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BOREL MAPS ON SETS OF VON NEUMANN ALGEBRAS

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1. INTRODUCTION

In [2], E. Effros showed how to make the collection of closed subsets of a Polish space into a standard Borel space. Applying this idea in [3], he introduced a standard Borel structure on the collection $\mathscr{A}$ of von Neumann algebras acting on a fixed separable Hilbert space H. The subcollection $\mathscr{F}$ of factor von Neumann algebras in $\mathscr{A}$ is easily seen to be Borel, and it makes sense to ask whether the various subcollections of $\mathscr{F}$ connected with type classification theory are Borel as well. In the follow-up paper [4], Effros provided affirmative answers to most of these questions; in particular he showed that the collection $\mathscr{T}$ of finite factors on H is Borel, but did not resolve the issue for the collection $\mathscr{S}$ of semi-finite factors.

Since a projection e in a factor A is finite if and only if eAe supports a finite trace, it is easy to see that $\mathscr{S}$ is analytic. In [11], O. Nielsen applied the Tomita-Takesaki theory of modular automorphism groups to show that $\mathscr{F}\setminus\mathscr{S}$ is also analytic, thereby proving that $\mathscr{S}$ is Borel. A second proof that $\mathscr{S}$ is Borel, outlined on pages 136–7 of [12] is based on a representation-theoretic argument of G. Pedersen [13].

The main result of the present paper, Theorem 5.3, states that there is a Borel function defined on $\mathscr{S}$ which selects a non-zero finite projection from each factor belonging to $\mathscr{S}$. The key idea in the proof is the application of a selection theorem which asserts that each Borel set in a product of Polish spaces all of whose sections are σ-compact admits a Borel uniformization. The paper uses only classical results from the theory of von Neumann algebras. In particular, a priori knowledge that $\mathscr{S}$ is Borel is not required, and this fact is established independently.

It is the theme of this paper that descriptive set theory, especially those parts of it dealing with set-valued maps, can be profitably applied to the study of the Effros Borel structure. Conversely, the existence of a standard Borel structure on the collection of closed subsets of a Polish space suggests a reformulation and reinterpretation of some of the classical results. An expository account of these topics, including corollaries of the above selection theorem, is presented in Section 2. This account is intended to be readable by non-experts in either operator algebras or descriptive set theory; it is supplemented in § 3 by historical comments and references to the literature.

The material in § 2 leads to a quick proof, at the beginning of Section 4, that the space $\mathscr{A}$ of von Neumann algebras is standard. Section 4 also contains several new results, most notably that the set-valued function sending each $A \in \mathscr{A}$ to its (*-strongly closed) set of partial isometries is Borel. The proof of the main theorem is given in § 5, and this is followed in §§ 6 and 7 by several applications to the finer structure of semi-finite factors.

The main result of § 6 implies that there is a Borel choice of unitary equivalences between type II_∞ factors and tensor products of type II_1 factors with $L(H)$. This amounts to choosing, in a Borel fashion, a supplementary orthogonal family of mutually equivalent, finite projections from each type II_∞ factor. Since Theorem 5.3 chooses one such projection from each factor, the basic problem is one of exhaustion; the required arguments are based on a somewhat unusual application of an optimal selection theorem.

It is an immediate consequence of Theorem 5.3 that there is a Borel choice of traces for semi-finite factors. In Section 7, it is shown that there is a Borel choice of operators in $L(H)$ which induce these traces; the proof again relies on the exhaustion arguments of § 6. The final section of the paper raises three open problems.

In closing this introductory section, I would like to thank Dan Mauldin for first bringing Theorem 2.4 to my attention, and the referee for his expository suggestions.

2. THE BOREL SPACE OF CLOSED SUBSETS OF A POLISH SPACE

This section is an expository presentation of slight variations of known results. Its topics are (1) a brief review of the general theory of Borel spaces, (2) a short description of the Hausdorff Borel structure, and (3) an amalgamation of the Hausdorff Borel structure with certain topics in descriptive set theory. A good reference for (1) is provided by K. Kuratowski and A. Mostowski's book [8]; Section 16 of O. Nielsen's monograph [12] contains a more leisurely treatment, including omitted proofs, of the Hausdorff Borel structure than is given here. Historical comments and further references will be given in the next section.

A *Borel structure* on a set X is a distinguished σ-algebra $\mathscr{B}$ of subsets of X; the pair $(X, \mathscr{B})$ is called a *Borel space* and if $\mathscr{B}$ is understood, its elements are referred to as *Borel subsets* of X. Subspaces and cartesian products of Borel spaces are defined in a natural fashion. A map between Borel spaces is said to be *Borel measurable* (Borel for short) if its inverse images of Borel sets are themselves Borel.

If X is equipped with a metric, the Borel structure on X will be taken to be the one generated by the metric topology on X. A simple fact, which we will often exploit, is that if $\{f_n: X \to Y_n\}$ is a countable family of Borel functions between separable metric spaces, then the cartesian product $\times f_n: X \to \times Y_n$ is also Borel. Thus if the $\{Y_n\}$ coincide, the domain of agreement of the $\{f_n\}$, being the inverse image of the diagonal under $\times f_n$, will be Borel; in particular, the set of fixed points of a Borel map on a separable metric space is always a Borel subset of the space.

A *Polish space* is a complete separable metric space; a Borel space which is (Borel) isomorphic to such a space is called *standard*. Every one-to-one Borel map between standard spaces is automatically an isomorphism. This explains why Borel structures are often more "canonical" than topological structures: if $\tau_2 \subseteq \tau_1$ are topologies induced by complete separable metrics on X, then the identity map: $(X, \tau_1) \to (X, \tau_2)$ is Borel, so τ_1 and τ_2 generate the same (standard) Borel structure on X. Standard Borel spaces are ubiquitous: the relative Borel structure on a Borel subset of a standard space is itself standard, and countable products of standard spaces are standard; somewhat paradoxically, all uncountable standard spaces are isomorphic.

The direct image of one Polish space in another under a Borel map is said to be *analytic*. Not every analytic set is Borel, and many of the deepest results of the theory rely on efforts to circumvent this difficulty. For example, the classical result that disjoint analytic sets can be separated by Borel sets plays a major role in establishing the assertions of the preceding paragraph. As mentioned in the Introduction, Nielsen's proof [11] that the space $\mathscr{S}$ of semi-finite factors is standard is also based on this classical result.

Let E be a subset of the cartesian product of the standard spaces X and Y. The projection D of E on X is called the *domain* of E. By a *uniformization* of E is meant a subset ψ of E which is (the graph of) a function mapping D into Y. Since the map $x \to (x, \psi(x))$ is one-to-one, requiring ψ to be a Borel measurable function with Borel domain is the same as requiring ψ to be a Borel subset of $X \times Y$. In particular, showing that E has a Borel uniformization is one way of guaranteeing that it has a Borel domain. This is the way $\mathscr{S}$ will be proven standard in the present paper.

Let X be a Polish space and write $\mathscr{C}(X)$ for the collection of non-empty closed subsets of X. Our next goal is to make $\mathscr{C}(X)$ into a standard Borel space. Let (Y, d) be a metrizable compactification of X. The *Hausdorff metric* ρ on $\mathscr{C}(Y)$ is defined by

$$\rho(S_1, S_2) = \max\{\sup_{y_1 \in S_1} d(y_1, S_2), \sup_{y_2 \in S_2} d(y_2, S_1)\}.$$

Under this metric, $\mathscr{C}(Y)$ is itself compact, and we equip it with the subordinate Borel structure, which is of course standard. Let j be the one-to-one map from $\mathscr{C}(X)$ into $\mathscr{C}(Y)$ which sends each set $S \in \mathscr{C}(X)$ to its Y-closure. Then j induces metric, topological, and Borel structures on $\mathscr{C}(X)$. There is nothing unique about the first two of these, but the induced Borel structure on $\mathscr{C}(X)$ is independent of the choice of Y. We will always regard $\mathscr{C}(X)$ as equipped with this *Hausdorff Borel structure*, which makes it a standard Borel space. (We are following [12] in reserving the term "Effros Borel structure" for spaces of von Neumann algebras.)

Given an open subset U of X, we write $\langle U \rangle$ for $\{S \in \mathscr{C}(X) \mid S \cap U \neq \emptyset\}$ and $[U]$ for $\{S \in \mathscr{C}(X) \mid S \subseteq U\}$.

PROPOSITION 2.1. *The family* $\{\langle U \rangle \mid U$ *open in* $X\}$ *generates the Hausdorff Borel structure on* $\mathscr{C}(X)$.

Proof. It is easy to see that the family $\{[V], \langle V \rangle \mid V$ open in $Y\}$ is a subbasis for the topology on $\mathscr{C}(Y)$. Let V be open in Y. Then we can write $Y \setminus V = \bigcap_{n=1}^{\infty} V_n$ where the $\{V_n\}$ are open and for each n, the closure of V_{n+1} is contained in V_n. By compactness, we thus have $[V] = \mathscr{C}(Y) \setminus \bigcap_{n=1}^{\infty} \langle V_n \rangle$. This shows that the family $\{\langle V \rangle \mid V$ open in $Y\}$ generates the Borel structure on $\mathscr{C}(Y)$. The proof is completed by the observation that for V open in Y, we have $j^{-1}(\langle V \rangle) = \langle V \cap X \rangle$. ▨

The advantages of Proposition 2.1 are analogous to those of knowing that the open sets generate the Borel structure on **R**. The following proposition and corollary can often be used to apply knowledge of X-valued maps to the study of $\mathscr{C}(X)$-valued ones.

PROPOSITION 2.2. *Let X be Polish. Then there is a sequence $\{\psi_n\}_{n=1}^{\infty}$ of Borel functions from $\mathscr{C}(X)$ into X such that $\{\psi_n(S)\}_{n=1}^{\infty}$ is a dense subset of S for each $S \in \mathscr{C}(X)$.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be dense in X and $r > 0$. Define $\eta_1 : \mathscr{C}(X) \to X$ by setting $\eta_1(S) = x_k$ where k is the smallest integer for which S intersects the ball of radius $r/2$ about x_k. Assuming η_n to be defined, let $\eta_{n+1} : \mathscr{C}(X) \to X$ by taking $\eta_{n+1}(S)$ to be the x_k of smallest index satisfying the two conditions:

(1) the distance from x_k to $\eta_n(S)$ is less than $\dfrac{r}{2^n}$, and

(2) the ball of radius $\dfrac{r}{2^{n+1}}$ about x_k intersects S.

The sequence $\{\eta_n\}_{n=1}^{\infty}$ converges (uniformly) to a Borel function $\psi : \mathscr{C}(X) \to X$ such that $\psi(S) \in S$ for every $S \in \mathscr{C}(X)$. Note that $\psi(S)$ is within $2r$ of x, whenever the ball of radius $r/2$ about x_1 intersects S.

Repeat the construction of ψ for each sequence obtained from $\{x_k\}_{k=1}^{\infty}$ by interchanging x_1 and some other x_j, and for every rational $r > 0$. The resulting sequence $\{\psi_n\}_{n=1}^{\infty}$ of functions has the desired properties. ▨

Suppose X is Polish, Y is standard Borel, and $\Phi\colon Y \to \mathscr{C}(X)$. By a *selector* of Φ is meant a function $\varphi\colon Y \to X$ satisfying $\varphi(y) \in \Phi(y)$ for all $y \in Y$. A *dense sequence of selectors* for Φ is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of selectors for Φ such that $\{\varphi_n(y)\}_{n=1}^{\infty}$ is dense in $\Phi(y)$ for all $y \in Y$.

COROLLARY 2.3. *A map from Y into $\mathscr{C}(X)$ is Borel iff it has a dense sequence of Borel selectors.*

Proof. If $\Phi\colon Y \to \mathscr{C}(X)$ is Borel, set $\varphi_n = \psi_n \circ \Phi$ where the $\{\psi_n\}$ are from Proposition 2.2; then the $\{\varphi_n\}$ are a dense sequence of Borel selectors for Φ.

Suppose conversely, that we have a dense sequence $\{\varphi_n\}_{n=1}^{\infty}$ of Borel selectors for Φ. Then if V is open in X, we have $\Phi^{-1}(\langle V \rangle) = \bigcup_{n=1}^{\infty} \varphi_n^{-1}(V)$ which is Borel in Y. ▨

The following is the selection theorem mentioned in the Introduction. Given a subset E of a product space $X \times Y$, we employ the usual sectional notation $E_x \equiv \{y \mid (x, y) \in E\}$ for each $x \in X$. The domain of E is $\{x \in X \mid E_x \neq \emptyset\}$. The *section map* associated with E sends x in the domain of E to the section E_x.

THEOREM 2.4. *Let X and Y be Polish. Suppose E is a Borel subset of $X \times Y$ such that each section E_x is σ-compact. Then the domain D of E is Borel and there exists a Borel function $\psi\colon D \to Y$ whose graph is contained in E.*

COROLLARY 2.5. *Suppose in Theorem 2.4 that Y is compact. Then there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of Borel functions : $D \to Y$ such that $\{\varphi_n(x)\}_{n=1}^{\infty}$ is dense in E_x for each $x \in X$.*

If, in addition, each E_x is compact, then the associated section map from D to $\mathscr{C}(Y)$ is Borel.

Proof. Let V be open in Y. Then $E \cap (X \times V)$ satisfies the hypotheses of Theorem 2.4. Thus the domain D_V of this relation is Borel and there is a Borel function $\varphi_V\colon D_V \to Y$ whose graph is contained in E and satisfies $\varphi_V(x) \in V$ whenever $x \in D_V$. Using the function ψ of Theorem 2.4, it is easy to extend φ_V to a Borel function on all of D. We obtain the desired sequence $\{\varphi_n\}_{n=1}^{\infty}$ by repeating this construction for each V in some countable basis for the topology of Y. The final statement of the corollary now follows from Corollary 2.3. ▨

PROPOSITION 2.6. *Let X be Polish. Then "(closed) countable union" is a Borel map from $\times_{n=1}^{\infty} \mathscr{C}(X)$ to $\mathscr{C}(X)$.*

If X is compact, then "countable intersection" is also Borel.

Proof. If V is open in X, and the sequence $\{S_n\}_{n=1}^{\infty} \in \times_{n=1}^{\infty} \mathscr{C}(X)$, then $\bigcup_{n=1}^{\infty} S_n$ meets V iff some S_n meets V. The Borel measurability of "countable union" now follows from Proposition 2.1.

The precise meaning of the final assertion of the Proposition is that $D \equiv \left\{ \{S_n\}_{n=1}^{\infty} \in \bigtimes_{n=1}^{\infty} \mathscr{C}(X) \,\middle|\, \bigcap_{n=1}^{\infty} S_n \neq \emptyset \right\}$ is Borel and the map from D into $\mathscr{C}(X)$ defined by $\{S_n\}_{n=1}^{\infty} \to \bigcap_{n=1}^{\infty} S_n$ is Borel. Choose a countable basis $\{U_m\}_{m=1}^{\infty}$ for the topology on X and note that $x \in \bigcap_{n=1}^{\infty} S_n$ iff every U_m which meets $\{x\}$ also meets each S_n. This shows that the set of ordered pairs $(\{S_n\}_{n=1}^{\infty}, x)$ in $\bigtimes_{n=1}^{\infty} \mathscr{C}(X) \times X$ for which $x \in \bigcap_{n=1}^{\infty} S_n$ is Borel. The proof is therefore completed by applying Corollary 2.5. ▨

J. P. R. Christensen has shown [1, Theorem 5] that "intersection" can fail to be Borel when X is not compact. The following result will be used to avoid this difficulty in § 6. The *graph* of a function $\Phi: Y \to \mathscr{C}(X)$ is $\{(y, x) \in Y \times X \mid x \in \Phi(y)\}$. This notion is dual to that of section map.

PROPOSITION 2.7. *Let $\Phi: Y \to \mathscr{C}(X)$ be Borel, where Y is standard and X is Polish. Let E be a Borel subset of the graph of Φ having the property that E_y is relatively open in $\Phi(y)$ for each $y \in Y$. Then the domain D of E is Borel as is the map $\Psi: D \to \mathscr{C}(X)$ sending $y \in Y$ to the closure of E_y.*

Proof. Let $\{\varphi_n\}$ be a dense sequence of selectors for Φ. Note that the pair (y, x) belongs to the graph of Φ iff $\inf_n d(\varphi_n(y), x) = 0$, so the graph of Φ is Borel. Thus for each n, the intersection ψ_n of the graph of φ_n and E is the graph of a Borel function; in particular ψ_n has a Borel domain D_n. By the relative openness hypotheses, $D = \bigcup_{n=1}^{\infty} D_n$, so D is Borel. Using the $\{\psi_k\}$ for $k \neq n$, we can extend each ψ_n to a Borel function on D, at which point the $\{\psi_n\}$ will be a dense sequence of selectors for Ψ. ▨

PROPOSITION 2.8. *Let Y be standard, X Polish, and $\Phi: Y \to \mathscr{C}(X)$ Borel. Suppose θ is a bounded real-valued Borel function defined on the graph of Φ, which is continuous in its second variable. Then there is a Borel selector φ of Φ satisfying*

$$\theta(y, \varphi(y)) \geqslant \frac{1}{2} \sup_{x \in \Phi(y)} \theta(y, x).$$

Proof. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a dense sequence of selectors for Φ. Take $\varphi(y)$ to be the $\varphi_n(y)$ of lowest subscript satisfying $\theta(y, \varphi_n(y)) \geqslant \frac{1}{2} \sup_m \theta(y, \varphi_m(y))$. ▨

3. COMMENTS ON THE PRECEDING SECTION

Let X be Polish. Although Effros [2] is responsible for equipping $\mathscr{C}(X)$ with a standard Borel structure, there is an extensive literature on methods of topologizing $\mathscr{C}(X)$ and on notions of measurability for $\mathscr{C}(X)$-valued maps. The purpose of this section is to make some of the connections between these concepts explicit. This material will not be used in the sequel. General surveys of topologies on $\mathscr{C}(X)$ and of measurability for set-valued functions can be found in the papers [10] and [6] of E. Michael and C. J. Himmelberg respectively. The state of the art concerning measurable selection theorems is catalogued in D. H. Wagner's papers [16] and [17].

Proposition 2.1, which is implicit in [2], is the bridge to the literature on measurability of set-valued maps. A map Φ defined on a (standard) Borel space Y and taking values in $\mathscr{C}(X)$ is said to be *weakly measurable* [6] if $\{y \mid \Phi(y) \cap U \neq \emptyset\}$ is Borel for each open subset of U of X. By Proposition 2.1, this is the same as requiring Φ to be measurable as a function when $\mathscr{C}(X)$ is equipped with the Hausdorff Borel structure. It seems quite natural when speaking of "measurable maps" to be referring to a fixed Borel structure on the range space, but this often seems to have been overlooked in the study of set-valued maps. As a practical matter, having any Borel structure on $\mathscr{C}(X)$ encourages composition of set-valued maps; knowing that such a structure is standard is a bonus which allows the application of the deep classical theory. The paper [2] thus singles out weak measurability of set-valued maps as being more natural than the competing notions.

Wagner refers to Corollary 2.3 as the "Fundamental Measurable Selection Theorem" because of its importance; pages 867 and 901 of his first survey paper [16] give a detailed account of its origin. Although we derived Corollary 2.3 from Proposition 2.2, the reverse implication is equally transparent. In fact, our proof of the latter is essentially the one used by K. Kuratowski and C. Ryll-Nardzewksi to establish their main result [8, page 458].

There are many names associated with the development of Theorem 2.4. W. J. Arsenin, K. Kunugui, P. Novikov, and E. Stchegolkov worked in the classical setting ($X = Y = \mathbf{R}$), and other mathematicians generalized their results to arbitrary Polish spaces. A. D. Ioffe's supplement [7] to Wagner's survey articles nicely documents this history. An interesting, self-contained proof of Theorem 2.4 has been given by J. Saint-Raymond [14].

The proofs of 2.5 through 2.8 given above are slight variations of arguments in [6]. When the function Φ of Proposition 2.8 is compact-valued, it has a Borel selector φ for which $\theta(y, \varphi(y)) = \max_{x \in \Phi(y)} \theta(y, x)$ [16, Section 9]. Such optimal selection theorems are important in dynamic programming. The applications of Proposition 2.8 in § 6 have a somewhat different flavor, being concerned with exhaustion rather than with optimization.

We close this section with a comparison of topologies. The family $\{\langle U\rangle \mid U \text{ open in } X\}$ forms a sub-basis for the (*lower*) *semi-finite* topology on $\mathscr{C}(X)$; the *finite* topology on $\mathscr{C}(X)$ has the larger family $\{[U], \langle U\rangle \mid U \text{ open in } X\}$ as a sub-basis [10, Section 9]. On first thought, it might seem that the finite topology is the more natural of the two: it always separates points and is the topology induced by the Hausdorff metric when X is compact. The overriding fact however (Proposition 2.1) is that the semi-finite topology generates the Hausdorff Borel structure on $\mathscr{C}(X)$ even when X fails to be compact. On the other hand, it can happen that the finite topology on $\mathscr{C}(X)$ is not even contained in this Borel structure. This follows from Theorem 8 of [1]; it is also a consequence of an example of J. Kaniewski [17, Example 2.4], namely a weakly measurable, i.e. Borel, function into $\mathscr{C}(X)$ which is not measurable in the sense of [6].

4. THE BOREL SPACE OF VON NEUMANN ALGEBRAS

Fix a separable infinite-dimensional Hilbert space H, and denote by C the set of its contraction operators, i.e., those bounded linear operators on H of norm less than or equal to one. We equip C with the weak operator topology, under which it becomes a compact metric space. We will use the fact that operator multiplication is a Borel map from $C\times C$ into C. This can be seen either by noting that multiplication is weakly continuous in each variable separately, or by realizing that the Borel structure on C is the same as that generated by the strong operator topology and that multiplication is jointly strongly continuous on bounded sets.

PROPOSITION 4.1. *The following are Borel maps on* $\mathscr{C}(C)$*:*

(1) $S \to S^*$ *(the set of adjoints of operators in* S*)*
(2) $S \to S'$ *(the set of contractions commuting with each operator in* S*).*

Proof. Let $\{\psi_n\}_{n=1}^{\infty} : \mathscr{C}(C) \to C$ be as in Proposition 2.2.

(1) Since taking adjoints is continuous in the weak operator topology the $\{\psi_n^*\}_{n=1}^{\infty}$ form a dense sequence of Borel selectors for the map in question, and we have only to apply Corollary 2.3.

(2) Let $\mathscr{E}$ be the set of ordered pairs (S, a) in $\mathscr{C}(C)\times C$ such that a commutes with the $\{\psi_n(S)\}_{n=1}^{\infty}$. Since each of the maps $(S, a) \to a\psi_n(S)-\psi_n(S)a$ is Borel, we see that $\mathscr{E}$ is Borel and has compact sections. An application of Corollary 2.5 thus completes the proof. ∎

COROLLARY 4.2. *The collection* $\mathscr{A}$ *of von Neumann algebras on* H *and its subcollection* $\mathscr{F}$ *of factors are Borel subsets of* $\mathscr{C}(C)$.

Proof. The von Neumann algebras are the common fixed points of the Borel maps $S \to S^*$ and $S \to S''$ on $\mathscr{C}(C)$ so $\mathscr{A}$ is Borel. By Proposition 2.6, the map $A \to A \cap A'$ is Borel on $\mathscr{A}$, and $\mathscr{F}$ is the inverse image of a singleton under this map. ▨

In Corollary 4.2, and in the sequel, we identify von Neumann algebras with their unit balls. This point of view, due originally to O. Maréchal [9], enables us to apply Theorem 2.4 and its consequences.

Recall that by spectral theory, every bounded Borel function λ on $\mathbf{R}$ induces a function on the positive operators on H. In the next, well-known result (and only there), the latter function will be denoted by $\tilde{\lambda}$ to distinguish it from λ. Lemma 4.3 is the last result from this section needed in § 5.

LEMMA 4.3. *Suppose $\lambda : [0, 1] \to \mathbf{R}$ is bounded and Borel. Then $\tilde{\lambda}$ is strongly Borel. If λ is continuous, then $\tilde{\lambda}$ is strongly continuous.*

Proof. Since the range of $\tilde{\lambda}$ may contain non-contractions, the lemma refers to the Borel structure on the space of all bounded operators which is subordinate to the strong operator topology; the relativizations of this structure to each bounded ball is standard. With this understanding, we have only to note that $\tilde{\lambda}$ is strongly continuous whenever λ is a polynomial. This completes the proof since $\{\lambda \mid \tilde{\lambda}$ is strongly continuous$\}$ is closed under uniform limits, while $\{\lambda \mid \tilde{\lambda}$ is Borel$\}$ is closed under bounded pointwise limits. ▨

It will be convenient to have fixed notations for certain subsets of C. We adopt:

P for the set of positive (semi-definite) operators in C,

W for the set of partial isometries in C, and

E for the set of (self-adjoint) projections in C.

We will equip P with the weak operator topology, under which it is compact. Unfortunately, W and E are not weakly closed; we equip them with the relative *-strong operator topology, under which they are Polish, but not compact. See the beginning of Chapter 2 of J. Ernest's memoir [5] for an exposition of the basic properties of this topology. Of course, the strong and *-strong topologies agree on P. On occasion, we will regard C or P as equipped with the *-strong topology; this will be indicated by the notations $\hat{C}$ and $\hat{P}$ respectively. It is perhaps well to point out that the identity map: $\hat{C} \to C$ is continuous, so the Borel structures on C and $\hat{C}$ coincide.

PROPOSITION 4.4. *Let $\mathscr{J}$ denote the set of ordered triples (A, e, f) where A is a von Neumann algebra on H while e and f are projections in A. The maps which send (A, e, f) to*

(1) *the set of positive operators in eAe*

(2) *the set of partial isometries in fAe*

(3) *the set of projections in eAe*

are Borel maps from $\mathscr{J}$ into $\mathscr{C}(P)$, $\mathscr{C}(W)$, and $\mathscr{C}(E)$ respectively.

COROLLARY 4.5. *The following are Borel maps from $\mathscr{A}$ into $\mathscr{C}(P)$, $\mathscr{C}(W)$, and $\mathscr{C}(E)$ respectively:*

(1) $A \to A \cap P$

(2) $A \to A \cap W$

(3) $A \to A \cap E$.

The corollary follows immediately from the proposition by composing the obviously Borel correspondence $A \to (A, i, i)$ with the maps of the latter. (We write i for the identity operator on H.) Also, since the map $(A, e, f) \to eAe$ clearly has a dense sequence of Borel selectors, Proposition 4.4 (1) follows directly from Proposition 2.6. By contrast, the non-compactness of W and E makes the rest of Proposition 4.4 more difficult to establish.

As motivation for the following lemmas, note that every continuous map between Polish spaces induces a Borel set-valued function. In proving Proposition 4.4, we start with maps which enjoy only a vestigal form of continuity.

LEMMA 4.6. *Let λ be the map from $\hat{P}$ to E corresponding to the characteristic function of the interval* $[1/2, 1]$.

Then λ is Borel, idempotent, and continuous at each $e \in E$.

Proof. That λ is Borel follows from the preceding lemma. Also λ maps P into E and since $\lambda(e) = e$ for any projection e, we see $\lambda \circ \lambda = \lambda$, i.e. that λ is idempotent. It remains to check that λ is continuous at each $e \in E$. Let μ, ν: $[0, 1] \to [0, 1]$ by

$$\mu(s) = \begin{cases} 0 & \text{if } s \in [0, 1/4] \\ 4(s - 1/4) & \text{if } s \in [1/4, 1/2] \\ 1 & \text{if } s \in [1/2, 1] \end{cases} \qquad \nu(s) = \begin{cases} 0 & \text{if } s \in [0, 1/2] \\ 4(s - 1/2) & \text{if } s \in [1/2, 3/4] \\ 1 & \text{if } s \in [3/4, 1]. \end{cases}$$

Then μ and ν are continuous, so if $\{a_n\}_{n=1}^{\infty}$ is a sequence in P converging strongly to a projection e, then both of the sequences $\{\mu(a_n)\}_{n=1}^{\infty}$ and $\{\nu(a_n)\}_{n=1}^{\infty}$ converge strongly to e. Now for any $h \in H$, and any n,

$$\|[\lambda(a_n) - \nu(a_n)]\, h\| \leqslant \|[\mu(a_n) - \nu(a_n)]h\|.$$

Thus $\{\lambda(a_n)\}_{n=1}^{\infty}$ converges strongly to e as well, so λ is continuous at e and the proof is complete. ▨

Recall the polar decomposition of an operator a is given by $a = w|a|$ where $|a| = \sqrt{a^* a}$ and the null space of w contains the null space of $|a|$. These conditions determine w uniquely, and it is automatically a partial isometry.

LEMMA 4.7. *The map μ which sends each contraction to the partial isometry appearing in its polar decomposition is Borel.*

Proof. For each $a \in \hat{C}$ and integer n, set $\mu_n(a) = a\lambda_n(a^* a)$ where $\lambda_n : [0, 1] \to \mathbf{R}^+$ by

$$\lambda_n(s) = \begin{cases} \dfrac{1}{\sqrt{s}} & \text{if } s \geqslant \dfrac{1}{n^2} \\ 0 & \text{if } s \leqslant \dfrac{1}{n^2} \cdot \end{cases}$$

Consideration of $\mu_n(a)^* \mu_n(a)$ shows each $\mu_n(a)$ is a partial isometry, so μ_n is a Borel map from $\hat{C}$ to W. Let $a \in \hat{C}$. If the vector h is in the null space of $|a|$, we have $\lim_{n\to\infty} \mu_n(a)h = 0$ while for $h = |a|k$ in the range of $|a|$, we have

$$\lim_{n\to\infty} \mu_n(a)h = \lim_{n\to\infty} \mu_n(a)\,|a|k = ak.$$

This shows $\{\mu_n(a)\}_{n=1}^{\infty}$ converges strongly to $\mu(a)$ and completes the proof. ▨

LEMMA 4.8. *Let λ and μ be as in the preceding two lemmas and define $\nu : \hat{C} \to W$ by $\nu(a) = \mu[\lambda(aa^*)a\lambda(a^*a)]$. Then ν is Borel, idempotent, and continuous at each $w \in W$.*

Proof. ν is Borel since λ and μ are, and ν maps into W since μ does. If $w \in W$, then $\nu(w) = \mu(ww^*ww^*w) = \mu(w) = w$ so we see that ν is idempotent. To establish the assertion concerning continuity, let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\hat{C}$ converging (*-strongly) to the partial isometry w. Set $e_n = \lambda(a_n^* a_n)$, $f_n = \lambda(a_n a_n^*)$, $e = w^*w$, and $f = ww^*$. It follows from Lemma 4.6 that the $\{e_n\}$ and $\{f_n\}$ are sequences of projections converging strongly to e, f respectively.

Write $b_n = f_n a_n e_n$, and let $b_n = w_n|b_n|$ be its polar decomposition. We must show $\{w_n\}$ converges *-strongly to w. We begin by noting that $\{|b_n|^2\}$ and hence $\{|b_n|\}$ converges strongly to e. Now for h in the null space of w, we have $\lim_{n\to\infty} e_n h = 0$ so

$$\lim_{n\to\infty} w_n h = \lim_{n\to\infty} w_n e_n h = 0 = wh.$$

On the other hand, if h belongs to the initial space of w, then

$$\lim_{n\to\infty} w_n h = \lim_{n\to\infty} w_n |b_n| h = \lim_{n\to\infty} a_n h = wh.$$

This shows that $w_n \to w$ strongly.

In particular $w_n \to w$ and hence $w_n^* \to w^*$ weakly. But then expansion of $\|(w_n^* - w^*)h\|^2$ in terms of inner products shows $w_n^* \to w^*$ strongly as well and the proof is complete. ▨

Proof of Proposition 4.4. We have already proved (1). To establish (2), let $\{\psi_n\}_{n=1}^{\infty}$ be a (weakly) dense sequence of Borel selectors for the map $(A, e, f) \to fAe$, taking $\mathscr{F}$ into $\mathscr{C}(C)$. Since this map is convex-valued, by augmenting the $\{\psi_n\}_{n=1}^{\infty}$ by all of their rational convex combinations if necessary, we can assume that they are $*$-strongly dense selectors for $(A, e, f) \to fAe$ viewed as a map into $\mathscr{C}(\hat{C})$. But then taking v from Lemma 4.8, we see that the $\{v \circ \psi_n\}_{n=1}^{\infty}$ are a dense sequence of selectors for the map $(A, e, f) \to (fAe) \cap W$ and the proof of (2) is complete.

We establish (3) by composing the map λ of Lemma 4.6 with a $*$-strongly dense sequence of Borel selectors for the map $(A, e, f) \to (eAe) \cap P$ of (1). ▨

5. A BOREL CHOICE OF FINITE PROJECTIONS

The main result of the paper appears as Theorem 5.3 below. As mentioned in the Introduction, the key idea in the proof is use of the selection result, Theorem 2.4; before this can be done, we need two observations concerning von Neumann's comparison theory for projections. We use the standard notations and terminology, as found for example in D. Topping's notes [15].

PROPOSITION 5.1. *The following sets are Borel.*

(1) *The set of pairs (A, e) such that A is a factor on H and e is a finite projection in A.*

(2) *The set of triples (A, e, f) such that A is a factor on H and $e \precsim f$ are projections in A.*

Proof. (1) We follow Nielsen [12, pages 89 f]. The set of triples $(A, e, w) \in \mathscr{F} \times E \times W$ with $e \in A$ and $w \in A$ satisfying $ww^* = e$ and $w^*w < e$ is Borel, so the set of (1) is at least coanalytic.

Let T denote the set of trace class operators in P, and write τ for the canonical trace. Since τ is weakly lower semi-continuous, we conclude that T is a σ-compact subset of C and that τ is weakly Borel on T. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a dense sequence of Borel selectors for the identity map on $\mathscr{F}$. The set of ordered triples

$$\{(A, e, t) \in \mathscr{F} \times E \times T \mid e \in A, \quad \tau(t) = 1, \quad \text{and} \quad \tau(te\varphi_n(A)e\varphi_m(A)e) = \\ = \tau(te\varphi_m(A)e\,\varphi_n(A)e) \ \text{for all} \ n, m\}$$

is Borel. Since e is a finite projection in A precisely when eAe admits a finite trace, the projection of this set of triples on $\mathscr{F} \times E$ is the set of (1). This shows the set of (1) is also analytic, hence Borel.

(2) Since $e \precsim f$ always holds when f is infinite, but never holds when e is infinite and f is finite, it suffices to restrict attention to those ordered triples having e and f finite. Now each of the sets

$$\{(A, e, f, w) \in \mathscr{F} \times E \times E \times W \mid e, f \text{ finite projections in } A, \ w \in A, \text{ with } ww^* < e \text{ and } w^*w = f\}$$

$$\{(A, e, f, w) \mid A, e, f, w \text{ as above, but } ww^* = e, \ w^*w = f\}$$

$$\{(A, e, f, w) \mid A, e, f, w \text{ as above, but } ww^* = e, \ w^*w < f\}$$

is Borel, so their projections on $\mathscr{F} \times E \times E$ are analytic. Since these projected sets are disjoint and have a Borel union they are in fact themselves Borel, and the proof is complete. ▨

We use the notation τ_A for a trace on the semi-finite factor A. When A is finite we will assume $\tau_A(i) = 1$, which determines τ_A uniquely. When A is infinite, τ_A has no convenient normalization, but the quotient appeaiing below does not reflect this arbitrariness.

PROPOSITION 5.2. *The map* $(A, e, f) \to \dfrac{\tau_A(e)}{\tau_A(f)}$ *is Borel on the set of ordered triples in* $\mathscr{F} \times E \times E$ *for which it is defined.*

Proof. Set

$$\mathscr{D} = \{(A, e, f) \mid A \in \mathscr{F}, e, f \text{ are finite projections in } A, f \neq 0\}.$$

Then $\mathscr{D}$ is Borel and is the domain of the map in question. For convenience of notation, write $\sigma(A, e, f) = \dfrac{\tau_A(e)}{\tau_A(f)}$. For any positive real number r, we have

$$\sigma^{-1}([0, r)) = \left\{(A, e, f) \in \mathscr{D} \mid \text{there exist integers } m \text{ and } n \text{ with } \frac{m}{n} < r \text{ such that } e \text{ and } f \text{ admit orthogonal decompositions in } A \text{ of the form } e = e_1 + \dots + e_m \text{ and } f = f_1 + \dots + f_n \text{ where the } \{f_j\} \text{ are mutually equivalent and each } e_i \precsim f_1\right\},$$

which is clearly an analytic set. Similarly, $\sigma^{-1}([r, \infty))$ is also analytic, whence σ is Borel. ▨

THEOREM 5.3. *The collection* $\mathscr{S}$ *of semi-finite factors on* H *is Borel and there is a Borel function which selects a non-zero finite projection from each of its elements.*

Proof. Write $\mathscr{R}$ for the set of ordered pairs (A, t) such that A is a factor on H and t is a non-zero, positive trace-class contraction in A. Since any trace on a semi-finite factor is lower semi-continuous, for any factor A and integer n, the set

$\{t \in A \cap P \mid \tau_A(t) \leqslant n\}$ is (weakly) compact. This shows the vertical sections of $\mathscr{R}$ are σ-compact. The main idea of this proof is to apply Theorem 2.4 to $\mathscr{R}$; in order to do so, we must only show that $\mathscr{R}$ is Borel.

For each positive integer n, let $\lambda_n: [0, 1] \to [0, 1]$ be the characteristic function of the interval $(1/2^n, 1/2^{n-1}]$. According to Proposition 4.3, the $\{\lambda_n\}$ can be regarded as Borel functions on P. Now if $(A, t) \in \mathscr{R}$ then all of the projections $\{\lambda_n(t)\}_{n=1}^{\infty}$ are finite, there is one of lowest index, $\mu(t)$, which is non-zero, and the sum $\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\tau_A(\lambda_n(t))}{\tau_A(\mu(t))}$ converges; these conditions are also sufficient for membership in $\mathscr{R}$. Proposition 5.1 implies that

$$\mathscr{R}_0 \equiv \{(A, t) \in \mathscr{F} \times P \mid t \neq 0 \text{ and } \lambda_n(t) \text{ is finite for all } n\}$$

is Borel. Moreover, since μ is Borel and composition preserves Borel measurability, we can apply Proposition 5.2 to conclude that for each n the map

$$(A, t) \to (A, \lambda_n(t), \mu(t)) \to \frac{\tau_A(\lambda_n(t))}{\tau_A(\mu(t))}$$

is Borel on $\mathscr{R}_0$. This completes the proof that $\mathscr{R}$ is Borel since we have characterized $\mathscr{R}$ as

$$\left\{(A, t) \in \mathscr{R}_0 \mid \text{the partial sums of } \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\tau_A(\lambda_n(t))}{\tau_A(\mu(t))} \text{ are bounded}\right\}.$$

We are now in a position to apply Theorem 2.4. This tells us that the projection of $\mathscr{R}$ on $\mathscr{F}$, namely $\mathscr{S}$, is Borel, and we get a Borel function $\varphi: \mathscr{S} \to P \setminus \{0\}$ such that $\varphi(A)$ has finite A-trace for each $A \in \mathscr{S}$. With μ as above, set $\varepsilon(A) = \mu(\varphi(A))$ to get the desired selection of projections. ∎

NOTE. In future references to $\varepsilon(A)$, we will assume it has been redefined to satisfy $\varepsilon(A) = i$ whenever A is finite.

COROLLARY 5.4. *There is a Borel choice of traces for semi-finite factors. More precisely, the set*

$$Q = \{(A, t) \in \mathscr{S} \times C \mid t \text{ is of trace class in } A\}$$

is Borel and there is a Borel function $\sigma: Q \to \mathbf{C}$ *such that for each* $A \in \mathscr{S}$, *the functional* $\sigma(A, \cdot)$ *is a trace on* A.

Proof. Note that t is of trace class in A iff the positive and negative parts of $\operatorname{Re} t$, $\operatorname{Im} t$ are. This shows Q is Borel. Let $\varepsilon: \mathscr{S} \to E$ be the selection of finite projections constructed in Theorem 5.3. Take $\sigma(A, t) = \lim_{k \to \infty} \sum_{n=2}^{k} \frac{n}{k} \frac{\tau_A(\lambda_{n,k}(t))}{\tau_A(\varepsilon(A))}$ for $t \geqslant 0$, where $\lambda_{n,k}$ is the characteristic function of $(n - 1/k, n/k]$ and extend σ by linearity to all of Q. ▨

COROLLARY 5.5. *The classification of factors is Borel.*

Proof. We already know the sets of finite, semi-finite and type III factors on H are Borel. The type I factors are those with abelian projections, while a semi-finite factor A fails to be of type I iff it contains a sequence of non-zero projections with $\tau_A(e_n) \downarrow 0$; thus both of these sets are analytic and hence Borel. Since a finite factor A is of type I_n iff it contains an abelian projection e with $\tau_A(e) = 1/n$, these classes are also analytic and hence Borel. ▨

6. EXHAUSTION ARGUMENTS IN SEMI-FINITE FACTORS

Let A be a type II_∞ factor with e_1 a finite non-zero projection in A. Then A contains a sequence $\{w_n\}_{n=1}^\infty$ of partial isometries having e_1 as their common initial projection, and final projections $\{e_n\}_{n=1}^\infty$ which are mutually orthogonal and supplementary. This is a precise way of saying that A is (unitarily equivalent to) the tensor product of the type II_1 factor e_1Ae_1 with the type I_∞ factor $L(H)$; to a large extent, this reduces the study of semi-finite factors to that of finite factors. This principle can be applied to the measurability considerations of the present paper by making the $\{w_n\}$ Borel functions of A.

Theorem 5.3 tells us how to make a Borel choice of $e_1(A)$ and we may as well take $w_1(A) = e_1(A)$. Consider the problem of constructing $w_2(A)$. We can apply Proposition 4.4 (2) to get a non-zero partial isometry $v_1(A)$ with initial projection $f_1(A) \leqslant e_1(A)$ and final projection $g_1(A) \perp e_1(A)$. This could be repeated to get $v_2(A)$ with initial projection $f_2(A) \leqslant e_1(A) - f_1(A)$ and final projection $g_2(A) \perp$ $\perp (e_1(A) + g_1(A))$. If this process were continued transfinitely, the sum of the v's obtained would provide a candidate for $w_2(A)$. The only problem with this construction is that the limit of an uncountable net of Borel functions can fail to be Borel. What is required is a version of this exhaustion process which does not depend on transfinite induction; the technical means of achieving this is provided by Proposition 2.8.

PROPOSITION 6.1. *Let $\mathscr{J}$ be the set of ordered triples (A, e, f) where A is a semi-finite factor on H, and e and f are projections in A with e finite and $e \precsim f$. There is a Borel function $\omega: \mathscr{J} \to W$ satisfying:*

(1) *the initial projection of $\omega(A, e, f)$ is e, and*

(2) *the final projection of $\omega(A, e, f)$ is a subprojection of f.*

Proof. Let $\Phi\colon \mathscr{J} \to \mathscr{C}(W)$ by

$$\Phi(A, e, f) = \{w \in A \cap W \mid w^*w \leqslant e \text{ and } ww^* \leqslant f\}.$$

By Proposition 4.4 (2), Φ is Borel. Define θ on the graph of Φ by

$$\theta(A, e, f, w) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|w^* h_n\|^2 \equiv \eta(w)$$

where the $\{h_n\}_{n=1}^{\infty}$ form a norm-dense sequence in the unit ball of H. Let $\varphi\colon \mathscr{J} \to W$ be one of the selections of Φ guaranteed by Proposition 2.8. Define a sequence $\{\varphi_n\}_{n=1}^{\infty}\colon \mathscr{J} \to W$ inductively as follows:

$$\varphi_1 = \varphi$$

and

$$\varphi_{n+1}(\cdot) = \varphi\left(A, e - \sum_{k=1}^{n} \varphi_k^*(\cdot)\,\varphi_k(\cdot), f - \sum_{k=1}^{n} \varphi_k(\cdot)\,\varphi_k^*(\cdot)\right).$$

Note that both the initial and final projections of the partial isometries $\{\varphi_n(A, e, f)\}_{n=1}^{\infty}$ are mutually orthogonal for each (A, e, f) so the infinite series $\sum_{n=1}^{\infty} \varphi_n(\cdot)$ converges pointwise $*$-strongly to a Borel function $\omega(\cdot)$. Fixing (A, e, f), it is clear that the initial and final projections e_0 and f_0 of $\omega(A, e, f)$ are subprojections of e and f respectively, so it only remains to check that e_0 actually equals e.

If $f_0 = f$, then $e_0 \sim f_0 = f \gtrsim e$, so $e = e_0$ by the finiteness of e. On the other hand, if $e_0 < e$ and $f_0 < f$ both held, then there would be a non-zero partial isometry $w \in A$ orthogonal to all of the $\{\varphi_n(A, e, f)\}_{n=1}^{\infty}$. (We say two partial isometries are *orthogonal* if they have orthogonal initial projections and orthogonal final projections.) Now, by definition of φ, we have $\eta(w) \leqslant 2\eta(\varphi_n(A, e, f))$ for all n. Moreover, since the $\{\varphi_n(A, e, f)\}_{n=1}^{\infty}$ are mutually orthogonal, we have $\lim_{n\to\infty} \eta(\varphi_n(A, e, f)) = 0$. But this leads to the contradiction $\eta(w) = 0$, so w does not exist and the proof is complete. ∎

COROLLARY 6.2. *There is a sequence of Borel functions which associate with each semi-finite but infinite factor A on H the partial isometries $\{\omega_n(A)\}_{n=1}^{\infty}$. For each A, these partial isometries have a common finite initial projection $\varepsilon_1(A)$, and their final projections $\{\varepsilon_n(A)\}_{n=1}^{\infty}$ are mutually orthogonal and supplementary.*

Proof. Let $\varepsilon\colon \mathscr{S} \to E$ be the selection of finite projections constructed in Theorem 5.3. For each $A \in \mathscr{S} \setminus \mathscr{T}$, set $\omega_1(A) = \varepsilon_1(A) = \varepsilon(A)$. Adopting the notations of the proof of Theorem 6.1, define $\omega_n(A)$ inductively by the formula

$$\omega_{n+1}(A) = \omega\left(A, \varepsilon(A), i - \sum_{k=1}^{n} \omega_k(A)\,\omega_k^*(A)\right).$$

The range projections $\{\varepsilon_n(A)\}_{n=1}^{\infty}$ of the $\{\omega_n(A)\}_{n=1}^{\infty}$ are clearly orthogonal; thus by definition of η, we have $\lim_{n\to\infty} \eta(\omega_n(A)) = 0$. Since the construction of ω in the proof of Theorem 6.1 begins with an application of the selector φ, we conclude that the only partial isometry in A whose final projection is orthogonal to $\sum_{n=1}^{\infty} \varepsilon_n(A)$ is zero. This means the $\{\varepsilon_n(A)\}_{n=1}^{\infty}$ are supplementary and completes the proof. ▨

Corollary 6.2 is the result promised in the opening paragraph of the section. The iteration of ω occurring in the proof can be carried out in finite factors as well; although this could have been incorporated in the statement of Corollary 6.2, it seems less awkward to formalize it in a separate corollary.

COROLLARY 6.3. *Let $\mathscr{K}$ denote the set of ordered pairs (A, e) with A a finite factor and e a projection in A. There is a sequence $\{\omega_n\}_{n=1}^{\infty}$ of Borel functions: $\mathscr{K} \to W$ such that for each $(A, e) \in \mathscr{K}$,*

(1) *the initial projections of the $\{\omega_n(A, e)\}$ are all subprojections of e,*

(2) *the final projections of the $\{\omega_n(A, e)\}$ are mutually orthogonal and supplementary, and*

(3) *all but finitely many of the $\{\omega_n(A, e)\}$ are zero.*

Proof. Let ω be as in Proposition 6.1. Set $\omega_1(A, e) = e$. Assuming $\omega_n(A, e)$ to be defined, set

$$\varepsilon_n(A, e) = i - \sum_{k=1}^{n} \omega_k(A, e)\ \omega_k^*(A, e)$$

and define

$$\omega_{n+1}(A, e) = \begin{cases} \omega(A, e, \varepsilon_n(A, e), e) & \text{if } e \precsim \varepsilon_n(A, e) \\ \omega^*(A, \varepsilon_n(A, e), e) & \text{otherwise.} \end{cases}$$

As long as $e \precsim \varepsilon_n(A, e)$, the initial projections of the $\{\omega_n(A, e)\}$ will all be e. Since A is finite and the final projections of the $\{\omega_n(A, e)\}$ are clearly mutually orthogonal, this will stop after finitely many steps, and then there can be at most one more non-zero $\omega_n(A, e)$. ▨

The following lemma on linear functionals is included in this section because its proof depends on an exhaustion argument. Write $\mathscr{G}$ for the set of ordered pairs (A, a) where A is a finite factor and $a \in A$. Suppose $\eta: \mathscr{G} \to \mathbf{C}$ is Borel and for each A, the functional $a \to \eta(A, a)$ is linear, self-adjoint, contractive, and ultraweakly continuous. For example, η might be given by $\eta(A, a) = \langle ah, h \rangle$ for some fixed $h \in H$, or $\eta(A, a) = \tau_A(a)$, or (the application we will make in § 7) a real linear combination of these types.

LEMMA 6.4. *Let $\eta: \mathscr{G} \to \mathbf{C}$ be as above, and suppose $\eta(A, i) = 0$ for all finite factors A. Then there is a Borel function $\varepsilon: \mathscr{T} \to E\setminus\{0\}$ such that for each finite factor $A \in \mathscr{T}$, the projection $\varepsilon(A)$ has the property that $\eta(A, a) \leqslant 0$ for all $a \in A$* with $0 \leqslant a \leqslant \varepsilon(A)$.

Proof. Let

$$\mathscr{L} = \{(A, e) \in \mathscr{G} \mid e \in E\},$$

and define $\Phi: \mathscr{L} \to \mathscr{C}(E)$ by $\Phi(A, e) = \{f \in A \cap E \mid f \leqslant e\}$. That Φ is Borel is a consequence of Proposition 4.4 (3). Now let

$$\mathscr{E} = \{(A, e, f) \text{ in the graph of } \Phi \mid \eta(A, f) > 0\}.$$

Applying Proposition 2.7, we conclude that the map Ψ sending (A, e) to the $*$-strong closure of $\mathscr{E}_{(A,e)}$ is Borel. In order to make sure the domain of Ψ is all of $\mathscr{L}$, we replace it by its union with the constant set-valued map $(A, e) \to \{0\}$. Recapitulating then, Ψ is the Borel map from $\mathscr{L}$ into $\mathscr{C}(E)$ which sends (A, e) to the $*$-strong closure of $\{f \in A \cap E \mid \text{either } f = 0 \text{ or } f \text{ is a subprojection of } e \text{ with } \eta(A, f) > 0\}$. Define θ on the graph of Ψ by $\theta(A, e, f) = \eta(A, f)$, and let $\psi: \mathscr{L} \to E$ be a selector for Ψ given by Proposition 2.8.

Define $\delta_n: \mathscr{T} \to E$ inductively by $\delta_1(A) = \psi(A, i)$, and $\delta_{n+1}(A) = \psi\left(A, i - \sum_{k=1}^{n} \delta_k(A)\right)$. Set

$$\varepsilon(A) = \begin{cases} i, & \text{if } \eta(A, \delta_1(A)) = 0 \\ i - \sum_{n=1}^{\infty} \delta_n(A), & \text{if } \eta(A, \delta_1(A)) > 0 . \end{cases}$$

Note that by definition of ψ, if e is a projection in the finite factor A which is a subprojection of $\varepsilon(A)$, we must have $\eta(A, e) \leqslant 0$, so by spectral theory, $\eta(A, a) \leqslant 0$, for all $0 \leqslant a \leqslant \varepsilon(A)$. Finally we note that $\varepsilon(A)$ is non-zero: indeed, if $\eta(A, \delta_1(A)) > 0$, then $\eta(A, \varepsilon(A)) = -\sum_{n=1}^{\infty} \eta(A, \delta_n(A))$ is strictly negative. ∎

7. EXTENSION OF TRACES

Every trace on a semi-finite factor is induced by a bounded linear operator on the underlying Hilbert space. In this section, we make a Borel choice of these operators.

PROPOSITION 7.1. *There is a Borel map φ from $\mathcal{T}$ into the finite rank positive contractions of H satisfying $\tau_A(a) = \tau(a\varphi(A))$ for all $a \in A \in \mathcal{T}$.*

Proof. Fix a unit vector $h \in H$. With $\mathcal{G}$ as in Lemma 6.4, take $\eta: \mathcal{G} \to \mathbf{C}$ by $\eta(A, a) = \tau_A(a) - \langle ah, h \rangle$; let $\varepsilon: \mathcal{T} \to E \setminus \{0\}$ be as in the conclusion of that lemma. Next, with $\{\omega_n\}_{n=1}^{\infty}$ as in Corollary 6.3, define $v_n: \mathcal{T} \to W$ by $v_n(A) = \omega_n(A, \varepsilon(A))$. We then have $v_n^*(A) v_n(A) \leqslant \varepsilon(A)$ for each n, and $\sum_{n=1}^{\infty} v_n(A) v_n^*(A) = i$ for each $A \in \mathcal{T}$. An elementary computation shows that for $a \in A \cap P$, with $A \in \mathcal{T}$, we have

$$\tau_A(a) = \sum_{n=1}^{\infty} \tau_A(v_n^*(A) a v_n(A)) \leqslant \sum_{n=1}^{\infty} \langle v_n^*(A) a v_n(A) h, h \rangle.$$

Since this inequality continues to hold when a is replaced by $i - a$, we actually have equality. The proof is thus completed by taking $\varphi(A) = \sum_{n=1}^{\infty} v_n(A) h \otimes v_n(A) h$; this operator is always finite rank since all but finitely many of the $\{v_n(A)\}$ vanish. ▨

It remains to generalize Proposition 7.1 to the semi-finite case. We first consider the problem for a single semi-finite factor A, i.e., we do not worry about measurability. As mentioned in the introduction to § 6, there is a sequence $\{w_n\}_{n=1}^{\infty}$ of partial isometries in A all having a common finite initial projection e_1 and final projections $\{e_n\}_{n=1}^{\infty}$ which are mutually orthogonal and supplementary. Then $e_1 A e_1$ is a finite factor acting on the Hilbert space $e_1 H$. Let $\pi: e_1 A e_1 \to L(H)$ by $\pi(a) = \sum_{n=1}^{\infty} w_n a w_n^*$. (It is helpful to think matricially here: a is an infinite matrix with a single non-zero entry in the (1,1) position, while $\pi(a)$ has this entry repeated all along the diagonal.) Then π is an algebraic isomorphism onto a finite factor B on H. Write $\tau_A(\tau_B)$ for the trace on $A(B)$ normalized by taking $\tau_A(e_1) = 1$ (respectively $\tau_B(i) = 1$).

LEMMA 7.2. *Suppose t is a positive trace class operator in $L(H)$ which induces τ_B in the sense that $\tau_B(b) = \tau(bt)$ for all $b \in B$. Let $s = \sum_{i,j} w_i w_j^* t w_j w_i^*$. Then s is a positive contraction satisfying the following conditions:*

(1) *The domain of τ_A consists of precisely those operators $a \in A$ with $\sqrt{s}\, a \sqrt{s}$ of trace class in $L(H)$.*

(2) *For a in the domain of τ_A, we have $\tau_A(a) = \tau(\sqrt{s}\, a \sqrt{s})$.*

Proof. The operator $\sum_j w_j^* t w_j$ is positive, has trace less than or equal to $\tau(t)$, and is supported on $e_1 H$. Since the projections $w_i w_i^*$ are mutually orthogonal, the sum defining s converges strongly to a positive contraction.

Suppose first that $a \in e_k(A \cap P)e_k$. Then $as = aw_k(\sum_j w_j^* t w_j) w_k^*$. On the other hand, $\tau_A(a) = \tau_A(w_k^* a w_k) = \tau_B(\pi(w_k^* a w_k))$. Since $\pi(w_k^* a w_k)$ commutes with each e_j, we thus have $\tau_A(a) = \tau\left(\sum_{j=1}^{\infty} \pi(w_k^* a w_k) e_j t e_j\right)$ whence it follows that $\tau_A(a) = \tau(as)$. Since s has an orthogonal basis of eigenvectors, we conclude that $\tau(as) = \tau(\sqrt{s} a \sqrt{s})$. Thus formula (2) holds for all a in the linear span of the $\{e_i A e_i\}$.

Now let a be an arbitrary element of $A \cap P$. The projections $\left\{\sum_{i=1}^{n} e_i\right\}_{n=1}^{\infty}$ converge strongly to the identity and thus by normality of τ_A and τ, we conclude $\tau_A(a) < \infty$ iff $\tau(\sqrt{s} a \sqrt{s}) < \infty$, with equality hodling in the finite case. The proof is completed by the observation that the map $a \to \sqrt{s} a \sqrt{s}$ preserves decompositions into real and imaginary and positive and negative parts. ▨

THEOREM 7.3. *There is a Borel function $\psi: \mathscr{S} \to C$ such that for any $A \in \mathscr{S}$,*

(1) *The trace class operators of A are precisely those operators $a \in A$ with $\psi(A) a \psi(A)$ of trace class in $L(H)$.*

(2) *The map $a \to \tau(\psi(A) a \psi(A))$ is a trace in A.*

Proof. Write $\mathscr{G}$ for the set of ordered pairs (A, a) with A semi-finite and $a \in A$, and let $\sigma: \mathscr{G} \to C$ by $\sigma(A, a) = \sum_j \omega_j(A) a\, \omega_j(A)^*$. Then for fixed A, the map $a \to \sigma(A, a)$ takes dense sequences of A to dense sequences of a von Neumann algebra $\pi(A)$, so π is a Borel map from $\mathscr{S}$ into $\mathscr{T}$. Let $\varphi: \mathscr{T} \to C$ be the function given by Proposition 7.1. Take $\psi(A)$ to be the positive square root of $\sum_{ij} \omega_i(A) \omega_j(A)^* \varphi(\pi(A))\, \omega_j(A) \omega_i(A)^*$. Then ψ is Borel and, by virtue of Lemma 7.2, it satisfies properties (1) and (2). ▨

8. SOME OPEN PROBLEMS

The following problems are suggested by the body of this paper.

PROBLEM 8.1. Prove directly that the set of semi-finite von Neumann algebras on H is Borel.

PROBLEM 8.2. For X Polish, find general conditions on Borel maps into $\mathscr{C}(X)$ guaranteeing that their intersection is also Borel.

PROBLEM 8.3. Is there a Borel method of choosing a maximal projection from each closed subset of E?

In this paper, we have dealt with the classification and structure of factor von Neumann algebras. It is natural to ask whether this can be carried out in non-factors. One way to approach this is to employ direct integral theory. Actually all proofs of the fact that the set of global semi-finite von Neumann algebras is Borel rely on this technique. This is somewhat awkward, and suggests Problem 8.1.

It is interesting to observe how the proof of Theorem 5.3 fails in this connection. Although there is no canonical trace on a semi-finite factor, its trace class is independent of which trace one chooses. This fact was used in the proof of Theorem 5.3 to show that the set

$$\mathscr{R} = \{(A, t) \mid A \in \mathscr{F},\ t \in A \cap P,\ \tau_A(t) < \infty\}$$

is Borel and has σ-compact sections — without having to make a choice of τ_A. By contrast, when A is not a factor, $\{t \in A \cap P \mid \tau_A(t) < \infty\}$ depends on the choice of τ_A. Now, on the one hand, a haphazard choice of trace classes $\{T_A\}$ may leave $\{(A, t) \mid A \in \mathscr{A},\ t \in \mathscr{A} \cap P \cap T_A\}$ non-Borel, while

$$\{(A, t) \mid A \in \mathscr{A}, t \in A \cap P,\ \ t \text{ belongs to any (faithful) trace class}\},$$

which can be shown to be Borel, no longer has σ-compact sections. The hope of resolving Problem 8.1 along the lines of this paper thus lies in making a different kind of choice of $\mathscr{R}$ and/or finding a different selection theorem to apply.

Problem 8.2 is suggested by the ad hoc nature of the proofs of Propositions 2.7 and 4.4. Although Christensen [1] shows that intersection is not globally well-behaved — his Theorem 8 even implies the intersection of the identity map on $\mathscr{C}(X)$ with a constant map can fail to be Borel — it should be possible to obtain some positive results.

Problem 8.3 is motivated by the exhaustion arguments of Section 6. Proposition 2.8 is clearly a primitive tool for these, and it would be useful to have more powerful techniques for choosing maximal projections and partial isometries.

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