(B_{x_0, r})$ be the volume of $B_{x_0, r}$. The right hand side of (48) in which $\|\chi_{x, t\mu_0} f\|_2$ is substituted by

$\int\limits_{B_{x_0,r}} \|\chi_{x,t_0\mu_0} f\|_2 \, dx \leqslant m(B_{x_0,r}) \|\chi_{x_0,r+t_0\mu_0} f\|_2$ gives an estimate from above for this expression that is

$$\left| m^{-1}(B_{x_0,r}) \int\limits_{B_{x_0,r}} \hat{\varphi}(t_0 A_0^{\frac{1}{2}}) f(x) \, dx \right| \leqslant \tag{49}$$

$$\leqslant \gamma_0 t_0^{-\frac{m}{2}} \theta(\tilde{K}_{x_0,t_0,r}) \cosh(t_0 \tilde{k}_{x_0,t_0,r}^{\frac{1}{2}}) \|\chi_{x_0,r+t_0\mu_0} f\|_2$$

for every $x_0 \in \mathbf{R}^m$ and $r > 0$. By Lebesgue differentiation theorem the left hand side converges to $|\hat{\varphi}(t_0 A_0^{\frac{1}{2}}) f(x)|$ a.e. in $\mathbf{R}^m$ $(r \to 0)$. By passing to the limit in (49) as $r \to 0$ and changing x_0, t_0 to x, t we come to (41).

The main result of the present section lies in extending Proposition 3 to potentials $V(x) = V_+(x) - V_-(x)$,

$$0 \leqslant V_\pm(x) \in L^\infty_{\text{comp}}(\mathbf{R}^m); \tag{50}$$

here $L^\infty_{\text{comp}}(\mathbf{R}^m)$ stands for the set of all $g \in L^\infty(\mathbf{R}^m)$ with compact supports. Later we use it in the case $V(x) \in C_0^\infty(\mathbf{R}^m)$, $V_+(x) = \max\{V(x), 0\}$, $V_-(x) = -\min\{V(x), 0\}$.

PROPOSITION 4. *Let all conditions of Proposition* 3 *be fulfilled except* (38) *now substituted by* (50). *Then the assertion of Proposition* 3 *remains true.*

Proof. It is possible to construct the sequences $0 \leqslant V_\pm^{(n)}(x) \in C_0^\infty(\mathbf{R}^m)$, $n = 1,2,\ldots$, converging to $V_\pm(x)$ in $L^q(\mathbf{R}^m)$ for every $1 \leqslant q < \infty$ (use e. g. Friedrichs mollifier [15, Chapter 5, Section 2]). Consider the minimal operators $A_0^{(n)}$ corresponding to the expressions $T + V_+^{(n)} - V_-^{(n)}$. By using the bound from below of $\inf A_0^{(n)}$ given in Proposition 1 we see that there exists a constant $c \leqslant \inf A_0^{(n)}$, $n = 1,2,\ldots$. Take an arbitrary continuous bounded function $h(\lambda)$, $c \leqslant \lambda < \infty$. Since $A_0^{(n)} f \to A_0 f$ in $L^2(\mathbf{R}^m)$ on the core $C_0^\infty(\mathbf{R}^m)$ of $A_0^{(n)}$, A_0 the operators $h(A_0^{(n)})$ converge to $h(A_0)$ in the strong sense [14, Section 8.7]. In particular

$$\operatorname*{s-lim}_{n\to\infty} \hat{\varphi}(t(A_0^{(n)})^{\frac{1}{2}}) f = \hat{\varphi}(t A_0^{\frac{1}{2}}) f \tag{51}$$

for every $f \in L^2(\mathbf{R}^m)$ at any moment $t > 0$. Applying Proposition 3 to $A_0^{(n)}$ we receive the estimate (41) for $A_0 = A_0^{(n)}$ with $V_\pm^{(n)}(x)$ instead of $V_\pm(x)$ in formulae (39), (40). By passing to the limit as $n \to \infty$ and using (51) together with L^p-convergence of $V_-^{(n)}(x)$ we obtain the assertion of Proposition 4.

6. REMOVING RESTRICTIONS ON DIFFERENTIAL EXPRESSION

Now we want to remove restrictions imposed on differential expression in sections 2—5. Let S be of the form (1) with

$$V(x) = V_+(x) - V_-(x) \in C^\infty(\mathbf{R}^m),$$

$$V_+(x) = \max\{0, V(x)\},$$

$$V_-(x) = -\min\{0, V(x)\}$$

and with a matrix $a(x) = \{a_{jk}(x)\}$ satisfying the conditions mentioned at the beginning of Section 1. Denote an arbitrary self-adjoint extension of the minimal operator connected with S by A.

PROPOSITION 5. *The estimate obtained in Proposition* 4 *for* $\hat{\varphi}(tA_0^{\frac{1}{2}})f$, $f \in L^2(\mathbf{R}^m)$, *remains true also for* $\hat{\varphi}(tA^{\frac{1}{2}})f$, $f \in L^2_{\text{comp}}(\mathbf{R}^m)$.

REMARK. The operators $\hat{\varphi}(tA^{\frac{1}{2}})$, $t > 0$, are unbounded if A is not bounded from below.

Proof. Fix the ball $B_r = \{x\colon |x| < r\}$, $r > 0$, and $t_0 > 0$. Let $a^0(x) = \{a^0_{jk}(x)\}$ be a matrix satisfying the conditions of Section 2 with the same bounds $0 < \nu_0^2 \leqslant \mu_0^2$ for the eigenvalues as in the case of $a(x)$. We also impose on $a^0(x)$ the following condition:

$$a^0(x) = a(x), \quad x \in B_{r+2t_0\mu_0}. \tag{52}$$

Consider the solution $u(t, x)$ of the Cauchy problem

$$\frac{\partial^2 u}{\partial t^2} + S_0[u] = 0, \quad u|_{t=0} = f \in C_0^\infty(\mathbf{R}^m),\ \operatorname{supp} f \subset B_r,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0,$$

$$S_0 = -\operatorname{div} a^0(x) \operatorname{grad} + V^0(x), \tag{53}$$

$$V^0(x) = V(x), \quad x \in B_{r+2t_0\mu_0}, \quad V^0(x) \in C_0^\infty(\mathbf{R}^m).$$

The function $u(t, x)$ is in $C^\infty([0, \infty) \times \mathbf{R}^m)$ and satisfies

$$\operatorname{supp} u(t, x) \subset B_{r+t_0\mu_0}, \quad 0 \leqslant t < t_0 \tag{54}$$

(see the proof of Lemma 9). The conditions (52), (53) imply that $u(t, x)$ is a solution of

$$\frac{\partial^2 u}{\partial t^2} + S[u] = 0, \quad u|_{t=0} = f, \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

at the time-interval $0 \leqslant t < t_0$. Introducing the vector valued function $U(t) = u(t, \cdot)$ with values in $L^2(\mathbf{R}^m)$ we see that $U'' + A_0 U = 0$, $U'' + AU = 0$, $0 \leqslant t < t_0$, $U|_{t=0} = f$, $U'|_{t=0} = 0$ (the derivatives are taken in L^2-norm). It follows (see [10,Section 3]) that $f \in \mathscr{D}(\cos(tA^{\frac{1}{2}}))$, $U(t) = \cos(tA^{\frac{1}{2}})\, f = \cos(tA_0^{\frac{1}{2}})f$, $0 \leqslant t < t_0$. This formula implies $f \in \mathscr{D}(\hat{\varphi}(tA^{\frac{1}{2}}))$. Moreover

$$\hat{\varphi}(tA^{\frac{1}{2}})f = \hat{\varphi}(tA_0^{\frac{1}{2}})f, \quad 0 \leqslant t < t_0, \tag{55}$$

where $f \in C_0^\infty(\mathbf{R}^m)$, $\operatorname{supp} f \subset B_r$, by (29) and by the same representation for $\hat{\varphi}(tA^{\frac{1}{2}})\, f$ (in the case $\cos(tA^{\frac{1}{2}})$ are unbounded operators the proof of such representation may be found in [10, Section 4]). It is easily seen that (54) implies

$$\operatorname{supp} \hat{\varphi}(tA^{\frac{1}{2}})f \subset B_{r+t_0\mu_0}, \quad 0 \leqslant t < t_0.$$

The right hand side of (55) may be estimated by Proposition 4 with $V^0_-(x) = -\min\{0, V^0(x)\}$ the same estimate being true also for $\hat{\varphi}(tA^{\frac{1}{2}})f(x)$, $0 \leqslant t < t_0$. In the case $x \in B_{r+t_0\mu_0}$, the expressions $k_{x,t}$, $K_{x,t}$ depend on the meaning of $V^0_-(x)$ in the ball $|y - x| < t\mu_0 \subset B_{r+2t_0\mu_0}$. We may substitute V^0_- by V_- in these expressions according to the restriction (53). Since $\hat{\varphi}(tA^{\frac{1}{2}})f(x) = 0$, $|x| \geqslant r + t_0\mu_0$, we obtain Proposition 4 for $f \in C_0^\infty(\mathbf{R}^m)$, $\operatorname{supp} f \subset B_r$, $0 \leqslant t < t_0$. This result may be easily extended to every $f \in L^2(\mathbf{R}^m)$, $\operatorname{supp} f \subseteq B_r$ passing to limit in (55), if one takes into account that the operators $\hat{\varphi}(tA_0^{\frac{1}{2}})$ are bounded and that $\hat{\varphi}(tA^{\frac{1}{2}})$ are closed. Since $r > 0$, $t_0 > 0$ are arbitrary, Proposition 4 is proved.

7. POINTWISE ESTIMATES FOR SPECTRAL FAMILY

Denote the spectral family of A by E_λ, $-\infty \leqslant \lambda \leqslant \infty$, $E_\infty = 1$ (the identity operator), $E_{-\infty} = 0$. Our tool in proving Theorem 1 will be

PROPOSITION 6. *Fix $p \geqslant 1, p > \frac{m}{2}$ $(1 \leqslant m \leqslant 3)$. Let $f \in L^2(\mathbf{R}^m)$, $\delta \in (0, 4^{-1}\pi\mu_0)$. For every $\lambda \geqslant 1$ and λ_0 satisfying $-\infty \leqslant \lambda_0 < \lambda$ the following estimate holds a.e. in $\mathbf{R}^m$:*

$$|(E_\lambda - E_{\lambda_0})f(x)| \leqslant \gamma_1 \delta^{-\frac{m}{2}} \left[1 + \delta \|\chi_{x,\delta} V_-\|_p^{\frac{p}{2p-m}}\right]^{\frac{m}{2}} \|(E_\lambda - E_{\lambda_0})f\|_2.$$

Here γ_1 is a constant depending only on m, p, μ_0, ν_0.

REMARK. A special case of Proposition 6 is proved in [11, Theorem 2].

Proof. Proposition 5 implies that

$$|\hat{\varphi}(tA^{\frac{1}{2}})f(x)| \leqslant c^2(t, x) \int_{B_{x,t\mu_0}} |f(y)|^2 \, dy \quad \text{a.e. in } \mathbf{R}^m \tag{56}$$

$f \in L^2_{\text{comp}}$, $t > 0$, with some $c(t, x)$ depending only on $m, p, \mu_0, v_0, \varphi(\tau), V_-(x)$. It follows that the operators $\hat{\varphi}(tA^{\frac{1}{2}})$ may be extended by continuity from $L^2_{\text{comp}}(\mathbf{R}^m$ to $L^2_{\text{loc}}(\mathbf{R}^m)$ (the convergence of $g_n(x)$ in $L^2_{\text{loc}}(\mathbf{R}^m)$ means that $\chi_{y,\rho}(x)g_n(x)$ converges in $L^2(\mathbf{R}^m)$ for every $y \in \mathbf{R}^m$, $\rho > 0$). Let $\varphi_{\text{loc}}(tA^{\frac{1}{2}})$ be the extention of $\hat{\varphi}(tA^{\frac{1}{2}})$ to $L^2_{\text{loc}}(\mathbf{R}^m)$. The estimate (56) extends (by continuity) to

$$|\varphi_{\text{loc}}(tA^{\frac{1}{2}})f(x)|^2 \leqslant c^2(t, x) \int_{B_{x,t\mu_0}} |f(y)|^2 dy \tag{57}$$

a.e. in $\mathbf{R}^m$, $f \in L^2_{\text{loc}}(\mathbf{R}^m)$. Fix $\lambda \geqslant 1$, $\varepsilon \in (0, 4^{-1}\pi\mu_0)$. The constants $a = \varepsilon\mu_0^{-1}$ $t_0 = a\lambda^{-\frac{1}{2}}$ satisfy

$$a \in \left(0, \frac{\pi}{4}\right), \; t_0 \leqslant a, \; t_0 \in \left(0, \frac{\pi}{4}\right), \; t_0\lambda^{\frac{1}{2}} \in \left(0, \frac{\pi}{4}\right). \tag{58}$$

It follows from (22), (23) that $\hat{\varphi}(t_0\sigma^{\frac{1}{2}}) \geqslant 1$, $\sigma \leqslant 0$; $\hat{\varphi}(t_0\sigma^{\frac{1}{2}}) \geqslant \cos a \geqslant 2^{-\frac{1}{2}}$, $0 < \sigma \leqslant \lambda$.

Fix $\lambda_0 < \lambda$. Denoting the characteristic function of the interval $\lambda_0 < \sigma \leqslant \lambda$ by $e_{\lambda_0\lambda}(\sigma)$ we conclude that $h(A)$ is a correctly defined bounded operator provided

$$h(\sigma) = [\hat{\varphi}(t_0\sigma^{\frac{1}{2}})]^{-1} e_{\lambda_0\lambda}(\sigma).$$

For every $f \in L^2(\mathbf{R}^m)$ we have:

$$\begin{aligned}(E_\lambda - E_{\lambda_0})f &= \hat{\varphi}(t_0A^{\frac{1}{2}})h(A)f = \\ &= \hat{\varphi}_{\text{loc}}(t_0 A^{\frac{1}{2}})h(A)(E_\lambda - E_{\lambda_0})f.\end{aligned} \tag{59}$$

The second of these equalities was proved in [11, Section 3], so we omit the details. Since $|h(\lambda)| \leqslant 2^{\frac{1}{2}}$ one finds from (59) and (57) that

$$|(E_\lambda - E_{\lambda_0})f(x)| \leqslant 2^{\frac{1}{2}} c(t_0,x) \| (E_\lambda - E_{\lambda_0})f\|_2$$

a.e. in $\mathbf{R}^m$ for every $f \in L^2(\mathbf{R}^m)$. By Proposition 5

$$c(t_0, x) = \gamma_0 t_0^{-\frac{m}{2}} \theta(K_{x,t_0}) \cosh(k_{x,t_0}^{\frac{1}{2}}),$$

where γ_0 is a constant depending only on $m, p, \mu_0, \nu_0, \varphi(\tau)$. The expressions k_{x,t_0}, K_{x,t_0} are defined by (39), (40). Noting that

$$k_{x,t_0} \leqslant k_{x,a} = k_{x,\varepsilon\mu_0^{-1}}, \quad K_{x,t_0} \leqslant K_{x,\varepsilon\mu_0^{-1}}, \quad t_0 = \varepsilon\mu_0^{-1}\lambda^{-\frac{1}{2}}$$

(see (58)), one obtains

$$c(t_0, x) \leqslant \tilde{\gamma}_1 \varepsilon^{-\frac{m}{2}} \lambda^{\frac{m}{4}} \theta(K_{x,\varepsilon\mu_0^{-1}}) \cosh(k_{x,\varepsilon\mu_0^{-1}}^{\frac{1}{2}}) = \tilde{c}(\varepsilon, x), \quad \tilde{\gamma}_1 = \gamma_0 \mu_0^{\frac{m}{2}};$$

(60)

$$|(E_\lambda - E_{\lambda_0})f(x)| \leqslant 2^{\frac{1}{2}} \tilde{c}(\varepsilon, x) \|(E_\lambda - E_{\lambda_0})f\|_2$$

a.e. in $\mathbf{R}^m$, $f \in L^2(\mathbf{R}^m)$, $\lambda \geqslant 1$, $-\infty \leqslant \lambda_0 < \lambda$.

Fix $f \in L^2(\mathbf{R}^m)$. Let μ be the Lebesgue measure in $\mathbf{R}^m$. For every $\varepsilon > 0$ there exists a set $\Omega_\varepsilon \subseteq \mathbf{R}^m$, $\mu(\mathbf{R}^m \setminus \Omega_\varepsilon) = 0$, such that (60) holds for every $x \in \Omega_\varepsilon$. Let Ω be the intersection of the sets Ω_ε corresponding to all rational $\varepsilon > 0$. Then $\mu(\mathbf{R}^m \setminus \Omega) = 0$ and the continuity of $\tilde{c}(\varepsilon, x)$, both on ε and x, implies that (60) is true if $x \in \Omega$ and $\varepsilon > 0$ are arbitrary. Thus for every $x \in \Omega$ one may choose

$$\varepsilon = \varepsilon(x) = \delta[1 + \delta\|\chi_{x,\delta} V_-\|_p^{\frac{p}{2p-m}}]^{-1},$$

where $\delta \in (0, 4^{-1}\pi\mu_0)$ is some fixed number. Since $0 < \varepsilon(x) < 4^{-1}\pi\mu_0$, $\varepsilon(x) < \|\chi_{x,\delta} V_-\|_p^{-\frac{p}{2p-m}}$, $\varepsilon(x) < \delta$, it is easily seen that $k_{x,\varepsilon(x)\mu_0^{-1}}$, $K_{x,\varepsilon(x)\mu_0^{-1}}$ are bounded from above by a constant (denote it c) depending only on m, p, μ_0, ν_0. This fact together with the estimate (60) ($\varepsilon = \varepsilon(x)$, $x \in \Omega$) yields Proposition 6.

8. PROOF OF THEOREM 1

To prove Theorem 1 we choose a sequence $1 = \lambda_0 < \lambda_1 < \ldots < \lambda_j < \ldots$, $\lim_{j\to\infty} \lambda_j = \infty$, with the property

(61)
$$\sum_{j=1}^{\infty} \lambda_{j-1}^{-1} \lambda_j^{\frac{m}{4}} < \infty$$

(e.g. $\lambda_j = (j+1)^\alpha$, $\alpha > 4(4-m)^{-1}$). Let $\tau \neq 0$, $\operatorname{Im}\tau = 0$. Fix $\delta \in (0, 4^{-1}\pi\mu_0)$ and $f \in L^2(\mathbf{R}^m)$. By Proposition 6

$$|(E_{\lambda_j} - E_{\lambda_{j-1}})(A - i\tau)^{-1} f(x)| \leqslant \tag{62}$$

$$\leqslant \gamma_0 \delta^{-\frac{m}{2}} \lambda_j^{\frac{m}{4}} \left[1 + \delta \|\chi_{x,\delta} V_-\|_p^{\frac{p}{2p-m}}\right]^{\frac{m}{2}} \|(E_{\lambda_j} - E_{\lambda_{j-1}})(A - i\tau)^{-1} f\|_2$$

a.e. in $\mathbf{R}^m$ and by well known properties of spectral families

$$\|(E_{\lambda_j} - E_{\lambda_{j-1}})(A - i\tau)^{-1} f\|_2 \leqslant (\lambda_{j-1}^2 + \tau^2)^{-\frac{1}{2}} \|f\|_2, \quad j \geqslant 1.$$

In the case $j = 0$ the estimate (62) also remains true if $\lambda_{-1} = -\infty$; note that $\|E_{\lambda_0}(A - i\tau)^{-1} f\|_2 \leqslant |\tau|^{-1}\|f\|_2$. These facts imply for every $k = 1, 2, \ldots$

$$|E_{\lambda_k}(A - i\tau)^{-1} f(x)| = |E_{\lambda_0}(A - i\tau)^{-1} f(x) +$$

$$+ \sum_{j=1}^{k} (E_{\lambda_j} - E_{\lambda_{j-1}})(A - i\tau)^{-1} f(x)| \leqslant \tag{63}$$

$$\leqslant c_0(|\tau|)\gamma_1 \delta^{-\frac{m}{2}} \left[1 + \delta \|\chi_{x,\delta} V_-\|_p^{\frac{p}{2p-m}}\right]^{\frac{m}{2}} \|f\|_2$$

a.e. in $\mathbf{R}^m$, $c_0(r) = r^{-1} + \sum_{j=1}^{\infty} (\lambda_{j-1}^2 + r^2)^{-\frac{1}{2}} \lambda_j^{\frac{m}{4}}$, $r > 0$. The function $c_0(r)$ is finite because of the restriction (61). It is easily seen that $c_0(r) \to 0$ as $r \to \infty$. Passing $k \to \infty$ in the inequality obtained above and noting that $E_{\lambda_k}(A - i\tau)^{-1} f$ converges in L^2-norm to $(A - i\tau)^{-1} f$, one finds that the estimate (63) holds also if we substitute its left hand side by $|(A - i\tau)^{-1} f(x)|$. It is known [8] that the estimate $|Bf(x)| \leqslant c(x)\|f\|_2$ a.e. in $\mathbf{R}^m$ ($f \in L^2(\mathbf{R}^m)$, B a bounded operator in $L^2(\mathbf{R}^m)$, and $c(x)$ a fixed function) is equivalent to the integral character of B together with the estimate $\int_{\mathbf{R}^m} |B(x,y)|^2 \, dy \leqslant c^2(x)$ a.e. in $\mathbf{R}^m$, $B(x,y)$ being the kernel of B (remind that integral operators with such property are called Carleman operators). This remark concludes the proof of Theorem 1.

9. REMARK ON THE CASE $m \geqslant 4$

Here we briefly discuss the result for dimensions $m \geqslant 4$, analogous to Theorem 1. It reads as follows.

THEOREM 1′. *Let* $m \geqslant 4$. *Fix* $p > \frac{m}{2}$, $\tau \neq 0$, $\operatorname{Im}\tau = 0$, *and an integer* $l > \frac{m}{4}$.

Then

(64) $$|(A - i\tau)^{-l} f(x)| \leqslant c_0(|\tau|)\delta^{-\frac{m}{2}} \left[1 + \delta \| \chi_{x,\delta} V_- \|_p^{\frac{p}{2p-m}}\right]^{\frac{m}{2}} \|f\|_2$$

a.e. in $\mathbf{R}^m$ *for every* $f \in L^2(\mathbf{R}^m)$, $\delta \in (0, 4^{-1}\tau\mu_0)$, *the function* $c_0(r) > 0$, $\lim_{r\to\infty} c_0(r) = 0$, *depending only on* m, p, μ_0, ν_0, l.

In Theorem 1′, A is the same self-adjoint operator as in Section 1. We noted at the end of Section 8 that the estimate (64) is equivalent to the Carleman estimate of $\int_{\mathbf{R}^m} |R_l(x, y, \tau)|^2 \, dy$, $R_l(x, y, \tau)$ being the kernel of $(A - i\tau)^{-l}$. To prove Theorem 1′ one should make certain changes in the proof of Theorem 1. Instead of (14), we have now

(65) $$\|(A_0 + k)^{-m_0} f\|_\infty \leqslant ck^{-\left(m_0 - \frac{m}{4}\right)} \left[1 - \left(\frac{k_0}{k}\right)^{\frac{2p-m}{2p}}\right]^{-m_0} \|f\|_2,$$

$f \in L^2(\mathbf{R}^m)$, $m_0 = \left[\frac{m}{4}\right] + 1$, c, k_0, p being the same as in Proposition 1. This estimate is a consequence of the formula

(66) $$(A_0 + k)^{-m_0} f = \prod_{j=1}^{m_0} P_k^{(j)} (1 + \mathscr{H}_k^{(j)})^{-1} f,$$

$f \in L^1(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$ in which $P_k^{(j)}$ are integral operators of the form (17) considered as operators from $L^{w_j}(\mathbf{R}^m)$ to $L^{w_{j+1}}(\mathbf{R}^m)$ with some $w_{j+1} > w_j$, $\mathscr{H}_k^{(j)}$ is defined by formula (15) as an operator in $L^{w_j}(\mathbf{R}^m)$. By using Lemma 2 and the condition $p > \frac{m}{2}$ it is possible to find a sequence $1 = w_1 < w_2 < \ldots < w_{m_0} = 2$ such that every operator in the right hand side of (66) is bounded, this representation being true for sufficiently large $k > 0$ ($k > k_0$). The proof of these facts delivers also estimates for k_0 and for the norms of the operators $P_k^{(j)}$, $(1 - \mathscr{H}_k^{(j)})^{-1}$ which implies (65) for $k > k_0$. Formula (65) implies an estimate differing from (25) only by the form of $\theta(\rho)$. More exactly the estimate (25) holds for dimensions $m \geqslant 4$ with $\theta(\rho) = \theta_m(\rho) = (1 + \rho)^{m_0} \rho^{\frac{m}{4} - m_0}$. The same change must be done also in Propositions 3 and 4 to make them true for dimensions $m \geqslant 4$. The $\mathbf{R}^m$-analogues ($m \geqslant 4$) to Propositions 2—6 may be obtained by making insignificant changes in their proofs.

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Received July 13, 1981.