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CALCULUSES, ANNIHILATORS AND HYPERINVARIANT SUBSPACES

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0. INTRODUCTION

A large amount of recent achievements in operator theory is connected with the problem of invariant subspaces. It remains unknown however whether every bounded operator T in a Banach (or even in a Hilbert) space B has a nontrivial invariant subspace, i.e. a closed subspace X such that $TX \subset X$ and $\{0\} \neq X \neq B$.

One way to an invariant subspace (IS) goes through a functional calculus. We shall also follow this pattern. The words "functional calculus for an operator T" usually mean a continuous homomorphism φ of a function algebra, containing z (= the identity map of C) into $\mathscr{L}(B)$ (= the algebra of all bounded operators) such that $\varphi(z) = T$.

The Riesz-Dunford calculus is defined on the set of functions analytic in a neighbourhood of $\sigma(T)$ (= the spectrum of T) by the formula

$$\text{(R)} \qquad \varphi(f) = f(T) = -\frac{i}{2\pi}\int_{\Gamma} f(\lambda)\mathrm{R}(\lambda, T)\,\mathrm{d}\lambda,$$

Γ being a contour surrounding $\sigma(T)$ and $\mathrm{R}(\lambda, T) \stackrel{\mathrm{def}}{=} (T-\lambda I)^{-1}$. This calculus enables us to find invariant subspaces corresponding to the isolated parts of the spectrum. Quite an opposite position is occupied by the $L^{\infty}(\sigma(T))$-calculus for normal operators, the richest one from the point of view of functions and the poorest one from the point of view of operators. In the common practice intermediate cases are considered.

First of all we should like to mention a calculus based on the Cauchy-Green formula, developed by E. M. Dyn'kyn [11]. It is defined on the set of functions analytic in a domain Ω containing $\sigma(T)$ in its closure, and sufficiently smooth up to $\partial\Omega$. The rate of smoothness depends on the rate of growth of the resolvent near the domain. Some results of this paper can be obtained on the base of this calculus.

J. Wermer has constructed a calculus (the "$W(T)$-calculus") applicable to operators with spectrum on the unit circumference $\mathbf{T}$:

$$W(T) \stackrel{\text{def}}{=} \{f \in L^1(\mathbf{T}) : \sum_{-\infty}^{\infty} |\hat{f}(n)| \, \|T^n\| < +\infty\};$$

$$\varphi(f) = f(T) \stackrel{\text{def}}{=} \sum_{-\infty}^{\infty} \hat{f}(n) T^n.$$

We shall also use the algebra $W_A(T) = W(T) \cap H^\infty$. (Throughout the paper we use the technique of spaces H^p and $E^p(\Omega)$. For references see [4].) With the aid of the $W(T)$-calculus J. Wermer proved a theorem which originated the approach we discuss. Namely, T has a nontrivial IS provided

$$\text{(W)} \qquad \sum_{-\infty}^{\infty} \frac{\log \|T^n\|}{1+n^2} < \infty$$

and $\sigma(T)$ contains more than a single point. (Note that (W) implies $\sigma(T) \subset \mathbf{T}$.

Then it was proved by Y. I. L'ubich and V. I. Matsaev [14] that if $\sigma(T)$ lies on a smooth curve γ, it contains more than one point and if T satisfies the following nonquasianalyticity condition:

$$\int_0 \log_+ \log_+ \max\{\|R(\lambda, T)\| ; \operatorname{dist}(\lambda, \gamma) \geqslant t\} \, dt < \infty$$

then T has a nontrivial IS. The proofs of these theorems virtually show much more, namely, that T is decomposable (see [2]), and the IS's they yield are in fact maximal spectral subspaces corresponding to an arbitrary closed subset of $\sigma(T)$. These IS's are hyperinvariant, i.e. they are invariant for all operators commuting with T. We shall write HIS for a hyperinvariant subspace.

When $\sigma(T)$ is a single point the decomposability does not work. A. Atzmon [1] was the first to apply a functional calculus in this case. The main result of [1] is

THEOREM A1. *Let T be an invertible operator in a Banach space, such that for some $k \in \mathbf{N}$, $c_1, c_2 > 0$,*

$$\text{(A)} \qquad \|T^n\| \leqslant cn^k, \quad n > 0, \ k \in \mathbf{N};$$

$$\text{(AA)} \qquad \|T^{-n}\| \leqslant c_1 \exp(c_2 n^{1/2}), \quad n > 0, \ c_1, c_2 > 0.$$

Then either $T = \lambda I$, $\lambda \in \mathbf{C}$, or T has a nontrivial HIS. *If also $\sigma(T) = \{\lambda_0\}$ then either $(T - \lambda_0 I)^{k+1} = \mathbf{O}$, or T has an uncountable chain of nontrivial* HIS'*s. (Note that in the case $\sigma(T) \neq \{\lambda_0\}$ the statement of Theorem* A1 *follows from Wermer's theorem.)*

A. Atzmon employs the following scheme: calculus → annihilator → HIS. By an annihilator we mean an $f \not\equiv 0$ such that $f(T) = \mathbf{O}$ in some calculus. If it exists we write $T \in C_0(\mathfrak{A})$, where $\mathfrak{A}$ is the algebra of the calculus. (We denote by C_0 the class $C_0(H^\infty)$.) The passage annihilator → HIS is interpreted by N. K. Nikolskiĭ in the following way:

LEMMA. *Let $T \in C_0(\mathfrak{A})$. Suppose that every closed ideal $A \subset \mathfrak{A}$ has the property: either $A = \{f \in \mathfrak{A} : f(\lambda) = 0\}$, $\lambda \in \mathbf{C}$, or there exist $f \notin A$, $g \notin A$ such that $fg \in A$. Then either T has a nontrivial* HIS, *or $T = \lambda I$, $\lambda \in \mathbf{C}$.*

Proof. Consider $A = \{f \in \mathfrak{A} : f(T) = \mathbf{O}\} \neq \{0\}$. It is a closed ideal in $\mathfrak{A}$ (by the continuity and multiplicativity of the calculus). If there exist $f, g \notin A$ such that $fg \in A$, we have $\{0\} \subsetneq \operatorname{Im} g(T) \subset \operatorname{Ker} f(T) \subsetneq B$ and $\operatorname{Ker} f(T)$ is the required HIS. The alternative implies that $z - \lambda \in A$ for some $\lambda \in \mathbf{C}$ and then $T = \lambda I$. ▨

Our paper is in a sense a development of Atzmon's results. They are extended onto a wider class of operators. An important step is the choice of a calculus. It is convenient for our purposes to define the "calculus" on some linear set $\mathscr{F}$ (which is not an algebra). An analog of the equality $z(T) = T$ holds in a "limit" sense. The multiplicativity holds for some pairs $f, g \in \mathscr{F}$ with $fg \in \mathscr{F}$. The proposed calculus is based on the formula (R) and works when the spectrum is "thin". The definition of an annihilator remains unchanged. The passage annihilator → HIS is based on considerations close to those of Lemma.

Let us now turn to an approximate description of our results. Let $\sigma(T) = \{\lambda_0\}$ and suppose there is a Jordan domain Ω with a rectifiable boundary $\partial\Omega$ such that

$$\|R(\lambda, T)\| \leqslant \varphi_1(|\lambda - \lambda_0|), \quad \lambda \notin \Omega;$$

$$\|R(\lambda, T)\| \leqslant \varphi_2(|\lambda - \lambda_0|), \quad \lambda \in \Omega.$$

The growth of φ_2 corresponds to the critical Phragmén-Lindelöf growth in λ_0 for Ω; the growth of φ_1 is essentially weaker. Then either T has a nontrivial HIS, or the upper estimate for $\|R(\lambda, T)\|$ inside Ω is also $c\varphi_1(|\lambda - \lambda_0|)$. Thus, to be sure of the existence of a HIS we have to exclude the cases of symmetrical growth of the resolvent, either imposing an extra condition or using the fact that $T - \lambda_0 I$ must be nilpotent when $\varphi_1 = x^{-n}$. Theorem A1 gives an example of application of such a scheme with $\Omega = \mathbf{D}$ (the unit circle) and $\lambda_0 \in \mathbf{T}$.

The existence of an IS can be sometimes deduced from a "weak" estimate in Ω:

$$|(R(\lambda, T)x, y)| \leqslant \varphi_2(|\lambda - \lambda_0|), \quad \lambda \in \Omega,$$

where $x \in B$, $y \in B^*$ are fixed and nonzero. The existence of an annihilator does not always imply the existence of an IS. The paper [1] contains the following

PROPOSITION A. *Let $\|T^n\| = O(\exp c|n|^\alpha)$, $n \to \pm\infty$, $\alpha < 1/2$. Then $T \in C_0(W_A(T))$.*

The $W_A(T)$-algebra has prime nonmaximal ideals and Lemma cannot be applied.

We extend the result of this Proposition up to its natural bounds. We also produce an example, answering in the negative two questions posed by A. Atzmon in [1].

I wish to express my deep gratitude to N. K. Nikolskiĭ for his constant guidance and encouragement. I am thankful to G. Y. Bomash, E. D. Gluskin and V. P. Havin for helpful discussions.

1. FUNCTIONAL CALCULUS FOR OPERATORS WITH THIN SPECTRUM ON $\partial\Omega$

1°. DEFINITION 1. Let $\Omega \subset \mathbf{C}$ be a star-shaped domain (with the centre λ_0) with rectifiable boundary. Let $T \in \mathscr{L}(B)$, $\sigma(T) \subset \bar{\Omega}$ and suppose the set $\sigma(T) \cap \partial\Omega$ has zero linear measure. The function

$$K(\lambda) \stackrel{\text{def}}{=} \sup_{r \geqslant 1} \|\mathrm{R}(\lambda_0 + r(\lambda - \lambda_0), T)\|, \quad \lambda \in \partial\Omega,$$

is measurable and a.e. finite. We assume that there exists an $h \in H^\infty(\Omega)$ such that

$$(1) \qquad |h(\lambda)K(\lambda)| \leqslant c \quad \text{a.e. on } \partial\Omega.$$

The 1-*calculus* is defined for functions $f = hg$, $g \in E^1(\Omega)$ by the Riesz formula; we write $\partial\Omega$ for the boundary contour with the natural orientation.

$$(2) \qquad f(T) \stackrel{\text{def}}{=} -\frac{i}{2\pi} \int_{\partial\Omega} f(\lambda)\mathrm{R}(\lambda, T)\, d\lambda.$$

NOTE. We may suppose that h is an outer function (for definition see [4]).

PROPOSITION 1.1. *The 1-calculus has the following properties: (for the sake of convenience we consider $\lambda_0 = 0$)*

(i) "*Continuity*": $\|f(T)\| \leqslant (2\pi)^{-1}c\|g\|_1 \stackrel{\text{def}}{=} \frac{c}{2\pi} \int_{\partial\Omega} |g(\lambda)|\,|d\lambda|$.

(ii) "*Permanence*": $f_r(T) \to f(T)$, $r \to 1-$.

Here $f_r(\lambda) \stackrel{\text{def}}{=} f(\lambda r) \in E^1(r^{-1}\Omega)$, *so* $f_r(T)$ *is defined by the Riesz-Dunford calculus.*

(iii) "*Multiplicativity*":

(a) $(hg)(T)\,p(T) = [h(gp)](T)$; $\quad p \in \mathscr{P}_A, g \in E^1(\Omega)$.

(b) $(hg_1)(T)\,(hg_2)(T) = (h(g_1hg_2))\,(T)$; $g_1, g_2, g_1g_2 \in E^1(\Omega)$.

(By $\mathscr{P}_A$ we denote the set of polynomials.)

(iv) $\sigma(f(T)) \supset f(\sigma(T) \cap \Omega)$.

Proof. Since (i) follows immediately from Definition 1, we start with the property (ii).

Choose F_ε to be relatively open in $\partial\Omega$, containing $\sigma(T) \cap \partial\Omega$, and with $|F_\varepsilon| < \varepsilon$. The set $S \stackrel{\text{def}}{=} \mathbf{C} \setminus (\Omega \cup \bigcup_{\rho>0} \rho F_\varepsilon)$ is closed and does not intersect $\sigma(T)$, therefore $\|\mathrm{R}(\lambda, T)\| \leqslant c_\varepsilon$, $\lambda \in S$. Further, when $r > 1/2$

$$(hg)(T) - (h_r g_r)(T) = \frac{\mathrm{i}}{2\pi} \int_{\partial\Omega} h(\lambda) g(\lambda) [\mathrm{R}(\lambda, T) - r^{-1}\mathrm{R}(r^{-1}\lambda, T)]\, \mathrm{d}\lambda$$

and thus

$$2\pi \|(hg)(T) - (h_r g_r)(T)\| \leqslant \int_{\partial\Omega \setminus F_\varepsilon} |h(\lambda) g(\lambda)|\, \|\mathrm{R}(\lambda, T) - r^{-1}\mathrm{R}(r^{-1}\lambda, T)\|\, |\mathrm{d}\lambda| +$$

$$+ \int_{F_\varepsilon} |h(\lambda) g(\lambda)| (\|\mathrm{R}(\lambda, T)\| + 2\|\mathrm{R}(r^{-1}\lambda, T)\|)\, |\mathrm{d}\lambda| \leqslant$$

$$\leqslant c_\varepsilon^2 \|T\| (r^{-1} - 1)\, \|g\|_1 \|h\|_\infty + 3c \int_{F_\varepsilon} |g(\lambda)|\, |\mathrm{d}\lambda|.$$

Choosing first ε and then r we obtain (ii).

The "multiplicativity" properties follow easily from (ii) and from the multiplicativity of the Riesz-Dunford calculus.

Finally, if $T - \lambda I$ has no inverse and $\lambda \in \Omega$, then $f_r(T) - f_r(\lambda)$ has no inverse and so does $f(T) - f(\lambda)$ (make $r \to 1$ and consider the limit in the uniform topology). ▨

Now we shall introduce typical situations when the 1-calculus can be constructed.

DEFINITION. $V_{a,m,\varepsilon} \stackrel{\text{def}}{=} \{x + \mathrm{i}y \in \mathbf{C} : 0 \leqslant x \leqslant \varepsilon,\ |y| \leqslant ax^m\}$, $a, \varepsilon > 0$, $m \in \mathbf{N}$. We say that a domain Ω belongs to the *class* (m) if it satisfies the requirements of Definition 1 for $\lambda_0 = 0$ and

$$\exists\, a, \varepsilon > 0,\ \forall\, \lambda \in \partial\Omega,\ V_{a,m,\varepsilon}\, \mathrm{e}^{\mathrm{i} \arg \lambda} + \lambda \cap \Omega = \emptyset$$

(see Figure 1).

PROPOSITION 1.2. *Let Ω be of the class (m), $\sigma(T) \subset \bar{\Omega}$ and let $\mathscr{E} = \sigma(T) \cap \partial\Omega$ be the set of nonuniqueness for the class $C_A^{mn}(\Omega) \stackrel{\text{def}}{=} \{f \in C_A(\Omega) : f^{(mn)} \in C(\bar{\Omega})\}$. Let also*

$$\|\mathrm{R}(\lambda, T)\| \leqslant c\, \mathrm{dist}^{-n}(\lambda, \mathscr{E}), \quad \lambda \notin \Omega.$$

Then T admits a 1-calculus in Ω.

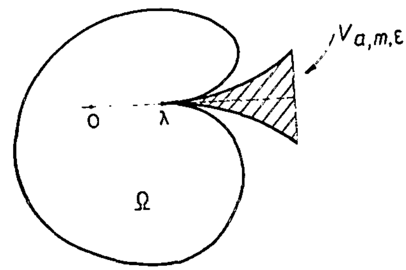

Fig. 1.

Proof. For $r > 1$,

$$\|R(r\lambda,T)\| \leqslant c\,\mathrm{dist}^{-n}(r\lambda, \mathscr{E}) \leqslant \tilde{c}\,\mathrm{dist}^{-mn}(\lambda, \mathscr{E}).$$

The last estimate can be deduced from simple geometrical observations. Thus $K(\lambda) \leqslant \tilde{c}\,\mathrm{dist}^{-mn}(\lambda, \mathscr{E})$. There is an $h \in C_A^{mn}(\Omega)$ such that $h^{(k)}|\mathscr{E} = 0$; $k = 0, \ldots, mn-1$. Then $|h(\lambda)| \leqslant \mathrm{const}\ \mathrm{dist}^{mn}(\lambda, \mathscr{E})$ and (1) is satisfied which is all that is needed for the 1-calculus. ▨

PROPOSITION 1.3. *Let Ω be the unit disc* $\mathbf{D}$, $\sigma(T) \subset \bar{\mathbf{D}}$, $\sigma(T) \cap \mathbf{T} = \{1\}$; $\|R(\lambda, T)\| \leqslant \Phi(|\lambda - 1|)$, $\lambda \notin \mathbf{D}$. *Suppose Φ is decreasing and* $\int_0 \log \Phi(x)\,dx < \infty$.

Then T admits a 1-calculus in Ω.

Proof. Clearly $K(\lambda) \leqslant \Phi(|\lambda - 1|)$. There exists an outer function $h \in H^\infty$ such that $|h(e^{it})| = 1/\Phi(|e^{it} - 1|)$ and (1) is satisfied. ▨

2°. DEFINITION 2. Let $\Omega \subset \mathbf{C}$ be a domain of class (S) (V.I. Smirnov's class [4]) and $\partial\Omega$ — a simple rectifiable curve. Let

$$T \in \mathscr{L}(B), \quad \sigma(T) \subset \bar{\Omega}, \quad \sigma(T) \cap \partial\Omega = \{\lambda_0\}$$

and

$$\|R(\lambda, T)\| \leqslant c\,|\lambda - \lambda_0|^{-n}, \quad \lambda \notin \Omega. \tag{3}$$

The *2-calculus* is defined for functions $f = (z - \lambda_0)^n g$, $g \in E^1(\Omega)$ by the Riesz formula (2).

NOTE. The 2-calculus imposes more restrictions on the operator and less on the domain than the 1-calculus. However when $\Omega - \lambda_0 \in (m)$ one can choose $h(\lambda) = (\lambda - \lambda_0)^n$ and both calculuses will coincide. In fact we shall deal only with such cases, but it is convenient to consider the 2-calculus separately because the estimate (3) takes place rather often and makes possible a slightly different approach.

PROPOSITION 1.4. *The 2-calculus has the following properties. (We put $\lambda_0 = 0$.)*

(i) *"Continuity": $\|f(T)\| \leqslant c\,(2\pi)^{-1}\|g\|_1$;*

(ii) *"Permanency": for functions analytic in a neighbourhood of $\bar{\Omega}$ the 2-calculus agrees with the Riesz-Dunford calculus;*

(ii′) *The weak convergence $g_k \to g$ in $E^1(\Omega)$ implies the weak operator convergence $(z^n g_k)(T) \xrightarrow{w} (z^n g)(T)$.*

(iii) *Multiplicativity:*

$$\text{(a)}\ (z^n gp)(T) = (z^n g)(T)p(T);\quad p \in \mathscr{P}_A,\ g \in E^1(\Omega);$$

$$\text{(b)}\ [z^n(g_1 z^n g_2)](T) = (z^n g_1)(T)\,(z^n g_2)(T);\quad g_1 \in E^1(\Omega),\ g_2 \in H^\infty(\Omega);$$

(iv) $\sigma(f(T)) \supset f(\sigma(T) \cap \Omega)$.

Proof. (i) is straightforward so we pass to the property (ii). In the Riesz-Dunford calculus $f^{(R)}(T) = \frac{i}{2\pi}\int_\Gamma f(\lambda)\,R(\lambda, T)\,d\lambda$, where Γ surrounds the spectrum. For Γ choose Γ_ε, the contour $\partial(\Omega \cup B(\lambda_0, \varepsilon))$ with the natural orientation where $B(\lambda_0, \varepsilon) = \{z : |z - \lambda_0| < \varepsilon\}$. We have:

$$\frac{2\pi}{i}(f^{(R)}(T) - f(T)) = \Big(\int_{\Gamma_\varepsilon \cap \partial B(\lambda_0, \varepsilon)} - \int_{\partial\Omega \cap B(\lambda_0, \varepsilon)}\Big) f(\lambda)\,R(\lambda, T)\,d\lambda,$$

hence

$$2\pi\|f^{(R)}(T) - f(T)\| \leqslant (2\pi\varepsilon + |\partial\Omega \cap B(\lambda_0, \varepsilon)|)c\|g\|_\infty \xrightarrow{\varepsilon \to 0} 0.$$

(ii′) is the consequence of the fact that $(R(\lambda, T)x, y)\lambda^n \in L^\infty(\partial\Omega)$.

(iii). We check only (b) as a less trivial case. Since Ω belongs to the class (S), g_1 can be approximated by polynomials p_k in E^1 (see [4]). $\partial\Omega$ being a simple rectifiable curve, we can choose a sequence $\{q_k\}$ of polynomials, converging to g_2 in the weak-star topology of $H^\infty(\Omega)$ (see [5], Chapter VI, 5°). Then $z^n p_k q_k \to z^n g_1 g_2$ weakly in $E^1(\Omega)$. Making use of the properties (ii′), (i), we obtain $(z^n q_k)(T) \xrightarrow{w} (z^n g_2)(T)$ and $(z^n p_k)(T) \to (z^n g_1)(T)$ (in norm). Hence

$$(z^n p_k)(T)\,(z^n q_k)(T) \xrightarrow{w} (z^n g_1)(T)(z^n g_2)(T).$$

But

$$(z^n p_k)(T)\,(z^n q_k)(T) = [z^n(p_k z^n q_k)](T) \xrightarrow{w} [z^n(g_1 z^n g_2)](T),$$

so the required equality is established.

(iv). We have $f = z^n g$, $g \in E^1(\Omega)$. As above, we choose $q_k \in \mathscr{P}_A$ such that $q_k \to g$ in $E^1(\Omega)$ and then $(z^n q_k)(T) \to f(T)$. From this point the argument repeats that of the Proposition 1.1 (iv). ▨

2. ANNIHILATORS

Let $T \in \mathscr{L}(B)$ and suppose that T admits a 1- or 2-calculus. Question arises whether a function $g \in E^1(\Omega)$, $g \not\equiv 0$, exists such that $(hg)(T) = \mathbf{O}$. If the answer is affirmative, hg is naturally called an *annihilator* for T. (In the 2-calculus we always set $h(z) = (z - \lambda_0)^n$.)

Let hg be an annihilator. $\partial\Omega$ being rectifiable, $g \in E^1(\Omega)$ can be represented in the form g_1/g_2, where $g_i \in H^\infty(\Omega)$. Thus, using the multiplicativity (b) of the calculus we obtain that $h(hg_1)$ is also an annihilator. So we shall always choose an annihilator hg with $g \in H^\infty(\Omega)$.

THEOREM 1. *Let $g \in H^\infty(\Omega)$. The following statements are equivalent:*

(i) $(hg)(T) = \mathbf{O}$.

(ii) $\forall x \in B, y \in B^*$, $\mathscr{F}_{x,y} \overset{\text{def}}{=} hg(\mathrm{R}(\cdot, T)x, y) \in H^\infty(\Omega)$.

(iii) $|h(\lambda)g(\lambda)| \, \|\mathrm{R}(\lambda, T)\| \leqslant \text{const}, \quad \lambda \in \Omega\setminus\sigma(T)$.

Proof. (iii) $\Rightarrow$ (ii). Since hg is analytic in Ω, it vanishes on the spectrum. Hence T can have only isolated points of spectrum in Ω. Now (ii) follows from the standard removable singularity theorem.

(ii) $\Rightarrow$ (i). We have $((hg)(T)x, y) = \int_{\partial\Omega} \mathscr{F}_{x,y}(\lambda)\, \mathrm{d}\lambda = 0$ (see [9], Chapter X, § 5, Theorem 1).

(i) $\Rightarrow$ (iii). The function $\mathscr{F}_{x,y}$ is analytic in $\Omega\setminus\sigma(T)$ and has nontangential limits a.e. on $\partial\Omega$, for $\sigma(T) \cap \partial\Omega$ is a closed set of zero linear measure. The boundary function $\mathscr{F}_{x,y}|\partial\Omega$ is summable (and even bounded) (see (1)).

$$\int_{\partial\Omega} \lambda^n \mathscr{F}_{x,y}(\lambda)\, \mathrm{d}\lambda = \frac{2\pi}{\mathrm{i}} (T^n \cdot (hg)(T)x, y) = 0, \qquad n \in \mathbf{N},$$

using the multiplicativity (a) of the calculuses. Thus we conclude ([9], Chapter X, § 4, Theorem 2) that $\mathscr{F}_{x,y}$ is the Cauchy integral of its boundary values and hence $\mathscr{F}_{x,y} \in E^1(\Omega)$ ([9], Chapter X, § 5, Theorem 2). (More precisely: $\mathscr{F}_{x,y}$, defined on $\Omega\setminus\sigma(T)$ can be extended on the whole of Ω, resulting in an $E^1(\Omega)$-function.) Finally, using the maximum principle:

$$\sup_{\lambda\in\Omega\setminus\delta(T)} |hg(\lambda)| \, \|\mathrm{R}(\lambda, T)\| \leqslant \sup_{\substack{\lambda\in\Omega \\ \|x\|\leqslant 1, \|y\|\leqslant 1}} |\mathscr{F}_{x,y}(\lambda)| \leqslant c \|g\|_\infty. \qquad \blacksquare$$

COROLLARY.

$$h(T) = \mathbf{O} \Leftrightarrow \sup_{\lambda\in\Omega\setminus\delta(T)} |h(\lambda)| \, \|\mathrm{R}(\lambda, T)\| < \infty.$$

NOTE. In the case when $h(T) = \mathbf{O}$ the calculus is trivial. On the other hand the estimate $\sup_{\lambda \in \partial\Omega} |h(\lambda)| \, \|R(\lambda, T)\| < \infty$ holds according to Definitions 1 and 2. It expresses the fact that our calculuses can be used only when the behaviour of the resolvent is different on different directions approaching the spectrum point.

Now we pass to our "typical situations".

PROPOSITION 2.1. *Let $\sigma(T) = \{0\}$ and for some β, $0 < \beta \leqslant \pi$,*

$$\|R(\lambda, T)\| \leqslant k|\lambda|^{-n}, \quad |\arg \lambda| \geqslant \beta; \tag{4}$$

$$\|R(\lambda, T)\| \leqslant k_1 \exp(k_2|\lambda|^{-\pi/2\beta}), \quad \lambda \in \mathbf{C}.$$

Then T has an annihilator.

Proof. For $\beta < \pi$ we consider $\Omega = \mathbf{D} \cap \{z\colon |\arg z| < \beta\}$. (We set $\arg z \in (-\pi, \pi]$.) It is clear that the calculus (1 or 2) is well-defined. Put $g(z) = \exp(-k_2 z^{-\pi/2\beta}) \in H^\infty(\Omega)$. Now $z^n g(z)\,(R(z, T)x, y)$ is analytic in Ω and bounded on the rays $\arg z = \pm\beta$, $\arg z = 0$. The Phragmén-Lindelöf principle works with a huge "margin of safety" and thus our function is in $H^\infty(\Omega)$. Making use of Theorem 1, we obtain that $z^n g$ is an annihilator.

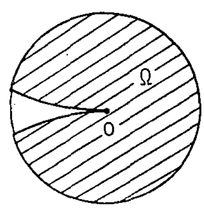

Fig. 2.

For $\beta = \pi$ we consider $\Omega = \mathbf{D} \setminus \left\{x + iy : y \leqslant \frac{1}{2k}|x|^n, x \leqslant 0\right\}$. By the Hilbert formula $R(\mu, T) = R(\lambda, T)\,(I - (\mu - \lambda)\,R(\lambda, T))^{-1}$, so for $|\mu - \lambda| \leqslant \frac{1}{2}\|R(\lambda, T)\|^{-1}$, we have: $\|R(\mu, T)\| \leqslant 2\|R(\lambda, T)\|$. The domain is chosen so that $\|R(\lambda, T)\| \leqslant 2k|\lambda|^{-n}$, $\lambda \notin \Omega$. The calculus can be defined and we again consider $g(z) = \exp(-k_2 z^{-1/2}) \in H^\infty(\Omega)$ and similarly conclude that $z^n g$ is an annihilator. ▨

PROPOSITION 2.1′. *The conclusion of Proposition* 2.1 *holds if* (4) *is replaced by a weaker condition*

$$\|R(\lambda, T)\| \leqslant k \operatorname{dist}^{-n}(\lambda, \partial\Omega), \quad \lambda \notin \Omega, \tag{4'}$$

where $\Omega = \mathbf{D} \cap \{z\colon |\arg z| < \beta\}$, $\beta < \pi$.

Proof. We can consider a slightly enlarged domain $\tilde{\Omega} \supset \Omega$ (see Figure 3).

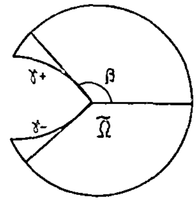

Fig. 3.

γ_+ and γ_- are chosen so that $|\exp(-z^{-\pi/2\beta})| = \text{const} > 1$ on them. A calculation shows that γ_+ and γ_- have the tangency of a finite order with the corresponding rays $\{\arg z = \pm\beta\}$ and therefore we can find $m \in \mathbf{N}$ such that

$$\|R(\lambda, T)\| \leqslant \tilde{k}\,|\lambda|^{-m}, \quad \lambda \notin \tilde{\Omega}.$$

The calculus can be defined and $z^m \exp(-k_2 z^{-\pi/2\beta})$ is an annihilator (this can be proved similarly). ▨

Let us mention that we use (and shall do so constantly) the Phragmén-Lindelöf principle for curvilinear angles (a precise reference is [12], Theorem 3.4.3) but with "nearly" a double "margin of safety".

As a rule we have to deal with conditions like (4′). Consequently we have to distort Ω to obtain an inequality similar to (4) in its inferior. How far one can go in this direction is illustrated by the following proposition.

In this proposition we impose a purely technical requirement of "regularity" on the function Φ limiting the growth of resolvent, namely

$$(5)\qquad x\cdot\left|\frac{(\log\Phi)'}{\log\Phi}\right| \leqslant \text{const}.$$

PROPOSITION 2.2. *Let* $\sigma(T) = \{0\}$,

(i) $\|R(\lambda, T)\| \leqslant c_1 \exp(c_2|\lambda|^{-1}), \quad \lambda \in \mathbf{C};$

(ii) $\|R(\lambda, T)\| \leqslant \Phi(\operatorname{Re}\lambda), \qquad 0 < \operatorname{Re}\lambda < A.$

Let $\Phi \in C^1(0, A)$, $\Phi\downarrow$, $\Phi > 1$ *and suppose* (5) *is satisfied. If*

$$(a)\qquad \int_0 \left(\frac{\log\Phi(x)}{x}\right)^{1/2} dx < \infty,$$

then T admits a 1*-calculus in some domain* Ω *and has an annihilator. If in addition*

(iii) $\varliminf_{\substack{\lambda\to 0\\ \operatorname{Re}\lambda<0}} \log\|R(\lambda, T)\|\cdot|\lambda| > 0$

the calculus in nontrivial.

The proof is rather long but does not involve new ideas. We write c for an arbitrary positive constant, not necessarily the same in different formulas.

Mention that $\Phi \downarrow$ and (a) imply

$$\text{(b)} \qquad x \log \Phi(x) \to 0, \quad x \to 0.$$

1°. First of all we must "distort" the half-plane and estimate the corresponding conformal map.

LEMMA 2.1. *Set* $\Omega = \{z = x + iy : |z| < \delta,\ x < 0 \text{ or } |y| > \varphi(x)\}$, *where* $\varphi \in C^1(0, \delta)$, $\varphi(x) \xrightarrow{x \to 0} 0$, $\varphi'(x) \xrightarrow{x \to 0} +\infty$,

$$(*) \quad \int_0 \frac{dx}{\varphi(x)} < +\infty,$$

$$(**) \quad x \frac{\varphi'}{\varphi} \leqslant \text{const.}$$

Let $\omega \colon \Omega \to \{z \colon \operatorname{Re} z < 0\}$ *be the conformal map such that* $\omega(0) = 0$, $\omega(a \pm i\varphi(a)) = \pm i$, *where* $|a + i\varphi(a)| = \delta$.

Then for some k_1, k_2

$$k_1|z| \geqslant |\omega(z)| \geqslant k_2|z|, \quad z \in \Omega.$$

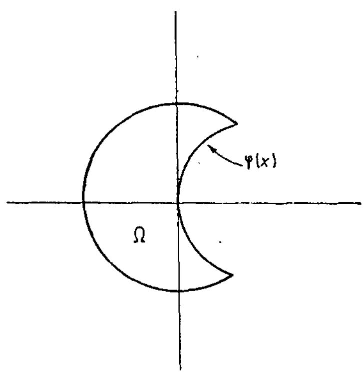

Fig. 4.

The proof will be given at the end of the section.

In the proposition we naturally put $\varphi(x) = \left(\dfrac{x}{\log \Phi(x)}\right)^{1/2}$. Clearly, $\varphi \in C^1(0,A)$, $\varphi(X) \xrightarrow{x \to 0} 0$ and $(*)$ holds.

$$\varphi'(x) = \frac{1}{2}\left(\frac{\log \Phi(x)}{x}\right)^{1/2} \frac{\log \Phi(x) - x\,[\log \Phi(x)]'}{\log^2 \Phi(x)},$$

so by (5) and as $[\log \Phi(x)]' < 0$,

$$\frac{1}{c}\left(\frac{1}{x \log \Phi(x)}\right)^{1/2} < \varphi'(x) < c\left(\frac{1}{x \log \Phi(x)}\right)^{1/2} = c\,\frac{\varphi(x)}{x}.$$

Now we have (**) and (b) implies that $\varphi'(x) \xrightarrow{x \to 0} +\infty$. All the requirements of Lemma 2.1 are checked.

2°. Here we show that there exists an outer function $h \in H^\infty(\Omega)$ such that h is continuous on $\bar{\Omega} \setminus \{0\}$ and

$$|h(\lambda)| \leqslant [\Phi(\operatorname{Re}\lambda)]^{-1}, \quad \lambda = x + i\varphi(x).$$

The function $\Phi(\operatorname{Re}\omega^{-1}(iy))$, defined on $[-1,1]$ has summable logarithm. Indeed, $|\omega^{-1}(iy)| > c|y|$ by Lemma 2.1 and for y close to 0, $|\omega^{-1}(iy)| < 2\varphi(\operatorname{Re}\omega^{-1}(iy))$, so $\operatorname{Re}\omega^{-1}(iy) > \varphi^{-1}(c|y|/2)$ and since Φ decreases, for some $r > 0$,

$$\int_{-r}^{r} \log \Phi(\operatorname{Re}\omega^{-1}(iy))\,dy \leqslant 2\int_{0}^{r} \log \Phi(\varphi^{-1}(cy/2))\,dy \leqslant$$

$$\leqslant \frac{4}{c}\int_{0}^{a} \log \Phi(u)\,\varphi'(u)\,du \leqslant \text{const}\int_{0}^{a} \frac{\log \Phi(u)}{u}\,\varphi(u)\,du = \text{const}\int_{0}^{a} \frac{du}{\varphi(u)} < +\infty.$$

So there exists an outer function $\tilde{h} \in H^\infty(\{z : \operatorname{Re} z < 0\})$ such that

$$|\tilde{h}(iy)| = [\Phi(\operatorname{Re}\omega^{-1}(iy))]^{-1}, \quad y \in [-1,1],$$

and continuous on $\{z : \operatorname{Re} z < 0\} \setminus \{0\}$. Then $h = \tilde{h} \circ \omega$ is the function we are looking for. This very h provides the 1-calculus for T. We choose δ sufficiently small so that Ω should be star-shaped. This is possible owing to the fact that $\varphi'(x) \xrightarrow{x \to 0} +\infty$. Obviously, $K(\lambda) \leqslant \max(\text{const}, \varphi(\operatorname{Re}\lambda))$, hence $hK \in L^\infty(\partial\Omega)$ and the calculus is well-defined.

3°. Now we prove that T has an annihilator. For $z = x + i\varphi(x)$,

$$|\exp(c_2 z^{-1})| = \exp(c_2 x(x^2 + \varphi^2(x))^{-1}) \leqslant$$

$$\leqslant \exp(c_2 x(\varphi(x))^{-2}) = \exp(c_2 \log \Phi(x)) \leqslant \Phi^m(\operatorname{Re} z), \quad m \in \mathbf{N}, \ m > c_2.$$

It means that $h^m(z)\exp(c_2 z^{-1})$ is bounded on $\partial\Omega$. Since it is also bounded in the left half-plane, $h^m(z)\exp(c_2 z^{-1}) \in H^\infty(\Omega)$ by the Phragmén-Lindelöf principle The same principle gives us that $h^{m+1}(z)\,(R(z, T)x, y)\exp(c_2, z^{-1})$ is bounded in Ω for this is the case on $\{z : \operatorname{Im} z = 0,\ \operatorname{Re} z < 0\}$ and on $\partial\Omega$. (Here we use (i).) So we apply Theorem 1 and conclude that $h^{m+1}(z)e^{c_2/z}$ annihilates T.

4°. It remains to check that provided (iii) holds, the calculus is nontrivial, i.e. $h(T)\neq\mathbf{O}$. We make use of the following well known result.

LEMMA 2.2. *Let g be an outer function in* $\{z : \operatorname{Re} z > 0\}$. *Then*

$$\lim x \log |g(x)| = 0, \ x \to 0_+. \qquad ▨$$

Thus we have: $\lim |x| \log |\tilde{h}(x)| = 0, \ x \to 0_-$. For $z = \mathrm{i}y$, sufficiently small, as at the beginning of 2°,

$$|z|\,|\log \tilde{h}(z)| \leqslant |y| \log \Phi(\operatorname{Re} \omega^{-1}(\mathrm{i}y)) \leqslant |y| \log \Phi(\varphi^{-1}(c|y|)) =$$

$$= \frac{1}{c}[\varphi^{-1}(c|y|) \log \Phi(\varphi^{-1}(c|y|))]^{1/2} \xrightarrow{y\to 0} 0.$$

(We used (b).) Applying the usual Phragmén-Lindelöf method we obtain that $|\lambda|\,|\log|\tilde{h}(\lambda)| \to 0$; $\lambda \to 0$, $\operatorname{Re}\lambda < 0$.

Let $z \in \Omega$. By Lemma 2.1,

$$|z|\,|\log|h(z)|| = |z|\,|\log|\tilde{h}(\omega(z))|| \leqslant \frac{1}{k_2}\cdot|\omega(z)|\cdot|\log|\tilde{h}(\omega(z))|| \xrightarrow{z\to 0} 0.$$

If $h(T) = \mathbf{O}$, by Theorem 1, $\|\mathrm{R}(\lambda, T)\| \leqslant c\,|h(\lambda)|^{-1}$ and

$$|\lambda| \log \|\mathrm{R}(\lambda, T)\| \leqslant \text{const}\,|\lambda| + |\lambda| |\log|h(\lambda)|| \to 0; \quad \lambda \to 0, \ \lambda \in \Omega,$$

which contradicts (iii).

5°. Now we prove Lemma 2.1. The proof is a straightforward application of Warshawskiĭ's asymptotics.

First make a proper substitution: $z_1 = \log(-z^{-1})$ $(\arg z \in (-\pi, \pi])$. The left half-plane is mapped onto the standard stripe $G = \{z : |\operatorname{Im} z| < \pi/2\}$ and Ω is mapped onto a curvilinear stripe Ω', bounded from the left by $\{z : \operatorname{Re} z = \log 1/\delta\}$. Ω' is symmetrical; denote its boundary curves by $\{s \pm \mathrm{i}/2\, \theta(s),\ s > b \overset{\text{def}}{=} \log 1/\delta\}$ and by W the conformal map of Ω' onto G arising from ω. Note that $\theta(s)$ can be written implicitly:

$$s = -\log|x + \mathrm{i}\varphi(x)| = -\frac{1}{2}\log(x^2 + \varphi^2(x)),$$

$$\frac{1}{2}\theta(s) = \operatorname{arc\,tg}\frac{x}{\varphi(x)} + \frac{\pi}{2}.$$

We want to write the asymptotics of W when s tends to infinity in the form of [12], Theorem 2.6.5, p. 161. The preliminary requirements are satisfied: Ω' is symmetrical and $\theta(s) \xrightarrow{s\to\infty} \pi$ as $\dfrac{x}{\varphi(x)} \leqslant \dfrac{c}{\varphi'(x)} \to 0,\ x \to 0$. We have to check also that

$$\int^{\infty} \theta'^2(s)\,\frac{\mathrm{d}s}{\theta(s)} < \infty$$

where $1/\theta(s)$ can be dropped since $\theta(s) > \pi$.

$$\frac{1}{4}\int^{\infty} \theta'^2(s)\,\mathrm{d}s = \int_0 \frac{(\varphi - x\varphi')^2}{(\varphi^2 + x^2)^2}\,\frac{\varphi^2 + x^2}{x + \varphi\varphi'}\,\mathrm{d}x \leqslant c\int_0 \frac{\mathrm{d}x}{\varphi\varphi'},$$

according to (**). Now by (*) and since φ' is separated from zero, we obtain the convergence of the required integral.

Thus,

$$\operatorname{Re} W(\xi + \mathrm{i}y) = \pi\int_a^{\xi} \frac{\mathrm{d}s}{\theta(s)} + \mathrm{O}(1).$$

Note that

$$\pi\int_a^{\xi} \frac{\mathrm{d}s}{\theta(s)} = \pi\int_a^{\xi}\left(\pi + 2\operatorname{arctg}\frac{x}{\varphi(x)}\right)^{-1}\mathrm{d}s = \xi + \mathrm{O}(1),$$

since

$$\int^{\infty} \operatorname{arctg}\frac{x}{\varphi(x)}\,\mathrm{d}s < c\int^{\infty}\frac{x}{\varphi(x)}\,\mathrm{d}s = c\int_0 \frac{x}{\varphi(x)}\cdot\frac{x + \varphi\varphi'}{x^2 + \varphi^2}\,\mathrm{d}x \leqslant c'\int_0 \frac{\mathrm{d}x}{\varphi(x)} < \infty.$$

So we have: $\operatorname{Re} W(\xi + \mathrm{i}y) = \xi + \mathrm{O}(1)$ or, equivalently,

$$\log|z^{-1}| - c \leqslant \log|(\omega(z))^{-1}| \leqslant \log|z^{-1}| + c$$

and

$$\mathrm{e}^{-c}|z| \leqslant |\omega(z)| \leqslant \mathrm{e}^{c}|z|.$$ ▨

3. HIS'S AND IS'S

1°. THEOREM 2. *Suppose that* $T \in \mathscr{L}(B)$ *admits a nontrivial* 1- *or* 2-*calculus and that there exists an annihilator* hg, $g \in E^1(\Omega)$. *Then either* $T = cI$, *or* T *has a nontrivial* HIS. *In the case of the* 2-*calculus either* $p(T) = \mathbf{O}$ *for some polynomial* p, *or* T *has an uncountable set of HIS's.*

Proof. We can assume h is an outer function. Consider $\operatorname{Ann} T = \{g \in E^1(\Omega) : :(hg)(T) = \mathbf{O}\}$. This is a subspace of $E^1(\Omega)$, closed by the property (i) of the calculuses, nontrivial and z-invariant by the multiplicativity (a). Our domain is good enough, so that the Beurling classification of z-invariant subspaces of $E^1(\Omega)$ is valid. ($\partial\Omega$ is a simple rectifiable curve, so polynomials are weak-star dense in $H^\infty(\Omega)$ which implies that the operators of multiplication by z and by the conformal map of Ω in $H^1(\mathbf{D})$ have the same lattice of IS's.) So, there is an inner function φ, such that $\operatorname{Ann} T = \varphi E^1(\Omega)$. If $\varphi = \varphi_1\varphi_2$, $\varphi_i \not\equiv 1$ is inner, then $\operatorname{Ker}(h\varphi_1)(T)$ is a nontrivial HIS. Indeed, since Ω is of class (S), $\varphi_i \notin \varphi E^1(\Omega)$, and by the multiplicativity (b) (Propositions 1.1 and 1.4) we have

$$\{0\} \subsetneq \operatorname{Im}(h\varphi_2)(T) \subset \operatorname{Ker}(h\varphi_1)(T) \subsetneq B.$$

Now we prove that different factorizations of φ lead to different HIS's. Suppose that $\varphi = \varphi_1\varphi_2 = \varphi_3\varphi_4$, φ_i are inner and $\not\equiv 1$. If $\operatorname{Ker}(h\varphi_1)(T) = \operatorname{Ker}(h\varphi_3)(T)$ then $\operatorname{Im}(h\varphi_2)(T) \subset \operatorname{Ker}(h\varphi_3)(T)$ and $h\varphi_2\varphi_3 \in \operatorname{Ann} T = \varphi E^1(\Omega)$. Then φ_4 divides φ_2. Similarly, φ_2 divides φ_4 which yields $\varphi_2 = \varphi_4$, $\varphi_1 = \varphi_3$.

If φ does not admit any factorization, it is a prime Blaschke factor corresponding to some $c \in \mathbf{C}$, and then $z - c \in \operatorname{Ann} T$ and $[h(z)(z-c)](T) = =h(T)(T - cI) = \mathbf{O}$. Now $\overline{\operatorname{Im} h(T)}$ is hyperinvariant and nonzero. If this subspace is equal to B, then $T = cI$.

Now note that if φ is not a finite Blaschke product, it has uncountably many different factorizations and thus T has an uncountable set of HIS's. If φ is a finite Blaschke product and $h(z) = (z - z_0)^n$ as in the 2-calculus then $p(T) = \mathbf{O}$ for some polynomial p. ▨

NOTE. If φ in the proof of Theorem 2 has a singular component, T has an uncountable chain of HIS's.

PROPOSITION 3.1. *Under conditions of Proposition* 2.1, *either* $T^n = \mathbf{O}$, *or* T *has an uncountable chain of* HIS'*s*.

Proof. We can apply Theorem 2. The calculus is constructed in the proof of Proposition 2.1. It remains to check that $h(T) \neq \mathbf{O}$. But $h(\lambda) = \lambda^n$, so $h(T) = \mathbf{O}$ means $T^n = \mathbf{O}$ If this is not the case we note that $\exp(-k_2 z^{-\pi/2\beta}) \in \operatorname{Ann} T$. So φ ("the least annihilator") can be but singular and we have an uncountable chain of HIS's. ▨

PROPOSITION 3.2. *Let* $\sigma(T) = \{1\}$,

$$\|\mathrm{R}(\lambda, T)\| \leqslant c_1 \exp(c_2|\lambda - 1|^{-\alpha}), \quad \alpha < 1;\ \lambda \notin \mathbf{D};$$

$$\|\mathrm{R}(\lambda, T)\| \leqslant d_1 \exp(d_2|\lambda - 1|^{-1}), \quad \lambda \in \mathbf{D}.$$

We suppose also that

$$(*)\qquad \overline{\lim_{\lambda\to 1,\, \lambda\in \mathbf{D}}}\ (|\lambda - 1| \cdot \log \|R(\lambda, T)\|) > 0.$$

Then T has a nontrivial HIS.

Proof. The 1-calculus can be defined according to Proposition 1.3 with $h(\lambda) = (\lambda - 1)\exp(-(1-\lambda)^{-\alpha'})$, $\alpha < \alpha' < 1$. That $h(\lambda)\exp(-(1-\lambda)^{-1})$ is an annihilator is proved by the Phragmén-Lindelöf principle, as in Proposition 2.1. If $h(T) = \mathbf{O}$, applying Theorem 1 we have that $\|R(\lambda, T)\| \leqslant \exp(c|\lambda - 1|^{-\alpha'})$, $\lambda \in \mathbf{D}$, which contradicts $(*)$.

2°. It turns out that "weak-type" estimates inside Ω are often sufficient for the existence of an IS.

THEOREM 3. *Suppose that $T \in \mathscr{L}(B)$ admits a nontrivial 1- or 2-calculus for functions from $hE^1(\Omega)$ and that there exist nonzero $x \in B$, $y \in B^*$ such that $(R(\lambda, T)x, y)$ is a meromorphic function of class $N(\Omega)$ (a ratio of 2 bounded analytic functions). Then T has a nontrivial invariant subspace.*

Proof. For some $f \in H^\infty(\Omega)$ we have $hf(R(\cdot, T)x, y) \in H^\infty(\Omega)$. Then also $(hf)(\lambda)\lambda^n(R(\lambda, T)x, y) \in H^\infty(\Omega)$. Applying the Cauchy's theorem we obtain that

$$((hf)(T)T^n x, y) = 0, \quad n = 0, 1, \ldots .$$

Put $\mathscr{F} \stackrel{\text{def}}{=} \{u \in B: ((hf)(T)T^n u, y) = 0,\ n = 0, 1, \ldots\}$. $\mathscr{F}$ is linear and closed being an intersection of closed subspaces. What is more, $\mathscr{F}$ is T-invariant and $x \in \mathscr{F}$. It remains to examine the case when $\mathscr{F} = B$, but then $\overline{\operatorname{Im}(hf)(T)}$ is a HIS "orthogonal" to $y \neq 0$. If the latter is zero, $(hf)(T) = 0$ and we can apply Theorem 2. ▨

4. ATZMON'S THEOREMS AND ATZMON'S QUESTIONS

1°. In the paper of A. Atzmon [1] restrictions on operator are imposed in terms of its powers. They can be rewritten easily in terms of the resolvent.

LEMMA 4.1. (i) $\|T^n\| = O(n^k)$, $n \to +\infty$ *implies*

$$\|R(\lambda, T)\| = O((|\lambda| - 1)^{-k-1}), \quad |\lambda| \to 1_+.$$

(ii) $\log|(T^n x, y)| = O(|n|^\alpha)$, $\alpha < 1$; $n \to +\infty$ $(n \to -\infty)$, *if and only if*

$$\log|(R(\lambda, T)x, y)| = O(||\lambda| - 1|^{-1/\beta}), \quad \beta = \alpha^{-1} - 1, \quad |\lambda| \to 1+ \quad (|\lambda| \to 1-),$$

with absolute constants in corresponding estimates.

(ii) *Both "O"-s in* (ii) *can be replaced by "o"-s.*

Proof is based on the formulas $R(\lambda, T) = -\sum_0^\infty T^n\lambda^{-n-1}$ for $|\lambda| > 1$ and $R(\lambda, T) = \sum_0^\infty T^{-n-1}\lambda^n$ for $|\lambda| < 1$. Further calculations can be found in [1]. ▨

When $\sigma(T) = \{1\}$ these estimates are insufficient and one must apply the Domar's Lemma [3]. We quote the result in the form of [7] and in the case when $E = \{1\}$. In § 5 we give the proof of some generalized version of this lemma.

LEMMA 4.2. (Domar). *Let u be subharmonic in* $\mathbf{C}\setminus\{1\}$ *and* $u(z) \leqslant ||z| - 1|^{-1}$ *Then* $u(z) \leqslant \text{const}\, |z - 1|^{-1}$.

Proof of the Theorem A1 (Atzmon). If $\sigma(T) = \{1\}$ and (A), (AA) hold we obtain, applying Lemmas 4.1 and 4.2, that

$$\log\|R(\lambda, T)\| = O(|\lambda - 1|^{-1}), \quad \lambda \to 1, \ \lambda \in \mathbf{D};$$

$$\|R(\lambda, T)\| = O((|\lambda| - 1)^{-k-1}), \quad |\lambda| \to 1_+$$

hence

$$\|R(\lambda, T)\| = O(|\lambda - 1|^{-2k-2}), \quad \lambda \to 1, \ \operatorname{Re}\lambda > 1.$$

The operator $I - T$ satisfies conditions of Proposition 3.1 for $\beta = \pi/2$ and the theorem is proved. ▨

2°. In [1] the following result is announced:

THEOREM A2. *Let $T \in \mathscr{L}(B)$, suppose B is reflexive,* $\sigma(T) = \{1\}$, (A) *holds and for some nonzero $x \in B$, $y \in B^*$*

$$|(T^n x, y)| = O(\exp(c|n|^{1/2})), \ n \to -\infty.$$

Then T has a nontrivial IS.

Proof can be derived from Theorem 3. Applying Lemmas 4.1. and 4.2 we conclude that

$$\|R(\lambda, T)\| \leqslant c|\lambda - 1|^{-2k-2}, \ \operatorname{Re}\lambda \geqslant 1;$$

$$|(R(\lambda, T)\,x, y)| \leqslant d_1 \exp(d_2|\lambda - 1|^{-1}), \quad \lambda \in \mathbf{C}.$$

The first estimate guarantees the 2-calculus in $\Omega = \{\lambda : |\lambda| < 2, \operatorname{Re}\lambda < 1\}$. By the Phragmén-Lindelöf principle, $(R(\lambda, T)x, y)(\lambda - 1)^{2k+2}\exp(d_2(\lambda - 1)^{-1}) \in H^\infty(\Omega)$. It remains to apply Theorem 3. ▨

3°. THEOREM 4. *Let $\sigma(T) = \{1\}$ and suppose that*

(NQ) $$\sum_{n \geqslant 1} n^{-3/2}\log\|T^n\| < \infty,$$

(i) $$\log\|T^n\| = O(|n|^{1/2}), \ n \to -\infty.$$

Then T has an analytic annihilator. If in addition

(ii) $$\overline{\lim_{n\to-\infty}} |n|^{-1/2}\log\|T^n\| > 0,$$

then T has an uncountable chain of HIS's.

The second statement was announced in the paper of A. Atzmon [1]; the first statement answers a question posed in [1].

Proof. An important step is to "regularize" the sequence $\{\|T^n\|\}$.

DEFINITION. We shall say that a positive sequence $\{c_n\}$ is s-*regular* (*strictly sublinear and regular*) if there exist N, δ, $0<\delta<1$, such that $n^{\delta-1}c_n\downarrow$, $n>N$.

CRITERION OF S-REGULARITY. Easy calculations show that $\{c_n\}$ is s-regular if and only if for some N_1, δ_1, $0<\delta_1<1$,

$$c_n - c_{n-1} \leqslant (1-\delta_1)n^{-1}c_n, \quad n > N_1.$$

LEMMA 4.3. *Suppose $\{a_n\}_1^\infty$ is a positive sequence, such that $a_{m+n}\leqslant a_m + a_n$, $\forall\, m, n\in\mathbf{N}$, and $\sum_{n\geqslant 1} n^{-3/2}a_n<\infty$. Then there exists a concave* s-*regular sequence $\{c_n\}_1^\infty$ such that $a_n\leqslant c_n$, $n\in\mathbf{N}$ and*

(6) $$\sum_{n\geqslant 1} n^{-3/2}c_n < \infty.$$

The proof of Lemma 4.3 is enclosed in Appendix.

Next some technical preparatory work has to be done.

LEMMA 4.4. *Suppose $\{c_n\}_0^\infty$ is* s-*regular and concave. Put $\mu(x)=\sup_n (1-x)^n\exp c_n$, $0<x\leqslant 1$, $\psi(x)=\sum_0^\infty (1+x)^{-n}\exp c_n$, $x>0$. Then for some k, $N>0$, $h<1$:*

(a) $\psi(x) < c\mu(x/2)x^{-k}$, $x<h$;

(b) $u\mapsto \log\mu(1-\mathrm{e}^u)$ *is convex and piecewise linear on the interval* $(\log(1-h), 0)$;

(c) $\forall n>N, \exists\, x:\ \mu(x)=(1-x)^n\exp c_n$;

(d) $x\,|(\log\mu(x))'|\leqslant c\log\mu(x)$, $x<h$.

(The proof follows further.)

LEMMA 4.5. (N. K. Nikolskiĭ). *Suppose θ is a positive function on* $[0, 1)$, *$u\mapsto\log\theta(\mathrm{e}^u)$ is convex; $d_n = \inf_{0\leqslant x<1} x^{-n}\theta(x)$, $n\geqslant 0$. Then*

$$\int_0^1 \left(\frac{\log\theta(x)}{1-x}\right)^{1-\varepsilon}\mathrm{d}x \leqslant c(\varepsilon)\sum_{n=1}^\infty n^{-1-\varepsilon}\log d_n, \quad 0<\varepsilon<1.$$

(See [15], § 2.6, Lemma 2).

Now we make use of these lemmas to deduce the required result. We intend to apply Proposition 2.2 and Theorem 2 (with the Note), so it is necessary to choose a proper majorant.

We apply Lemma 4.3 with $a_n = \log\|T^n\|$, $n \geqslant 1$, and fix the obtained sequence $\{c_n\}_1^\infty$. Since $\{c_n\}$ is s-regular, we have: $n^{-3/2}c_n \downarrow$, $n>N$, and (6) implies $c_n = o(\sqrt{n})$, so $\log\|T^n\| = O(|n|^{1/2})$, $n \to \pm\infty$. Applying Lemma 4.1 we obtain that $\log\|R(\lambda, T)\| = O(||\lambda| - 1|^{-1})$, $|\lambda| \to 1$. Since $\log|(R(\cdot, T)x, y)|$ is subharmonic in $\mathbf{C}\setminus\{1\}$, Lemma 4.2 gives

$$\log\|R(\lambda, T)\| = O(|\lambda - 1|^{-1}), \quad \lambda \to 1.$$

For $|\lambda| > 1$,

$$\|R(\lambda, T)\| \leqslant \sum_0^\infty |\lambda|^{-n-1}\|T^n\| \leqslant \Psi(|\lambda| - 1),$$

where $\Psi(x) = \sum_0^\infty (1+x)^{-n}\exp c_n$, $c_0 \overset{\text{def}}{=} 0$. We consider $\mu(x) = \sup_n (1-x)^n \exp c_n$, $x \in (0,1]$, apply Lemma 4.4 (b), and choose $v \in C^1(\log(1-h), 0)$ to be increasing and such that

$$\log\mu(1-e^u) < v(u) < 2\log\mu(1-e^u), \tag{7}$$

$$v'(u) \leqslant (\log\mu(1-e^u))'. \tag{8}$$

Our majorant will be $\Phi(x) = cx^{-k}\exp v(\log(1-x/2))$, $x < h$. By Lemma 4.4 (a) and (7), $\|R(\lambda, T)\| \leqslant \Phi(|\lambda| - 1)$, $1 < |\lambda| < 1 + h$. Our goal is to check all the hypotheses of Proposition 2.2 for $T_1 = T - I$. Clearly Φ is smooth enough and

$$\|R(\lambda, T_1)\| \leqslant \Phi(\operatorname{Re}\lambda), \quad |\lambda| < h, \operatorname{Re}\lambda > 0.$$

Since v increases, Φ decreases.

1) To prove that $\int_0 \left(\frac{\log\Phi(x)}{x}\right)^{1/2} dx < \infty$, it is sufficient to show that $\int_0 \left(\frac{\log\mu(x)}{x}\right)^{1/2} dx < \infty$. Put $\theta(x) = \mu(1-x)$, $0 \leqslant x < 1$. Lemma 4.5 can be applied and we obtain that our integral has an upper bound $c\sum_1^\infty n^{-3/2}\log d_n$, where $d_n = \inf_{0\leqslant x<1} x^{-n}\theta(x)$. But by Lemma 4.4 (c) there is an x_0 such that $\theta(x_0) = x_0^n \exp c_n$ and $\log d_n \leqslant c_n$, $\forall n > N$. So the desired estimate follows from (6).

2) Condition (5): $x|(\log\Phi(x))'| \leqslant c\log\Phi(x)$ follows easily from (7), (8) and the similar estimate for μ, stated in Lemma 4.4 (d).

Now we are in the position to apply the first part of Proposition 2.2 which claims the existence of an annihilator.

3) If we have (ii) we prove that $\varliminf_{\lambda\to 1,\,\lambda\in\mathbf{D}} |\lambda-1|\log\|R(\lambda, T)\| > 0$. If not, $\log\|R(\lambda, T)\| = o(|\lambda-1|^{-1})$, $\lambda\to 1$, $\lambda\in\mathbf{D}$ and by Lemma 4.1 $\log\|T^n\| = o(|n|^{1/2})$ $n\to -\infty$, which is a contradiction. So (NQ), (i), (ii) make possible to use Proposition 2.2 and then Theorem 2. It remains to prove Lemma 4.4 and we shall be done.

Proof of Lemma 4.4. Put $\varphi(n) = n^{\delta-1}c_n$.

(a) $\psi(x) = \sum_0^\infty (1+x)^{-n}\exp c_n$, put $\tilde\mu(x) = \sup_n (1+x)^{-n}\exp c_n$.

We show that $\psi(x) < c\tilde\mu(x)\,x^{-k}$, $x < h < 1$, and clearly $\tilde\mu(x) < \mu(x/2)$, $x < 1$. For N big enough

$$\sum_N^\infty (1+x)^{-n}\exp c_n \leqslant \sum_N^\infty (1+x)^{-n}\exp(n^{1-\delta}\varphi(N)) \overset{\text{def}}{=} \sum_N^\infty b_n.$$

Estimating b_{n+1}/b_n we see that it is less than $(1+x/2)^{-1}$ for $n\geqslant \text{const}\,x^{-1/\delta}$. So for $N\asymp \text{const}\,x^{-1/\delta}$,

$$\sum_N^\infty b_n \leqslant b_N\sum_0^\infty (1+x/2)^{-n} \leqslant \text{const}\,x^{-1}(1+x)^{-N}\exp c_N.$$

Therefore $\psi(x) \leqslant (\text{const}\,x^{-1/\delta}+\text{const}\,x^{-1})\tilde\mu(x) \leqslant cx^{-k}\tilde\mu(x)$.

(b) The function $u\mapsto \log\mu(1-e^u)$ is the upper envelope of linear functions $u\mapsto c_n+nu$, so it is convex and piecewise linear.

(c) $\{c_n\}_1^\infty$ is concave so for x such that

$$c_{k+1}-c_k \leqslant \log\left(\frac{1}{1-x}\right) \leqslant c_k-c_{k-1},$$

$$\mu(x) = \exp\left(\max_n\left(c_n - n\log\left(\frac{1}{1-x}\right)\right)\right) = (1-x)^k\exp c_k.$$

(d) $\log\mu(x) = c_n - n\log\left(\frac{1}{1-x}\right)$ for some n. If there are several such n's we choose the largest. Then the same formula holds for x such that

$$c_{n+1}-c_n \leqslant \log\left(\frac{1}{1-x}\right) \leqslant c_n-c_{n-1}.$$

We see that $(\log\mu(x))' = -\frac{n}{1-x}\cdot$ Now we have

$$\log\mu(x) = c_n - n\log\left(\frac{1}{1-x}\right) \geqslant c_n - \frac{nx}{1-x} \geqslant$$

$$\geqslant n(1+\delta_1)(c_n - c_{n-1}) - \frac{nx}{1-x} \geqslant n\left((1+\delta_1)\log\left(\frac{1}{1-x}\right) - \frac{x}{1-x}\right) \geqslant$$

$$\geqslant \delta_2\frac{nx}{1-x} = \delta_2|(\log\mu(x))'|x,$$

for x sufficiently small. (We used the criterion of s-regularity.) ▨

4°. In the paper [1], two questions are posed:

1) *Suppose that T is a completely non-unitary contraction acting on a Hilbert space, suppose T satisfies condition* (AA) *and assume that $\sigma(T)$ is of measure zero (with respect to Lebesgue measure on* $\mathbf{T}$*). Is T a C_0-operator?*

2) *Let T be an operator acting on a Banach space and assume that T satisfies conditions* (A), (AA) *and that $\sigma(T)$ is a Carleson set. Is T in $C_0(H_m^\infty)$ for some positive integer m?*

PROPOSITION 4.2. *Let E be an infinite closed subset of* $\mathbf{T}$. *There exists a c.n.u. Hilbert space contraction T, satisfying* (AA) *and such that $\sigma(T) = E$, $T \notin C_0$.*

Thereby the answer to both questions is negative.

Proof. Consider $T_\theta = P_k S$, $K = H^2 \ominus \theta H^2$, $\theta(z) = \exp\frac{z+1}{z-1}$, S is the shift operator in H^2. It is well known (see [16], Lecture III) that $\|T_\theta\| = 1$, $\sigma(T_\theta) = \{1\}$; for $\varphi \in H^\infty$, $\varphi(T_\theta)f \overset{\text{def}}{=} P_k\varphi f$; $\varphi(T_\theta) = \mathbf{O}$ if and only if $\varphi \in \theta H^\infty$; T_θ is completely non-unitary.

Since $\|T_\theta\| \leqslant 1$, this operator admits the 2-calculus in $\Omega = \{\lambda : |\lambda| < 2, \operatorname{Re}\lambda < 1\}$ for functions from $(\lambda - 1)^2 E^1(\Omega)$. The properties of permanency imply that $(\lambda - 1)^2\,\theta(\lambda)$ is the annihilator in the 2-calculus as well. Then by Theorem 1,

$$\|R(\lambda, T_\theta)\| \leqslant c|\lambda - 1|^{-2}\exp(2|\lambda - 1|^{-1}) \leqslant \exp(d|\lambda - 1|^{-1}).$$

According to Lemma 4.1 (ii), $\log\|T_\theta^{-n}\| = O(n^{1/2})$, $n \to \infty$. Thus T_θ satisfies the condition (AA), is c.n.u., and $\sigma(T_\theta) = \{1\}$.

Let $\mathscr{E}$ be a dense countable subset of E. Consider $T = \sum_{\lambda\in\mathscr{E}} \oplus\, \lambda T_\theta$. This operator is also a c.n.u. contraction and satisfies the condition (AA), $\sigma(T) = E$. Suppose that $\varphi(T) = \mathbf{O}$ for some $\varphi \in H^\infty$. Then $\varphi(\lambda T_\theta) = \mathbf{O}$, $\lambda \in \mathscr{E}$. So $\varphi(\lambda z)(T_\theta) = \mathbf{O}$ and we obtain that $\varphi(\lambda z) \in \theta H^\infty$, $\varphi(z) \in e^{(z+\lambda)/(z-\lambda)}H^\infty$ for all $\lambda \in \mathscr{E}$. But $\mathscr{E}$ is infinite, so $\varphi(z) \equiv 0$, and we may conclude that $T \notin C_0$. ▨

5. ANNIHILATORS AND NONQUASIANALYTICITY

In this section we discuss the existence of an annihilator for operators with 1-point spectrum. It is always connected with some nonquasianalyticity condition.

1°. We shall need some facts concerning (non)quasianalytic function classes. Suppose $\{m_n\}_0^\infty$ is a positive sequence. Let $C_q\{m_n\} := \{f \in C^\infty(\mathbf{T}): |f^{(n)}(\zeta)| \leqslant \leqslant \text{const}\, q^n m_n\}$, where $f^{(n)}(\zeta)$ denotes the complex variable derivative: $f'(e^{it}) := -ie^{-it}\frac{df(e^{it})}{dt}$. We also use

$$C_{A,q}\{m_n\} := C_A^\infty \cap C_q\{m_n\},$$

$$\tilde{C}_q\{m_n\} := \left\{ f \in C^\infty(\mathbf{T}): \left|\frac{d^n f(e^{it})}{dt^n}\right| \leqslant \text{const}\, q^n m_n \right\}$$

and

$$\tilde{C}_{A,q}\{m_n\} := C_A^\infty \cap \tilde{C}_q\{m_n\}.$$

A function class $\mathscr{F}$ is called nonquasianalytic if there exists an $f \in \mathscr{F}$, $f \not\equiv 0$, such that $f^{(k)}(1) = 0$, $k \geqslant 0$; otherwise $\mathscr{F}$ is called quasianalytic.

THEOREM COSK. (T. Carleman, A. Ostrovskii, R. Salinas, B. Korenblum, see [6], Chapter IV, [13]). Suppose $m_n > 0$, $m_n \uparrow$.

1) *The following are equivalent:*

(i) $C_q\{m_n\}$ *is quasianalytic.*

(ii) $\sum_1^\infty \beta_n^{-1} = \infty$, *where* $\beta_n = \inf_{k \geqslant n} m_k^{1/k}$.

(iii) $\int^\infty \frac{\ln \theta(r)}{r^2}\, dr = \infty$, *where* $\theta(r) = \sup_{n \geqslant 1} \frac{r^n}{m_n}$.

2) *The following are equivalent:*

(i)′ $C_{A,q}\{m_n\}$ *is quasianalytic.*

(ii)′ $\tilde{C}_{A,q}\{m_n\}$ *is quasianalytic.*

(iii)′ $C_q\{\sqrt{m_n}\}$ *is quasianalytic.*

Other function classes are characterized in terms of Fourier coefficients. Suppose $\{\rho_n\}_{n\in\mathbf{Z}}$ is a positive sequence. Let

$$\mathscr{B}\{\rho_n\} := \{f \in L^1(\mathbf{T}): |\hat{f}(n)| < c\rho_n^{-1},\ n \in \mathbf{Z}\},$$

$$\mathscr{B}_A\{\rho_n\} := \mathscr{B}\{\rho_n\} \cap H^1.$$

LEMMA 5.1. 1) *Suppose* $\rho_n > 0$ *and for some integer* k *and* $c > 0$, $\rho_{n+m} \leqslant \leqslant c\rho_{kn}\rho_{km}$; $n, m \in \mathbf{Z}$. *If* $\sum_{n\in\mathbf{Z}} (1+n^2)^{-1}\log \rho_n < \infty$, *there exists an* $f \in \mathscr{B}\{\rho_n\}$ *vanishing on some arc but* $\not\equiv 0$.

2) *Suppose* $r_n \uparrow$, $n \geqslant 0$, *and* $\lim_{n\to+\infty} (n^{-m} r_n) = \infty$, $\forall\, m \in \mathbf{N}$ *(so that* $\mathscr{B}_A\{r_n\} \subset C_A^\infty$*). If* $\sum_{n\geqslant 1} n^{-3/2}\log r_n < \infty$, *the class* $\mathscr{B}_A\{r_n\}$ *is nonquasianalytic.*

The author does not know whether such a result was ever stated explicitly but its proof is a mere compilation and will be hinted only.

Proof. 1) We put $\tilde{\rho}_n = \rho_{kn}$ and observe that $\sum_{n\in\mathbf{Z}} (1+n^2)^{-1}\log\tilde{\rho}_n < \infty$. Next we follow the proof of Lemma 2.10, [2], Chapter 5 which treats the case $k = 1$. We replace the estimate $\rho_{n-m}\rho_m \geq \rho_n$ (used there only once) by $\tilde{\rho}_{n-m}\tilde{\rho}_m \geq \frac{1}{c}\rho_n$, and obtain that the constructed "cap-function" is in $\mathscr{B}\{\rho_n\}$.

2) can be derived from 1). The idea of the argument is due to A. Atzmon [1], § 4, proof of Proposition 2. Put $\rho_n = r_{n^2}$, $n\in\mathbf{Z}$. Monotony and $\sum_{n\geq 1} n^{-3/2}\log r_n < \infty$ imply $\sum_{n\in\mathbf{Z}} (1+n^2)^{-1}\log\rho_n < \infty$. Since $\rho_n\uparrow$, $n \geq 0$, $\rho_{2n}\rho_{2m} \geq \rho_0\rho_{n+m}$ and 1) can be applied. So we construct $f\in\mathscr{B}\{\rho_n\}$, $f(-1)\neq 0$, $f(e^{it}) = 0$, $|t| \leq \varepsilon$. We can assume $f(e^{-it}) = f(e^{it})$ so $\hat{f}(-n) = \hat{f}(n)$. Consider $g(z) = \sum_{n\geq 0}\hat{f}(n)z^{n^2}$. Evidently $g\in\mathscr{B}_A\{r_n\}$. Since $W_A\{r_n\}\subset C_A^\infty$ we have for $k\geq 1$

$$\frac{d^k g(e^{it})}{dt^k}(0) = \sum_{n\geq 1}\hat{f}(n)(in^2)^k = \frac{i^{-k}}{2}\sum_{n\in\mathbf{Z}}\hat{f}(n)(in)^{2k} = \frac{i^{-k}}{2}\frac{d^{2k}f(e^{it})}{dt^{2k}}(0) = 0,$$

which implies the nonquasianalyticity.

2°. Consider

$$\mathscr{F}\{\omega_k\} = \left\{f\in C^\infty(\mathbf{T}) : \|f\|_{\mathscr{F}} \stackrel{\text{def}}{=} \sum_{k\geq 0} M_k(f)\frac{\omega_k}{k!} < \infty\right\},$$

where $M_k(f) = \sup_{|\zeta|=1}|f^{(k)}(\zeta)|$. It is supposed that $\omega_k > 0$, $\omega_{k+l}\leq\omega_k\omega_l$.

PROPERTIES OF $\mathscr{F}\{\omega_k\}$.

1) $\mathscr{F}\{\omega_k\}$ supplied with the norm $\|\cdot\|_{\mathscr{F}}$ is a Banach algebra.

2) $\mathscr{F}\{\omega_k\}$ is separable (Fejer sums converge to f in the norm $\|\cdot\|_{\mathscr{F}}$).

3) $\mathscr{F}_A\{\omega_k\} \stackrel{\text{def}}{=} \mathscr{F}\{\omega_k\}\cap C_A^\infty = \{f\in\mathscr{F}\{\omega_k\} : \hat{f}(n) = 0,\ n < 0\}$ is a closed subalgebra in $\mathscr{F}\{\omega_k\}$.

4) If $f\in\mathscr{F}_A\{\omega_k\}$, then $f_r \xrightarrow{r\to 1-} f$ in $\mathscr{F}_A\{\omega_k\}$, $f_r(z)\stackrel{\text{def}}{=} f(rz)$. (It follows from the fact that $M_n(f_r)\leq M_n(f)$ and $f_r\to f$ in C_A^∞.)

5) $\mathscr{F}\{\omega_k\}$ is nonquasianalytic if and only if $\sum_{n\geq 1}\alpha_n < \infty$, where $\alpha_n = \sup_{k\geq n} k^{-1}\omega_k^{1/k}$ (see Theorem COSK).

6) $\mathscr{F}_A\{\omega_k\}$ is nonquasianalytic if and only if $\sum_{n\geq 1}\sqrt{\alpha_n} < \infty$ (see Theorem COSK).

DEFINITION OF THE $\mathscr{F}$-CALCULUS. Let $\sigma(T) = \{1\}$, $\|(T-I)^k\|\leq c\omega_k$. Put

$$f(T) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty}\frac{f^{(k)}(1)}{k!}(T-I)^k,\quad f\in\mathscr{F}\{\omega_k\}.$$

PROPERTIES OF THE $\mathscr{F}$-CALCULUS.

1) $\|f(T)\| \leqslant c\|f\|_{\mathscr{F}}$.

2) The $\mathscr{F}$-calculus coincides with the Riesz-Dunford calculus for functions analytic in a neighobourhood of the unit circle.

3) $(\alpha f+\beta g)(T) = \alpha f(T)+\beta g(T)$; $f, g \in \mathscr{F}\{\omega_k\}$.

4) $(fg)(T) = f(T)g(T)$; f, $g \in \mathscr{F}\{\omega_k\}$.

Proof. Consider

$$p_n = \sum_0^n f^{(k)}(1)(x-1)^k/k!,$$

$$q_n = \sum_0^n g^{(k)}(1)(x-1)^k/k!$$

By definition, $f(T) = \lim p_n(T)$, $g(T) = \lim q_n(T)$. For polynomials we have the multiplicativity, so it remains to check that $(p_nq_n)(T) \to (fg)(T)$.

$$\|(fg)(T)-(p_nq_n)(T)\| \leqslant \sum_{k=n+1}^{\infty}\left|\frac{(fg)^{(k)}(1)}{k!}\right| \|(T-I)^k\| +$$

$$+c\sum_{k=n+1}^{2n}\sum_{l=0}^{k}\frac{f^{(l)}(1)}{l!}\frac{|g^{(k-l)}(1)|}{(k-l)!}\omega_l\omega_{k-l} = S_1+S_2,$$

$$S_2 \leqslant c\left(\|f\|_{\mathscr{F}}\sum_{k=[n/2]}^{\infty}\frac{|g^{(k)}(1)|}{k!}\omega_k + \|g\|_{\mathscr{F}}\sum_{k=[n/2]}^{\infty}\frac{|f^{(k)}(1)|}{k!}\omega_k\right).$$

Since f, g and fg belong to the algebra $\mathscr{F}\{\omega_k\}$ both summands tend to zero. ▨

5) If $\mathscr{F}\{\omega_k\}$ is nonquasianalytic, there is an $f \in \mathscr{F}\{\omega_k\}$, $f \not\equiv 0$, such that $f(T) = \mathbf{O}$

6) If $\mathscr{F}_A\{\omega_k\}$ is nonquasianalytic, there is an $f \in \mathscr{F}_A\{\omega_k\}$, $f \not\equiv 0$, such that. $f(T) = \mathbf{O}$. For this f we have also:

$$f_r(T) \to \mathbf{O}, \quad r \to 1-.$$

PROPOSITION 5.1. *Suppose* $\sigma(T) = \{1\}$, $\|(T-I)^k\| \leqslant \alpha_k$, $\{\log(\alpha_k^{-1})\}_{k_0}^{\infty}$ *is convex and*

$$(*) \quad \sum_{k\geqslant 1}(\alpha_k^{1/k}k^{-1})^{1/2} < \infty.$$

Then there exists an $f \in C_A$, $f \not\equiv 0$, $\lim_{r\to 1-} f_r(T) = \mathbf{O}$.

Proof. Consider

$$\mathscr{F}_A(T) \overset{\text{def}}{=} \mathscr{F}_A\{\|(T-I)^k\|\} \supset C_{A,1/2}\left\{\frac{k!}{\alpha_k}\right\}.$$

The sequence $\{\log (k!\, \alpha_k^{-1})\}_{k_0}^{\infty}$ is convex which implies the monotony of $(k!\, \alpha_k^{-1})^{1/k}$, $k > k_1$. Applying Theorem COSK we see that $\mathscr{F}_A(T)$ is nonquasianalytic and it remains to use the property 6) of the $\mathscr{F}$-calculus. ▨

NOTES. 1) (*) is satisfied for example if $T-I \in \mathfrak{S}_{1,1/2}$ (Lorentz space, $\mathfrak{S}_{1,1/2} = \{T \in \mathfrak{S}_\infty : \sum_{n \geq 1} (s_n(T) n^{-1})^{1/2} < \infty\}$). It follows from the estimate obtained in [17].

2) For the existence of a nonanalytic annihilator it is sufficient that $\|(T-I)^k\| \leqslant \alpha_k$, $\{\log (\alpha_k^{-1})\}$ is convex and $\sum_{k \geq 1} \alpha_k^{1/k} k^{-1} < \infty$. These conditions are satisfied for example if $T - I \in \mathfrak{S}_\omega$ (Matsaev ideal, $\mathfrak{S}_\omega = \{T \in \mathfrak{S}_\infty : \sum_{n \geq 1} s_n(T)\, n^{-1} < \infty\}$).

3°. In Proposition 2, § 4 of [1] an analytic annihilator for T is found provided $\log \|T^n\| = O(|n|^\alpha)$, $\alpha < 1/2$, $n \to +\infty$. The next theorem gives an extension of this result on the class of operators, satisfying (NQ) with its inverse. On the other hand one can hardly hope to get an annihilator if (NQ) fails.

THEOREM 5. *Let* $\sigma(T) = \{1\}$,

$$\sum_{n \in \mathbf{Z}} (1 + n^{3/2})^{-1} \log \|T^n\| < \infty.$$

Then T has an annihilator in the $W_A(T)$-calculus.

Proof. The main step of the proof is the following

LEMMA 5.2. *Under hypotheses of Theorem 5 there exists a sequence $\{\alpha_k\}_0^\infty \downarrow$ such that $\{\log \alpha_k^{-1}\}$ is convex, $\|(T-I)^k\| \leqslant \alpha_k$, $k \geqslant 0$, and* $\sum_{k \geq 1} (k^{-1} \alpha_k^{1/k})^{1/2} < \infty$.

We apply Lemma 4.3 with $a_n = \max (\log \|T^n\|, \log \|T^{-n}\|)$ and fix the obtained sequence $\{c_n\}_1^\infty$ (in the proof of Lemma 5.2 as well).

We can assume that $n^{-k} \exp c_n \to +\infty$, $\forall k \in \mathbf{N}$, otherwise $(T-I)$ is nilpotent.

First we shall deduce the theorem. Lemma 5.2 and Proposition 5.1 imply that $\mathscr{F}_A(T)$ contains some nonquasianalytic class $C_{A,1}\{m_n\}$, $m_n \uparrow$. Put $M_n = 2^{-n} m_n$. One can easily check that $\tilde{C}_{A,1}\{M_n\} = \tilde{C}_{A,1/2}\{m_n\} \subset C_{A,1}\{m_n\}$. Put $\rho_k = k^2 \sup_n (k^n / M_n)$. By Theorem COSK, $\sum_{k \geq 1} k^{-2} \log \theta(k) < \infty$, where $\theta(k) = \sup_n (k^n / \sqrt{M_n})$. But $\rho_{k^2} = k^4 \theta^2(k)$, so $\{\rho_k\}$ being monotone, $\sum_{k \geq 1} k^{-3/2} \log \rho_k < \infty$. We have $\mathscr{B}_A\{\rho_n\} \subset \tilde{C}_{A,1}\{M_n\}$ because for $f \in \mathscr{B}_A\{\rho_n\}$,

$$\left| \frac{\mathrm{d}^n f(\mathrm{e}^{it})}{\mathrm{d}t^n} \right| \leqslant \sum_{k \geq 0} k^n |\hat{f}(k)| \leqslant c \sum_{k > 0} \frac{k^n}{\rho_k} \leqslant$$

$$\leqslant c\, M_n \sum_{k \geq 1} 1/k^2 = c' M_n.$$

On the other hand $W_A(T) \supset \mathscr{B}_A\{n^2 \exp c_n\}$. Bringing all together we see that

$$W_A(T) \cap \mathscr{F}_A(T) \supset \mathscr{B}_A\{\rho_n n^2 \exp c_n\}.$$

The latter class satisfies the hypotheses of Lemma 5.1.2) and so we can take $f \not\equiv 0$, $f^{(k)}(1) = 0$, $k \geqslant 0$ and $f \in W_A(T) \cap \mathscr{F}_A(T)$. By the property 6) of the $\mathscr{F}$-calculus $\lim_{r\to 1-} f_r(T) = \mathbf{O}$, so $f(T) = \mathbf{O}$ in the Wermer calculus.

Proof of Lemma 5.2. As usual, power norm estimates can be transferred into resolvent estimates with a majorant depending on $|\lambda|$. To make use of the fact that $\sigma(T) = \{1\}$, we prove a generalized version of Domar's Lemma for the majorant in question. The estimates of the resolvent now depend on $|\lambda - 1|$ which makes possible the return to power norm estimates again but with the powers of $(T-I)$ (instead of T).

$$\|R(\lambda, T)\| \leqslant 2\Psi(|\lambda| - 1), \quad |\lambda| < 2,$$

where $\Psi(x) = \sum_{n\geqslant 0} (1+x)^{-n} \exp c_n$. In its turn $\Psi(x) \leqslant c\mu(x)x^{-k} \stackrel{\text{def}}{=} \Phi(x)$ by Lemma 4.4 (a), where $\mu(x) = \sup_n (1-x)^n \exp c_n$. Put $\theta(x) = \max(1, \log\Phi(x))$, $x > 0$. Lemma 4.4 (d) implies $x|\theta'(x)| \leqslant c\,\theta(x)$, hence $x^c\theta(x)\uparrow$ and

$$\theta(x/2) \leqslant 2^c\theta(x),$$

SUBLEMMA. *Let u be subharmonic in $\mathbf{C}\setminus E$, suppose E is closed, $E \subset \mathbf{T}$,*

(a) $$u(z) \leqslant \varphi(||z| - 1|),$$

$\varphi \downarrow$ and

(b) $$\varphi(x/2) \leqslant c\varphi(x).$$

Then $u(z) \leqslant$ const $\cdot\, \varphi(\rho(z, E))$, where $\rho(z, E) =$ dist(z, E).

The proof is given a bit further and now we make use of Sublemma: $u_{x,y}(\lambda) = \log|(R(\lambda, T)x, y)|$ is subharmonic in $\mathbf{C}\setminus\{1\}$ and $u_{x,y}(\lambda) \leqslant \text{const}\,\theta(||\lambda| - 1|)$; $\|x\|, \|y\| \leqslant 1$. We apply Sublemma with $\varphi = \theta$ and obtain:

$$\|R(\lambda, T)\| \leqslant \text{const}\,\Phi^l(|\lambda - 1)|,$$

for some positive l and λ close to 1.

Put $M(r) = \sup \|R(\lambda, T)\|$, $|\lambda - 1| = r$. As in Theorem 4 applying Lemma 4.5 we conclude that $\int_0 (r^{-1}\log M(r))^{1/2}\,dr < \infty$ hence

(9) $$\int_0 (\log M(r^2))^{1/2}\,dr < \infty.$$

Since $(T-I)^k = \dfrac{i}{2\pi}\displaystyle\int_{|\lambda-1|=r} (\lambda-1)^k R(\lambda, T)\,d\lambda,$

$$\|(T-I)^k\| \leqslant \alpha_k \stackrel{\text{def}}{=} \inf_{0<r\leqslant 1} (M(r)r^k), \quad k \geqslant 0.$$

Evidently, $\alpha_k \downarrow$, $\log(\alpha_k^{-1}) = \max_{1 \leqslant u < \infty} (ku - \log M(e^u))$ is convex. Let τ be the inverse function for $r \mapsto (\log M(r^2))^{1/2}$, so that $M(\tau^2(w)) = \exp(w^2)$. Clearly (9) implies $\int^{\infty} \tau(w) dw < \infty$ and $\sum_{k=1}^{\infty} \tau(\sqrt{k})\, k^{-1/2} < \infty$ since τ decreases. Thus,

$$\alpha_k^{1/k} \leqslant M^{1/k}(\tau^2(\sqrt{k}))\tau^2(\sqrt{k}) = (\exp k)^{1/k}\tau^2(\sqrt{k}) = e\ \tau^2(\sqrt{k}).$$

So we have that $\sum_{k=1}^{\infty} (k^{-1}\alpha_k^{1/k})^{1/2} < \infty$.

Proof of Sublemma. It follows the argument of Taylor and Williams [7], where $\varphi(x) = x$.

For technical reasons replace E by $E \cup \{0\}$. Put $\psi = \varphi^{-1}$. First we prove that conditions $z_0 \in \mathbf{C} \setminus E$, $|z_0| \geqslant 1/2$, $u(z_0) \geqslant c^v$ imply that a disc with the centre z_0 and radius $R > 4c^2(c-1)^{-1}\psi(c^{v-1})$ either contains a z such that $u(z) > c^{v+1}$ or a $z \in E$.

Indeed, if $\{|z - z_0| \leqslant R\} \subset \mathbf{C} \setminus E$, $u(z) \leqslant c^{v+1}$, then u being subharmonic,

$$c^v \leqslant u(z_0) \leqslant \frac{1}{\pi R^2} \int_{|\xi| \leqslant R} u(z_0 + \xi)\, d\lambda(\xi) \leqslant$$

$$\leqslant c^{v-1} + \frac{c^{v+1}}{\pi R^2} \operatorname{mes}\{z : u(z) > c^{v-1},\ |z - z_0| \leqslant R\}.$$

The second summand does not exceed

$$\frac{c^{v+1}}{\pi R^2} \operatorname{mes}\{z = re^{i\theta} : |z - z_0| \leqslant R,\ \ \varphi(|1 - r|) > c^{v-1}\} \leqslant$$

$$\leqslant \frac{c^{v+1}}{\pi R^2} \operatorname{mes}\left\{z = re^{i\theta} : |1 - r| \leqslant \psi(c^{v-1}),\ \ |\theta - \arg z_0| \leqslant \arcsin \frac{R}{|z_0|}\right\}.$$

(Here we use that $R < |z_0|$ because $0 \in E$.) So we see that the measure of our set does not exceed $2\psi(c^{v-1}) \dfrac{\pi R}{|z_0|}$ and bringing all together, $c^v \leqslant c^{v-1} + c^{v+1}\psi(c^{v-1})\, 2R^{-1}|z_0|^{-1}$, hence $R \leqslant 2c^2\psi(c^{v-1})\,|z_0|^{-1}(c-1)^{-1}$ which is a contradiction.

The proved statement implies that provided $u(z_0) \geqslant c^v$, $\rho(z_0, E) \leqslant \text{const} \sum_{k \geqslant v} \psi(c^{k-1})$. The majorant φ decreases, so does ψ and thus $x/2 \geqslant \psi(c\varphi(x))$ and $\psi(cy) \leqslant \dfrac{1}{2}\psi(y)$. Therefore $\rho(z_0, E) \leqslant \text{const}\, \psi(c^{v-1})$ hence

$$\varphi(\rho(z_0, E)) \geqslant \varphi(\text{const}\ \psi(c^{v-1})) \geqslant \text{const}\ c^{v-1} \geqslant \text{const}\ u(z_0). \quad ▨$$

NOTE. Mention that an analytic annihilator is constructed in Theorem 4 which can be partly applied to the operator in Theorem 5. However it may prove difficult to choose this annihilator from $W_A(T)$, so we have preferred to develop the approach of this section.

6. EXAMPLES

1° Let K be the operator in $L^2(0,1)$: $Kf(x) = \int_0^x f(t)\,dt$. Consider $T = \sum \oplus K | \mathscr{L}$, where $\mathscr{L}$ is an invariant subspace of the direct sum of some number (may be infinite) copies of the operator K. Then T has an uncountable chain of HIS's.

Proof. It is well known (see [10]) that $\sigma(K) = \{0\}$,

$$\|R(\lambda, K)\| \leqslant c|\operatorname{Re} \lambda|^{-1}, \quad \operatorname{Re} \lambda < 0;$$

$$\|R(\lambda, K)\| \leqslant c_1 \exp(c_2|\lambda|^{-1}), \quad \lambda \in \mathbf{C}.$$

These properties clearly persist for T. So T satisfies the conditions of Proposition 2.1, $\beta = \pi/2$ and Theorem 2 can be applied. Since $K^n \neq \mathbf{O}$, also $T^n \neq \mathbf{O}$ and T has an uncountable chain of HIS's. ☐

2°. Let Ω be of class (m) (see Definition, § 1). Consider a Banach algebra B such that $C_A^k(\bar{\Omega}) \hookrightarrow B$ is a continuous inclusion. Let $\mathfrak{A}$ be a prime ideal in B, corresponding to a point $\lambda_0 \in \partial\Omega$. Denote $A = B/\mathfrak{A}$. Let T be the factor-operator of multiplication by z. What can we say about T? First of all, $\sigma(T) = \{\lambda_0\}$. The invariant subspaces of T correspond to closed ideals in A. For $\lambda \notin \Omega$,

$$\|R(\lambda, T)\| \leqslant \|(z-\lambda)^{-1}\|_B \leqslant \text{const}\, \|(z-\lambda)^{-1}\|_{C^k(\Omega)} \leqslant \text{const dist}^{-k-1}(\lambda, \partial\Omega).$$

We can consider an enlarged domain $\tilde{\Omega} \supset \Omega$ such that

$$\|R(\lambda, T)\| \leqslant \text{const}\, |\lambda - \lambda_0|^{-m(k+1)}, \quad \lambda \notin \tilde{\Omega}.$$

So the 2-calculus is well-defined in $\tilde{\Omega}$.

PROPOSITION 6.1. *Suppose that there exists a $\theta \in \mathfrak{A}$ such that $\theta \in C_A^{k+1}(\tilde{\Omega})$ (it can often be $\theta(z) = z^p \exp(\alpha(z - \lambda_0)^{-\beta})$). Then either A is finite-dimensional or it has an uncountable chain of closed ideals.*

Proof. For $\lambda \in \tilde{\Omega}$, $R(\lambda, T) = -\dfrac{1}{\theta(\lambda)}\left[\dfrac{\theta(z)-\theta(\lambda)}{z-\lambda}\right](T)$. Here $\left[\dfrac{\theta(z)-\theta(\lambda)}{z-\lambda}\right](T)$ denotes the factor-operator of multiplication. So,

$$\|R(\lambda, T)\| \cdot |\theta(\lambda)| \leqslant \left\|\frac{\theta(z)-\theta(\lambda)}{z-\lambda}\right\|_B \leqslant \text{const} \left\|\frac{\theta(z)-\theta(\lambda)}{z-\lambda}\right\|_{C^k(\tilde{\Omega})} \leqslant$$

$$\leqslant \text{const}\, \|\theta'\|_{C^k(\tilde{\Omega})}, \quad \lambda \in \tilde{\Omega}.$$

Applying Theorem 1 we obtain that $(z-\lambda_0)^{m(k+1)}\theta(z)$ is the annihilator in the 2-calculus and the required result now follows from Theorem 2. ▨

3°. Let T be a nonzero operator such that $\sigma(T)=\{0\}$,

$$\|R(\lambda, T)\| \leqslant c_1 \exp(c_2|\lambda|^{-1}), \quad \lambda \in \mathbf{C};$$

$$\|R(ix, T)\| \leqslant c\,|x|^{-n}, \quad x \in \mathbf{R}.$$

Then T has a nontrivial HIS.

Proof. We shall prove that T^2 has a nontrivial HIS (assume $T^2 \neq \mathbf{O}$). Check that $-T^2$ satisfies the conditions of Proposition 3.1, $\beta=\pi$. Indeed, $\sigma(T^2)=\{0\}$,

$$\|R(a, T^2)\| \leqslant \|R(i\sqrt{a}, T)\| \cdot \|R(-i\sqrt{a}, T)\| \leqslant c^2 a^{-n}, \quad a>0.$$

$$\|R(\lambda, T^2)\| \leqslant \|R(\sqrt{\lambda}, T)\| \cdot \|R(-\sqrt{\lambda}, T)\| \leqslant$$

$$\leqslant c_1^2 \exp(2c_2|\lambda|^{-1/2}), \quad \lambda \in \mathbf{C}.$$ ▨

As an example of such an operator we can suggest a part of direct sum $\sum_{l\in I} \oplus K_l$, where $K_l=K$ or $K_l=-K$; K is the operator introduced in 1°.

APPENDIX

Proof of Lemma 4.3.

1) First we construct an s-regular sequence $\{b_n\}_1^\infty$, satisfying (6) and such that $a_n \leqslant cb_n, n \geqslant 1$. Fix $\delta \in (0, 1/2)$ and put

$$b_n = n^{1-\delta} \sum_{i=[n/4]}^{\infty} i^{\delta-2} a_i.$$

Obviously $n^{\delta-1} b_n \downarrow$ and

$$\sum_{n=1}^{\infty} n^{-3/2} b_n \leqslant \sum_{i=1}^{\infty} i^{\delta-2} a_i \sum_{n=1}^{4(i+1)} n^{-(1/2)-\delta} \leqslant c \sum_{i=1}^{\infty} i^{-3/2} a_i < \infty.$$

Finally,

$$a_n \leqslant cn^{1-\delta} n^{\delta-2} \sum_{i=[n/4]}^{[n/2]-1} (a_i + a_{n-i}) \leqslant cn^{1-\delta} \sum_{i=[n/4]}^{n} i^{\delta-2} a_i \leqslant cb_n.$$

2) It remains to obtain the concavity. Put $c_n = \sum_1^n i^{-1} b_i$. We have: $i^{-1} b_i \downarrow$ since $i^{\delta-1} b_i \downarrow$, hence $c_n \geqslant n(n^{-1} b_n) = b_n$. The sequence $\{c_n\}_1^\infty$ is concave because

$(c_n - c_{n-1}) \downarrow$. Simple verification (as for $\{b_n\}_1^\infty$) shows that (6) persists for $\{c_n\}_1^\infty$. To prove the s-regularity we use the criterion from 4.3). We can assume $c_n \to \infty$, otherwise $a_n \leqslant cc_n \leqslant$ const and everything is trivial. We have for n big enough: $b_n - b_{n-1} \leqslant (1 - \delta_1)n^{-1}b_n$, hence $b_n \leqslant A + (1 - \delta_1)c_n \leqslant (1 - \delta_2)\, c_n$. Thus,

$$c_n - c_{n-1} \leqslant (1 - \delta_2)\, n^{-1}\, c_n,$$ ☐

and $\{c_n\}_1^\infty$ is s-regular.

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