J. OPERATOR THEORY
9(1983), 147–162

SOME NORM BOUNDS AND QUADRATIC FORM INEQUALITIES FOR SCHRÖDINGER OPERATORS

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1. INTRODUCTION

We present a number of new norm bounds on some operators involving powers of the resolvent of

$$H = H_0 + V$$

on $L^2(\mathbf{R}^N)$, where $H_0 = -\Delta$. We assume that V lies in the class $\mathscr{G}$ of potentials for which

$$0 \leqslant V \in L^1_{\text{loc}}(\mathbf{R}^n \setminus \Omega)$$

where Ω is a closed set of zero Lebesgue measure, depending on V. The operator H is then well-defined as a form sum, with form domain

$$\text{Quad}(H) \supseteq C_c^\infty(\mathbf{R}^N \setminus \Omega)$$

and one may use the quadratic form version of the Trotter product formula due to Kato [4, p. 121]. This implies by the methods of [3; 4, p. 179] or [8, p. 186] that the integral kernels of $(H+\lambda)^{-\beta}$ are non-negative and pointwise dominated by those of $(H_0+\lambda)^{-\beta}$ for all $\beta, \lambda > 0$. It is probable that many of the estimates we obtain could be extended to the case where V has a negative part V_- which is small enough (V_- would have to have form bound less than one with respect to H_0), but we have not attempted such an extension.

We start with an elementary result.

LEMMA 1. *If $V, W \in \mathscr{G}$ then*

$$\|(H_0 + V + 1)^{-1} - (H_0 + W + 1)^{-1}\| \leqslant \left\| \frac{V - W}{(V+1)^{1/2}(W+1)^{1/2}} \right\|_\infty .$$

Proof.

$$\|(H_0 + V + 1)^{-1} - (H_0 + W + 1)^{-1}\| =$$

$$= \|(H_0 + V + 1)^{-1}(V - W)(H_0 + W + 1)^{-1}\| \leqslant$$

$$\leqslant \|(H_0 + V + 1)^{-1}(V + 1)^{1/2}\| \cdot \|(H_0 + W + 1)^{-1}(W + 1)^{1/2}\| \cdot$$

$$\cdot \|(V + 1)^{-1/2}(V - W)(W + 1)^{-1/2}\|.$$

The result follows since

$$\|(V + 1)^{1/2}(H_0 + V + 1)^{-1/2}\| \leqslant 1$$

$$\|(W + 1)^{1/2}(H_0 + W + 1)^{-1/2}\| \leqslant 1.$$

Corollary 2. *Let $H_n = H_0 + V_n$ and $H = H_0 + V$ where $V_n, V \in \mathcal{G}$. Then H_n converges in the norm resolvent sense to H if either*

$$\lim_{n\to\infty} \|V_n - V\|_\infty = 0$$

or

$$1 + V_n = (1 + V)(1 + X_n)$$

where

$$\lim_{n\to\infty} \|X_n\|_\infty = 0.$$

Proof. The first statement is elementary. For the second we need only note that if $\|X_n\|_\infty \leqslant 1/2$ then

$$\left\| \frac{V - V_n}{(1 + V)^{1/2}(1 + V_n)^{1/2}} \right\|_\infty \leqslant 2^{1/2}\|X_n\|_\infty .$$

We next note that it is by no means the case that one always gets norm resolvent convergence when $V_n \to V$. For example if $N = 1$ and

$$V(x) = 1 + \cos(2\pi x)$$

and if

$$V_n(x) = \begin{cases} V(x) & \text{if } |x| \leqslant n \\ 2 & \text{otherwise}, \end{cases}$$

then V_n converges locally uniformly and boundedly to V. Now $f(H_n)$ is a compact operator for any continuous function f with support in $(0,2)$ but $f(H)$ is not compact, so $f(H_n)$ does not converge in norm to $f(H)$. Thus H_n does not converge in the norm resolvent sense to H, by [4, p. 114].

The next lemma yields a rather attractive criterion for norm resolvent convergence, but also an estimate of the rate of convergence.

LEMMA 3. *If* $0 < a, b, \lambda < \infty$ *and*

$$(V+\lambda)^2 \leqslant a^2 (H_0 + V + \lambda)^2$$

$$(W+\lambda)^2 \leqslant b^2 (H_0 + W + \lambda)^2$$

then

$$\|(H_0 + V + \lambda)^{-1} - (H_0 + W + \lambda)^{-1}\| \leqslant ab\|(V+\lambda)^{-1} - (W+\lambda)^{-1}\|.$$

Proof. The hypotheses imply that

$$\mathrm{Dom}(H_0 + V + \lambda) \subseteq \mathrm{Dom}(V+\lambda)$$

and

$$\|(V+\lambda)(H_0 + V + \lambda)^{-1}\| \leqslant a.$$

Similarly

$$\|(W+\lambda)(H_0 + W + \lambda)^{-1}\| \leqslant b.$$

Thus

$$\|(H_0 + V + \lambda)^{-1} - (H_0 + W + \lambda)^{-1}\| =$$

$$= \|(H_0 + V + \lambda)^{-1}(V - W)(H_0 + W + \lambda)^{-1}\| \leqslant$$

$$\leqslant ab\,\|(V+\lambda)^{-1}(V - W)(W+\lambda)^{-1}\| =$$

$$= ab\,\|(V+\lambda)^{-1} - (W+\lambda)^{-1}\|.$$

We conclude that $H_0 + V_n$ converges in the norm resolvent sense to $H_0 + V$ provided $(V_n + \lambda)^{-1}$ converges uniformly to $(V+\lambda)^{-1}$ and one has a uniform quadratic form bound

$$0 \leqslant (V_n + \lambda)^2 \leqslant a^2(H_0 + V_n + \lambda)^2.$$

2. SOME COMMUTATOR ESTIMATES

Lemma 3 provides one reason for attempting to determine the potentials $V \in \mathscr{G}$ for which there exists $a < \infty$ such that

$$0 \leqslant V^2 \leqslant a(H_0 + V)^2.$$

This question was studied by Glimm and Jaffe [7] in an abstract operator context for application in constructive quantum field theory, and they mentioned the possibility of applying it to Schrödinger operators. We initially assume that V is a strictly positive C^∞ potential which is bounded together with all of its partial derivatives. We may then write

$$(H_0 + V)^2 = H_0^2 + \sum_{|\alpha|<4} a_\alpha(x) D^\alpha = H_0^2 + X$$

where

$$D^\alpha = D_1^{\alpha(1)} \dots D_N^{\alpha(N)}$$

and

$$|\alpha| = \alpha(1) + \dots + \alpha(N)$$

and each $a_\alpha(x)$ is a bounded C^∞ function. Now X is a perturbation of H_0^2 with relative bound zero so

$$\mathrm{Dom}(H_0 + V)^2 = \mathrm{Dom}(H_0^2)$$

and $C_c^\infty(\mathbf{R}^N)$ is a core for $(H_0 + V)^2$.

LEMMA 4. *If V satisfies the above regularity conditions and*

$$|\nabla V(x)|^2 \leqslant \alpha V(x)^3 \tag{1}$$

for some $0 < \alpha < 2$ and all $x \in \mathbf{R}^N$, then

$$0 \leqslant V^2 \leqslant (1 - \alpha/2)^{-1}(H_0 + V)^2.$$

Proof. Using the identity

$$H_0 V + V H_0 = 2V^{1/2} H_0 V^{1/2} - \frac{|\nabla V|^2}{2V}$$

of [7], we see that

$$(H_0 + V)^2 = H_0^2 + H_0 V + V H_0 + V^2 =$$

$$= H_0^2 + 2V^{1/2} H_0 V^{1/2} - \frac{|\nabla V|^2}{2V} + V^2 \geqslant$$

$$\geqslant (1 - \alpha/2) V^2$$

as a form inequality on $C_c^\infty(\mathbf{R}^N)$. The result follows upon observing that $C_c^\infty(\mathbf{R}^N)$ is a form core of $(H_0 + V)^2$, or equivalently an operator core of $H_0 + V$.

NOTE 5. If $N \geqslant 3$ we may derive a weaker condition on V by utilising the lower bound

$$2V^{1/2} H_0 V^{1/2} \geqslant \frac{N^2 - 4N + 2}{4} \cdot \frac{V}{Q^2}$$

proved for $N = 3$ in [8, p. 169)].

NOTE 6. The condition (1) also arose in proving JWKB bounds on Schrödinger eigenfunctions by a completely different method in [5].

We now say that V lies in the class $\mathscr{G}_\alpha$ for $0 < \alpha < \infty$ if we may write

$$V(x) = v(x)^{-2}$$

where the function $v: \mathbf{R}^N \to [0, \infty)$ satisfies

(2) (i) $|v(x) - v(y)| \leqslant \frac{1}{2}\alpha^{1/2}|x - y|$ for all $x, y \in \mathbf{R}^N$;

(ii) The closed set

$$\Omega = \{x \in \mathbf{R}^N : v(x) = 0\}$$

has zero Lebesgue measure.

We see that $\mathscr{G}_\alpha \subseteq \mathscr{G}$ and that any $V \in \mathscr{G}_\alpha$ is actually strictly positive and coninuous outside Ω. The condition (2) is a slight generalization of

$$|\nabla v(x)| \leqslant \frac{1}{2}\alpha^{1/2}$$

which is formally equivalent to (1).

THEOREM 7. *If $V \in \mathscr{G}_\alpha$ for some $\alpha < 2$ then*

$$0 \leqslant V^2 \leqslant (1 - \alpha/2)^{-1}(H_0 + V)^2.$$

Proof. Standard approximation techniques allow one to construct a sequence of functions $v_n: \mathbf{R}^N \to [0, \infty)$ satisfying

(i) $m^{-1} \leqslant v_n(x) \leqslant n$ for all $x \in \mathbf{R}^N$, and some fixed $m > 0$;

(ii) Each v_n is C^∞ with all its partial derivatives bounded on $\mathbf{R}^N$;

(iii) $|\nabla v_n(x)| \leqslant \frac{1}{2}\alpha^{1/2}$ for all n and all $x \in \mathbf{R}^N$;

(iv) $v_n(x) \to \max(m^{-1}, v(x))$ is locally uniformly in $\mathbf{R}^N$.

If

$$V_n(x) = v_n(x)^{-2}$$

then we see that

$$V_n(x) \to \min(m^2, V(x)) \equiv W_m$$

locally uniformly in $\mathbf{R}^N$. Thus

$$\lim_{n\to\infty} (H_0 + V_n)f = (H_0 + W_m)f$$

for all $f \in C_c^\infty(\mathbf{R}^N)$. Since such f form a core for $H_0 + W_m$ we conclude that $H_0 + V_n$ converge to $H_0 + W_m$ in the strong resolvent sense. Now $H_0 + V_n$ satisfy the conditions of Lemma 4 so

$$\| V_n(H_0 + V_n + \varepsilon)^{-1} f\| \leqslant (1 - \alpha/2)^{-1/2} \| f \|$$

for all $\varepsilon > 0$ and all $f \in L^2(\mathbf{R}^N)$. Letting $n \to \infty$ we conclude that

$$\| W_m(H_0 + W_m + \varepsilon)^{-1} f \| \leqslant (1 - \alpha/2)^{-1/2} \| f \|.$$

The monotonicity of the integral kernels with respect to the potential now enables us to conclude that

$$\| W_m(H_0 + V + \varepsilon)^{-1} \| \leqslant (1 - \alpha/2)^{-1/2}$$

and an application of the monotone convergence theorem finally yields

$$\| V(H_0 + V + \varepsilon)^{-1} \| < (1 - \alpha/2)^{-1/2}$$

for all $\varepsilon > 0$, as required.

Corollary 8. *If* $V \in \mathcal{G}_\alpha$ *for some* $\alpha < 2$ *and*

$$V_n(x) = \min(V(x), n)$$

then

$$\|(H_0 + V_n + \lambda)^{-1} - (H_0 + V + \lambda)^{-1}\| \leqslant (1 - \alpha/2)^{-1} (n + \lambda)^{-1}$$

for all $n, \lambda > 0$. *In particular* $H_0 + V_n$ *converges to* $H_0 + V$ *in the norm resolvent sense as* $n \to \infty$.

Proof. We first observe that $V + \lambda$ and $V_n + \lambda$ lie in $\mathcal{G}_\alpha$ for all $n, \lambda > 0$ by (3). So

$$\|(V + \lambda)(H_0 + V + \lambda)^{-1}\| \leqslant (1 - \alpha/2)^{-1/2}$$

$$\|(V_n + \lambda)(H_0 + V + \lambda)^{-1}\| \leqslant (1 - \alpha/2)^{-1/2}.$$

Therefore

$$\|(H_0 + V_n + \lambda)^{-1} - (H_0 + V + \lambda)^{-1}\| \leqslant$$

$$\leqslant (1 - \alpha/2)^{-1} \|(V_n + \lambda)^{-1} - (V + \lambda)^{-1}\|_\infty \leqslant$$

$$\leqslant (1 - \alpha/2)^{-1} (n + \lambda)^{-1}$$

by Lemma 3.

Although the condition $\alpha < 2$ in Theorem 7 is in general necessary, the following trick sometimes enables one to circumvent it. If $\lambda > 0$ we define v_λ by

$$V + \lambda = v_\lambda^{-2}.$$

An easy computation establishes that

$$0 \leqslant v_\lambda \leqslant \lambda^{-1/2}$$

and

$$\nabla v_\lambda' = \frac{\nabla v}{(1 + \lambda v^2)^{3/2}} \tag{3}$$

so that v_λ converges to zero uniformly as $\lambda \to \infty$ and

$$\lim_{\lambda \to \infty} \nabla v_\lambda (x) = 0$$

at all points $x \in \mathbf{R}^N$ where $v(x)$ is differentiable.

LEMMA 9. *If v is differentiable and*

$$\lim_{\lambda \to \infty} \| \nabla v_\lambda \|_\infty = 0$$

then

$$\|V(H_0 + V + \lambda)^{-1}\| < \infty$$

for all large enough $\lambda > 0$, and hence all $\lambda > 0$.

It is elementary to check that the conditions of Lemma 9 are satisfied for any non-negative polynomial potential in one dimension. We shall see in Section 4 that V may even have infinite local singularities. We next extend the above arguments to obtain higher order quadratic form inequalities.

THEOREM 10. *If $V \in \mathcal{G}_\alpha$ for some α such that*

$$0 < \alpha < \frac{2}{(2n - 1)^2} \tag{4}$$

then

$$0 \leqslant V^{2n} \leqslant c_n(H_0 + V)^{2n}$$

where

$$0 < c_n = \prod_{m=1}^{n} (1 - (2m - 1)^2 \alpha/2)^{-1} < \infty. \tag{5}$$

Proof. The regularization method of Theorem 7 allows us to reduce to the case where V is a strictly positive C^∞ function which is bounded together with all of its partial derivatives, and satisfies

$$|\nabla V(x)|^2 \leqslant \alpha V(x)^3$$

for all $x \in \mathbf{R}^N$. For similar reasons it it enough to prove the form inequality

$$\langle V^{2n} f, f \rangle \leqslant c_n \langle (H_0 + V)^{2n} f, f \rangle$$

for all $f \in C_c^\infty(\mathbf{R}^N)$, and for this purpose formal manipulations suffice.

We suppose inductively that

$$V^{2(m-1)} \leqslant c_{m-1}(H_0 + V)^{2(m-1)}$$

which holds for $m = 1$ with $c_0 = 1$. Then

$$c_{m-1}(H_0 + V)^{2m} \geqslant (H_0 + V)\, V^{2(m-1)}(H_0 + V) =$$

$$= H_0 V^{2(m-1)} H_0 + H_0 V^{2m-1} + V^{2m-1} H_0 + V^{2m} \geqslant$$

$$\geqslant V^{2m} - \frac{|(2m-1)\, V^{2m-2}\, \nabla V|^2}{2V^{2m-1}} =$$

$$= V^{2m} - \frac{(2m-1)^2}{2} V^{2m-3} |\nabla V|^2 \geqslant b_m^{-1} V^{2m}$$

provided

$$b_m^{-1} = 1 - (2m-1)^2 \alpha/2.$$

The hypothesis (4) ensures that $0 < b_m < \infty$ for all $1 \leqslant m \leqslant n$. Thus $c_m = b_m c_{m-1}$ and the theorem holds for the value

$$c_n = \prod_{m=1}^{n} b_m.$$

The next two theorems taken together allow us to weaken the conditions on the potential V in Theorems 7 and 10.

THEOREM 11. *Suppose that* $V \in \mathcal{G}$, *that* $W \in \mathcal{G}_\alpha$ *where*

$$0 < \alpha < \frac{2}{(2n-1)^2}$$

and that

$$0 \leqslant W(x) \leqslant V(x)$$

for all $x \in \mathbf{R}^N$. *Then*

$$0 \leqslant W^{2n} \leqslant c_n (H_0 + V)^{2n} \tag{6}$$

where c_n *is defined by* (5).

Proof. Theorem 7 implies that

$$\|W^n (H_0 + W + \varepsilon)^{-n}\| \leqslant c_n^{1/2}$$

for all $\varepsilon > 0$. But the integral kernel of the L.H.S. is non-negative and pointwise dominates that of $W^n(H_0 + V + \varepsilon)^{-n}$ by [3], so

$$\|W^n(H_0 + V + \varepsilon)^{-n}\| \leqslant c_n^{1/2}$$

for all $\varepsilon > 0$, which is equivalent to (6).

THEOREM 12. *If $V \in \mathscr{G}$ and $0 < \alpha < \infty$ and $\varepsilon > 0$ then there exists a largest potential $W \in \mathscr{G}_\alpha$ satisfying*

$$0 \leqslant W(x) \leqslant V(x) + \varepsilon$$

for all $x \in \mathbf{R}^N$.

Proof. Defining

$$w(x) = \begin{cases} W(x)^{-1/2} & \text{if } 0 < W(x) < \infty \\ 0 & \text{if } W(x) = \infty \end{cases}$$

we are claiming the existence of a smallest function w satisfying

$$(V(x) + \varepsilon)^{-1/2} \leqslant w(x) < \infty$$

and

$$|w(x) - w(y)| \leqslant \frac{1}{2}\alpha^{1/2}|x - y|.$$

The class $\mathscr{F}$ of all such w is non-empty since the constant function $\varepsilon^{-1/2}$ lies in it. Moreover the class $\mathscr{F}$ is closed under taking minima and under taking decreasing pointwise limits [5].

NOTE 13. If V is a non-negative polynomial in one dimension then

$$\lim_{|x|\to\infty} V'(x^2)\, V(x)^{-3} = 0$$

and the potential W constructed in Theorem 12 coincides with $V(x) + \varepsilon$ for all large enough $|x|$.

These theorems may be used to show that eigenfunctions of $H_0 + V$ vanish at any strong enough local singularities of V, at least in the L^2 sense. For pointwise results of this type see [5], and references there.

COROLLARY 14. *If $V \in \mathscr{G}$ and*

(7) $$\lim_{|x|\to\infty} x^2 V(x) = +\infty$$

then any eigenfunction f of $H_0 + V$ satisfies

$$\|\,|Q|^{-n} f\|_2 < \infty$$

for all $n > 0$.

Proof. If we put

$$W(x) = \frac{4}{\alpha x^2}$$

then a simple computation shows that $W \in \mathscr{G}_\alpha$. Moreover there exists a constant $a_\alpha > 0$ such that

$$0 \leqslant W(x) \leqslant V(x) + a_\alpha$$

for all $x \in \mathbf{R}^N$, by (7). By choosing α small enough we deduce by Theorem 11 that

$$\|W^n f\|^2 = \langle W^{2n} f, f \rangle \leqslant c_n \langle (H_0 + V + a_\alpha)^2 f, f \rangle < \infty.$$

3. LOCAL SINGULARITIES

We apply the above results to the analysis of local singularities of a non-negative potential. For expository reasons we confine attention to the case where H is defined as the form sum

$$H = H_0 + V$$

on $L^2(\mathbf{R}^3)$, where

$$V(x) = c|x|^{-\alpha} \tag{8}$$

for some $c, \alpha > 0$. For earlier results concerning this problem see the note added in proof. Since H is defined as a form sum we have

$$\mathrm{Quad}(H) = \left\{ f \in \mathscr{W} : \int V(x)\, |f(x)|^2 \, dx < \infty \right\}$$

where $\mathscr{W}$ is the Sobolev space

$$\mathscr{W} = W^{1,2}(\mathbf{R}^3) = \mathrm{Quad}(H_0).$$

The exact specification of the domain of H is a more difficult problem. It is a general consequence of the definition of form sums that

$$\mathrm{Dom}(H_0 + V) \supseteq \mathrm{Dom}(H_0) \cap \mathrm{Dom}(V)$$

and it is obvious that the right-hand side contains $C_c^\infty(\mathbf{R}^3 \setminus 0)$. The following proposition is essentially due to Glimm and Jaffe [7].

PROPOSITION 15. *The following conditions are equivalent*

(9) (i) $$\mathrm{Dom}(H_0 + V) = \mathrm{Dom}(H_0) \cap \mathrm{Dom}(V);$$

(ii) $$\mathrm{Dom}(H_0 + V) \subseteq \mathrm{Dom}(V)$$

and

$$\|V(H_0 + V + 1)^{-1}\| < \infty;$$

(iii) $$0 \leqslant (V + \lambda)^2 \leqslant c(H_0 + V + \lambda)^2$$

for some $c, \lambda \geqslant 0$.

We now apply this proposition to the particular potential (8). If $0 < \alpha < 3/2$ then

$$V \in L^2(\mathbf{R}^3) + L^\infty(\mathbf{R}^3)$$

so V is a perturbation of H_0 with relative bound zero, whence

$$\mathrm{Dom}(H_0 + V) = \mathrm{Dom}(H_0).$$

If $0 < \alpha < 2$ then one knows [8, p. 170] that V has relative form bound zero with respect to H_0, so that

(10) $$\mathrm{Quad}(H_0 + V) = \mathrm{Quad}(H_0).$$

If $\alpha > 2$ we have the following result. (See note added in proof.)

THEOREM 16. *If* $2 < \alpha < \infty$ *then*

$$\mathrm{Dom}(H) = \mathrm{Dom}(H_0) \cap \mathrm{Dom}(V).$$

Indeed

(11) $$\|V^n(H_0 + V + \lambda)^{-n}\| < \infty$$

for all $\lambda, n > 0$.

Proof. Putting

$$\lambda + c\,|x|^{-\alpha} = v_\lambda(x)^{-2}$$

we see by (3) that if $x \in \mathbf{R}^3 \setminus 0$ then

$$|\nabla v_\lambda(x)| = \frac{\alpha}{2} c^{-1/2} |x|^{\alpha/2-1} (1 + \lambda |x|^\alpha / c)^{-3/2}.$$

This is a non-negative continuous function on $\mathbf{R}^3$ which vanishes at 0 and ∞. Moreover it decreases monotonically to zero as $\lambda \to \infty$ for each $x \in \mathbf{R}^3$, so

$$\lim_{\lambda \to \infty} \|\nabla v_\lambda\|_\infty = 0$$

by Dini's theorem. We may now apply Theorem 11, to obtain (11) for large enough $\lambda > 0$, and hence for all $\lambda > 0$.

A similar method making use of Note 5, enables one to prove the following result.

THEOREM 17. *If* $\alpha = 2$ *and* $3/2 < c < \infty$ *then*

$$\|V(H_0 + V + 1)^{-1}\| < \infty$$

so

$$\text{Dom}(H) = \text{Dom}(H_0) \cap \text{Dom}(V).$$

We now consider the cases $3/2 \leqslant \alpha < 2$. The formula (9) is not then valid, but to establish this we must use PDE methods. We shall construct an explicit spherically symmetric function f on $\mathbf{R}^3$ such that $f \in \text{Dom}(H)$ but

$$\int |V(x) f(x)|^2 \, dx = +\infty .$$

The difficulty is to obtain a sufficiently precise definition of Dom(H) to be sure that the function f we construct really does lie in Dom(H). This is achieved by the following sequence of propositions.

PROPOSITION 18. *Let*

$$\mathcal{D} \subseteq \text{Dom}(H_0) \cap \text{Dom}(V)$$

be a quadratic form core of $H_0 + V$, *and suppose that* $f \in \text{Quad}(H)$, $g \in L^2(\mathbf{R}^3)$ *satisfy*

$$\langle f, Hh \rangle = \langle g, h \rangle \tag{12}$$

for all $h \in \mathcal{D}$. *Then* $f \in \text{Dom}(H)$ *and* $Hf = g$.

PROPOSITION 19. *The linear subspace* $\mathcal{D} = C_c^\infty(\mathbf{R}^3 \setminus 0)$ *is a quadratic form core for* H_0, *and hence by* (10) *also for* H.

PROPOSITION 20. *Let* $f: (0, \infty) \to \mathbf{C}$ *be twice continuously differentiable and define* $\tilde{f}: \mathbf{R}^3 \setminus 0 \to \mathbf{C}$ *by*

$$\tilde{f}(x) = |x|^{-1} f(|x|). \tag{13}$$

Then $\tilde{f} \in \text{Dom}(H)$ *if and only if the following conditions all hold.*

(i) $$\int_0^\infty |f(r)|^2 \, dr < \infty;$$

(ii) $$\int_0^\infty |f'(r)|^2 \, dr < \infty;$$

(iii) $$\lim_{r\to 0} f(r) = 0;$$

(iv) $$\int_0^\infty |-f''(r) + V(r) f(r)|^2 \, dr < \infty.$$

Proof. Conditions (i), (ii), (iii) are needed to ensure that f lies in $\mathcal{W}$, which equals Quad(H) by (10). Condition (iv) coincides with (12), once one chooses $\mathcal{D} := C_c^\infty(\mathbf{R}^3 \setminus 0)$ and notes that distributional derivatives coincide with ordinary derivaties when the latter exist.

THEOREM 21. *If* $3/2 \leqslant \alpha < 2$ *and* $0 < c < \infty$ *then*

$$\mathrm{Dom}(H_0 + V) \neq \mathrm{Dom}\,(H_0) \cap \mathrm{Dom}(V).$$

Proof. Let $f\colon (0, \infty) \to \mathbf{C}$ be a C^∞ function which vanishes for $x \geqslant 2$ and satisfies

$$f(x) = x^{1/2} J_{(2-\alpha)^{-1}} \left(\frac{2ic^{1/2}}{2 - \alpha} \, x^{(2-\alpha)/2} \right) =$$

$$=: x \sum_{n=0}^{\infty} a_n x^{(2-\alpha)n}$$

for $0 < x < 1$. Then

$$-f''(x) + cx^{-\alpha} f(x) = 0$$

for $0 < x < 1$ by [1, p. 362]. It follows by a routine calculation that the function $\tilde{f}$ defined by (13) lies in Dom(H) but not in Dom(V).

NOTE 22. A similar but simpler calculation, based upon putting $f(x) := x^\beta$ for small x, establishes the same conclusion for $\alpha = 2$ and $0 < c \leqslant 3/4$. B. Simon has informed the author that a separate analysis in each angular momentum sector establishes that Theorem 17 holds for $\alpha = 2$ and *all* $3/4 < c < \infty$.

4. HILBERT-SCHMIDT ESTIMATES

We now turn to the question of finding conditions on V, $W \in \mathcal{G}$ under which

(14) $$W(H_0 + V + 1)^{-\beta} \in \mathcal{J}_2$$

where $\mathcal{J}_p$ are the various trace ideals [9]. For simplicity we shall consider only the case $p = 2$, which is of importance in the Kato-Birman approach to scattering theory; one shows that operators of the form

$$(H_0 + V' + 1)^{-\gamma} X(H_0 + V + 1)^{-\beta}$$

lie in $\mathcal{J}_1$ by writing them as products $A^* B$, where A, B are of the form (14).

This problem has been extensively studied [2, 8, 9, 10] and it is known that (14) holds if $W \in L^2(\mathbf{R}^N)$, $V \in \mathcal{G}$ and $\beta > N/4$. Moreover the condition on W cannot be weakened if one takes no account of the potential V. Our theorems above suggest, however, that it might be possible to extend the results by allowing W to have singularities wherever V has strong enough positive singularities. Our results below confirm such hopes.

In our first theorem, which can probably also be proved using Dirichlet decoupling or functional integration [6, 10], we write χ_R for the characteristic function of the ball with centre 0 and radius R. We do not attempt to find the smallest β in any of these theorems.

THEOREM 23. *If $\beta > N/4 + 1$ then there exists a constant $c < \infty$ independent of V such that*

$$\|\chi_R V^{1/2}(H_0 + V + 1)^{-\beta}\|_2 \leqslant c$$

for all $V \in \mathcal{G}$.

Proof. Since the constant c is independent of V a standard approximation technique shows that it is sufficient to treat the case where V is bounded. We suppose that $f \in C_c^\infty(\mathbf{R}^N)$ equals 1 if $|x| \geqslant R$ and 0 if $|x| \geqslant 2R$. Then

$$\chi_R V^{1/2}(H_0 + V + 1)^{-\beta} = \chi_R f\, V^{1/2}(H_0 + V + 1)^{-\beta} =$$

$$= \chi_R V^{1/2}(H_0 + V + 1)^{-1} f(H_0 + V + 1)^{-\beta+1} +$$

$$+ \chi_R V^{1/2}(H_0 + V + 1)^{-1}[H_0 + V + 1, f](H_0 + V + 1)^{-\beta} =$$

$$= \chi_R V^{1/2}(H_0 + V + 1)^{-1} f(H_0 + V + 1)^{-\beta+1} +$$

$$+ \chi_R V^{1/2}(H_0 + V + 1)^{-1}(g_0 + \sum D_i g_i)(H_0 + V + 1)^{-\beta}$$

where g_i are combinations of partial derivatives of f. The fact that this is Hilbert-Schmidt follows from the bounds

$$\|V^{1/2}(H_0 + V + 1)^{-1/2}\| \leqslant 1, \quad \|(H_0 + V + 1)^{-1/2} D_i\| \leqslant 1,$$

$$\|f(H_0 + V + 1)^{-\beta+1}\|_2 \leqslant \|f(H_0 + 1)^{-\beta+1}\|_2 < \infty,$$

$$\|g_i(H_0 + V + 1)^{-\beta}\|_2 \leqslant \|g_i(H_0 + 1)^{-\beta}\|_2 < \infty,$$

as does the fact that the constant c is independent of V.

THEOREM 24. *Suppose $V \in \mathcal{G}_\alpha$ where*

$$0 < \alpha < \frac{2}{(2n-1)^2}.$$

Then

$$\|\chi_R V^{n/2}(H_0 + V + 1)^{-\beta}\|_2 < \infty$$

provided $\beta > 0$ is large enough.

Proof. We observe that

$$\|\chi_R V^{n/2}(H_0 + V + 1)^{-\beta}\|_2^2 =$$
$$= \mathrm{tr}[(H_0 + V + 1)^{-\beta} \chi_R V^n (H_0 + V + 1)^{-\beta}] \leqslant$$
$$\leqslant \|(H_0 + V + 1)^{-\beta} \chi_R\|_1 \|V^n(H_0 + V + 1)^{-\beta}\|.$$

The first term is finite by [2] (see also [10, p. 266]) while the second is finite for $\beta \geqslant n$ by Theorem 10.

THEOREM 25. *Suppose $V \in \mathcal{G}$ and*

$$\lim_{|x| \to 0} x^2 V(x) = +\infty.$$

Then for every $n > 0$ there exists $\beta > 0$ such that

$$\|\chi_R |Q|^{-n}(H_0 + V + 1)^{-\beta}\|_2 < \infty. \tag{15}$$

Proof. If $c > 0$ and $W(x) = c/x^2$, then there exists $\lambda > 0$ such that

$$0 \leqslant W(x) \leqslant V(x) + \lambda$$

for all $x \in \mathbf{R}^N$. Now by taking c large enough and applying Theorem 24 we obtain

$$\|\chi_R W^{n/2}(H_0 + V + \lambda + 1)^{-\beta}\|_2 \leqslant$$
$$\leqslant \|\chi_R W^{n/2}(H_0 + W + 1)^{-\beta}\|_2 < \infty$$

which implies (15).

We finally comment that it should be possible to derive all the results of this section by use of functional integration [10], if one first estimated the probability of Brownian paths getting very close to the origin sufficiently carefully.

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Received January 4, 1982.

Note added in proof. Professor D. W. Robinson has drawn to our attention his paper in *Ann. Inst. H. Poincaré*, **21**(1974), 185–215, in which the first half of our Theorem 16 and our Theorem 17 are proved by a similar method.