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INFINITE TRACES OF AF-ALGEBRAS AND CHARACTERS OF $U(\infty)$

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INTRODUCTION

The representation theory of approximately finite-dimensional (AF) C^*-algebras was vigorously developed by Strătilă and Voiculescu. Their main objective was a study of the unitary representations of the unitary group $U(\infty)$, the direct limit of the classical unitary groups $U(n)$; such a study had been previously suggested by Kirillov [7]. Having introduced an AF-algebra $\mathfrak{A} = A(U(\infty))$ such that the factor representations of $U(\infty)$ naturally correspond to those of $\mathfrak{A}$, Strătilă and Voiculescu proceeded to investigate various families of factor representations of $U(\infty)$. In particular, they discovered certain traceable factor (= normal) representations [11, V.2]; and, subsequently, Voiculescu [13] found a list of finite factor representations of $U(\infty)$, which he conjectured to be complete.

The purpose of our work was to determine all the normal representations of $U(\infty)$ and to examine to what extent the canonical bijection established by Pukanszky and Green (see [9], [6]) between the primitive ideal space and the quasi-equivalence classes of normal representations of (the C^*-algebra of) a connected locally compact group holds for $U(\infty)$. In order to achieve this goal, we had to extend the „dynamical system" characterization of finite traces of AF-algebras of Strătilă and Voiculescu to infinite traces: roughly speaking, we show that any infinite trace is fully determined by its values on the Bratteli diagram [2] of the AF-algebra. This is carried out in Section 1. In Section 2, after giving an explicit description of the Bratteli diagram of $\mathfrak{A}$ and its primitive ideals and quotients, we show how the solution of a problem posed by I. J. Schoenberg [10] in 1948 and solved by A. Edrei [5] in 1953, may be used to demonstrate the completeness of Voiculescu's list of finite characters (i.e., finite factor traces). We then show in Section 3, by a detailed analysis, that a primitive ideal of $\mathfrak{A}$ arises as the kernel of a normal representation if and only if the corresponding quotient contains an ideal B stably isomorphic to an ideal of a finite primitive quotient of $\mathfrak{A}$. (The ideal B may be characterized as the one generated by "trace-class" elements.) The faithful characters of the finite quotient correspond exactly to the faithful characters of the ideal, and hence the primitive quotient under consideration. After this work was completed, A. J. Wassermann proved that the finite primitive quotients of $\mathfrak{A}$ admit no faithful infinite characters (this is included in an Appendix). Thus we obtain a complete list of the characters of $U(\infty)$. The Pukanszky-Green correspondence is very poor for $U(\infty)$; in particular, we exhibit primitive quotients of $\mathfrak{A}$ with no faithful characters. We close by giving realizations of the normal representations as subrepresentations of a tensor product of a finite factor representation with a traceable irreducible representation.

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1. INFINITE TRACES ON AF-ALGEBRAS

The purpose of this section is to extend to infinite traces the Voiculescu-Strătilă characterization of finite traces on AF-algebras in terms of dynamical systems. We must first briefly review the Voiculescu-Strătilă approach to AF-algebras. By an AF-algebra, we mean a unital C^*-algebra A which is the inductive limit of a (directed) sequence (A_n, φ_{mn}), $m \leqslant n$, of finite-dimensional C^*-algebras such that $A_0 = \mathbf{C}\cdot I$ and $\varphi_{mn}\colon A_m \to A_n$, $m \leqslant n$, forms a consistent family of unital imbeddings. We often identify A_m with its image under φ_{mn} in A_n and its image in the limit algebra A, so we may write $A = \left(\bigcup_{n=0}^{\infty} A_n\right)^-$.

We define $C_n \subset A$, $n \geqslant 1$, to be the maximal abelian subalgebra generated by C_{n-1} and E_n, where E_n is any maximal abelian subalgebra in $(A_n)' \cap A_{n+1}$ and $C_0 = \mathbf{C}$. Let $D_n = \hat{A}_n$, with corresponding minimal central projections $q^{(n)}(\pi)$, $\pi \in D_n$, in A_n. We also let $\{q^{(n)}(\omega)\,;\, \omega \in \Omega_n\}$ be a maximal orthogonal family of minimal projections in C_n, so that $C_n \cong \mathscr{C}(\Omega_n)$.

We set $\mathscr{U}_n$ to be the subgroup of the full unitary group $U(A_n)$ of A_n given by:

$$u \in \mathscr{U}_n \text{ iff } \mathrm{Ad}(u)C_n = C_n.$$

If $N = \{u \in \mathscr{U}_n\,;\ \mathrm{Ad}(u)|C_n = \mathrm{Id}\}$, then $N = \mathscr{U}_n \cap C_n$ and $\mathscr{U}_n$ is the semidirect product of U_n and N, where U_n is a finite subgroup of $U(A_n)$ [11, I.1.9]. The group U_n induces an isomorphic group of homeomorphisms Γ_n on Ω_n. Finally, we define a faithful $\mathscr{U}_n$-invariant conditional expectation $P_n\colon A_n \to C_n$ by:

$$P_n(x) = \sum \{q^{(n)}(\omega)xq^{(n)}(\omega)\,;\ \omega \in \Omega_n\}, \quad [11, \text{I.1.2}].$$

Voiculescu and Strătilă [11, I.1.10] now associate to the AF-algebra A a dynamical system (Ω, Γ) such that A is isomorphic to a concrete representation of the crossed-product of Γ with $\mathscr{C}(\Omega)$, where

$\Omega = \varprojlim \Omega_n$ is a compact Hausdorff space;

$\Gamma = \bigcup_{n=1}^{\infty} \Gamma_n$ is a countable discrete group of homeomorphisms of Ω. Note that a point ω of Ω is given by a sequence (ω_n), $\omega_n \in \Omega_n$, such that $q^{(n+1)}(\omega_{n+1}) \leqslant q^{(n)}(\omega_n)$, [11, I.1.11].

We also associate to A the diagonalization (C, P), where

$C =$ maximal abelian *-subalgebra of A generated by $\bigcup_{n=1}^{\infty} C_n$;

$P = \mathscr{U}$-invariant conditional expectation of A onto C induced by the consistent family of conditional expectations (P_n);

$\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$;

Note that $C \cong \mathscr{C}(\Omega)$.

REMARK. The Bratteli diagram [2] of the AF-algebra A is determined by the finite sets D_n and the multiplicity $[\pi; \pi']$ of the imbeddings of $q^{(n)}(\pi)A_n$ into $q^{(n+1)}(\pi')A_{n+1}$ under $\varphi_{n,n+1}$ where $\pi \in D_n$, $\pi' \in D_{n+1}$. Note that $[\pi; \pi'] = \mathrm{Tr}(q^{(n+1)}(\pi')q^{(n)}(\omega))$, where $q^{(n)}(\pi)q^{(n)}(\omega) \neq 0$, $\omega \in \Omega_n$. Note that the Bratteli diagram is an isomorphism invariant only then the imbeddings $\varphi_{n,n+1}$ are all unital.

1.1. Suppose t is a trace on the subalgebra $\bigcup_{n=0}^{\infty} A_n$ of A. If $t_n = t \mid A_n$, then we have:

$$t_n = \sum \{a^{(n)}(\pi, t)\mathrm{Tr}^{(n)}(\pi)\,;\, \pi \in D_n\}, \tag{1.1.1}$$

where $\mathrm{Tr}^{(n)}(\pi)$ denotes the unnormalized trace on the matrix algebra $q^{(n)}(\pi)A_n$ and $0 \leqslant a^{(n)}(\pi, t) \leqslant \infty$; moreover,

$$a^{(n)}(\pi, t) = \sum \{[\pi, \pi']a^{(n+1)}(\pi', t)\,;\, \pi' \in D_{n+1}\}. \tag{1.1.2}$$

1.2. DEFINITION. A system $\{a^{(n)}(\pi)\,;\, \pi \in D_n,\ n \geqslant 0,\ 0 \leqslant a^{(n)}(\pi) < \infty\}$ is called a *trace coefficient system* for $\bigcup_{n=0}^{\infty} A_n$ if it satisfies (1.1.2).

Given a trace coefficient system, there is an associated trace t on $\bigcup_{n=0}^{\infty} A_n$ and a finitely additive Γ-invariant measure μ on $(\Omega, \mathscr{F})$. Here $\mathscr{F}$ is the ring of sets on Ω generated by $p_k^{-1}(B_k)$, where $p_k\colon \Omega \to \Omega_k$ is the canonical map of the projective limit and B_k is the ring of Borel sets of Ω_k. $\mu = \varprojlim \mu_n$, where μ_n is the measure on Ω_n given by:

$$\mu_n(\omega) = a^{(n)}(\pi), \quad \text{where } \omega \in \Omega_n, \ \pi \in D_n, \ q^{(n)}(\omega)q^{(n)}(\pi) \neq 0.$$

For a thorough discussion of projective limit measures, see Yamasaki [15]. If t is a trace on $\cup A_n$ with trace coefficient system $\{a^{(n)}(\pi, t)\}$, then $\mu_n(\omega) = a^{(n)}(\pi, t)$, $\omega \in \Omega_n$.

If t is a finite trace, then t is uniquely determined by $\{a^{(n)}(\pi, t)\}$. The main theme of this section is to develop a criterion when this is true for infinite traces. The main difficulty will be deciding when the measure μ on Ω is countably additive.

1.3. NOTATION. Let t be a trace on a C^*-algebra A. Denote by $n_t = \{x \in A\ ;\ t(x^*x) < +\infty\}$, the ideal of Hilbert-Schmidt elements (with respect to t) in A and $m_t = (n_t)^* n_t$, the ideal of definition of t [4, Chapter 6]. If s is a bitrace on A, let n_s denote its ideal of definition.

The obvious interpretations of these notions will apply when the trace is only defined on a dense $*$-subalgebra of A, for example on a generating nest of an AF-algebra.

For the remainder of this section, A will always denote an AF-algebra with generating nest $\bigcup_{n=0}^{\infty} A_n$, dynamical system (Ω, Γ), and diagonalization (C, P).

1.4. THEOREM. *Let t be a (faithful) semifinite lower-semicontinuous trace on A. Then $t|C$ is a (faithful) $\mathscr{U}$-invariant lower-semicontinuous trace on C.*

Proof. It is clear that $t|C$ is $\mathscr{U}$-invariant and faithful when t is. For technical convenience, we shall work with bitraces. Let s be the maximal bitrace associated to t with ideal of definition n. Let $n' = n \cap C$ and $s' = s|n' \times n'$. It suffices to show that s' is a maximal bitrace with associated trace $t' = t|C$, by [4, 6.4.5]. To establish this, it is enough to show that s' is a bitrace whose associated trace t' satisfies $n_{t'} = n'$, by [4, 6.4.3]. We proceed in steps.

(a) $P(n) \subset n'$. Fix $x \in A^+$. Then

$$t(P_n(x)) = t(\{\textstyle\sum q^{(n)}(\omega)xq^{(n)}(\omega)\ ;\ \omega \in \Omega\}) = t(x).$$

By lower-semicontinuity,

$$t(P(x)) = t(\lim P_n(x)) \leqslant t(x).$$

If $y \in n$, then we have:

$$t(P(y)P(y)^*) \leqslant t(P(yy^*)) = s(y, y),$$

so, $y \in n'$, and

$$(*) \qquad s'(P(y), P(y)) \leqslant s(y, y).$$

(b) Fix an approximate identity (u_α) for n. If $(v_\alpha) = (P(u_\alpha))$, then (v_α) is an approximate unit both in norm and the s-inner product.

The assertion is a consequence of (u_α) being an approximate identity and the inequalities:

$$\| P(u_\alpha)x - x\| = \|P(u_\alpha x - x)\| \leqslant \|xu_\alpha - x\|, \quad x \in n,$$

and

$$s'(v_\alpha x - x,\ v_\alpha x - x) = s(u_\alpha x - x,\ u_\alpha x - x) \leqslant$$

$$\leqslant 2[t(x^* x) - t(x^* u_\alpha x)], \quad x \in n,$$

by (*) and [4, 6.4.1].

(c) s' is a bitrace. By [4, 6.2.1], it is enough to show that $(n')^2$ is dense in n' with respect to the s-inner product. This follows immediately from (b).

(d) The norm closures of n' and n'' coincide, where $n'' = n_{t'}$. Clearly, $n' \subset n''$. Since t' is lower-semicontinuous, $t'|M$, $M = (n_{t'})^-$, is given uniquely by integration against a Radon measure on the spectrum of M [8,5.6.7]. If (d) were false, there is an $x \in M^+$ whose support is disjoint from the spectrum of $(n')^-$. In particular, x would be orthogonal to n' with respect to the inner product given by the maximal bitrace $(s')^{\sim}$ which extends s'. By [4; 6.4.3, 5.3.1], $n'' = n_{(s')^{\sim}}$ and n' is dense in n'' in the inner product norm. Contradiction.

(e) t and t' coincide on $[(n')^-]^+$. Let $x \in [(n')^-]^+$. Then x may be written as yy^*, $y \in (n')^-$. Note that (v_α^2) is still an approximate unit for $(n')^-$. Then $y_\alpha \uparrow xx^*$, in norm, and $y^{1/2}v_\alpha \in n'$, where $y_\alpha = (y^{1/2}v_\alpha)^*(y^{1/2}v_\alpha)$. Thus,

$$t'(x) = t'(yy^*) = \lim t'(y_\alpha) =$$

$$= \lim s(y^{1/2}v_\alpha,\ y^{1/2}v_\alpha) = \lim t(y_\alpha) = t(x).$$

Now (d) and (e) together imply that $n' = n''$, which completes the proof.

REMARK. Theorem 1.4 is analogous to [11, I.3.11] except that we are working on the C^*-algebra level.

1.5. LEMMA. *Let t be a $\mathcal{U}$-invariant semifinite lower-semicontinuous trace on C. Then*

$$t(x) = \int_{\Omega} x(\omega)\mathrm{d}\mu(\omega), \quad x \in C^+, \tag{1.5.1}$$

where μ is a σ-finite Γ-invariant Borel measure on Ω such that (i) $\mu|\Omega_0$ is Radon, and (ii) $\mu(\Omega - \Omega_0) = 0$, where $\Omega_0 \subset \Omega$ is the spectrum of $(m_t)^-$.

Proof. Let $J = (m_t)^-$. By [8,5.6.7], $t|J$ is given by integration against a Radon measure ν. Trivally extend ν to Ω by (ii) to form μ. Then (1.5.1) holds immediately if $x \in J^+$. To complete the proof, we need to show that $\int x \, d\mu = +\infty$, whenever $x \in C^+ - J^+$. Consider the basis of open sets $\{E^{(n)}(\omega) ; n \geqslant 0, \omega \in \Omega_n\}$ for Ω, where $E^{(n)}(\omega) = \text{supp}(q^{(n)}(\omega))$. Here $q^{(n)}(\omega)$ is viewed as a function on Ω. If $x(\omega_0) \neq 0$, $\omega_0 \in \Omega_0$, choose an open set E such that $x(\omega) > \frac{1}{2} x(\omega_0) \neq 0$, for all $\omega \in E$. Let $E^{(n)}(\tilde{\omega})$ be a basis open set such that $\omega_0 \in E^{(n)}(\tilde{\omega}) \subset E$. Then $x(\omega) \geqslant \frac{1}{2} q^{(n)}(\tilde{\omega})(\omega)$, $\omega \in \Omega$; moreover, $t(q^{(n)}(\tilde{\omega})) = +\infty$, since $q^{(n)}(\tilde{\omega}) \notin J$. If e_m is the unit in $m_t \cap C_m$, then $e_m q^{(n)}(\tilde{\omega})$ is a projection in m_t and the sequence $\{e_m q^{(n)}(\tilde{\omega})\}_{m=1}^{\infty}$ converges pointwise almost everywhere to $q^{(n)}(\tilde{\omega})$. Hence, $\int e_m q^{(n)}(\tilde{\omega}) d\mu \to \int q^{(n)}(\tilde{\omega}) d\mu$. But $\int e_m q^{(n)}(\tilde{\omega}) d\mu = t(e_m q^{(n)}(\tilde{\omega})) \to t(q^{(n)}(\tilde{\omega}))$. The proof is now complete.

REMARK. If a measure μ satisfies (i) and (ii) of 1.5, then the integral formula (1.5.1) defines a semifinite lower-semicontinuous trace on C.

1.6. PROPOSITION. *Let t' be a $\mathcal{U}$-invariant (faithful) semifinite lower-semicontinuous trace on C. Then $t = t' \circ P$ is a (faithful) semifinite lower-semicontinuous trace on A.*

Proof. By Lemma 1.5, t' is given by a Γ-invariant Borel measure μ^* such that $\mu^* | \Omega_0$ is Radon and $\mu^*(\Omega - \Omega_0) = 0$. If μ is a Γ-quasi-invariant probability measure equivalent to μ^*, then the representation π_μ of A, given by the Kreiger construction [11; I.3, esp. I.3.11], is semifinite and faithful (if t' is) with normal trace t^* given on $(\pi_\mu(A)'')^+$ by:

$$t^*(x) = \int_\Omega P^\mu(x)(\omega) \, d\mu(\omega), \quad x \in (\{\pi_\mu(A)\}'')^+,$$

where P^μ is the conditional expectation given in [11, p. 40] such that $P^\mu(x) = \pi_\mu(P(x))$, $x \in A$. If $t = t' \circ P$ on A^+, then $t(x) = t^*(\pi_\mu(x))$, $x \in A^+$. By [4, 6.1.5], to establish the proposition it suffices to construct a sequence e_m in $m_{t'}$ such that $\{\pi_\mu(e_m)\}$ is a strong approximate unit for $\{\pi_\mu(A)\}''$. We define e_m to be the unit in $m_{t'} \cap C_m$. Then e_m converges monotonically to the characteristic function of Ω_0. Since $\mu^*(\Omega - \Omega_0) = 0$, $\pi_\mu(e_m)$ must converge strongly to the identity.

REMARK. It is possible to show that e_m is an approximate identity for $(m_t)^-$.

1.7. COROLLARY. *Let t be a semifinite lower-semicontinuous trace on a unital* AF*-algebra A, with diagonalization (C, P). Then: $t = (t | C) \circ P$.*

Proof. We first set $t' = (t | C) \circ P$, which is semifinite and lower-semicontinuous by 1.6. By definition, $t' = t$ on $\cup (C_m \cap m_t)$ and so also on $\cup (A_m \cap m_t)$. Hence, $t' = t$ on $J = (m_t)^-$ by [4, 6.3.5]. It now follows that $t' = t$ on A since

t admits a unique semifinite lower-semicontinuous extension from its ideal of definition to A. To see this, let (f_α) be any increasing approximate identity for J, and let t_1 be any such extension. If $x \in A^+ - J$, then $x^{1/2} f_\alpha x^{1/2} (\leqslant x)$ is an increasing sequence in J^+. Thus, $t_1(x) \geqslant \lim t_1(x^{1/2} f_\alpha x^{1/2}) = \lim t(x^{1/2} f_\alpha x^{1/2}) = t(x)$, so $m_{t_1} = m_t$ and $t_1 = t$ everywhere.

The following definition singles out the property for a trace defined on a generating nest to possess a semifinite lower-semicontinuous extension to the full AF-algebra. The technical problem is that of guaranteeing that an associated projective limit measure is σ-additive.

1.8. DEFINITION. Let t be a trace on the dense subalgebra $\cup A_n$ of A. We say that t satisfies the *trace extendibility condition* if for any projection e in $\cup A_n$,

$$t(e) = \sup\{t(f)\ ;\ f \leqslant e,\ t(f) < \infty,\ f = \text{projection in } \cup A_n\}.$$

REMARKS. (i) The proof of Lemma 1.5 shows that any semifinite lower-semicontinuous trace on A satisfies the trace extendibility condition.

(ii) The trace extendibility condition is an almost word-by-word translation of condition (FC) of Yamasaki concerning projective limit measures to the context of AF-algebras.

1.9. THEOREM. *Let t be an infinite trace on $\cup A_n$ satisfying the trace extendibility condition. Then t admits a unique semifinite lower-semicontinuous extension to A.*

Proof. Consider the finitely additive Γ-invariant measure $\mu = \varprojlim \mu_n$ which is associated to t. Since the trace extendibility condition insures that the sequence (μ_n) of measures on Ω_n satisfies condition (FC) of Yamasaki, the family (μ_n) admits a unique minimal projective limit measure μ. If $t' = \int x \, d\mu$, $x \in C^+$, we will show that $t^* = t' \circ P$ is the desired extension. By Proposition 1.6, it suffices to show that t' is semifinite and lower-semicontinuous. In fact, it is enough to check that $\mu|\Omega_0$ is Radon and $\mu(\Omega - \Omega_0) = 0$, where Ω_0 is the spectrum of $(C \cap m_t)^-$. Let e_m denote the unit in $m_t \cap C_m$ and set $E_m = \operatorname{supp}(e_m)$. Then E_m are open subsets of Ω, with finite μ-measure and $\cup E_m = \Omega_0$. Note that $\mu(\Omega - \Omega_0) = 0$, by the definition of μ, [15, p. 401, 6.4]. $\mu|\Omega_0$ is Radon since if $K \subset \Omega_0$ is compact, then $K \subset \bigcup_{m=1}^{\infty} E_m$, so there is an index i such that $K \subset E_i$. Hence, $\mu(K) < +\infty$. Uniqueness of the extension follows since any semifinite extension determines the same projective limit measure on Ω which, by 1.7, uniquely specifies the trace.

NOTATION. Let e and f be two projections in a unital C^*-algebra. If e is unitarily equivalent to f, i.e., there is a unitary u such that $u^*eu = f$, we write: $e \sim_u f$.

1.10. THEOREM. *Let $A = \lim(A_n, \varphi_{m,n})$ be an AF-algebra with diagonalization (C, P), and let B be a closed ideal A such that:*

(†) given any projection $e \in C_N - B$, there is an increasing sequence (J_i) of positive integers such that $f(i,j) \in B$, $1 \leqslant j \leqslant J_i$, are projections such that the following conditions hold:

$$\varphi_{N,N+i}(e) \geqslant f(i,1) + \dots + f(i,J_i); \tag{1.10.1}$$

$$f(i,j) \sim_u \varphi_{N,N+1}(f(1,1)), \quad 1 \leqslant i \leqslant J_i. \tag{1.10.2}$$

Then if t is any faithful semifinite lower-semicontinuous trace on A, we have $(m_t)^- \subset B$.

Proof. We first show that $\cup(A_n \cap m_t)$ is norm-dense in m_t. By [11, 1.2], it suffices to show that their respective norm-closures intersected with C coincide. Because the closures of the trace-class and Hilbert-Schmidt elements with respect to a given trace agree, $(m_t)^- \cap C = (m_t \cap C)^-$, by 1.4(d). Finally, by an elementary topological argument (similar to 1.5), $\{\cup(m_t \cap C_n)\}^- = (m_t \cap C)^-$. Hence, $\cup(m_t \cap C_n) = \cup(m_t \cap A_n) \cap C$ has identical closure to $(m_t)^- \cap C$.

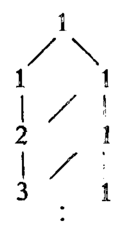
The above paragraph reduces the proof to checking that $\cup(m_t \cap A_n)^- \subset B$. To verify this inclusion, it suffices to work with projections. Let $e \in C_N - B$ be a projection. Then we have:

$$t(e) = t(\varphi_{n,N+1}(e)) \sum_{j=1}^{J_i} t(f(i,j)) =$$

$$= \sum_{j=1}^{J_i} t(\varphi_{N,N+1}(f(1,1))) = J_i \cdot t(f(1,1)).$$

Hence, $t(e) = +\infty$ and $C_N \cap m_t \subset B$, $n \geqslant 0$. Since any projection in A_N is unitarily equivalent to one in C_N, we see that $A_N \cap m_t \subset B$. Therefore, $(m_t)^- \subset B$.

EXAMPLE. Let $A = \mathscr{K} + \mathbf{C} \cdot I$, where $\mathscr{K}$ is the algebra of compact operators on a separable Hilbert space. By [2, 1.9], $A = (\cup A_n)^-$, where $A_n = M_n(C) \oplus \mathbf{C}$, and has Bratteli diagram:



The ideal B of the theorem is given by $(\cup M_n(\mathbf{C}))^-$. Note: $M_n(\mathbf{C}) \subset M_{n+1}(\mathbf{C})$. If t is a faithful semifinite lower-semicontinuous trace on A, we sketch why $t(e) = +\infty$, where e is the central projection of A_N not in B. Let $\varphi_{m,n}: A_m \to A_n$ indicate the imbedding of A_m into A_n. Then $\varphi_{N,N+1}(e) = e_1 + f_1$, where e_1 is a minimal projection in B and f_1 is the central projection of A_{N+1} not in B; in other words, e splits apart into two pieces in A_{N+1}: one piece lying in B and the other not. Now $\varphi_{N,N+2}(e) = \varphi_{N+1,N+2}(e_1 + f_1) = e_1 + e_2 + f_2$, where $\varphi_{n+1,N+2}(f_1) = e_2 + f_2$. Iterating this process, we have that $\varphi_{N,N+i}(e) = e_1 + e_2 + \ldots + e_i + f_i$, where $e_1, \ldots, e_i$ are orthogonal minimal projections in B_{N+i}. Hence, $t(e) = +\infty$.

2. THE C^*-ALGEBRA OF U(∞) — ITS PRIMITIVE IDEALS AND FINITE CHARACTERS

In this section we first briefly review and amplify the results of Strătilă and Voiculescu [11, Chapters II, III] so that the theorems of Section 1 may smoothly be applied to $U(\infty)$. In particular, we explicitly give the Bratteli diagrams of the C^*-algebra of $U(\infty)$ and its primitive quotients. In the last half of this section, we indicate how classification problem of the finite characters of $U(\infty)$ is equivalent to the known classification of the totally positive sequences. As a consequence, Voiculescu's list [13, Proposition 2] of finite factor traces is complete.

2.1. If $U(n)$ denotes the group of all unitary operators on $\mathbf{C}^n$, we let $U(\infty)$ be the direct limit group of the unitary groups $U(0) \subset U(1) \subset U(2) \subset \ldots$, endowed with the direct limit topology. (Here, $U(0) = \{e\}$.)

In [11, II.1], a C^*-algebra $\mathfrak{A} = A(U(\infty))$ is associated to the group $U(\infty)$. Let $M(U(\infty))$ be the direct limit of the Banach $*$-algebras $M(U(n))$, which is the measure algebra of $U(n)$. Since $L_1(U(j)) \subset M(U(n)) \subset M(U(\infty))$, $0 \leqslant j \leqslant n$, $L_{(n)} = \sum_{k=0}^{n} L_1(U(k))$ forms a closed $*$-subalgebra of $M(U(\infty))$. It follows that $L(U(\infty)) \equiv \left(\bigcup_{n=0}^{\infty} L_{(n)}\right)^- \subset M(U(\infty))$ forms a Banach $*$-subalgebra whose C^*-completion is defined to be $A(U(\infty))$. In [11, II.1.5], it is shown that there is a canonical bijection between the factor representations of $U(n)$, $0 \leqslant n \leqslant \infty$, and those of $\mathfrak{A} = A(U(\infty))$. As a consequence, we may identify a (factor) representation of $U(\infty)$ with the corresponding one of $\mathfrak{A}$.

2.2. $\mathfrak{A}$ can be explicitly described as an AF-algebra. We give a somewhat more explicit description than [11, II.2].

Fix an increasing, resp. decreasing, sequence K_n^+, resp. K_n, of integers. Consider the sequence $(\Delta_n)_{n=0}^{\infty}$ $(=: (\Delta_n ; K_n^+, K_n)_{n=0}^{\infty})$ of finite sets such that:

(2.2.1) $$\Delta_n \subset \bigcup_{k=0}^{n} U(k)^\wedge$$

where

(2.2.2) $$\pi \in \Delta_n \cap U(k)^\wedge =: \Delta_{n,k} \text{ iff } K_n \leqslant m_i(\pi) \leqslant K_n^+, \quad 1 \leqslant i \leqslant k.$$

(cf. [11, p. 64]).

Let $P(\pi)$, $\pi \in U(j)^\wedge$, denote the central projection corresponding to π in $M(U(j))$, and let

$$B(\pi) := P(\pi)L_1(U(j)) =: P(\pi)L_{(j)}.$$

Define X_n to be the algebra generated by $B(\pi)$, $\pi \in \Delta_n$, and let $\mathfrak{A}_n$ be its C^*-completion. X_n admits the central decomposition (11, [II.2.3]):

$$X_n := \oplus \{Q^{(n)}(\pi)\, B(\pi) ; \pi \in \Delta_n\},$$

where

(2.2.3) $$P_j^{(n)} := \sum \{P(\pi) ; \pi \in \Delta_{n,j}\},$$

(2.2.4) $$Q^{(n)}(\pi) := P(\pi)\,(1 - P_{j+1}^{(n)}), \quad \pi \in \Delta_{n,j};$$

and so,

$$\mathfrak{A}_n =: \oplus \{M(d(\pi));\ \pi \in \Delta_n\}$$

where $d(\pi) :=$ rank of π, and $M(d) =:$ matrix algebra of rank d.

Let $q^{(n)}(\pi)$, resp. $p_j^{(n)}$, $p(\pi)$, denote the image of $Q^{(n)}(\pi)$, resp. $P_j^{(n)}$, $P(\pi)$, in $\mathfrak{A}_n$. Note that, for $\pi \in \Delta_{n,j}$,

$$p(\pi) = q^{(n)}(\pi) + q^{(n)}(\pi) \sum \{p(\pi) ; \pi \in \Delta_{n,j+1}\};$$

in particular, when $j = n$,

$$p(\pi) = q^{(n)}(\pi).$$

Thus, the central projections of $\mathfrak{A}_n$ are expressible in terms of the $p(\pi)$'s. In 2.9, we will see that this implies that a finite trace on $\mathfrak{A}$ is uniquely determined by its values on the projections $p(\pi)$, $\pi \in \Delta_n$, $n \geqslant 0$.

2.3. The following lemma determines the Bratteli diagram of $\mathfrak{A}$. Although the result is implicit in [11, Chapter II, esp. II.2.8], it never appears in a form suitable to apply the results of Section 1.

LEMMA. (a) *Let* $\pi \in \Delta_{n,j}, 0 \leqslant j \leqslant n$. *Then:*

$$q^{(n)}(\pi) = q^{(n+1)}(\pi) +$$

$$+ q^{(n)}(\pi) \cdot \sum \{q^{(n+1)}(\tilde{\pi}) \,;\, \tilde{\pi} \in \Delta_{n+1,i} - \Delta_{n,i},\ \pi \prec \tilde{\pi},\ j+1 \leqslant i \leqslant n+1\};$$

(b) *For* $\pi \in \Delta_{n,j}$, $\tilde{\pi} \in \Delta_{n+1,k}$, *let* $[\pi; \tilde{\pi}]_\Delta =$ *the partial multiplicity of* $q^{(n)}(\pi)\mathfrak{A}_n$ *in* $q^{(n+1)}(\tilde{\pi})\,\mathfrak{A}_{n+1}$. *Then:*

$$[\pi; \tilde{\pi}]_\Delta = \#\{(\pi_{j+1}, \ldots, \pi_{k-1}) \,;\, \pi_i \in \Delta_{n+1,i} - \Delta_{n,i},\ \pi \prec \pi_{j+1} \prec \ldots \prec \pi_{k-1} \prec \tilde{\pi}\}.$$

Proof. (a) Since $1 = \sum \{q^{(n+1)}(\tilde{\pi}) \,;\, \tilde{\pi} \in \Delta_{n+1}\}$, it suffices to check when $q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \neq 0$. Assume $\tilde{\pi} \in \Delta_{n+1,i}$ We consider three cases.

If $i < j$, then: $q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) = 0$, since

$$0 \leqslant p(\pi)(1 - p_{i+1}^{(n+1)}) \leqslant p(\pi)\,(1 - p_j^{(n+1)}) = 0.$$

If $i = j$, then

$$q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) =$$

$$= p(\pi)p(\tilde{\pi})\,(1 - p_{j+1}^{(n+1)}) = \begin{cases} q^{(n+1)}(\tilde{\pi}), & \text{if } \pi = \tilde{\pi}\,; \\ 0, & \text{if } \pi \neq \tilde{\pi}. \end{cases}$$

If $i > j$, we will show that:

$$\pi \prec \tilde{\pi} \quad \text{and} \quad \tilde{\pi} \notin \Delta_{n,i} \ \text{ iff } \ q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \neq 0.$$

Assume that $q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \neq 0$. Clearly, $\pi \prec \tilde{\pi}$ since $p(\pi)p(\tilde{\pi}) \neq 0$. To see that $\tilde{\pi} \notin \Delta_n$, we argue by contradiction. If $\tilde{\pi} \in \Delta_n$, then $p(\tilde{\pi})\,(1 - p_i^{(n)}) = 0$, so that

$$q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \leqslant p(\pi)p(\tilde{\pi})(1 - p_{j+1}^{(n)})(1 - p_{i+1}^{(n+1)}) \leqslant$$

$$\leqslant p(\pi)p(\tilde{\pi})(1 - p_i^{(n)})(1 - p_{i+1}^{(n+1)}) = 0.$$

To establish the converse will involve the special construction of $(\Delta_n)_{n=0}^\infty$. Since $\pi \prec \tilde{\pi}$, there is a chain $\pi = \pi_j' \prec \pi_{j+1}' \prec \ldots \prec \pi_i' = \tilde{\pi}$, with $\pi_k' \in U(k)^\wedge$. By the construction of (Δ_n), each π_k' lies in $\Delta_{n+1,k}$. Since $\tilde{\pi} \notin \Delta_{n,i}$, either there is an index q such that $m_q(\tilde{\pi}) > K_n^+$ or $m_q(\tilde{\pi}) < K_n$. If $m_q(\tilde{\pi}) > K_n^+$, so is $m_1(\tilde{\pi})$. Define a new chain $\pi_k \in \Delta_{n+1,k} - \Delta_{n,k}$, $j+1 \leqslant k \leqslant i$, such that

$$m_1(\pi_k) = m_1(\tilde{\pi}), \quad m_p(\pi_k) = m_p(\pi_k'), \quad 2 \leqslant p \leqslant k.$$

Then $\pi \prec \pi_{j+1} \prec \ldots \prec \pi_i = \tilde{\pi}$. Choose $\tilde{\pi} \prec \pi' \in U(i+1)^\wedge$ but $\pi' \notin \Delta_{n+1,\, i+1}$. Then we have:

$$q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) = p(\pi)p(\tilde{\pi})(1 - p_{j+1}^{(n)})(1 - p_{i+1}^{(n+1)}) >$$

$$> p(\pi)p(\pi_{j+1}) \ldots p(\pi_i)(1 - p_{i+1}^{(n+1)}) >$$

$$> p(\pi)p(\pi_{j+1}) \ldots p(\pi_i)p(\pi') \neq 0,$$

since $p(\pi') \leqslant (1 - p_{i+1}^{(n+1)})$ and $p(\pi_{j+1}) \leqslant (1 - p_{j+1}^{(n)})$, by construction. In the situation where $m_q(\tilde{\pi}) < K_n$, define π_k by: $m_p(\pi_k) = m_p(\pi_k')$, $p \neq k$, $m_k(\pi_k) = m_k(\tilde{\pi})$. Then argue as before.

(b) We see that $[\pi; \tilde{\pi}]_\Delta = \mathrm{Tr}(q^{(n)}q^{(n+1)}(\tilde{\pi}))$, where $q^{(n)}$ is a minimal projection in $q^{(n)}C_n$. By [11, p. 65], $q^{(n+1)}(\tilde{\pi})C_{n+1}$ admits a family of orthogonal minimal projections of the form $p(\omega)q^{(n+1)}(\tilde{\pi})$, where

$$p(\omega) = p(\pi_1)p(\pi_2) \ldots p(\pi_n), \quad \omega\colon \pi_1 \prec \pi_2 \prec \ldots \prec \pi_n, \quad \pi_i \in U(i)^\wedge.$$

Now $\mathrm{Tr}(q^{(n)}q^{(n+1)}(\tilde{\pi}))$ = number of non-zero products $q^{(n)}p(\omega)q^{(n+1)}(\tilde{\pi})$. It follows that

iff $$q^{(n)}p(\omega)q^{(n+1)}(\tilde{\pi}) \neq 0$$

iff $$\pi_j = \pi, \quad \pi_i \notin \Delta_n, \; j+1 \leqslant i \leqslant k,$$

$$\pi \prec \pi_{j+1} \prec \pi_{j+2} \prec \ldots \prec \pi_{k-1} \prec \tilde{\pi}, \quad \pi_i \in \Delta_{n+1,\, i} - \Delta_{n,\, i}.$$

2.4. Definition. $\mathrm{Prim}(U(\infty))$ is the set of all $J \in \mathrm{Prim}(\mathfrak{A})$ such that J is the kernel of some irreducible representation of $U(\infty)$.

By [11, III.1.5], $J \in \mathrm{Prim}\,(U(\infty))$ is parametrized by a doubly-indexed signature $(U_j; L_j)_{j=1}^{\infty}$, where (U_j), resp. (L_j), is called the upper, resp. lower, signature. The entries of the signature must satisfy:

$$U_j \in \mathbf{Z} \cap \{+\infty\}, \quad L_j \in \mathbf{Z} \cup \{-\infty\},$$

$$U_j \geqslant U_{j+1} \geqslant L_{j+1} \geqslant L_j.$$

It is convenient to have more notation to describe J. We let:

$$r_J(= r) = \text{number of infinite entries in the upper signature;}$$

$$s_J(= s) = \text{number of infinite entries in the lower signature;}$$

$$U_\infty = \lim_{j\to\infty} U_j, \quad L_\infty = \lim_{j\to\infty} L_j.$$

Note that $0 \leqslant r_J,\, s_J \leqslant \infty$, and $-\infty \leqslant L_\infty \leqslant U_\infty \leqslant \infty$.

2.5. DEFINITION. If $J \in \mathrm{Prim}(U(\infty))$, we define the *index* of J to be the 4-tuple $(r_J, U_\infty; s_J, L_\infty)$.

2.6. We now describe $J \in \mathrm{Prim}(U(\infty))$ in terms of the central projections $q^{(n)}(\pi)$, $\pi \in \Delta_n$, which it contains. By [2, 3.3], this will uniquely characterize J; moreover, it will also give the Bratteli diagram of the quotient $\mathfrak{A}/J$.

CASE 1. $U_1 - L_1 < \infty$ (or $r_J = s_J = 0$).

Choose $K_n' \geqslant \max(n, |U_1|, |L_1|)$. Form $(\Delta_n;\ K_n^+, K_n)$ where $K_n^+ = K_n'$ and $K_n = -K_n'$. If $\pi \in \Delta_n$, we have $q^{(n)}(\pi) \notin J$ iff

$$\pi \in \Delta_{n,n};$$

$$U_i \geqslant m_i(\pi) \geqslant L_{n-i+1}, \quad 1 \leqslant i \leqslant n.$$

CASE 2. $0 < r_J + s_J$.

Choose $K_n^+ \geqslant n$, if $r_J = +\infty$, otherwise, $K_n^+ \geqslant \max(n, U_{r+1})$; and $K_n \leqslant -n$, if $s_J = \infty$, otherwise, $K_n \leqslant \min(-n, L_{s+1})$. For $\pi \in \Delta_{n,j}$, we have $q^{(n)}(\pi) \notin J$ iff

$$U_i \geqslant m_i(\pi),\ r+1 \leqslant i \leqslant j, \text{ holds when } r_J < +\infty,$$

and

$$m_{j-i+1}(\pi) \leqslant L_i,\ s+1 \leqslant i \leqslant j, \text{ holds when } s_J < \infty.$$

Note that $r_J = s_J = \infty$ implies $J = (0)$.

2.7. Fix $J \in \mathrm{Prim}(U(\infty))$ and consider $A = \mathfrak{A}/J$. If there is no danger of confusion, we denote the image of $q^{(n)}(\pi)$, $\pi \in \Delta_n$, in A by she same symbol, otherwise, we write $q^{(n)}(\pi, J)$. Then $A = \varinjlim A_n$ where $A_n = \mathfrak{A}_n/J \cap \mathfrak{A}_n$. By 2.6, we explicitly know the central decomposition of A_n; i.e.,

$$A_n = \oplus \{q^{(n)}(\pi)\mathfrak{A}_n/J \cap \mathfrak{A}_n;\ \pi \in \Delta_n(A)\},$$

$$\Delta_n(A) = \{\pi \in \Delta_n;\ q^{(n)}(\pi) \notin J\} \qquad (= \Delta_n(J), \text{ also}).$$

We note that when $U_1 - L_1 < +\infty$, then the partial multiplities of the imbeding of A_n into A_{n+1} are always either 0 or 1. We also observe that if $J_1, J_2 \in \mathrm{Prim}(U(\infty))$ have signatures whose entries all differ by a fixed finite constant, then $\mathfrak{A}/J_1 \cong \mathfrak{A}/J_2$. (This represents an algebraic analogue of multiplying a character of $U(n)$ by a power of the determinant.)

The remainder of the section is devoted to discussing the finite factor traces of $A(U(\infty))$ and the corresponding finite characters of $U(\infty)$, that is, the extremal normalized pozitive-definite class functions on $U(\infty)$.

2.8. Voiculescu [13] showed that the problem of determining all finite characters of $\mathfrak{G}$ on $U(\infty)$ is equivalent to characterizing all two-sided sequences $\{c_n\}_{n=-\infty}^{\infty}$ such that:

$$(2.8.1)\qquad \det\{c_{m_j+(i-i)}\}_{i,j=1}^{n} \geqslant 0, \quad m_1 \geqslant m_2 \geqslant \ldots \geqslant m_n; \quad \sum_{n=-\infty}^{+\infty} c_n = 1,$$

where c_n are the Fourier coefficients of $\mathfrak{G}|U(1) = f$; moreover, $\mathfrak{G}(V) = \det(f(V))$, $V \in U(\infty)$.

Let $\mathfrak{G}$ be any finite character. Consider $\mathfrak{G}_2 = \mathfrak{G}_1 \cdot \mathfrak{G}$, where $\mathfrak{G}_1(V) = \det(f_1(V))$, $f_1(z) = \exp(\varepsilon(z-1) + \varepsilon(z^{-1}-1))$, $\varepsilon > 0$. By [13, p. 8 and Proposition 2], $\mathfrak{G}_2$ is a finite character; moreover, $\mathfrak{G}_2$ is faithful by [11, III.1.5], [11, p. 18]. Let c_n be the Fourier expansion of $\mathfrak{G}_2|U(1)$. We will show that c_n is a totally positive sequence [10], [5], whose generating function is precisely given by [13, Proposition 2]. By [5, p. 367], it suffices to show that $A_m^{(n)} > 0$, for $m \in \mathbf{Z}$, $n \geqslant 0$, where $A_m^{(n)} = \det\{c_{m+(i-j)}\}_{i,j=0}^{n-1}$. Since $\mathfrak{G}_2$ is faithful and $A_m^{(n)}$ is just the transpose of (2.8.1) where $m_1 = \ldots = m_{n-1} = m$, $A_m^{(n)} > 0$. Because of the product structure of $f_2 = \mathfrak{G}_2|U(1)$, $\mathfrak{G}|U(1)$ must have the form of [13, Proposition 2]. Hence, we have the following:

THEOREM. (Edrei-Voiculescu). *Any finite character $\mathfrak{G}$ of $U(\infty)$ has the form:*

$$\mathfrak{G}(V) = \det(p(V));$$

where

$$p(z) = z^m \cdot \exp[\lambda(z-1) + \mu(z^{-1}-1)] \cdot$$

$$\cdot \prod_1^{\infty} p_{\mathrm{I}}(a_i\,;z) \prod_1^{\infty} p_{\mathrm{I}}(b_j\,;z^{-1}) \cdot$$

$$\cdot \prod_1^{\infty} p_{\mathrm{II}}(c_k\,;z) \prod_1^{\infty} p_{\mathrm{II}}(d_l\,;z^{-1}),$$

where $p_{\mathrm{I}}(a\,;z) = (1+az)/(1+a)$ and $p_{\mathrm{II}}(c\,;z) = (1-c)/(1-cz)$, $0 \leqslant \lambda, \mu, a_i, b_j < \infty$, $0 \leqslant c_k, d_l < 1$, $m \in \mathbf{Z}$, and $\sum (a_i + b_j + c_k + d_l) < \infty$.

REMARKS. The above representation of $p(z)$ is not unique; e.g., $z^{-1}p_{\mathrm{I}}(a;z) = p_{\mathrm{I}}(a^{-1};z)$. Also, note that $\{\rho^n\}_{n=-\infty}^{\infty}$, $\rho > 0$, is a totally positive sequence, but $\sum \rho^n = +\infty$.

2.9. PROPOSITION. *There is a bijective correspondence between the finite factor traces of $\mathfrak{A} = A(U(\infty))$, not supported on a single $U(n)$, and the finite characters of $U(\infty)$.*

Proof. If $\mathfrak{G}$ is a finite positive-definite function on $U(\infty)$, then $\mathfrak{G}|U(n) = \sum c_\pi \cdot \chi_\pi$, where χ_π is the character of $\pi \in U(n)^\wedge$. There is a unique trace t_n' on $M(U(n))$ such that $t_n'(P(\pi)) = c_\pi \cdot \operatorname{rank}(\pi)$. If $t_n = t_n'|L_{(n)}$, then the family (t_n)

induces a unique trace t on $\mathfrak{A}$. Conversely, let t be a finite trace on $\mathfrak{A}$, not supported on a single $U(n)$. Then t induces a consistent family of finite positive-definite functions $\mathfrak{G}_n$ on $U(n)$, where $\mathfrak{G}_n = \sum c_\pi \cdot \chi_\pi$, where $c_\pi = t(P(\pi))/\text{rank}(\pi)$, $\pi \in U(n)^\wedge$. Because the above two correspondences are inverses of one another, we have a bijection between the traces of $\mathfrak{A}$ and central positive-definite functions; moreover, this map preserves convex combinations. Hence, factor traces correspond to the finite characters.

By considering tensor products of the (corrected versions of the) concrete models of the finite traces given by [13, pp. 16—19], we can compute the primitive ideal which is the kernel of the finite representation corresponding to a given factor by trace [11, III.1.5].

2.10. THEOREM. *Let $J \in \text{Prim}(U(\infty))$. Then $\mathfrak{A}/J$ admits a faithful finite factor trace iff $U_{r+1} = U_\infty$ and $L_{s+1} = L_\infty$. In other words, $\mathfrak{A}/J$ is a finite C^*-algebra in the sense of Cuntz-Pedersen* [3] *iff all finite entries in the upper, resp. lower, signature of J are equal. If $U_1 = L_\infty$, then J is a maximal ideal and $\mathfrak{A}/J = \mathbf{C}$; otherwise, $\mathfrak{A}/J$ is antiliminal.*

Again, by consideration of the (corrected versions of the) concrete models of [13], we obtain the following:

2.11. PROPOSITION. *Let t be a finite factor trace of $\mathfrak{A}$ with generating function p, given by Theorem* 2.8. *Then if π is the finite representation of $\mathfrak{A}$ which corresponds to t, the signature $(U_j\,; L_j)$ of $J = \ker(\pi)$ is determined as follows:*

(i) *if $\lambda \neq 0$, resp. $\mu \neq 0$, let $U_j^{(1)} = +\infty$, resp. $L_j^{(1)} = -\infty$, $1 \leqslant j \leqslant \infty$; and* 0 *otherwise;*

(ii) *if $r = \#$ non-zero c's, resp. $s = \#$ non-zero d's, let $U_j^{(2)} = +\infty$, $1 \leqslant j \leqslant r$, resp. $L_j^{(2)} = -\infty$, $1 \leqslant j \leqslant s$, and* 0 *otherwise;*

(iii) *let $U_j^{(3)} = \#$ non-zero a's and $L_j^{(3)} = \#$ non-zero b's; then $U_j = U_j^{(1)} + U_j^{(2)} + U_j^{(3)} + m$ and $L_j = L_j^{(1)} + L_j^{(2)} + L_j^{(3)} + m$.*

REMARK. Conversely, to write down the generating functions for the traces of with given kernel J is elementary; however expression is somewhat awkward, so it is omitted.

3. INFINITE CHARACTERS OF $U(\infty)$

The main purpose of this section is to develop the character theory of the primitive quotients $A = \mathfrak{A}/J$, $J \in \text{Prim}(U(\infty))$, which do not admit faithful finite characters. Surprisingly, the classification of the faithful characters of such quotients can be effectively reduced to the study of the finite characters given in Section 2. The main tool in this reduction is the existence of an ideal B of A which is stably isomorphic to an ideal of a finite primitive quotient A_f. This allows a bridge to be built between the faithful character theory of A and A_f. This ideal B also possesses another special property: it is the norm-closure of the ideal of "trace-class" elements for any faithful trace of A. When the primitive quotient A is type I, the situation is even simpler since B is isomorphic $\mathscr{K}$, the algebra of compact operators. We close this section by showing that any normal representation π is a subrepresentation of a tensor product of a traceable irreducible representation with a finite factor representation. By making use of the explicit construction for the traceable irreducible representations given by Kirillov [7] and the (corrected versions of the) construction for the finite normal representations given by Voiculescu [13], we obtain a concrete construction for any normal representation.

Recall [1, 4.2] that we called $J \in \mathrm{Prim}(U(\infty))$ *elementary* if its signature contains only finitely many non-zero entries none of which is infinite; moreover, we showed that $\mathfrak{A}/J$ is type I. In fact, more is true.

3.1. THEOREM. *Let $J \in \mathrm{Prim}(U(\infty))$ be elementary. Then $A = \mathfrak{A}/J$ is a C^*-algebra with finite dual; in particular, A is type* I.

Proof. According to 2.7, $A = \varinjlim A_n$, where

$$A_n = \oplus \{M(\pi)\,;\pi \in U(n)^\wedge,\ U_i \geqslant m_i(\pi) \geqslant L_{n-i+1}\}$$

where $(U_j; L_j)$ is the signature of J. We set a_J, respectively, b_J, to be the first index such that U_{a_J+1}, resp. L_{b_J+1}, $= 0$. Define $B_n = M(\pi_n)$, $n > a_J + b_J$, where

(i) $m_i(\pi_n) = U_i$, $1 \leqslant i \leqslant a_J$ (omitted if $a_J = 0$);

(ii) $m_{n-i+1}(\pi_n) = L_i$, $1 \leqslant i \leqslant b_J$ (omitted if $b_J = 0$);

(iii) $m_i(\pi_n) = 0$, otherwise.

Now $q^{(n)}(\pi_n)q^{(n+1)}(\pi') \neq 0$ iff $\pi' = \pi_{n+1}, \pi' \in \Delta_{n+1}(A)$, by the branching rules. By [2, 3.3], $B = (\cup B_n)^-$ is a closed ideal of A such that $B \cong \mathscr{K}$, the algebra of compact operators on a separable infinite-dimensional Hilbert space.

We can directly verify that if $a_J = 1$, $b_J = 0$ or $a_J = 0$, $b_J = 1$, then $A = \mathscr{K} + \mathbf{C}\cdot I$. We shall prove the theorem by induction on the sum $\sum(U_j - L_j) = k$, called the height of J. Without loss of generality, we shall assume that $a_J > 0$. If $k = 1$, then A certainly has finite dual. Assume this is true for all A such that $\sum(U_j - L_j) = k$. Now consider J with height $k + 1$. We will examine $A/B = D$, where $D = (\cup D_n)^-$ and

$$D_n = \oplus\{M(\pi)\,;\ \pi \in U(n)^\wedge,\ U_i \geqslant m_i(\pi) \geqslant L_{n-i+1},\ \pi \neq \pi_n\} =$$

$$= \oplus\{M(\pi)\,;\ \pi \in U(n)^\wedge,\ U_i' \geqslant m_i(\pi) \geqslant L_{n-i+1}'\} = \oplus M(\pi_n') \oplus M(\pi_n''),$$

where

$$U'_i = U_i, \quad i \neq a_J, \quad U'_{a_J} = U_{a_J} - 1;$$

$$L'_i = L_i, \quad i \neq b_J, \quad L'_{b_J} = L_{b_J} + 1;$$

$$m_i(\pi'_n) = m_i(\pi_n), \quad i \neq a_J, \quad m_a(\pi'_n) = U'_{a_J};$$

$$m_{n-i+1}(\pi''_n) = m_{n-i+1}(\pi_n), \quad i \neq b_J, \quad m_{n-b_J+1}(\pi'') = L'_{b_J}.$$

If $b_J = 0$, then $D \cong \mathfrak{A}/J'$, where $J' \in \mathrm{Prim}(U(\infty))$ has signature $(U'_j; 0)$. If $b_J > 0$, then we shall show that there is an ideal E of D such that $E \cong \mathscr{K}$ and $D/E = \mathfrak{A}/J''$, where $J'' \in \mathrm{Prim}(U(\infty))$ has height $< k+1$. Let $E_n = M(\pi'_n)$, then arguing just as for B, $E = (\cup E_n)^- \cong \mathscr{K}$. Then $D/E = \mathfrak{A}/J''$, where J'' has signature $(U'_j; L'_j)$. Hence A has finite dual since $(A/B)/E$ has finite dual and $B \cong \mathscr{K}$ and $E \cong \mathscr{K}$.

3.2. THEOREM. *Let $J \in \mathrm{Prim}(U(\infty))$ be elementary. Then $A = \mathfrak{A}/J$ contains an ideal $B \cong \mathscr{K}$, which satisfies condition* 1.10(†). *Hence, if t is any faithful trace on A, $(m_t)^- = B$.*

Proof. We retain the notation of the proof of 3.1, except for π'_n, π''_n. Let $\{a_1, \ldots, a_J\}$ be the set of jump indices for the upper signature; i.e., $U_1 = \ldots = U_{a_1}$, $U_{a_1+1} \neq U_{a_1}$, $U_{a_1+1} = \ldots = U_{a_2}$, $U_{a_2} \neq U_{a_2+1}$, etc. Similarly, define jump indices for the lower signature. We now exhibit a collection of projections in A that satisfies condition 1.10(†). Fix a jump index a in $\{a_1, \ldots, a_J\}$. Define $\pi'_n \in \Delta_n(A)$, by $m_i(\pi'_n) = m_i(\pi_n)$, $i \neq a$, $m_a(\pi'_n) = U_a - 1$. By the branching rules, for $\pi \in \Delta_{n+1}(A)$,

$$q^{(n)}(\pi'_n) q^{(n+1)}(\pi) \neq 0 \quad \text{iff} \quad \pi = \pi_{n+1} \text{ or } \pi'_{n+1}.$$

Fix a minimal projection $q^{(n)}$ in $q^{(n)}(\pi'_n)C_n$. We let $\varphi_{n,n+1}$ denote the imbedding of A_n into A_{n+1}. Then we have:

$$\varphi_{n,n+1}(q^{(n)}) = q^{(n+1)}(\pi_{n+1})q^{(n)} + q^{(n+1)}(\pi'_{n+1})\, q^{(n)} = q_{\#}^{(n+1)} + q^{(n+1)},$$

where $q_{\#}^{(n+1)}$ and $q^{(n+1)}$ are minimal projections in $q^{(n+1)}(\pi_{n+1})C_{n+1}$, $q^{(n+1)}(\pi'_{n+1})C_{n+1}$, resp., since the imbedding of A_n into A_{n+1} has partial multiplicity at most one. Next we calculate $\varphi_{n,n+2}(q^{(n)})$, so

$$\varphi_{n,n+2}(q^{(n)}) = \varphi_{n+1,n+2}(q_{\#}^{(n+1)}) + \varphi_{n+1,n+2}(q^{(n+1)}).$$

Now $\varphi_{n+1,n+2}(q_{\#}^{(n+1)}) = q_{\#}^{(n+2)}$ is just a minimal projection in $B_{n+2} \cap C_{n+2}$, while $\varphi_{n+1,n+2}(q^{(n+1)})$ decomposes, similarly to $\varphi_{n,n+1}(q^{(n)})$, as $q^{(n+2)}(\pi_{n+2})\, q^{(n+1} + q^{(n+2)}$, which are minimal projections in $B_{n+2} \cap C_{n+2}$ and $q^{(n+2)}(\pi'_{n+2})C_{n+2}$, resp. Iterating this process, we obtain:

$$\varphi_{n,n+k}(q^{(n)}) = \sum_{i=1}^{k} q^{(n+k)}(\pi_{n+k}) \ldots q^{(n+i)}(\pi_{n+i})q^{(n+i-1)} + q^{(n+k)}$$

where $q^{(n+i-1)} \in q^{(n+i-1)}(\pi'_{n+i-1})C_{n+i-1}$, $q^{(n+k)} \in q^{(n+k)}(\pi_{n+k})C_{n+k}$ are minimal projections. Moreover, the terms in the above sum form an orthogonal family of minimal projections in B_{n+k}. Hence, any minimal projection $q^{(n)} \in q^{(n)}(\pi'_n)C_n$ satisfies 1.10(†). We may argue similarly for a minimal projection associated to the representation π''_n where $m_{n-i+1}(\pi''_n) = m_{n-i+1}(\pi_n)$, $i \neq b$, where b is some fixed jump index for the lower signature. Let Λ_n denote the set of all $\pi \in \Delta_n$ obtained from π_n by altering a jump index as above.

To establish 1.10(†) for any minimal projection $q \in C_n$, $q \notin B$, $q \in q^{(n)}(\tilde{\pi})C_n$, it suffices to show that there is an index k and $\pi \in \Lambda_{n+k}$ such that $\varphi_{n,n+k}(q) \cdot q^{(n+k)}(\pi) \neq 0$. For the sake of simplicity, we suppose there is a jump index a for the upper signature such that $m_a(\tilde{\pi}) < U_a$. Then take for π:

(i) $m_i(\pi) = U_i$, $1 \leqslant i \leqslant a_j$, $i \neq a$;

(ii) $m_a(\pi) = U_a - 1$;

(iii) $m_{2n+a_j+b_j-i}(\pi) = L_i$, $1 \leqslant i \leqslant b_j$,

(iv) all other entries are zero.

By the generalized branching law, $\tilde{\pi} \prec \pi$.

We next investigate $J \in \mathrm{Prim}(U(\infty))$ such that $U_{r+1} \neq U_\infty$ or $L_{s+1} \neq L_\infty$ and $-U_\infty$, $L_\infty \neq \infty$. Remarkably, there is a close analogue of Theorem 3.2 where the ideal B is stably isomorphic to an ideal of a finite primitive quotient of $\mathfrak{A}$. We were led to this theorem by a careful study of [11, V.2].

We need to introduce some notation. Let $J \in \mathrm{Prim}(U(\infty))$ have signature as above. Then we associate to J an ideal $J_f \in \mathrm{Prim}(U(\infty))$ such that $\mathfrak{A}/J_f$ admits a faithful finite factor trace. It suffices to specify J_f by its index (2.5, 2.10). J_f is characterized by having the same index as J itself.

3.3. THEOREM. *Let $J \in \mathrm{Prim}(U(\infty))$ such that either $U_{r+1} \neq U_\infty$ or $L_{s+1} \neq L_\infty$ and $-U_\infty$, $L_\infty \neq \infty$. Then $A = \mathfrak{A}/J$ contains an ideal B such that*

(1) *B is stably isomorphic to an ideal B_f of $A_f = \mathfrak{A}/J_f$;*

(2) *if t is a faithful semifinite lower-semicontinuous trace on A, then $(m_t)^- \subset B$.*

Proof. In the argument, two cases will be distinguished, viz., $U_1 - L_1 \prec +\infty$ and $U_1 - L_1 = +\infty$. We define a_j, resp. b_j, to be:

$$\inf\{j\,;\, U_{j+1} = U_\infty\}, \quad \text{resp.} \quad \inf\{j\,;\, L_{j+1} = L_\infty\}.$$

We make the convention that if $r_J = a_J$, resp. $s_J = b_J$, then any constraints on the upper, resp. lower, signature given below should be omitted. Similarly, if r_J, resp. s_J, is infinite, any constraint on the upper, resp. lower, signature is assumed to be omitted.

(a) Construction of B when $U_1 - L_1 < +\infty$.

If $n \geqslant a_J + b_J = c$, let

$$\Delta_n(B) = \{\pi \in U(n)^\wedge \,;\, m_i(\pi) = U_i,\ 1 \leqslant i \leqslant a_J,\ m_{n-i+1}(\pi) = L_i,\ 1 \leqslant i \leqslant b_J\},$$

and let $B_n = \oplus\{M(\pi)\,;\, \pi \in \Delta_n(B)\}$. Define $B = \left(\bigcup_{n=c}^{\infty} B_n\right)^-$. To show that B is an ideal, it suffices to check, for $\pi \in \Delta_n(B)$, $\tilde{\pi} \in \Delta_{n+1}(A)$,

$$q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \neq 0 \quad \text{iff} \quad \tilde{\pi} \in \Delta_{n+1}(B).$$

Since $[\pi; \pi]_\Delta \leqslant 1$, it is enough to show that $\pi \prec \tilde{\pi}$ iff $\tilde{\pi} \in \Delta_{n+1}(B)$. This is seen by the branching law, since

$$U_i \geqslant m_i(\tilde{\pi}) \geqslant m_i(\pi), \quad 1 \leqslant i \leqslant n;$$

$$L_i \leqslant m_{n-i+2}(\tilde{\pi}) \leqslant m_{n-i+1}(\pi), \quad 1 \leqslant i \leqslant n.$$

(b) Construction of B_f when $U_1 - L_1 < +\infty$.

Define $\Delta_n(B_f) \subset \Delta_n(A_f)$, $n \geqslant a + b$, by:

$$\pi \in \Delta_n(B_f) \quad \text{iff} \quad m_i(\pi) = U_{a+1},\ 1 \leqslant i \leqslant a \text{ and } m_{n-i+1}(\pi) = L_{b+1},\ 1 \leqslant i \leqslant b.$$

Then $B_f = (\cup\{M(\pi)\,;\,\pi \in \Delta_n(B_f)\})^-$ forms an ideal of A_f. We now give a map $\pi \mapsto \pi_f$ from $\Delta_n(B)$ onto $\Delta_n(B_f)$, where π_f is determined by:

$$m_i(\pi_f) = \begin{cases} U_{a+1}, & 1 \leqslant i \leqslant a, \\ m_i(\pi), & a+1 \leqslant i \leqslant n-b, \\ L_{b+1}, & n-b+1 \leqslant i \leqslant n. \end{cases}$$

By an easy application of the branching law, $\pi \prec \tilde{\pi}$ iff $\pi_f \prec \tilde{\pi}_f$, where $\pi \in \Delta_n(B_f)$, $\tilde{\pi} \in \Delta_{n+1}(B_f)$. Because the imbeddings in the generating nests for both B and B_f have partial multiplicity at most one, the Bratteli diagrams agree. Hence, B is stably isomorphic to B_f.

(c) Construction of B when $U_1 - L_1 = +\infty$.

We define $\Delta_{n,j}(B) \subset \Delta_{n,j}(A)$, $c \leqslant j \leqslant n$, by $\pi \in \Delta_{n,j}(B)$ if $m_i(\pi) = U_i$, $r + 1 \leqslant$ $\leqslant i \leqslant a_J$, and $m_{j-i+1}(\pi) = L_i$, $s + 1 \leqslant i \leqslant b_J$. Set $\Delta_n(B) = \cup\{\Delta_{n,j}(B)\,;\, c \leqslant j \leqslant n\}$. Define $B_n = \oplus\{M(\pi)\,;\,\pi \in \Delta_n(B)\}$, $B = \left(\bigcup_{n=c}^{\infty} B_n\right)^-$. B is an ideal if given $\pi \in \Delta_{n,j}(B)$,

$\tilde{\pi} \in \Delta_{n+1,k}(A)$ with $q^{(n)}(\pi)q^{(n+1)}(\tilde{\pi}) \neq 0$, then $\tilde{\pi} \in \Delta_{n+1,k}(B)$. By Lemma 2.4, $\pi \prec \tilde{\pi}$, so we have:

$$U_i \geqslant m_i(\tilde{\pi}) \geqslant m_i(\pi) = U_i, \quad r+1 \leqslant i \leqslant a_J;$$
$$L_i \leqslant m_{k-i+1}(\tilde{\pi}) \leqslant m_{j-i+1}(\pi) = L_i, \quad s+1 \leqslant i \leqslant b_J.$$

Hence, $\tilde{\pi} \in \Delta_{n+1}(B)$.

(d) Construction of B_f when $U_1 - L_1 = +\infty$.

We define $\pi \in \Delta_{n,j}(B_f) \subset \Delta_{n,j}(A_f)$, $c \leqslant j \leqslant n$, when:

(i) $m_i(\pi) \geqslant U_{r+1}$, $1 \leqslant i \leqslant r$; $m_{j-i+1}(\pi) \leqslant L_{s+1}$, $1 \leqslant i \leqslant s$;

(ii) $m_i(\pi) = U_{a+1}$, $r+1 \leqslant i \leqslant a$; $m_{j-i+1}(\pi) = L_{b+1}$, $s+1 \leqslant i \leqslant b$;

(iii) $\pi \in \Delta_{n,j}(A_f)$.

It is straightforward to verify that $B_f = (\cup \{M(\pi)\,;\, \pi \in \Delta_{n,j}(B_f)\})^-$ forms an ideal of A_f.

We now establish the stable isomorphism of B and B_f. It suffices to establish the equivalence of the Bratteli diagrams of B and B_f. Define a map $\pi \mapsto \pi_f$, from $\Delta_{n,j}(B)$ onto $\Delta_{n,j}(B_f)$ by:

(iv) $m_i(\pi_f) = m_i(\pi)$, $1 \leqslant i \leqslant r$; $a+1 \leqslant i \leqslant j-b$; $j-s+1 \leqslant i \leqslant j$;

(v) $m_i(\pi_f) = U_{a+1}$, $r+1 \leqslant i \leqslant a$;

(vi) $m_i(\pi_f) = L_{b+1}$, $j-b+1 \leqslant i \leqslant j-s$,

where $\mathfrak{A}/J$ and $\mathfrak{A}/J_f$ are formed with the same generating set $(\Delta_n\,;\, K_n,\, K_n^+)$ satisfying 2.8. This mapping establishes a bijection. By the extended branching law, we have, for $\pi \in \Delta_{n,j}(B)$ and $\tilde{\pi} \in \Delta_{n+1,k}(B)$,

(vii) $\pi \prec \tilde{\pi}$ iff $\pi_f \prec \tilde{\pi}_f$;

(viii) $\tilde{\pi}_f \in \Delta_{n,k}(A_f)$ iff $\tilde{\pi} \in \Delta_{n,k}(B)$.

These equivalences establish that the Bratteli diagrams of B and A_f agree up to partial multiplicities. In other words, it remains to show that $[\pi;\, \tilde{\pi}]_\Delta = [\pi_f;\, \tilde{\pi}_f]_\Delta$, where $\pi \in \Delta_{n,j}(B)$, $\tilde{\pi} \in \Delta_{n+1,k}(B)$, $c \leqslant j \leqslant k$. By Lemma 2.4 (b), $[\pi;\, \tilde{\pi}]_\Delta =$ number of distinct tuples $(\pi_{j+1}, \ldots, \pi_{k-1})$, where $\pi \prec \pi_{j+1} \prec \ldots \prec \pi_{k-1} \prec \tilde{\pi}$, $\pi_i \in \Delta_{n+1,i} - \Delta_{n,i}$, $j+1 \leqslant i \leqslant k-1$. $[\pi_f;\, \tilde{\pi}_f]_\Delta$ is analogously defined. But, by (vii) and (viii), the map $\pi \mapsto \pi_f$ induces a bijection between the two collection of such tuples.

We have now verified part (1) of the theorem. In order to establish part (2), we shall show that the ideal B satisfies condition (†) of Theorem 1.10. This will be somewhat lengthy. For the remainder of the argument, we shall join the cases $U_1 - L_1 < +\infty$ and $U_1 - L_1 = +\infty$ more closely together.

(e) Let $\{a_1, \ldots, a_J\}$ denote the jump indices for the upper signature where $a_{r+1} \leqslant a_i \leqslant a_J$. Here the jump indices are defined as in the proof of Theorem 3.2. For a fixed jump index a, define $\Lambda_n \subset \Delta_n(A)$ as follows: $\pi \in \Lambda_n$ if

(i) $\pi \in \Delta_{n,j}(A)$, $c \leqslant j \leqslant n$;

(ii) $m_i(\pi) = U_i$, $r+1 \leqslant i \leqslant a_J$, $i \neq a$;

(iii) $m_a(\pi) = U_a - 1$;

(iv) $m_{j-i+1}(\pi) = L_i, \quad s+1 \leqslant i \leqslant b_J$.

Now fix $\pi \in \Lambda_n$ with $\pi \in \Delta_{n,m}(A)$ and choose a minimal projection q from $q^{(n)}(\pi)C_n$. We will establish that q satisfies 1.10(†). To do this will require preparatory material in (f), (g), (h), and (i).

(f₁) given $\tilde{\pi} \in \Delta_{n+1,k}$, $\pi \prec \tilde{\pi}$, then either $\tilde{\pi} \in \Delta_{n+1}(B)$ or Λ_{n+1};

(f₂) if $\tilde{\pi} \in \Lambda_{n+1}$, then $\pi \prec \tilde{\pi}$ iff $\pi \prec \tilde{\pi}^T$, where $\tilde{\pi}^T \in \Delta_{n+1}(B)$ is determined by: $m_i(\tilde{\pi}^T) = m_i(\tilde{\pi})$, $i \neq a$, $m_a(\tilde{\pi}^T) = U_a$;

(f₃) $[\pi; \tilde{\pi}]_\Delta = [\pi; \tilde{\pi}^T]_\Delta$.

To see (1), observe that $\pi \prec \tilde{\pi}$ implies $m_{a-1}(\pi) \geqslant m_a(\tilde{\pi}) \geqslant m_a(\pi) \geqslant m_{a+1}(\tilde{\pi})$. Since $m_i(\pi) \leqslant U_i$, $m_a(\tilde{\pi}) = U_a$ or $U_a - 1$. If $r+1 \leqslant i \leqslant a_J$, $i \neq a$, then $U_i \geqslant m_i(\tilde{\pi}) \geqslant m_i(\pi) = U_i$, so $m_i(\tilde{\pi}) = U_i$. A similar argument gives $m_{k-i+1}(\tilde{\pi}) = L_i$, $s+1 \leqslant i \leqslant b_J$.

To prove (2), recall that $\pi \prec \tilde{\pi}$ iff $m_i(\tilde{\pi}) \geqslant m_i(\pi)$ and $m_{m-i+1}(\pi) \geqslant m_{k-i+1}(\tilde{\pi}^T)$, $1 \leqslant i \leqslant m$. By the definition of $\tilde{\pi}^T$, (2) holds if the inequalities $m_{a-1}(\pi) \geqslant m_a(\tilde{\pi})$ and $m_a(\tilde{\pi}^T) \geqslant m_a(\pi)$ are automatic. Since $m_i(\pi) = U_i$, $r+1 \leqslant i \leqslant a_J$, $i \neq a$, $m_a(\pi) = U_a - 1$, this is indeed the case.

It remains to show that $[\pi; \tilde{\pi}]_\Delta = [\pi; \tilde{\pi}^T]_\Delta$. Now $[\pi; \tilde{\pi}]_\Delta$ = number of tuples $(\pi_{m+1}, \ldots, \pi_{k-1})$ such that $\pi \prec \pi_{m+1} \prec \ldots \prec \pi_{k-1} \prec \tilde{\pi}$, $\pi_i \in \Delta_{n+1,i} - \Delta_{n,i}$, $m+1 \leqslant i \leqslant k-1$. Since $\pi \in \Lambda_n$ and $\tilde{\pi} \in \Lambda_{n+1}$, $\pi_i \in \Lambda_{n+1}$ as well. By (2), $[\pi; \tilde{\pi}]_\Lambda \leqslant [\pi; \pi^T]_\Delta$. A similar consideration yields $[\pi; \tilde{\pi}^T]_\Delta \leqslant [\pi; \tilde{\pi}]_\Delta$.

(g) We now wish to formalize (f₁) and (f₂). We observe that (f₂) induces a correspondence between projections of the form $q^{(n+1)}(\tilde{\pi})q$ in $q^{(n+1)}(\tilde{\pi})C_{n+1}$, $\pi \prec \tilde{\pi} \in \Lambda_{n+1}$, to the projections $q^{(n+1)}(\tilde{\pi}^T)q$ in $q^{(n+1)}(\tilde{\pi}^T)C_{n+1}$. If $U_1 - L_1 < +\infty$, note that these projections are minimal. This correspondence, in turn, induces a vector space isomorphism:

$$T_{n+1}: X_{n+1} \to Y_{n+1},$$

where X_{n+1}, resp. Y_{n+1}, is generated by finite linear combinations of $q^{(n+1)}(\tilde{\pi})q$, resp. $q^{(n+1)}(\tilde{\pi}^T)q$, where $\pi \prec \tilde{\pi} \in \Lambda_{n+1}$. More generally, we define an isomorphism T_{n+i}, $i \geqslant 1$,

$$T_{n+i}: X_{n+i} \to Y_{n+i},$$

where X_{n+i}, resp. Y_{n+i}, is generated by the products $q^{(n+i)}(\tilde{\pi}_i) \ldots q^{(n+1)}(\tilde{\pi}_1)q$, resp. $q^{(n+i)}(\tilde{\pi}_i^T)q^{(n+i-1)}(\tilde{\pi}_{i-1}) \ldots q^{(n+1)}(\tilde{\pi}_1)q$, where $\tilde{\pi}_j \in \Lambda_{n+j}$, $\pi \prec \tilde{\pi}_1 \prec \ldots \prec \tilde{\pi}_i$.

We now study some properties of the map T.

(h) Let $x \in X_{n+i}$, $i \geqslant 1$, then

(1) $e_B^{(n+i+1)} \cdot \varphi_{n+i,n+i+1}(x) \sim_u \varphi_{n+i,n+i+1}(T_{n+i}x)$;

(2) $e_B^{(n+i+1)} \cdot \varphi_{n+i,n+i+1}(x) = T_{n+i+1}((1 - e_B^{(n+i+1)}) \cdot \varphi_{n+i,n+i+1}(x))$.

where $e_B^{(k)} = \sum\{q^{(k)}(\pi') ; \pi' \in \Delta_k(B)\}$. It suffices to verify (1) and (2) when x has the form

$$q^{(n+i)}(\tilde{\pi}_i) \ldots q^{(n+1)}(\tilde{\pi}_1)q, \quad \pi \prec \tilde{\pi}_1 \prec \ldots \prec \tilde{\pi}_i, \quad \tilde{\pi}_j \in \Lambda_{n+j}.$$

Consider:

$$e_B^{(n+i+1)} \cdot \varphi_{n+i,n+i+1}(x) =$$

$$(*) \qquad = e_B^{(n+i+1)} \cdot \sum\{q^{(n+i+1)}(\pi')x ; \pi' \in \Delta_{n+i+1}(A)\} =$$

$$= \sum\{q^{(n+i+1)}(\pi')x ; \pi' \in \Delta_{n+i+1}(B)\}.$$

On the other hand, we see

$$\varphi_{n+i,n+i+1}(T_{n+i}x) =$$

$$= \sum\{q^{(n+i+1)}(\pi')q^{(n+i)}(\tilde{\pi}_i^T)q^{(n+i-1)}(\tilde{\pi}_{i-1}) \ldots$$

$$\ldots q^{(n+i)}(\tilde{\pi}_1)q ; \quad \pi' \in \Delta_{n+i+1}(B)\}.$$

By (f_2), $q^{(n+i+1)}(\pi')x \neq 0$ iff $q^{(n+i+1)}(\pi')T_{n+i}(x) \neq 0$, $\pi' \in \Delta_{n+i+1}(B)$. Since the terms in the above two sums are mutually orthogonal, it suffices to show that the individual terms are equivalent. If $U_1 - L_1 < +\infty$, this is immediate since the terms are minimal projections. More generally, we need to determine:

$$\mathrm{Tr}(q^{(n+i+1)}(\pi')q^{(n+i)}(\tilde{\pi}_i) \ldots q^{(n+1)}(\tilde{\pi}_1)q).$$

Since q is minimal, the value of this trace is

$$[\pi; \tilde{\pi}_1]_\Delta \ldots [\tilde{\pi}_{i-1}; \tilde{\pi}_i]_\Delta[\tilde{\pi}_i; \pi']_\Delta.$$

Note that this agrees with

$$\mathrm{Tr}(q^{(n+i+1)}(\pi')q^{(n+i)}(\tilde{\pi}_i^T)q^{(n+i-1)}(\tilde{\pi}_{i-1}) \ldots q^{(n+1)}(\tilde{\pi}_1)q) =$$

$$= [\pi; \tilde{\pi}_1]_\Delta \ldots [\tilde{\pi}_{i-1}; \pi_i^T]_\Delta[\pi_i^T; \pi']_\Delta,$$

by (f_3).

To verify (2), consider

$$(1 - e_B^{(n+i+1)}) \cdot \varphi_{n+i,n+i+1}(x) =$$

$$(**) \qquad = \sum\{q^{(n+i+1)}(\pi')x ; \pi' \in \Lambda_{n+i+1}\}.$$

By (f_2), the terms of ($**$) coincide under T_{n+i+1} with those of ($*$).

(i) We now establish some notation. Fix $\tilde{\pi} \in \Lambda_{n+1}$ with $\pi \prec \tilde{\pi}$ and set $q^{(n+1)} = q^{(n+1)}(\tilde{\pi})q$. We define: $f_1 = q^{(n+1)}$, $e_1 = T_{n+2}q^{(n+1)}$,

$$f_i = (1 - e_B^{(n+i)})\varphi_{n+i-1,n+i}(f_{i-1}), \quad i \geqslant 2;$$

$$e_i = e_B^{(n+i)}\varphi_{n+i-1,n+i}(f_{i-1}), \quad i \geqslant 2.$$

Note that $f_i \in Y_{n+i}$, $e_i \in X_{n+i}$. We now prove the crucial decomposition:

$$(*) \qquad \varphi_{n+1,n+k}(q^{(n+1)}) = \sum_{i=2}^{k} \varphi_{n+i,n+k}(e_i) + f_k, \quad k \geqslant 2,$$

where $e_1 \sim_u e_2 \sim_u \dots \sim_u e_k$.

If $k = 2$, (*) becomes:

$$\begin{aligned}
&\varphi_{n+1,n+2}(q^{(n+1)}) = \\
&= e_B^{(n+2)}\varphi_{n+1,n+2}(q^{(n+1)}) + (1 - e_B^{(n+2)})\varphi_{n+1,n+2}(q^{(n+1)}) = \\
&= e_B^{(n+2)}\varphi_{n+1,n+2}(f_1) + (1 - e_B^{(n+2)})\varphi_{n+1,n+2}(f_1) = e_2 + f_2.
\end{aligned}$$

By (h), $e_B^{(n+2)} \cdot \varphi_{n+1,n+2}(f_1) \sim_u \varphi_{n+1,n+2}(T_{n+1}f_1)$, so that $e_2 \sim_u e_1$. Arguing by induction, we assume (*) holds for $k - 1$, $k \geqslant 3$. Consider:

$$\begin{aligned}
&\varphi_{n+1,n+k}(q^{(n+1)}) = \\
&= \varphi_{n+k-1,n+k}\left(\sum_{i=2}^{k-1} \varphi_{n+i,n+k-1}(e_i) + f_{k-1}\right) = \\
&= \sum_{i=2}^{k-1} \varphi_{n+i,n+k}(e_i) + \varphi_{n+k-1,n+k}(f_{k-1}) = \\
&= \sum_{i=2}^{k-1} \varphi_{n+i,n+k}(e_i) + e_k + f_k.
\end{aligned}$$

We have: $T_{n+i}f_i = e_i$, since

$$\begin{aligned}
T_{n+i}f_i &= T_{n+i}((1 - e_B^{(n+i)})\varphi_{n+i-1,n+i}(f_{i-1})) = \\
&= e_B^{(n+i)}\varphi_{n+i-1,n+i}(f_{i-1}) = e_i,
\end{aligned}$$

by (h). On the other hand, we see that:

$$\begin{aligned}
e_k &= e_B^{(n+k)}\varphi_{n+k-1,n+k}(f_{k-1}) \sim_u \\
&\sim_u \varphi_{n+k-1,n+k}(T_{n+k-1}f_{k-1}) = \\
&= \varphi_{n+k-1,n+k}(e_{k-1}).
\end{aligned}$$

by (h). Hence, (*) is now proven.

We now summarize what we have shown: $\pi \in \Lambda_n$ was chosen together with a minimal projection q from $q^{(n)}(\pi)A_n$. Next we picked $\tilde{\pi} \in \Lambda_{n+1}$, $\pi \prec \tilde{\pi}$, and formed $q^{(n+1)} = q^{(n+1)}(\tilde{\pi})q$. Note: $\varphi_{n,n+1}(q) \geqslant q^{(n+1)}$. By (i), $q^{(n+1)}$ satisfies condition 1.10 (†), and, in turn, so will q itself.

(j) It is necessary to introduce jump indices $\{b_1, \ldots, b_J\}$ for the lower signature. Then for a fixed lower jump index b, form the set Λ_n', where $\pi \in \Lambda_n'$, if

(1) $\pi \in \Delta_{n,j}, \quad c \leqslant j \leqslant n,$

(2) $m_i(\pi) = U_i, \quad 1 \leqslant i \leqslant a_J,$

(3) $m_{j-b+1}(\pi) = L_b + 1,$

(4) $m_{j-i+1}(\pi) = L_i, \quad s+1 \leqslant i \leqslant b_J, \ b \neq b_J.$

As before, fix $\pi \in \Lambda_n'$ with a minimal projection q from $q^{(n)}(\pi)C_n$, then we can argue almost identically to the above that q satisfies 1.10 (†).

(k) It remains to verify 1.10 (†) for any minimal projection q' from $q^{(n)}(\pi')C_n$, $\pi' \notin \Delta_{n,j}(A) - \Delta_{n,j}(B)$, $c \leqslant j \leqslant n$. To do this, it suffices to find an index k and $\pi'' \in \Lambda_k$ or Λ_k' such that $\varphi_{n,k}(q)q^{(k)}(\pi'') \neq 0$. As usual, we will only check the upper signature case; viz, we assume there is a (upper) jump index a such that $m_a(\pi) < U_a$, $r+1 \leqslant a \leqslant a_J$. We take π'' to be:

(1) $\pi'' \in \Lambda_k \cap \Delta_{k,k}, \quad k = 2j + a_J + b_J;$

(2) $m_i(\pi'') = \min(K_k^+, U_i), \quad 1 \leqslant i \leqslant a_J; \ i \neq a_J;$

(3) $m_a(\pi'') = U_a - 1;$

(4) $m_i(\pi'') = U_\infty, \ a_J + 1 \leqslant i \leqslant a_J + j;$

(5) $m_i(\pi'') = L_\infty, \ a_J + j + 1 \leqslant i \leqslant a_J + 2j;$

(6) $m_{k-i+1}(\pi'') = \max(L_i, K_k), \ 1 \leqslant i \leqslant b_J.$

REMARK. If J is chosen as in Theorem 3.3 and $U_1 - L_1 < +\infty$, then the ideal B of $\mathfrak{A}/J$ is stably isomorphic to A_f.

Proof. By the theorem, B is stably isomorphic to the ideal B_f of A_f. We show that B_f is stably isomorphic to A_f. Define a bijection $\pi \mapsto \pi^\#$ from $\Delta_n(B_f)$, $n \geqslant c$, onto $\Delta_{n-c}(A_f)$, where $m_i(\pi^\#) = m_{i+a}(\pi)$, $1 \leqslant i \leqslant n-c$. This map yields an equivalence between the Bratteli diagrams of B_f and A_f, so that B_f and A_f are stably isomorphic.

3.4. COROLLARY. *Let $J \in \mathrm{Prim}(U(\infty))$ satisfy the conditions of Theorem 3.3. Then there is a bijection between the faithful characters of $\mathfrak{A}/J$ and the faithful characters of $\mathfrak{A}/J_f$.*

Proof. By [14, Corollary to Theorem 7], there is a bijection between the faithful characters of a separable primitive C^*-algebra and those of any non-zero ideal. Since there is a bijection between the (faithful) characters of any two stably isomorphic C^*-algebras, the corollary is established.

For the sake of completeness, we shall sketch a proof of the bijective correspondence of characters of stably isomorphic algebras. It is sufficient to work with a C^*-algebra A and $A \otimes \mathscr{K}$.

We first establish that every representation of $A \otimes \mathscr{K}$ on $\mathscr{H}$, say, has the form $\pi_1 \otimes \pi_2$, where π_1 is a representation of A and π_2 is the identity (irreducible) representation of $\mathscr{K}$. By the definition of the maximal tensor product, there exist representations σ_1 of A and σ_2 of $\mathscr{K}$ in $\mathscr{H}$ such that $\sigma_1(A)$ commutes with $\sigma_2(\mathscr{K})$ and $\pi(a \otimes b) = \sigma_1(a)\sigma_2(b)$, $a \in A$, $b \in \mathscr{K}$. By [4, Chapter 4], we know that σ_2 is a direct sum of copies of the identity representation of $\mathscr{K}$ on $\mathscr{H}_2$, say, so that $\mathscr{H} \equiv \mathscr{H}_1 \otimes \mathscr{H}_2$ with $\sigma_2(\mathscr{K})'' \equiv \mathbf{C} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{H}_2)$. Thus $\sigma_1(A) \subset \sigma_2(\mathscr{K})' = \mathscr{B}(\mathscr{H}_1) \mathbin{\overline{\otimes}} \mathbf{C}$, $\sigma_2(\mathscr{K}) \subset \mathbf{C} \mathbin{\overline{\otimes}} \mathscr{B}(\mathscr{H}_2)$; so that, regarding each σ_i as a representation π_i on $\mathscr{H}_i$, we see that π is equivalent to $\pi_1 \otimes \pi_2$ as claimed. Hence, we have correspondences: $\pi \mapsto \pi_1$ and $\pi_1 \mapsto \pi = \pi_1 \otimes \pi_2$ between representations π of $A \otimes \mathscr{K}$ and π_1 of A which preserve quasi-equivalence.

Next, we show: π_1 is normal iff π is normal. Assume π_1 is normal with trace τ. Choose $a \in A^+$ with $0 < \tau(\pi_1(A)) < +\infty$ and pick a minimal projection p from $\mathscr{K}$. Then $\tau \otimes \mathrm{Tr}(\pi_1(a) \otimes \pi_2(p)) = \tau(\pi_1(a)) < +\infty$. Conversely, assume π is normal. Then there exists a $c \in (A \otimes \mathscr{K})^+$ such that $0 < \tau \otimes \mathrm{Tr}(\pi(c)) < +\infty$. It will suffice to find c of the form $a \otimes b$ in order to establish that π_1 is normal. Now there exists a rank one projection p in $\mathscr{K}$ such that $\pi((1 \otimes p)c) \neq 0$. Then $(1 \otimes p)c(1 \otimes p)$ may be written as $a \otimes p$, so that $0 < \tau \otimes \mathrm{Tr}(a \otimes p) < +\infty$.

REMARK. The classification problem of the faithful characters of A, as in 3.3, is reduced by the above corollary to the consideration of the faithful characters of the finite primitive quotient A_f. In an Appendix, A. J. Wassermann shows that such A_f can never admit infinite faithful characters. Since the faithful finite characters of A_f were classified in Section 2, the character problem for A is solved.

3.5. THEOREM. *Let $J \in \mathrm{Prim}(U(\infty))$ be such that $U_\infty = -\infty$ or $L_\infty = +\infty$, then $A = \mathfrak{A}/J$ admits no faithful characters.*

Proof. The theorem relies heavily on the proof of Theorem 3.3. By symmetry considerations, we only treat the case when $U_1 = U_\infty = +\infty$ and $L_\infty = +\infty$. Now $A = \mathfrak{A}/J$ is also a quotient of the algebras of 3.3. In particular, $A \cong (\mathfrak{A}/J_m)/E_m$, where $J_m \in \mathrm{Prim}(U(\infty))$ and E_m is an ideal of $\mathfrak{A}/J_m$, $m \geqslant 1$. Here J_m has signature $(U_j', L_j')_{j=1}^{\infty}$ such that $U_1' = U_\infty' = +\infty$ and $L_j' = L_j$, $1 \leqslant j \leqslant b_m$, $L_j' = L_{b_m+1}$, $j > b_m$, where $\{b_1, b_2, \ldots\}$ denote the jump indices for the finite entries of the lower signature of J. The ideal E_m is determined by: for $\pi \in \Delta_{n,j}(J_m)$, $q^{(n)}(\pi, J_m) \in E_m$ iff $j > b_m$ and $m_{j-i+1}(\pi) < L_i$, $b_m + 1 \leqslant i \leqslant j$.

By part (i) of the proof of Theorem 3.3, if $\pi \in \Lambda_{n,j} \cup \Lambda_{n,j}'$ and $\tilde{\pi} \in \Lambda_{n+1} \cup \Lambda_{n+1}'$ such that:

$$(*) \qquad T_{n+1}(q^{(n+1)}(\tilde{\pi})q^{(n)}(\pi)) \notin E_m ,$$

then $q^{(n)}(\pi) \geqslant \sum_{i=1}^{k} p_i$, $k \geqslant 1$, where $\{p_i\}_{i=1}^{k}$ are mutually unitarily equivalent projections. We introduce the notation that $[x]$ denotes the canonical image in A of an element x of $\mathfrak{A}/J_m$. By the construction of p_1 $(=T_{n+1}(q^{(n+1)}(\tilde{\pi})q^{(n)}(\pi))$, $[p_1] \neq 0$; so that, by unitary equivalence, $[p_i]\neq 0$, $1 \leqslant i \leqslant k$. Hence, $t([q^{(n)}(\pi)]) = +\infty$, for any faithful character t on A.

To complete the proof, it suffices to show that given any $\pi' \in \Delta_{N,k}(J)$, there is an index m and $\pi \in \Delta_N(J_m)$, $\tilde{\pi} \in \Delta_{N+1}(J_m)$ which satisfy (*) and $q^{(N)}(\pi', J) \geqslant [q^{(N)}(\pi, J_m)]$. Choose m so that $b_m > N + k$. We determine π and $\tilde{\pi}$ by their signature entries. We set: $m_i(\pi) = K_N^+$, $1 \leqslant i \leqslant N - b_m$; $m_1(\tilde{\pi}) = K_{N+1}^+$; $m_i(\tilde{\pi}) = K_N^+$, $2 \leqslant i \leqslant N + 1 - b_m$. The entries $m_{N-i+1}(\pi)$ and $m_{N-i+2}(\tilde{\pi})$ are identical and are given by: $L_{b_m} + 1$, $i = b_m$; L_i, $s_J + 1 \leqslant i \leqslant b_m - 1$; K_N, $1 \leqslant i \leqslant s_J$. (Remember that $(\Delta_n; K_n^+, K_n)$ determines the diagram of $\mathfrak{A}$, 2.2, 2.6.) It is elementary to check that π and $\tilde{\pi}$ are the desired representations.

We now summarize our main results.

3.6. THEOREM. *Let* $J \in \operatorname{Prim}(U(\infty))$, *then:*

(1) *$\mathfrak{A}/J$ is finite and type* I *iff* $U_1 = L_\infty$ $(< +\infty)$;

(2) *$\mathfrak{A}/J$ is not finite and type* I *iff* $U_1 - L_1 < +\infty$, U_1 *or* $L_1 \neq 0$, *and* $U_\infty = L_\infty$;

(3) *$\mathfrak{A}/J$ is finite and antiliminary iff* $U_{r+1} = U_\infty$ *and* $L_{s+1} = L_\infty$, *and* $U_1 \neq L_\infty$;

(4) *$\mathfrak{A}/J$ is non-finite, semifinite, and antiliminary iff* $U_{r+1} \neq U_\infty$ *or* $L_{s+1} \neq L_\infty$, $-U_\infty$ *and* $L_\infty \neq \infty$, *and when* $r = s = 0$, $U_\infty \neq L_\infty$;

(5) *$\mathfrak{A}/J$ admits no faithful characters iff* $-U_\infty$ *or* $L_\infty = +\infty$.

REMARK. According to the terminology of Cuntz and Pedersen [3], the finite type II C^*-algebra $\mathfrak{A} = A(U(\infty))$ admits primitive quotients of types I, II, III. In particular, $\mathfrak{A}$ has quotients with no faithful semifinite lower-semicontinuous traces. In a sense, $\mathfrak{A}$ combines examples 4.12 and 4.15 of [3] together.

We next summarize our main results on the representation theory of $U(\infty)$.

3.7. THEOREM. (1) *Up to unitary equivalence, the finite-dimensional irreducible representations of* $U(\infty)$ *have the form:* $[\det(V)]^m$, $V \in U(\infty)$, $m \in \mathbf{Z}$;

(2) *Up to unitary equivalence, the traceable infinite-dimensional irreducible representations are precisely the ones given by Kirillov* [7] *together with their tensor products with integral powers of the determinant function;*

Assume that (π, t) *is a normal representation such that* $\{\pi(\mathfrak{A})\}''$ *is type* II.

(3) *Up to quasi-equivalence, the finite normal representations are the* (*corrected versions of*) *the ones given by Voiculescu* [13];

(4) *Suppose* $\{\pi(\mathfrak{A})\}''$ *is infinite. Then* $J = \ker(\pi)$ *must have the form of* 3.6.4. *The* (*infinite*) *characters of* $U(\infty)$ *with kernel J are in natural one-to-one correspondence with the* (*finite*) *characters with kernel* J_f, *where the ideals J and J_f have the same index.*

For the convenience of the reader, we isolate the following lemma which is used in the Appendix.

LEMMA. *Let t denote an infinite semifinite lower-semicontinuous trace of $U(\infty)$. Then t is uniquely determined by its values $t(p(\sigma))$, $\sigma \in U(n)^\wedge$, $n \geqslant 1$.*

Proof. Let t_1 and t_2 be two such traces which agree on the projections $p(\sigma)$, $\sigma \in U(n)^\wedge$. It follows at once that $t_1 = t_2$ on $C^*(U(n))$. Since any trace on $C^*(U(n))$ has a unique extension to $M(U(n))$, t_1 and t_2 must induce the identical trace on $L_{(n)} \subset M(U(n))$, where $L_{(n)}$ is given in 2.1. The result now follows from Theorem 1.9 since there are generating nests for $\mathfrak{A}$ that lie in $\cup L_{(n)}$.

3.8. THEOREM. *Up to quasi-equivalence, any normal representation may be realized as a subrepresentation of a tensor product of a traceable irreducible representation and a finite factor representation.*

Proof. We treat the type II_∞ case first. Let $J \in \mathrm{Prim}(U(\infty))$ satisfy 3.6.4. Set $A = \mathfrak{A}/J$ and choose B, B_f, and A_f as in 3.3. If t' is a faithful finite character of B_f given by restriction of a faithful finite character of A_f, then there is a unique faithful character t of A_f such that $t(p(\sigma))=t'(p(\sigma_f))$, for all $\sigma \in D_n(B) \equiv \bigcup_m \Delta_m(B) \cap \cap U(n)^\wedge$; moreover, $(m_t)^- = B$. We will still denote by t the pull-back of the character t to $\mathfrak{A}$.

We will construct a normal representation π of $\mathfrak{A}$ with $\ker(\pi) = J$ and with corresponding trace $\varphi = t$. Throughout the argument, we need to distinguish two situations: $U_1(J) - L_1(J) < +\infty$ and $U_1(J) - L_1(J) = +\infty$.

When $U_1 - L_1 < +\infty$, let π^* be the traceable irreducible representation with $\ker(\pi^*)$ having signature:

$$(U_1 - U_{a+1}, \ldots, U_a - U_{a+1}, 0, 0, \ldots),$$

$$(L_1 - L_{b+1}, \ldots, L_b - L_{b+1}, 0, 0, \ldots).$$

Here the indices "a" and "b" are the same as the ones used in the proof of Theorem 3.3. By [1, Section 4], π^* may be realized as a direct limit of irreducible representations; in particular, we may require that π^* act on a Hilbert space H with orthonormal basis $\{e(\alpha)\}$ such that $\pi^*|H_n \in U(n)^\wedge$, $n \geqslant a+b$, with signature $(U_1 - U_{a+1}, \ldots, U_a - U_{a+1}, \ldots, L_b - L_{t+1}, \ldots, L_1 - L_{b+1})$. The space H_n is just the linear span of $\{e(1), \ldots, e(k_n)\}$.

When $U_1 - L_1 = +\infty$, π^* is constructed exactly as above except that $\ker(\pi^*)$ has signature:

$$(U_{r+1} - U_{a+1}, \ldots, U_a - U_{a+1}, 0, 0, \ldots),$$

$$(L_{s+1} - L_{b+1}, \ldots, L_b - L_{b+1}, 0, 0, \ldots).$$

Now let $\pi^{\#}$ be a type II_1 factor representation of $\mathfrak{A}$ with index $(r_J, U_\infty(J); s_J, L_\infty(J))$ acting in standard fashion on a Hilbert space K. Suppose $\xi \in K$ is a unit trace vector. It follows that $\{\pi^*(\mathfrak{A}) \otimes \pi^{\#}(\mathfrak{A})\}''$ is a type II_∞-factor acting on $\mathrm{H} \otimes K$ with trace φ given by:

$$\varphi = \sum \omega_{e(\sigma)\otimes\xi},$$

where $\omega_{e(\alpha)\otimes\xi}$ denotes the vector state. We wish to establish the following formula:

$$(*) \qquad \varphi(\pi(p(\sigma'))) = c_\sigma \cdot \mathrm{rank}(\sigma'),$$

where $\sigma' \in D_n(B)$, $\sigma = \sigma'_f$, and $c_\sigma = \det(c_{m_i(\sigma)+(j-i)})$ (see 2.8).

For $\sigma' \in D_n(B)$, $\varphi(\pi(p(\sigma'))) = \sum_{\alpha=1}^{k_n} \omega_{e(\alpha)\otimes\xi}(\pi(p(\sigma')))$, just as in [11, pp. 152–153]. We shall treat the sum $\sum_{\alpha=1}^{k_n} \omega_{e(\alpha)\otimes\xi} \circ \pi | U(n)$ as a class function on $U(n)$ so that it may be expanded in terms of characters of $U(n)$. We have:

$$\sum_{\alpha=1}^{k_n} \omega_{e(\alpha)\otimes\xi} \circ \pi | U(n) = \chi(\pi_n^*) \sum \{c_\sigma \chi(\sigma)\,;\, \sigma \in D_n(A_f)\} =$$

$$= \chi(\pi_n^*) \sum \{c_\sigma \chi(\sigma)\,;\, \sigma \in D_n(B_f)\} + \psi_n,$$

where $\psi_n = \chi(\pi_n^*) \sum \{c_\sigma \chi(\sigma)\,;\, \sigma \in D_n(A_f) - D_n(B_f)\}$.

CLAIM. *Given $\sigma' \in D_n(B)$, then $\sigma' \prec \pi_n^* \otimes \sigma$, where $\sigma \in D_n(B_f)$ iff $\sigma = \sigma'_f$. Moreover, when $\sigma = \sigma'_f$, σ' appears with multiplicity one.*

To establish the claim, it is convenient to recast it in terms of weights; viz.,

(**) *let w' be the highest weight for σ', then w' is a highest weight of $\pi_n^* \otimes \sigma$, $\sigma \in B_f$, iff $\sigma = \sigma'_f$.*

$\Rightarrow$) Any weight of $\pi_n^* \otimes \sigma$ has the form $f_1 + f_2$, where f_1, resp. f_2, is a weight π_n^*, resp. σ. The requirement that $\sigma \in D_n(B_f)$ and $\sigma' \in D_n(B)$ yields:

$f_1 =$ highest weight for π_n^*,

$f_2 = (U_{a+1}, \ldots, U_{a+1}, *, \ldots, *, L_{b+1}, \ldots, L_{b+1})$,

when $U_1 - L_1 < +\infty$, otherwise,

$f_1 = (0^r, U_{r+1} - U_{a+1}, \ldots, U_a - U_{a+1}, 0, \ldots 0, L_b - L_{b+1}, \ldots, L_s - L_{b+1}, 0^s)$,

$f_2 = (*, \ldots, *, U_{a+1}, \ldots, U_{a+1}, *, \ldots, *, L_{s+1}, \ldots, L_{s+1}, *, \ldots, *)$.

Here a "$*$" entry means it is not yet specified. The fact that w' is a highest weight for $\pi_n^* \otimes \sigma$ forces f_2 to be the highest weight for σ. It now follows that σ must equal σ'_f.

$\Leftarrow$) Suppose $\sigma = \sigma_f'$. If f_1 and f_2 are chosen as above, it readily follows that $w' = f_1 + f_2$ is a highest weight for $\pi_n^* \otimes \sigma$.

Hence, claim (∗∗) is established. To check the statement about the multiplicity of σ', it suffices to verify that f_1 and f_2 have multiplicity one as weights of π_n^* and σ, resp. This is clear for f_2 and f_1 (if $U_1 - L_1 < +\infty$) since they are highest weights. When $U_1 - L_1 = +\infty$, f_1 has multiplicity one since it is obtained from the highest weight of π_n^* by the action of the Weyl group.

It is also elementary to check that σ' is disjoint from $\pi_n^* \otimes \sigma$ if $\sigma \in D_n(A_f) - D_n(B_f)$ using reasoning similar to the above paragraphs.

Hence, σ', $\sigma' \in D_n(B)$, is disjoint from every tensor product $\pi_n^* \otimes \sigma$, $\sigma \in D_n(A_f)$, except when $\sigma = \sigma_f'$, and in this case it appears with multiplicity one. This establishes formula (∗).

The remainder of the proof is identical to [11, p. 157]. Let $P_{(n)} = \sup\{\pi(p(\sigma'))\,;\ \sigma' \in D_n(B)\}$. Then $P_{(n)} \leqslant P_{(n+1)}$ and $P_{(n)}(H \otimes K)$ is invariant under the action of $U(n)$. Hence, $L = (\cup P_{(n)}(H \otimes K))^-$ is invariant under $U(\infty)$ and π restricted to L is the desired representation.

If π is a normal representation of finite type or type I_∞, then π is a tensor product of itself with the trivial representation. Since the trivial representation is simultaneously a traceable irreducible representation and a finite representation, π admits the desired form.

Remark. It would be interesting to compute how arbitrary tensor products of normal representations of $U(\infty)$ decompose (cf. [11, V.2.6]).

SUMMARY OF THE REPRESENTATION THEORY OF $U(n)$

(1) An irreducible representation π of $U(n)$ is determined up to unitarily equivalence by its signature $(m_1(\pi), \ldots, m_n(\pi))$, a non-increasing n-tuple of integers. We write the character of π, i.e., $\mathrm{Tr}\,\pi(g)$, as χ_π or $\chi(\pi)$.

(2) Let $m < n$. If $\pi \in U(m)^\wedge$ and $\pi' \in U(n)^\wedge$, write $\pi \prec \pi'$ if π appears as a subrepresentation of $\pi'|U(m)$. By [16, p. 188], we have the extended branching law:

$$\pi \prec \pi' \text{ iff}$$

$$(*) \qquad m_i(\pi') \geqslant m_i(\pi) \geqslant m_{i+(n-m)}(\pi'), \quad 1 \leqslant i \leqslant m.$$

This is well-known when $n = m + 1$. It is useful to rewrite (∗) as:

$$m_i(\pi') \geqslant m_i(\pi), \quad m_{n-i+1}(\pi') \leqslant m_{m-i+1}(\pi), \quad 1 \leqslant i \leqslant m.$$

Another short summary of the representation theory for $U(n)$ may be found in the appendix of [11].

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APPENDIX TO "CHARACTERS OF $U(\infty)$" by ANTONY WASSERMANN

In this appendix we shall prove that the list of normal representations of $U(\infty)$ obtained by Boyer in the preceding article is exhaustive. The argument relies on an idea used by the present author to prove a similar exhaustion theorem for normal representations of the infinite symmetric group $S(\infty)$; namely one may exploit the fact that in that case the dimension groups of the finite primitive quotients of $C^*(S(\infty))$ admit integral domain structures compatible with the order. In the case of $U(\infty)$ such a structure exists only on a suitable completion of $K_0(A(U(\infty)))$ which we shall not describe precisely here, although its presence will be clearly seen in the discussion below. (See [2].)

Let $\sigma \in U(m)^\wedge$ and $\pi \in U(n)^\wedge$. As usual we have an inclusion $U(m) \times U(n) \subset \subset U(m+n)$. We may then define the *Young product* of σ and π via

$$\sigma \cdot \pi \cong \operatorname{ind}_{U(m) \times U(n) \uparrow U(m+n)} \sigma \otimes \pi,$$

where $\sigma \otimes \pi$ denotes the *outer* Kronecker product of σ and π. We may expand in terms of irreducible representations of $U(m+n)$,

$$\sigma \cdot \pi \cong \oplus_{\tau \in U(m+n)^\wedge} \langle \sigma \cdot \pi, \tau \rangle_{U(m+n)} \tau$$

where $\langle \sigma \cdot \pi, \tau \rangle_{U(m+n)}$ is the multiplicity of τ in $\sigma \cdot \pi$. We recall that if π and π' are representations of a compact group G with normalised Haar measure dg, then the dimension of the space of intertwining maps between π and π' is given by the formula

$$\langle \pi, \pi' \rangle_G = \int \operatorname{Tr}(\pi(g)) \cdot \overline{\operatorname{Tr}(\pi'(g))}\, dg.$$

Now by Frobenius Reciprocity we have

$$\langle \sigma \cdot \pi, \tau) \rangle_{U(m+n)} = \langle \sigma \otimes \pi, \tau|_{U(m) \times U(n)} \rangle_{U(m) \times U(n)}$$

so that $\langle \sigma \cdot \pi, \tau \rangle$ is always finite.

Next suppose that t is a trace of $U(\infty)$. As established by Boyer, t is specified uniquely by its values on minimal projections of $C^*(U(n))$ for $n \geqslant 0$. Now $C^*(U(n)) \cong \cong \oplus_{\pi \in U(n)^\wedge} \operatorname{End}(V_\pi)$ and thus we denote by $t(\pi)$ the value of t on a minimal projection in End (V_π), where this value may possibly be infinite. To define a trace, t must satisfy the additivity conditions

$$t(\pi) = \sum_{\pi < \pi' \in U(n+1)^\wedge} t(\pi')$$

where $\pi' > \pi$ indicates that π occurs as an irreducible constituent of $\pi'|_{U(n)}$. We may extend t additively to arbitrary representations of $U(n)$; thus if t is finite, we find that t is a character if and only if $t(\sigma \cdot \pi) = t(\sigma) \cdot t(\pi)$ for all σ and π, provided of course that t is normalised, which we shall assume of all finite characters. In fact, this condition is just an integrated version of Proposition 1 of [1].

To establish the exhaustion theorem, it will suffice to show that no infinite character of $U(\infty)$ admits the same (primitive) kernel as a finite character. (We recall that two traces t and t' have the same kernel if and only if $K(t) = K(t')$, where $K(t) = \{\pi \mid t(\pi) = 0\}$.)

Indeed, suppose that $K(t) = K(t')$ for characters t and t', with t finite and t' infinite. Since t' is semifinite, we may select π such that $0 < t'(\pi) < \infty$. We define a new *finite* trace t'' on $U(\infty)$ by $t''(\sigma) = t'(\sigma \cdot \pi) \equiv \sum_\tau \langle \sigma \cdot \pi, \tau \rangle t'(\tau)$. We claim that $t'' \leqslant \dim(\pi)^{-1} \cdot t'$ and that t'' has kernel $K(t)$, which obviously contradicts the supposition that t' is an infinite character. There are two steps.

a) $t'' \leqslant \dim(\pi)^{-1} \cdot t'$. We have

$$t''(\sigma) = \sum_{\tau \in U(m+n)^\wedge} \langle \sigma \cdot \pi, \tau \rangle \cdot t'(\tau)$$

and

$$t'(\sigma) = \sum_{\tau \in U(m+n)^\wedge} \langle \sigma, \tau|_{U(m)} \rangle t'(\tau)$$

where $\sigma \in U(m)^\wedge$; so it will be enough to check that $\langle \sigma \cdot \pi, \tau \rangle \leqslant \dim(\pi)^{-1} \langle \sigma, \tau|_{U(m)} \rangle$. Now

$$\tau|_{U(m) \times U(n)} \cong \oplus_{\sigma', \pi'} n(\sigma', \pi') \, \sigma' \otimes \pi',$$

where

$$n(\sigma', \pi') = \langle \sigma' \otimes \pi', \tau|_{U(m) \times U(n)} \rangle = \langle \sigma' \cdot \pi', \tau \rangle$$

so that we have

$$\tau|_{U(m) \times \{1\}} \cong \oplus_{\sigma', \pi'} \dim(\pi') n(\sigma', \pi') \sigma'.$$

On the other hand, $\tau|_{U(m)} \cong \oplus_{\sigma'} n(\sigma')\sigma'$ where $n(\sigma') = \langle \sigma', \tau|_{U(m)} \rangle$. Comparing these two expressions, we obtain

$$\langle \sigma, \tau|_{U(m)} \rangle = \sum_{\pi'} \dim(\pi') \langle \sigma \cdot \pi', \tau \rangle.$$

Thus $\langle \sigma, \tau|_{U(m)} \rangle \geqslant \dim(\pi) \langle \sigma \cdot \pi, \tau \rangle$ as required.

b) $K(t'') = K(t')$. From a) we see that $K(t'') \supseteq K(t')$. Now suppose that $\sigma \notin K(t') = K(t)$; then we have $t(\sigma \cdot \pi) = t(\sigma) \cdot t(\pi) = \sum_\tau \langle \sigma \cdot \pi, \tau \rangle t(\tau) \neq 0$. Thus we may find $\tau \notin K(t)$ for which $\langle \sigma \cdot \pi, \tau \rangle \neq 0$. Hence

$$t''(\sigma) = \sum_{\tau'} \langle \sigma \cdot \pi, \tau' \rangle t'(\tau') \geqslant \langle \sigma \cdot \pi, \tau \rangle t'(\tau) > 0,$$

so that $\sigma \notin K(t'')$. Thus $K(t'') = K(t')$ and the proof is complete.

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