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POSITIVE SEMIGROUPS ON ORDERED BANACH SPACES

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0. INTRODUCTION

Let $\mathscr{B}$ be a Banach space ordered by a normal positive cone $\mathscr{B}_+$ with non-empty interior $\operatorname{int}\mathscr{B}_+$. First we give a simple proof of a recent theorem of Arendt, Chernoff and Kato; H generates a positive C_0-semigroup S on $\mathscr{B}$ if, and only if, $(I+\alpha H)^{-1}$ is a bounded positive operator for all small $\alpha > 0$. Second we demonstrate that $S_t\operatorname{int}\mathscr{B}_+ \subseteq \operatorname{int}\mathscr{B}_+$ and then relate a variety of stronger ergodic criteria of S. Next we examine C_0^*-semigroups. In this case it is not generally true that $S_t\operatorname{int}\mathscr{B}_+ \subseteq \operatorname{int}\mathscr{B}_+$ but this condition is equivalent to $D(H) \cap \operatorname{int}\mathscr{B}_+ \neq \emptyset$. Moreover H generates a positive C_0^*-semigroup such that $S_t\operatorname{int}\mathscr{B}_+ \subseteq \operatorname{int}\mathscr{B}_+$ if, and only if, $(I+\alpha H)^{-1}$ is a bounded positive operator for all small $\alpha > 0$ and $D(H) \cap \operatorname{int}\mathscr{B}_+ \neq \emptyset$. We conclude with some complementary remarks about semigroups on C^*- and W^*-algebras.

The aim of this paper is to characterize generators of positive C_0- and C_0^*-semigroups, i.e., continuous semigroups on ordered Banach spaces $\mathscr{B}$ which respect the order, and to discuss various ergodicity properties of these semigroups.

The infinitesimal characterization of continuous semigroups has been extensively studied (for recent reviews see, for example, [3], [5], [6]). The classic Hille-Yosida theorem demonstrates that an operator H generates a C_0-semigroup of contractions if, and only if, the resolvents $(I+\alpha H)^{-1}$ are well-defined contraction operators for all small $\alpha > 0$ and the Lumer-Phillips theorem re-expresses this property as a range condition $\mathbf{R}(I+\alpha H) = \mathscr{B}$ together with a condition of dissipativity of H. More generally the Feller-Miyadera-Phillips theorem establishes that bounds $\|(I+\alpha H)^{-n}\| \leqslant M(1-\alpha\omega)^{-n}$, $n = 1,2,\ldots$ are both necessary and sufficient for H to generate a general C_0-semigroup. Some of these characterizations then extend to C_0^*-semigroups by duality. The study of positive semigroups is much less complete.

In 1962 Phillips [15] demonstrated that H generates a positive C_0-semigroup on a Banach lattice if, and only if, $R(I+\alpha H) = \mathscr{B}$ and H also satisfies a condition of dispersivity similar to the condition of dissipativity. Various generalizations of this theorem were given in the lattice setting [8], [16] but it was not until 1980 that two of the present authors [4] rephrased the definition of dispersion for operators on a C^*-algebra with identity and obtained an analogue of Phillips theorem for semigroups acting on the algebra. Jørgensen [9] then announced a number of comeplmentary results for dispersive operators acting on algebras. Recently these algebraic investigations were clarified and extended by Arendt, Chernoff and Kato [1] who examined semigroups on a space $\mathscr{B}$ ordered by a positive cone $\mathscr{B}_+$ which is normal with non-empty interior. In particular the cone of positive elements of a C^*-algebra with identity has these properties. These latter authors unified the various notions of dissipativity and dispersion and also extended the Phillips style generator theorems to non-contractive, positive, C_0-semigroups. They also proved a version of a generator theorem, established by Evans and Hanche-Olsen [7], for norm-continuous positive semigroups. In particular it was shown in [1] that H generates a positive C_0-semigroup if, and only if, the resolvents $(I + \alpha H)^{-1}$ are bounded positive operators for all small $\alpha > 0$. This result is somewhat surprising insofar no explicit bounds on the norms $\|(I + \alpha H)^{-n}\|$ are required; these bounds follow from the positivity and the geometry of $\mathscr{B}_+$.

In Sections 2 and 3 this result is generalized for C_0-semigroups and C_0^*-semigroups respectively.

In Section 2 we demonstrate that a positive C_0-semigroup automatically maps the interior of $\mathscr{B}_+$ into itself and in Section 3 we characterize the generators of those C_0^*-semigroups which have this stronger positivity property. Various other notions of strict positivity, ergodicity and irreducibility are also compared. But first we begin by reviewing in Section 1 various properties of ordered Banach spaces.

1. ORDERED BANACH SPACES

Although the theory of partially ordered Banach spaces has been extensively studied it appears difficult to find a concise description of all the features relevant to the sequel of this paper. Therefore in this preliminary section we attempt to summarize the necessary material. Proofs of most of the subsequent statements can be found in the books by Krasnosel'skii [10], Peressini [14], Schaefer [17], or Namioka and Kelley [12]. Hence we only indicate the occasional proof.

Let $\mathscr{B}$ be a real Banach space. A subset $\mathscr{B}_+$ of $\mathscr{B}$ is defined to be a *proper closed convex cone* in $\mathscr{B}$ if

1. $\mathscr{B}_+$ is norm closed,
2. $\mathscr{B}_+ + \mathscr{B}_+ \subseteq \mathscr{B}_+$,
3. $\lambda\mathscr{B}_+ \subseteq \mathscr{B}_+$ for all $\lambda \geqslant 0$,
4. $\mathscr{B}_+ \cap -\mathscr{B}_+ = \{0\}$.

Each $\mathscr{B}_+$ determines a partial order $\geqslant$ on $\mathscr{B}$ by defining $a \geqslant b$ whenever $a - b \in \mathscr{B}_+$. Thus $a \geqslant 0$ is equivalent to $a \in \mathscr{B}_+$. We refer to elements of $\mathscr{B}_+$ as *positive elements* of $\mathscr{B}$, and to $\mathscr{B}_+$ as the *positive cone* of $\mathscr{B}$.

A real functional ω in the dual $\mathscr{B}^*$ of $\mathscr{B}$ is defined to be positive, $\omega \geqslant 0$, if $\omega(a) \geqslant 0$ for all $a \in \mathscr{B}_+$. A Hahn-Banach type argument proves the existence of positive functionals and then the set

$$\mathscr{B}_+^* = \{\omega ; \omega \in \mathscr{B}^*, \omega \geqslant 0\}$$

is called the *dual cone* of $\mathscr{B}_+$. It follows straightforwardly that $\mathscr{B}_+^*$ is a norm closed convex cone but it does not necessarily satisfy

$$\mathscr{B}_+^* \cap -\mathscr{B}_+^* = \{0\}.$$

This latter condition is true if, and only if, $\mathscr{B}_+$ is *weakly generating*, i.e., if, and only if, $\mathscr{B} = \overline{\mathscr{B}_+ - \mathscr{B}_+}$ where the bar denotes norm closure.

A positive cone $\mathscr{B}_+$ in the Banach space $\mathscr{B}$ is defined to be *normal* if there exists an $\alpha > 0$ such that

$$\|a + b\| \geqslant \alpha$$

for all $a, b \in \mathscr{B}_+$ with $\|a\| = 1 = \|b\|$. Normality can also be expressed as a compatibility condition between the order structure and the topology. Specifically the following conditions are equivalent:

1. $\mathscr{B}_+$ is normal;
2. there is a $\beta > 0$ such that $0 \leqslant a \leqslant b$ always implies

$$\beta\|a\| \leqslant \|b\|;$$

3. there is a $\gamma > 0$ such that $a \leqslant b \leqslant c$ always implies

$$\gamma\|b\| \leqslant \|a\| \vee \|c\|.$$

(See, for example, [14], Chapter 2.) Alternatively normality can be characterized by duality properties; *$\mathscr{B}_+$ is normal if, and only if, $\mathscr{B}_+^*$ generates $\mathscr{B}^*$*. Specifically the following conditions are equivalent:

1. $\mathscr{B}_+$ is normal;
2. $\mathscr{B}^* = \mathscr{B}_+^* - \mathscr{B}_+^*$.

This last statement also has a converse; *$\mathscr{B}_+^*$ is normal if, and only if, $\mathscr{B} = \mathscr{B}_+ - \mathscr{B}_+$*.

An element $u \in \mathscr{B}_+$ is called an *interior point* if $\mathscr{B}_+$ contains an open neighbourhood of u, i.e., u is an interior point of $\mathscr{B}_+$ if, and only if, there is an $\varepsilon > 0$ such that

$$\{a ; \|u - a\| < \varepsilon\} \subset \mathscr{B}_+.$$

There are two extreme cases. The set $\operatorname{int}\mathscr{B}_+$ of interior points is either empty or norm dense in $\mathscr{B}_+$ (see, for example, [14], Chapter 4, Proposition 4.5).

The existence of interior points is a property complementary to normality; *$\operatorname{int}\mathscr{B} \neq \emptyset$ if, and only if, norm bounded sets are order bounded.* To prove the first implication choose $u \in \operatorname{int}\mathscr{B}_+$ and $\varepsilon > 0$ such that

$$\{a ; \|u - a\| < \varepsilon\} \subset \mathscr{B}_+.$$

Therefore $u - \delta a/\|a\| \in \mathscr{B}_+$ for all $a \in \mathscr{B}$ and all $0 < \delta \leqslant \varepsilon$. (The possible equality $\delta = \varepsilon$ follows because $\mathscr{B}_+$ is norm closed.) Thus

$$-(\|a\|/\varepsilon)u \leqslant a \leqslant (\|a\|/\varepsilon)u$$

for all $a \in \mathscr{B}$, and norm bounded sets are order bounded. Conversely if there exists a $u \in \mathscr{B}$ such that $-u \leqslant a/\|a\| \leqslant u$ for all $a \in \mathscr{B}$ then the ball of radius one around u is contained in $\mathscr{B}_+$, i.e., u is an interior point in $\mathscr{B}_+$.

Some of the connections are nicely summarized in terms of the set $\mathrm{b}(\mathscr{B})$ of norm-bounded sets in $\mathscr{B}$ and the set $\mathrm{o}(\mathscr{B})$ of order-bounded sets in $\mathscr{B}$. Specifically one has

1. $\mathscr{B}_+$ normal implies $\mathrm{o}(\mathscr{B}) \subseteq \mathrm{b}(\mathscr{B})$.
2. $\mathrm{int}\,\mathscr{B}_+ \neq \emptyset$ is equivalent to $\mathrm{b}(\mathscr{B}) \subseteq \mathrm{o}(\mathscr{B})$.
3. If $\mathrm{int}\,\mathscr{B}_+ \neq \emptyset$ then $\mathscr{B}_+$ normal is equivalent to $\mathrm{o}(\mathscr{B}) = b(\mathscr{B})$.

Therefore $\mathscr{B}_+$ *is normal and* $\mathrm{int}\,\mathscr{B}_+ \neq \emptyset$ *if, and only if,* $\mathrm{o}(\mathscr{B}) = \mathrm{b}(\mathscr{B})$.

The geometric notion of interior point can also be expressed in another order-theoretic manner. An element $u \in \mathscr{B}$ is defined to be an *order unit* if for each $a \in \mathscr{B}$ there is a $\lambda \geqslant 0$ such that $a \leqslant \lambda u$. Thus u is an order unit if, and only if, $\mathscr{B} = \mathscr{B}_u$ where

$$\mathscr{B}_u = \{a\,;\, -\delta u \leqslant a \leqslant \delta u \text{ for some } \delta \geqslant 0\}.$$

More generally the following conditions are equivalent:

1. u is an interior point of $\mathscr{B}_+$,
2. u is an order unit,
3. $\mathscr{B} = \mathscr{B}_u$.

Conditions 2 and 3 are equivalent by the foregoing definition, and $3 \Rightarrow 1 \Rightarrow 2$ by the arguments of the previous paragraph.

Note that if $u \in \mathrm{int}\,\mathscr{B}_+$ and $\lambda \geqslant 0$ is chosen such that $a \leqslant \lambda u$ then $a = a_+ - a_-$ with $a_+ = \lambda u \geqslant 0$ and $a_- = \lambda u - a \geqslant 0$. Thus $\mathscr{B} = \mathscr{B}_+ - \mathscr{B}_+$, i.e., $\mathscr{B}_+$ generates $\mathscr{B}$ and, equivalently, $\mathscr{B}^*_+$ is normal.

There are various weaker notions of interior point and order unit. For example $u \in \mathscr{B}_+$ is called a quasi-interior (quint) point if

$$(*) \qquad \omega(u) > 0$$

for all $\omega \in \mathscr{B}^*_+ \setminus \{0\}$. Again the set $\mathrm{quint}\,\mathscr{B}_+$ is either empty or norm dense and if $\mathrm{int}\,\mathscr{B}_+ \neq \emptyset$ then $\mathrm{int}\,\mathscr{B}_+ = \mathrm{quint}\,\mathscr{B}_+$. Thus if $\mathrm{int}\,\mathscr{B}_+ \neq \emptyset$ then $(*)$ provides a criterion for u to be an interior point of $\mathscr{B}_+$.

Now assume that $u \geqslant 0$ and define

$$N_u(a) = \inf\{\lambda \geqslant 0\,;\, a \leqslant \lambda u\}$$

for all $a \in \mathscr{B}_u$. If u is an interior point and if $\varepsilon > 0$ is chosen such that

$$\{a\,;\, \|u - a\| < \varepsilon\} \subset \mathscr{B}_+$$

then it follows from the above discussion of interior points that

$$0 \leqslant N_u(a) \leqslant \|a\|/\varepsilon.$$

In particular if $u \in \operatorname{int}\mathscr{B}_+$ then N_u is a norm continuous function over $\mathscr{B}$ which is convex and positive homogeneous. (Arendt, Chernoff and Kato [1] call N_u a half-norm.) Note that

$$\mathscr{B}_+ = \{a\,;\, N_u(-a) = 0\}.$$

Moreover

$$a \leqslant N_u(a)u$$

for all $a \in \mathscr{B}$ because $\mathscr{B}_+$ is norm closed.

Next define

$$\|a\|_u = N_u(a) \vee N_u(-a).$$

It readily follows that $a \in \mathscr{B}_u \mapsto \|a\|_u$ is a norm and if $u \in \operatorname{int}\mathscr{B}_+$ the above continuity estimate for N_u also explicitly establishes that $\|\cdot\|_u$ is subordinate to $\|\cdot\|$. But conversely if $\|\cdot\|_u$ is subordinate to $\|\cdot\|$ then u must be an interior point of $\mathscr{B}_u \cap \mathscr{B}_+$. This follows because $\|a\|_u \leqslant c\|a\|$ implies $-cu \leqslant a/\|a\| \leqslant cu$ and hence the ball of radius $1/c$ around u, in $\overline{\mathscr{B}}_u$, is contained in $\overline{\mathscr{B}_u \cap \mathscr{B}_+}$. Thus *the two norms* $\|\cdot\|_u$ *and* $\|\cdot\|$ *are equivalent on* $\mathscr{B}$ *if, and only if,* $\mathscr{B}_+$ *is normal and* $u \in \operatorname{int}\mathscr{B}_+$. Moreover for each u the following conditions are equivalent:

1. $\mathscr{B}_+ \cap \mathscr{B}_u$ is normal;
2. there is a β' such that $0 \leqslant a \leqslant u$ always implies
$$\beta'\|a\| \leqslant \|u\|;$$
3. there is a γ' such that $-u \leqslant a \leqslant u$ always implies
$$\gamma'\|a\| \leqslant \|u\|;$$
4. there is a δ' such that
$$\delta'\|a\| \leqslant \|a\|_u \quad \text{for all } a \in \mathscr{B}_u.$$

The equivalence of Conditions 1, 2 and 3 is just an elaboration of the earlier conditions of normality. But $3 \Leftrightarrow 4$ by the following reasoning. Since $-N_u(-a)u \leqslant a \leqslant N_u(a)u$, Condition 3 implies that

$$\gamma'\|a\| \leqslant N_u(a) \vee N_u(-a) = \|a\|_u.$$

Thus the norm $\|\cdot\|$ is subordinate to $\|\cdot\|_u$. Conversely if $\|a\|_u \geqslant \varepsilon'\|a\|$ and $-u \leqslant a \leqslant u$ then one has $\|a\|_u \leqslant 1$ and Condition 3 is satisfied with $\delta' = \varepsilon'/\|u\|$.

Note that $\operatorname{int}\mathscr{B}_+ = \emptyset$ does not necessarily imply that $\operatorname{int}\mathscr{B}^*_+ = \emptyset$ nor conversely. For example if $\mathscr{B} = L^1(X)$ with $\mathscr{B}_+$ the pointwise positive L^1-functions then $\mathscr{B}^* = L^\infty(X)$ and $\mathscr{B}^*_+$ is the cone of positive L^∞-functions. But $\text{int}\,\mathscr{B}_+ = \emptyset$ and $\text{int}\,\mathscr{B}^*_+ \neq \emptyset$. In general *the existence or non-existence of interior points is not stable under duality.*

We complete this brief review with a comment about lattice ordering and interior points.

Assume that the partial order $\geqslant$ associated with $\mathscr{B}_+$ is a lattice ordering, i.e., each pair $a, b \in \mathscr{B}$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. Next let $a_\pm = (\pm a) \vee 0$ denote the positive and negative components of a and $|a| = a_+ + a_-$ the modulus. The norm on $\mathscr{B}$ is defined to be a lattice norm if $\| |a| \| = \|a\|$ and if $0 \leqslant a \leqslant b$ implies $\|a\| \leqslant \|b\|$, and then $(\mathscr{B}, \|\cdot\|)$ is called a Banach lattice. If, moreover, each order bounded set in $\mathscr{B}$ has a least upper bound in $\mathscr{B}$ then $\mathscr{B}$ is said to be order complete. For example $C_0(X)$ and $L^\infty(X)$ are Banach lattices, with the usual pointwise ordering and the usual norms, but note that only $\mathscr{B} = C(X)$ and $\mathscr{B} = L^\infty(X)$ have the property $\text{int}\,\mathscr{B}_+ \neq \emptyset$ (and $C(X)$ is not generally order complete). The statement about $\text{int}\,\mathscr{B}_+ \neq \emptyset$ has a general analogue. If $(\mathscr{B}, \|\cdot\|)$ is a Banach lattice the following conditions are equivalent:

1. $\text{int}\,\mathscr{B}_+ \neq \emptyset$;
2. $\mathscr{B}$ is lattice isomorphic to $C(X)$ for some compact Hausdorff space X.

The implication $1 \Rightarrow 2$ can be established by demonstrating that the norm $\|\cdot\|_u$ associated with an interior point u is a lattice norm with the special property that

$$\|a \vee b\|_u = \|a\|_u \vee \|b\|_u$$

for all $a, b \in \mathscr{B}_+$. The desired result then follows from a famous theorem of Kakutani, Kreĭn and Kreĭn (see, for example, [18], Chapter II, Section 7). The converse implication is evident and the equivalence demonstrates that the existence of interior points is incompatible with the lattice structure except in the simplest case of $C(X)$.

2. C_0-SEMIGROUPS

A C_0-semigroup $S = \{S_t\}_{t \geqslant 0}$ on a Banach space $\mathscr{B}$ is a family of bounded linear operators on $\mathscr{B}$ satisfying the semigroup property

$$S_s S_t = S_{s+t}$$

for $s, t \geqslant 0$ and the continuity property

$$\lim_{t \to 0} \| S_t a - a \| = 0$$

for all $a \in \mathscr{B}$. The generator H of S is the linear operator whose domain $D(H)$ consists of those $a \in \mathscr{B}$ for which there exists $a, b \in \mathscr{B}$ satisfying

$$\lim_{t \to 0} \|(I - S_t)a/t - b\| = 0.$$

If $a \in D(H)$ the action of H is defined by $Ha = b$. We say that H generates S and write $S_t = \exp\{-tH\}$.

If $\mathscr{B}$ is an ordered Banach space with a positive cone $\mathscr{B}_+$ a bounded operator on $\mathscr{B}$ is said to be positive if it maps $\mathscr{B}_+$ into $\mathscr{B}_+$. Moreover a semigroup S is said to be positive if each S_t maps $\mathscr{B}_+$ into $\mathscr{B}_+$.

Next assume that the cone $\mathscr{B}_+$ is normal and has non-empty interior, and let H be a densely defined operator on $\mathscr{B}$.

Under these circumstances it is known, [1], that the following three conditions are equivalent:

1. H generates a positive C_0-semigroup;

2. The range condition $\mathrm{R}(I + \alpha H) = \mathscr{B}$ holds for all small $\alpha > 0$, and H satisfies the property

(P) If $a \in D(H) \cap \mathscr{B}_+$ and $\omega \in \mathscr{B}_+^*$ is such that $\omega(a) = 0$, then

$$\omega(Ha) \leqslant 0;$$

3. The resolvent $(I + \alpha H)^{-1}$ exists as a positive operator for arbitrary small $\alpha > 0$.

Our first result is a version of this theorem in which we give a relatively straightforward proof of the equivalence of 1 and 3 without using Condition 2.

THEOREM 2.1. *Let $\mathscr{B}$ be a real ordered Banach space such that the cone $\mathscr{B}_+$ is normal and has nonempty interior. Let H be a densely defined operator on $\mathscr{B}$.*

It follows that H is the generator of a positive C_0-semigroup if, and only if,

$$\mathrm{R}(I + \alpha H) = \mathscr{B}$$

for all small $\alpha > 0$, and H satisfies one of the conditions:

a. *If $a \in D(H)$ and $(I + \alpha H)a \geqslant 0$, then $a \geqslant 0$, for all small $\alpha > 0$;*

b. *For some $u \in \operatorname{int}\mathscr{B}_+$ there exists a $\lambda \in \mathbf{R}$ such that*

$$N_u((I + \alpha H)a) \geqslant (1 - \alpha\lambda)N_u(a)$$

for all $a \in D(H)$ and all small $\alpha > 0$, where N_u is the half-norm defined by u;

c. *For all $u \in \operatorname{int}\mathscr{B}_+ \cap D(H)$ there exists a $\lambda \in \mathbf{R}$ such that*

$$N_u((I + \alpha H)a) \geqslant (1 - \alpha\lambda)N_u(a)$$

for all $a \in D(H)$ and all small $\alpha > 0$.

This theorem is a consequence of the following lemma.

LEMMA 2.2. *Let $\mathscr{B}$ be an ordered Banach space such that the cone $\mathscr{B}_+$ is normal with non-empty interior. Let H be an operator on $\mathscr{B}$, $\alpha > 0$, $\lambda \in \mathbf{R}$ and $u \in \operatorname{int}\mathscr{B}_+$. Consider the conditions:*

1. $(I + \alpha H)a \geqslant 0 \Rightarrow a \geqslant 0$ *for all* $a \in D(H)$;
2. $N_u((I + \alpha H)a) \geqslant (1 - \alpha\lambda)N_u(a)$.

Then 2. *implies* 1. *provided* $\alpha\lambda < 1$, *and conversely* 1 *implies* 2 *provided* $u \in \mathrm{R}(I + \alpha H) \cap$ $\cap D(H)$ *and* λ *is chosen such that*

$$Hu \geqslant -\lambda u,$$

e.g., $\lambda = N_u(-Hu)$, *and* $\alpha\lambda < 1$.

Proof. 2. $\Rightarrow$ 1. If $(I + \alpha H)a \leqslant 0$ then $N_u((I + \alpha H)a) = 0$, and by 2 one has $N_u(a) = 0$. Hence $a \leqslant 0$ and Condition 1 is satisfied.

1. $\Rightarrow$ 2. Since $u \in \mathrm{int}\mathscr{B}_+ \cap D(H)$ there exists a $\lambda \geqslant 0$ such that

$$Hu \geqslant -\lambda u,$$

the best choice being $\lambda = N_u(-Hu)$. Assume that α satisfies $\alpha\lambda < 1$ for this choice of λ. Let $a \in D(H)$ and set

$$c = (I + \alpha H)a.$$

As $u \in \mathrm{R}(I + \alpha H)$ there is a $b \in D(H)$ such that

$$u = (I + \alpha H)b$$

and $b \geqslant 0$ by 1. Since $Hu \geqslant -\lambda u$ one has

$$(I + \alpha H)u \geqslant (1 - \alpha\lambda)u = (1 - \alpha\lambda)(I + \alpha H)b$$

and Condition 1 implies

$$u \geqslant (1 - \alpha\lambda)b.$$

Hence

$$N_u(b) \leqslant (1 - \alpha\lambda)^{-1}.$$

The relations $N_u(c)u \geqslant c$, $u = (I + \alpha H)b$ and $c = (I + \alpha H)a$ together with Condition 1 then imply that

$$N_u(c)b \geqslant a.$$

Consequently

$$N_u(a) \leqslant N_u(b)N_u(c) \leqslant (1 - \alpha\lambda)^{-1}N_u(c),$$

i.e., Condition 2 is satisfied.

Proof of Theorem 2.1. If H generates a positive C_0-semigroup S it follows from the Laplace transform formula [3], [5], [6]

$$(I + \alpha H)^{-1} = \int_0^\infty dt\ e^{-t} S_{\alpha t}$$

hat $\mathrm{R}(I + \alpha H) = \mathscr{B}$ and $(I + \alpha H)^{-1}$ is positive, i.e., Condition a) holds. Lemma 2.2 then implies that Conditions b) and c) hold.

Conversely it follows from Lemma 2.2 that to complete the proof of Theorem 2.1 it suffices to consider case b), i.e., to assume that $R(I + \alpha H) = \mathscr{B}$ and

$$N_u((I + \alpha H)a) \geqslant (1 - \alpha\lambda)N_u(a)$$

for all $a \in D(H)$ and all small $\alpha > 0$. As $\|b\|_u = N_u(b) \vee N_u(-b)$ for all $b \in \mathscr{B}$ it follows that

$$\|(I + \alpha H)a\|_u \geqslant (1 - \alpha\lambda)\|a\|_u$$

for all $a \in D(H)$. But since $R(I + \alpha H) = \mathscr{B}$ the operator $(I + \alpha H)^{-1}$ exists and one has

$$\|(I + \alpha H)^{-1}\|_u \leqslant (1 - \alpha\lambda)^{-1}$$

where $\|A\|_u = \sup\{\|Ab\|_u ; \, b \in \mathscr{B}, \, \|b\|_u \leqslant 1\}$. Hence H generates a C_0-semigroup S on $\mathscr{B}$ with

$$\|S_t\|_u \leqslant e^{\lambda t},$$

[3], [5], [6]. Since $\|\cdot\|_u$ is equivalent to the original norm it follows that S is a C_0-semigroup on $\mathscr{B}$ with its usual norm. But $(I + \alpha H)^{-1}$ is positive by Lemma 2.2, and hence S is positive because

$$\lim_{n\to\infty} \|S_t a - (I + tH/n)^{-n}a\| = 0$$

for all $a \in \mathscr{B}$, [3], [6].

Note that the condition $(I + \alpha H)\, a \geqslant 0 \Rightarrow a \geqslant 0$ immediately implies that $(I+\alpha H)\, a=0 \Rightarrow \pm a \geqslant 0 \Rightarrow a=0$, and hence this condition, together with $R(I+\alpha H) = \mathscr{B}$, is equivalent to the statement that $(I + \alpha H)^{-1}$ exists as a positive operator. If the condition $\operatorname{int}\mathscr{B}_+ \neq \emptyset$ is replaced by the weaker condition that $\mathscr{B}_+$ is generating, i.e., $\mathscr{B} = \mathscr{B}_+ - \mathscr{B}_+$, the positivity of $(I + \alpha H)^{-1}$ still implies its boundedness, but it is unclear whether the positivity of $(I + \alpha H)^{-1}$ for small $\alpha > 0$ implies that H is a generator in this more general setting. In particular it is unclear whether this result is true on a Banach lattice*). A partial result is that if u is a positive element in $D(H)$ such that $Hu \geqslant -\lambda u$ for some $\lambda \geqslant 0$, then $(I + \alpha H)^{-1} \mathscr{B}_u \subseteq \mathscr{B}_u$ for small $\alpha > 0$, and if $a \in (I + \alpha H)^{-2}\mathscr{B}_u$ then the limits

$$S_t a = \lim_{n\to\infty} \left(I + \frac{t}{n} H\right)^{-n} a = \lim_{n\to\infty} e^{-tH\left(I + \frac{1}{n}H\right)^{-1}} a$$

exist in $\|\cdot\|_u$ (see, for example, [3], Theorem 3.1.10). Thus if $\mathscr{B}_+$ is normal the limits also exist in the original norm, because it is subordinate to $\|\cdot\|_u$. Moreover if $a \in (I + \alpha H)^{-3}\mathscr{B}_u$ then $t \mapsto S_t a$ is differentiable with derivative $-S_t Ha = -HS_t a$.

We next show that a positive C_0-semigroup maps the interior $\operatorname{int}\mathscr{B}_+$ of the cone $\mathscr{B}_+$ into itself. The strong continuity of the semigroup is essential for this

*) See Notes added in proofs, page 399.

result, and in Section 3 we give an example of a positive C_0^*-semigroup S which maps all interior points into non-interior points.

Recall that a is a quasi-interior point in $\mathscr{B}_+$, $a \in \operatorname{quint}\mathscr{B}_+$, if $\omega(a) > 0$ for all $\omega \in \mathscr{B}_+^* \setminus \{0\}$, and that $\operatorname{int}\mathscr{B}_+ = \operatorname{quint}\mathscr{B}_+$ whenever $\operatorname{int}\mathscr{B}_+ \neq \emptyset$.

PROPOSITION 2.3. *Let S be a positive C_0-semigroup with generator H. It follows that*

$$S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+ \quad \textit{for all } t \geq 0$$

$$(I + \alpha H)^{-1}(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+ \quad \textit{for all small } \alpha > 0$$

$$(I + \alpha H)^{-1}(\operatorname{quint}\mathscr{B}_+) \subseteq \operatorname{quint}\mathscr{B}_+ \quad \textit{for all small } \alpha > 0$$

but in general S_t does not map $\operatorname{quint}\mathscr{B}_+$ *into itself.*

Proof. We first show $S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$. If $\operatorname{int}\mathscr{B}_+ = \emptyset$, there is nothing to prove. Hence assume $\operatorname{int}\mathscr{B}_+ \neq \emptyset$ and choose $u \in \operatorname{int}\mathscr{B}_+$. Since $t \mapsto S_t u$ is norm continuous there is an $\varepsilon > 0$ such that $S_t u \in \operatorname{int}\mathscr{B}_+$ for $0 \leq t \leq \varepsilon$. If $v \in \operatorname{int}\mathscr{B}_+$, then $\lambda u \leq v$ for some $\lambda > 0$. Hence

$$\lambda S_t u \leq S_t v.$$

Since $\operatorname{int}\mathscr{B}_+ \neq \emptyset$, $\operatorname{int}\mathscr{B}_+$ consists of the points $v \in \mathscr{B}$ such that $\omega(v) > 0$ for all non-zero $\omega \in \mathscr{B}_+^*$. Thus, if $\omega \in \mathscr{B}_+^* \setminus \{0\}$ and $0 \leq t \leq \varepsilon$ one has

$$0 < \lambda\omega(S_t u) \leq \omega(S_t v)$$

and consequently, $S_t v \in \operatorname{int}\mathscr{B}_+$. This argument shows that $S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$ for $0 \leq t \leq \varepsilon$. The semigroup property

$$S_t = (S_{t/n})^n,$$

applied for $n > t/\varepsilon$, then shows that

$$S_t(\operatorname{int}\mathscr{B}_+) \subset \operatorname{int}\mathscr{B}_+$$

for all $t \geq 0$.

Since $\operatorname{int}\mathscr{B}_+ = \operatorname{quint}\mathscr{B}_+$ when $\operatorname{int}\mathscr{B}_+ \neq \emptyset$, the second statement of the proposition follows from the third:

$$(I + \alpha H)^{-1}(\operatorname{quint}\mathscr{B}_+) \subseteq \operatorname{quint}\mathscr{B}_+.$$

Therefore we prove this property. If $a \in \operatorname{quint}\mathscr{B}_+$ and $\omega \in \mathscr{B}_+^* \setminus \{0\}$, then $\omega(S_t a) > 0$ for $t = 0$ and $\omega(S_t a) \geq 0$ for all t. Since $t \mapsto \omega(S_t a)$ is continuous it follows that

$$\omega((I + \alpha H)^{-1}a) = \int_0^\infty dt\, e^{-t}\, \omega(S_{\alpha t} a) > 0$$

and hence $(I + \alpha H)^{-1} a \in \operatorname{quint}\mathscr{B}_+$.

The last statement of the proposition is a consequence of the next example.

EXAMPLE 2.4. Let

$$\mathscr{B} = \{f \in C[0, +\infty\rangle\,;\ f(0) = \lim_{x\to\infty} f(x) = 0\},$$

equipped with the supremum norm and let $\mathscr{B}_+$ be the positive functions in $\mathscr{B}$. Then

$$\operatorname{quint}\mathscr{B}_+ = \{f \in \mathscr{B}\,;\ f(x) > 0 \ \text{ for } x > 0\}.$$

Next define a semigroup S_t by

$$(S_t f)(s) = \begin{cases} 0 & \text{if } 0 \leqslant s \leqslant t \\ f(s-t) & \text{if } t \leqslant s. \end{cases}$$

It readily follows that S is a positive C_0-semigroup such that

$$S_t(\operatorname{quint}\mathscr{B}_+) \cap \operatorname{quint}\mathscr{B}_+ = \emptyset$$

for $t > 0$.

The property $S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$ derived in Proposition 2.3 is a mild form of ergodicity and next we analyze a variety of related conditions.

A bounded operator A on an ordered Banach space $(\mathscr{B}, \mathscr{B}_+)$ is defined to be strictly positive if $A\mathscr{B}_+ \subseteq \operatorname{quint}\mathscr{B}_+$. Similarly a C_0-semigroup S on $(\mathscr{B}, \mathscr{B}_+)$ is defined to be *strictly positive* if S_t is strictly positive for all $t > 0$, or, equivalently, for all small $t > 0$. Thus if $\operatorname{int}\mathscr{B}_+ \neq \emptyset$ then strict positivity of S means that each S_t maps $\mathscr{B}_+$ into its interior. Next we define a positive C_0-semigroup S on $(\mathscr{B}, \mathscr{B}_+)$ to be *irreducible* if the only closed hereditary S-invariant subcones of $\mathscr{B}_+$ are $\{0\}$ and $\mathscr{B}_+$. (A subcone $\mathscr{C} \subseteq \mathscr{B}_+$ is hereditary if $0 \leqslant a \leqslant b \in \mathscr{C}$ implies $a \in \mathscr{C}$.) Finally S is defined to be *ergodic* if the hereditary cone generated by $\{S_t a\,;\ t \geqslant 0\}$ is dense in $\mathscr{B}_+$ for all $a \in \mathscr{B}_+\setminus\{0\}$.

PROPOSITION 2.5. *Let $S_t = \exp\{-tH\}$ be a positive C_0-semigroup on an ordered Banach space $(\mathscr{B}, \mathscr{B}_+)$; assume*

either *$(\mathscr{B}, \mathscr{B}_+)$ is a Banach lattice,*

or $\operatorname{int}\mathscr{B}_+ \neq \emptyset$,

and consider the conditions:

1. *S is strictly positive;*
2. *S is irreducible;*
3. *S is ergodic;*
4. *for each $a \in \mathscr{B}_+\setminus\{0\}$, $\omega \in \mathscr{B}^*_+\setminus\{0\}$, there exists a $t > 0$ such that*

$$\omega(S_t a) > 0\,;$$

5. *the operators $(I + \alpha H)^{-1}$ are strictly positive for all small $\alpha > 0$;*
6. *the operator $(I + \alpha H)^{-1}$ is strictly positive for some $\alpha > 0$.*

It follows that $1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5 \Leftrightarrow 6$.

REMARK. The implications $1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ are valid without any continuity assumption on S, and for general semigroups of positive operators.

If $(\mathscr{B}, \mathscr{B}_+)$ is a Banach lattice the equivalences $2 \Leftrightarrow 3 \Leftrightarrow 4$ follow from [18], Chapter III, Proposition 8.3 and the other implications are proved by the arguments used below for the case $\operatorname{int}\mathscr{B}_+ \neq \emptyset$. For this latter case we need the following lemma.

LEMMA 2.6. *Let $(\mathscr{B}, \mathscr{B}_+)$ be a Banach space with* $\operatorname{int}\mathscr{B}_+ \neq \emptyset$. *If $\mathscr{C}$ is a hereditary subcone of $\mathscr{B}_+$ the following conditions are equivalent:*

1. *$\mathscr{C}$ is dense in $\mathscr{B}_+$;*
2. *for all $\omega \in \mathscr{B}^*_+\setminus\{0\}$ there exists an $x \in \mathscr{C}$ such that $\omega(x) > 0$.*

Proof. $1 \Rightarrow 2$. This is trivially true without the assumption that $\mathscr{C}$ is hereditary, and with the assumption $\operatorname{int}\mathscr{B}_+ \neq \emptyset$ replaced by the weaker condition $\mathscr{B} = \overline{\mathscr{B}_+ - \mathscr{B}_+}$.

$2 \Rightarrow 1$. Assume that $\mathscr{C}$ is not dense in $\mathscr{B}_+$. Then $\mathscr{C}$ contains no interior point of $\mathscr{B}$, because if $u \in \mathscr{C} \cap \operatorname{int}\mathscr{B}_+$ then

$$\mathscr{C} \supseteq \{a \in \mathscr{B}_+;\ a \leqslant \lambda u \ \text{for some}\ \lambda > 0\} = \mathscr{B}_+$$

by the hereditary property of $\mathscr{C}$. By [11], Theorem 1.2, there exists an $\omega \in \mathscr{B}^*\setminus\{0\}$ such that $\omega(x) \geqslant 0$ for all $x \in \mathscr{B}_+$ and $\omega(x) \leqslant 0$ for all $x \in \mathscr{C}$. But as $\mathscr{C} \subseteq \mathscr{B}_+$ it follows that $\omega \in \mathscr{B}^*_+$, $\omega \neq 0$, $\omega(x) = 0$ for all $x \in \mathscr{C}$. Therefore Condition 2 is false.

Proof of Proposition 2.5. Condition 1 is equivalent to the statement:

1′. For each $a \in \mathscr{B}_+\setminus\{0\}$, $\omega \in \mathscr{B}^*_+\setminus\{0\}$ and $t > 0$ one has

$$\omega(S_t a) > 0,$$

and hence $1 \Rightarrow 4$. If $\operatorname{int}\mathscr{B} \neq \emptyset$ the equivalence $3 \Leftrightarrow 4$ is evident from Lemma 2.6, together with the fact that the hereditary cone generated by $\{S_t a;\ t \geqslant 0\}$ is given by

$$\left\{b\ ;\ 0 \leqslant b \leqslant \sum_{k=1}^{n} \lambda_k S_{t_k} a \ \text{for some}\ \lambda_k > 0,\ t_k \geqslant 0,\ n \in \mathbf{N}\right\}.$$

$3 \Rightarrow 2$. If $\mathscr{C}$ is a closed hereditary subcone of $\mathscr{B}_+$ invariant under S and $\mathscr{C} \neq \{0\}$ then there exists a non-zero positive $a \in \mathscr{C}$. But then $\mathscr{C}$ contains the hereditary cone generated by $\{S_t a;\ t \geqslant 0\}$, and this cone is dense in $\mathscr{B}_+$ by Condition 3. It follows that $\mathscr{C} = \mathscr{B}_+$, and S is irreducible.

$2 \Rightarrow 4$. Assume $\operatorname{int}\mathscr{B}_+ \neq \emptyset$. If 4 does not hold, there is an $a \in \mathscr{B}_+\setminus\{0\}$ and an $\omega \in \mathscr{B}^*_+\setminus\{0\}$ such that

$$\omega(S_t a) = 0$$

for all $t \geqslant 0$. Define

$$\mathscr{C} = \{b \in \mathscr{B}_+\ ;\ \omega(S_t b) = 0 \ \text{for all}\ t \geqslant 0\}.$$

Then $\mathscr{C}$ is a closed, S-invariant hereditary subcone of $\mathscr{B}_+$, and $a \in \mathscr{C}$. As $\omega(x) > 0$ for all $x \in \operatorname{int}\mathscr{B}_+$, it follows that $\mathscr{C} \cap \operatorname{int}\mathscr{B}_+ = \emptyset$. Hence S is not irreducible.

$4 \Rightarrow 5$. This follows from the Laplace transform representation of the resolvent,

$$\omega((I + \alpha H)^{-1}a) = \int_0^\infty dt\, e^{-t}\omega(S_{\alpha t}a)$$

together with the continuity of $t \mapsto \omega(S_{\alpha t}a)$.

$5 \Rightarrow 6$. This is trivial.

$6 \Rightarrow 4$. This again follows from the Laplace transform representation.

The next example demonstrates that the implication $2 \Rightarrow 1$ is not valid, even when $\mathscr{B}_+$ has an interior point.

EXAMPLE 2.7. Let $\mathscr{B} = C(\mathbf{T})$ equipped with the supremum norm, where $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ is the circle and let $\mathscr{B}_+$ consist of the positive functions in $\mathscr{B}$. Define

$$(S_t f)(s) = f(s - t)$$

then S is a positive C_0-semigroup which is irreducible but not strictly positive.

3. C_0^*-SEMIGROUPS

Let $(\mathscr{B}_*, \mathscr{B}_{*+})$ be an ordered Banach space and $(\mathscr{B}, \mathscr{B}_+)$ its dual. A semigroup S of bounded operators on $\mathscr{B}$ is called a C_0^*-semigroup if each S_t is weak*-continuous, and the map $t \geqslant 0 \mapsto S_t a$ is weak*-continuous for each $a \in \mathscr{B}$. This is equivalent to the existence of a C_0-semigroup T on $\mathscr{B}_*$ such that $S_t = T_t^*$ for each t. Then S is positive if, and only if, T is positive. The generator H of S is defined in the weak*-topology, and H is then the adjoint of the generator of T, [3], [5], [6]. Note that H is weak*-closed, and is densely defined in the weak*-topology.

Next assume that the cone $\mathscr{B}_+$ is normal and has a non-empty interior in the norm topology. Even though H is densely defined it may happen that $D(H) \cap \cap \operatorname{int}\mathscr{B}_+ = \emptyset$.

EXAMPLE 3.1. Let $\mathscr{B} = L^\infty[0,1]$ and $\mathscr{B}_+$ be the positive L^∞-functions. Then $\mathscr{B}_* = L^1[0,1]$ and $\mathscr{B}_{*+}$ is the positive L^1-functions. Define S by

$$(S_t f)(s) = \begin{cases} 0 & \text{if } 0 \leqslant s < t \\ f(s-t) & \text{if } t \leqslant s \leqslant 1. \end{cases}$$

The dual semigroup T_t on $L^1[0, 1]$ is then given by

$$(T_t \eta)(s) = \begin{cases} \eta(s+t) & \text{if } 0 \leqslant s \leqslant 1-t \\ 0 & \text{if } 1-t \leqslant s \leqslant 1 \end{cases}$$

for $\eta \in L^1[0, 1]$. It follows that S is a positive C_0^*-semigroup and one computes that the resolvent of its generator is given by

$$(I + \alpha H)^{-1} f(s) = \int_0^{s/\alpha} dt\, e^{-t} f(s - \alpha t).$$

Let ω_ε be the state defined by

$$\omega_\varepsilon(g) = \frac{1}{\varepsilon} \int_0^{\varepsilon} dt\, g(t)$$

for $g \in L^\infty$. Then if $f \in L^\infty$ one has

$$|\omega_\varepsilon((I + \alpha H)^{-1} f)| = \frac{1}{\varepsilon} \left| \int_0^{\varepsilon} ds \int_0^{s/\alpha} dt\, e^{-t} f(s - \alpha t)\, dt \right| \leqslant$$

$$\leqslant \frac{\|f\|_\infty}{\varepsilon} \int_0^{\varepsilon} ds \int_0^{s/\alpha} dt = \frac{\varepsilon}{2\alpha} \|f\|_\infty .$$

But $g \in \operatorname{int}\mathscr{B}_+$ if, and only if, $g(x) \geqslant \delta$ for almost all $x \in [0,1]$ and some $\delta > 0$. Hence $\omega_\varepsilon(g) \geqslant \delta$ for all such states ω_ε. Therefore it follows from the above estimate that $(I + \alpha H)^{-1} f \notin \operatorname{int}\mathscr{B}_+$ for all $f \in \mathscr{B}$, i.e., $D(H) \cap \operatorname{int}\mathscr{B}_+ = \emptyset$.

This result shows that one cannot prove a C_0^*-version of Theorem 2.1 by the method used in the C_0-case. One can however extend the C_0-method to the case where $D(H)$ contains a point in $\operatorname{int}\mathscr{B}$. As a preliminary we characterize the positive C_0^*-semigroups which have this property.

THEOREM 3.2. *Let $(\mathscr{B}, \mathscr{B}_+)$ be the dual of an ordered Banach space, and assume that $\mathscr{B}_+$ is normal and contains an interior point (in the norm topology). Let $S_t = \exp\{-tH\}$ be a positive C_0-semigroup on $\mathscr{B}$. The following conditions are equivalent:*

1. *$D(H) \cap \operatorname{int}\mathscr{B}_+ \neq \emptyset$;*
2. *$(I + \alpha H)^{-1} u \in \operatorname{int}\mathscr{B}_+$ for all $u \in \operatorname{int}\mathscr{B}_+$ and all small $\alpha > 0$;*
3. *$(I + \alpha H)^{-1} u \in \operatorname{int}\mathscr{B}_+$ for some $u \in \operatorname{int}\mathscr{B}_+$ and some $\alpha > 0$;*
4. *$S_t u \in \operatorname{int}\mathscr{B}_+$ for all $u \in \operatorname{int}\mathscr{B}_+$ and all $t \geqslant 0$;*
5. *$S_t u \in \operatorname{int}\mathscr{B}_+$ for some $u \in \operatorname{int}\mathscr{B}_+$ and some $t > 0$.*

To prove the theorem we need the following

LEMMA 3.3. *Let $(\mathscr{B}, \mathscr{B}_+)$ be the dual of an ordered Banach space $(\mathscr{B}_*, \mathscr{B}_{*+})$ and assume that $\mathscr{B}_+$ has an interior point u. If $a \in \mathscr{B}_+ \setminus \operatorname{int}\mathscr{B}_+$ and $\varepsilon > 0$, there exists an $\omega \in \mathscr{B}_{*+}$ such that*

$$\omega(u) = 1, \quad \omega(a) < \varepsilon.$$

Proof. If $\mathscr{B}_+$ is a closed cone with nonempty interior and $a \in \partial(\mathscr{B}_+)$ there exists a positive functional φ with $\varphi(a) = 0$, [11], Theorem 1.3. Since $u \in \operatorname{int}\mathscr{B}_+$ one has $\varphi(u) > 0$, and hence one may normalize φ such that $\varphi(u) = 1$. But the cone $\mathscr{B}_{*+}$ is canonically embedded in $\mathscr{B}_+^*$ and $\mathscr{B}_+^*$ is the closure of $\mathscr{B}_{*+}$ in the weak*-topology [14]. Thus, approximating φ by a suitable $\omega \in \mathscr{B}_{*+}$, the lemma is established.

Proof of Theorem 3.2. 1 ⇒ 2. Choose $v \in D(H) \cap \operatorname{int}\mathscr{B}_+$. Let $u \in \operatorname{int}\mathscr{B}_+$ and assume that $\alpha > 0$ is small enough that $(I + \alpha H)^{-1}$ exists. Put $a = (I + \alpha H)v$. Since $\mathscr{B}_u = \mathscr{B}$ there exists a $\lambda > 0$ such that

$$a \leqslant \lambda u.$$

But $(I + \alpha H)^{-1}$ is positive and hence

$$v = (I + \alpha H)^{-1}a \leqslant \lambda(I + \alpha H)^{-1}u.$$

As $v \in \operatorname{int}\mathscr{B}_+$ and $\lambda > 0$ it follows that

$$(I + \alpha H)^{-1}\, u \in \operatorname{int}\mathscr{B}_+.$$

But 2 ⇒ 3 ⇒ 1 is trivial, and this establishes the equivalence 1 ⇔ 2 ⇔ 3.

4 ⇒ 5. Trivial.

5 ⇒ 4. Assume that $v \in \operatorname{int}\mathscr{B}_+$ and $t_1 > 0$ are such that

$$S_{t_1} v \in \operatorname{int}\mathscr{B}_+.$$

We prove this implication with the aid of two observations.

Observation 1. *If $S_{t_0} v \notin \operatorname{int}\mathscr{B}_+$ for some $t_0 \geqslant 0$ then $S_t v \notin \operatorname{int}\mathscr{B}_+$ for all $t \geqslant t_0$.*

Proof. If $S_{t_0} v \notin \operatorname{int}\mathscr{B}_+$ there exists a positive functional $\omega \in \mathscr{B}_+^* \setminus \{0\}$ such that $\omega(S_{t_0} v) = 0$. But as the norms $\|\cdot\|_v$ and $\|\cdot\|$ on $\mathscr{B}$ are equivalent (here the normality of $\mathscr{B}$ is used) and $\|S_t\| \leqslant M' \exp\{\lambda_0 t\}$, for suitable M' and λ_0, one deduces that

$$S_t v \leqslant M e^{\lambda_0 t} v$$

for all $t \geqslant 0$ and a suitable $M > 0$. But if $t \geqslant t_0$ this entails

$$\omega(S_t v) = \omega(S_{t_0} S_{t-t_0} v) \leqslant M \exp\{\lambda(t - t_0)\}\omega(S_{t_0} v) = 0.$$

Thus $S_t v \notin \operatorname{int}\mathscr{B}_+$ for all $t \geqslant t_0$.

Observation 2. *$S_{nt_1} v \in \operatorname{int}\mathscr{B}_+$ for $n = 1,2,3, \ldots$.*

Proof. For $n = 1$ this is just the hypothesis on v. Since $S_{t_1}v \in \operatorname{int}\mathscr{B}_+$ there exists an $\varepsilon > 0$ such that

$$S_{t_1}v \geqslant \varepsilon v.$$

Repeated application of S_{t_1} to both sides of this relation gives

$$S_{nt_1}v \geqslant \varepsilon^n v.$$

As $v \in \operatorname{int}\mathscr{B}_+$ this implies $S_{nt_1}v \in \operatorname{int}\mathscr{B}_+$ for all natural numbers n.

Observations 1 and 2 now imply that $S_t v \in \operatorname{int}\mathscr{B}_+$ for all $t \geqslant 0$. But if $u \in \operatorname{int}\mathscr{B}_+$ there exists an $\varepsilon > 0$ such that $u \geqslant \varepsilon v$, and then

$$S_t u \geqslant \varepsilon S_t v$$

for all $t \geqslant 0$. This implies that

$$S_t u \in \operatorname{int}\mathscr{B}_+$$

for all $t \geqslant 0$ and all $u \in \operatorname{int}\mathscr{B}_+$.

4 $\Rightarrow$ 2. Assume that $S_t u \in \operatorname{int}\mathscr{B}_+$ for all $t \geqslant 0$, and define

$$\varphi(t) = \sup\{\lambda\,;\ S_t u \geqslant \lambda u\}$$

for each $t \geqslant 0$. As $S_t u \in \operatorname{int}\mathscr{B}_+$ we have

$$\varphi(t) > 0$$

for each $t \geqslant 0$. But

$$\varphi(t) = \inf\{\omega(S_t u)\,;\ \omega \in \mathscr{B}_{*+},\ \omega(u) = 1\}$$

by the following reasoning:

First, if $\psi(t)$ denotes the right hand side of this equality, one has

$$\psi(t) = \inf\{\omega(S_t u)\,;\ \omega \in \mathscr{B}^*_+,\ \omega(u) = 1\}$$

because $\mathscr{B}_{*+}$ is weak*-dense in $\mathscr{B}^*_+$.

Second, if $\omega \in \mathscr{B}^*_+$, $\omega(u) = 1$, and $S_t u \geqslant \lambda u$, then $\omega(S_t u) \geqslant \lambda$ and hence

$$\psi(t) \geqslant \varphi(t).$$

But by the definition of $\varphi(t)$ one has

$$S_t u - \varphi(t)u \in \mathscr{B}_+ \setminus \operatorname{int}\mathscr{B}_+ = \partial(\mathscr{B}_+)$$

and hence there exists an $\omega \in \mathscr{B}^*_+ \setminus \{0\}$ such that

$$\omega(S_t u - \varphi(t)u) = 0.$$

Since $u \in \operatorname{int}\mathscr{B}_+$ we may normalize ω such that $\omega(u) = 1$. But then

$$\omega(S_t u) = \varphi(t)\omega(u) = \varphi(t).$$

This shows $\psi(t) \leqslant \varphi(t)$, and hence $\psi(t) = \varphi(t)$.

It follows that $\varphi(t)$ is the infimum of a family of positive continuous functions. Thus φ is upper semicontinuous and in particular Borel. The estimate

$$S_t u \geqslant \varphi(t)u$$

now implies

$$(I + \alpha H)^{-1}u = \int_0^\infty dt\, e^{-t} S_{\alpha t} u \geqslant \varepsilon u$$

where

$$\varepsilon = \int_0^\infty dt\, e^{-t} \varphi(\alpha t).$$

As $\varphi(\alpha t) > 0$ for all t one has $\varepsilon > 0$, and the estimate above implies

$$(I + \alpha H)^{-1}u \in \operatorname{int}\mathscr{B}_+.$$

$2 \Rightarrow 5$. Assume that 5 is false, i.e., $S_t u \in \mathscr{B}_+ \setminus \operatorname{int}\mathscr{B}_+$ for all $u \in \operatorname{int}\mathscr{B}_+$ and all $t > 0$. Fix $u \in \operatorname{int}\mathscr{B}_+$ and choose $t_0 > 0$, $\varepsilon > 0$. By Lemma 3.3, there exists an $\omega \in \mathscr{B}_{*+}$ such that $\omega(u) = 1$ and $\omega(S_{t_0}u) < \varepsilon$. By the proof of Observation 1 there are constants M, $\lambda_0 \geqslant 0$, such that

$$S_t u \leqslant M e^{\lambda_0 t} u$$

(where λ_0 is such that $\|S_t\| \leqslant M' \exp\{\lambda_0 t\}$.) But then

$$\omega(S_t u) = \omega(S_{t_0} S_{t-t_0} u) \leqslant M e^{\lambda_0 (t - t_0)} \omega(S_{t_0} u) \leqslant \varepsilon M e^{\lambda_0 t}$$

for $t \geqslant t_0$. If $\lambda > \lambda_0$ it follows that

$$\omega((\lambda I + H)^{-1}u) = \int_0^\infty dt\, e^{-\lambda t} \omega(S_t u) = \int_0^{t_0} dt\, e^{-\lambda t} \omega(S_t u) + \int_{t_0}^\infty dt\, e^{-\lambda t} \omega(S_t u) \leqslant$$

$$\leqslant \int_0^{t_0} dt\, M + \int_{t_0}^\infty dt\, \varepsilon M e^{-(\lambda - \lambda_0)t} \leqslant t_0 M + \frac{M}{\lambda - \lambda_0}\varepsilon.$$

Since t_0 and ε were arbitrary positive numbers it follows that for any $\delta > 0$ there exists an $\omega \in \mathscr{B}_{*+}$ such that $\omega(u) = 1$ and

$$\omega((\lambda I + H)^{-1}u) < \delta.$$

Hence, there does not exist a $\delta > 0$ such that

$$(\lambda I + H)^{-1}u \geqslant \delta u$$

and consequently $(\lambda I + H)^{-1}u \notin \operatorname{int} \mathscr{B}_+$, i.e., Condition 2 is false.

This ends the proof of Theorem 3.2.

The proof of the equivalence of 4 and 5 in the above theorem can be strengthened to show that any positive semigroup (disregarding continuity in t) which maps an interior point in $\mathscr{B}_+$ into a noninterior point has a property of instantaneous local collapse.

COROLLARY 3.4. *Let $\mathscr{B}_+$ be a closed cone with non-empty interior in a Banach space $\mathscr{B}$, and let S be a positive semigroup on $\mathscr{B}$. If there exists a $u \in \operatorname{int}\mathscr{B}_+$ and a $t > 0$ such that $S_t u \notin \operatorname{int}\mathscr{B}_+$, then there exists a positive non-zero functional ω on $\mathscr{B}$ such that*

$$\omega \circ S_t = 0$$

for all $t > 0$.

Proof. Fix $u \in \operatorname{int}\mathscr{B}_+$ such that $S_{t_0} u \notin \operatorname{int}\mathscr{B}_+$ for some $t_0 > 0$. There exists a $\varphi(t) > 0$ such that

$$S_t u \leqslant \varphi(t) u$$

for all $t \geqslant 0$, e.g., $\varphi(t) = N_u(S_t u)$. But then

$$S_t u = S_{t_0} S_{t-t_0} u \leqslant \varphi(t - t_0) S_{t_0} u$$

for $t \geqslant t_0$, and this shows that $S_t u \notin \operatorname{int}\mathscr{B}_+$ for all $t \geqslant t_0$. Combining this with Observation 2 in the proof of Theorem 3.2 we deduce that $S_t u \notin \operatorname{int}\mathscr{B}_+$ for all $t > 0$.

Define

$$E_t = \{\omega \in \mathscr{B}^*_+ \,;\, \omega(S_t u) = 0,\ \ \omega(u) = 1\}.$$

Then $E_t \neq \emptyset$ for all $t > 0$, and the estimate

$$S_{t_1} u \leqslant \varphi(t_1 - t_2) S_{t_2} u$$

for $t_1 \geqslant t_2$ shows that $t_1 \geqslant t_2$ implies $E_{t_1} \supseteq E_{t_2}$.

Since $u \in \operatorname{int}\mathscr{B}_+$, there exists a $\lambda > 0$ such that $\|a\| \leqslant 1$ implies

$$a \leqslant \lambda u$$

and hence if $\omega \in \mathscr{B}^*_+$,

$$\|\omega\| \leqslant \lambda \omega(u).$$

It follows that each E_t is contained in the closed ball of radius λ around the origin in $\mathscr{B}^*$. This ball is compact in the weak*-topology by Alaoglu's theorem, and the sets E_t are closed in this topology. It follows that

$$\bigcap_{t>0} E_t \neq \emptyset.$$

If $\omega \in \bigcap_{t>0} E_t$ then $\omega \geqslant 0$, $\omega(u) = 1$, and $\omega(S_t u) = 0$ for all $t > 0$. But if $a \in \mathscr{B}$ then

$$-\|a\|_u u \leqslant a \leqslant \|a\|_u u$$

and hence

$$0 = \omega(-\|a\|_u S_t u) \leqslant \omega(S_t a) \leqslant \omega(\|a\|_u S_t u) = 0,$$

i.e.,

$$\omega \circ S_t = 0$$

for $t > 0$.

We next state the announced result on generation of positive C_0^*-semigroups.

THEOREM 3.5. *Let $(\mathscr{B}, \mathscr{B}_+)$ be the dual of an ordered Banach space, and assume that $\mathscr{B}_+$ is normal and contains an interior point in the norm topology. Let H be a weak*-densely defined, weak*-weak* closed operator on $\mathscr{B}$.*

The following two conditions are equivalent:

1. a. $R(I + \alpha H) = \mathscr{B}$ *for small* $\alpha > 0$;
   b. $(I + \alpha H)a \geqslant 0 \Rightarrow a \geqslant 0$ *for all* $a \in D(H)$;
   c. $D(H) \cap \operatorname{int}\mathscr{B}_+ \neq \emptyset$.
2. *H is the generator of a C_0^*-semigroup S such that*

$$S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$$

for all $t > 0$.

REMARK. Note that it follows from Theorem 3.2 that $S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$ for all $t > 0$ is in fact equivalent to the seemingly weaker condition $S_t(\operatorname{int}\mathscr{B}_+) \cap \cap \operatorname{int}\mathscr{B}_+ \neq \emptyset$ for some $t > 0$.

Proof. $2 \Rightarrow 1$. This is a direct consequence of Theorem 3.2 and standard theory.

$1 \Rightarrow 2$. Conditions a) and b) imply that the resolvent $(I + \alpha H)^{-1}$ exists as a positive, weak*-continuous operator for small $\alpha > 0$. To prove that H is a generator it thus suffices to establish a resolvent bound of the form

$$\|(I + \alpha H)^{-n}\| \leqslant M(1 - \alpha\lambda)^{-n}$$

for some M, $\lambda > 0$, [3], [5], [6].

To this end, pick a $u \in D(H) \cup \operatorname{int}\mathscr{B}_+$ and set $\lambda = N_u(-Hu)$. Then

$$Hu \geqslant -\lambda u$$

and hence

$$(I + \alpha H)u \geqslant (1 - \alpha\lambda)u.$$

Since $(I + \alpha H)^{-1}$ is positive one then obtains

$$(I + \alpha H)^{-1}u \leqslant (1 - \alpha\lambda)^{-1}u$$

provided α is small enough, i.e., $\alpha\lambda < 1$. Iteration of this relation gives

$$(I + \alpha H)^{-n}u \leqslant (1 - \alpha\lambda)^{-n}u$$

for $n = 1,2, \ldots$. As $(I + \alpha H)^{-n}$ is positive this implies

$$\|(I + \alpha H)^{-n}\|_u \leqslant (1 - \alpha\lambda)^{-n}$$

where

$$\|A\|_u = \sup\{\|Aa\|_u;\ a \in \mathscr{B}\setminus\{0\},\ \|a\|_u = 1\}$$

for any operator $A \in \mathfrak{B}(\mathscr{B})$. But as $\mathscr{B}_+$ is normal, the norms $\|\cdot\|_u$ and $\|\cdot\|$ on $\mathscr{B}$ are equivalent, and hence the norms $\|\cdot\|_u$ and $\|\cdot\|$ on $\mathfrak{B}(\mathscr{B})$ are equivalent. In particular there exists an $M > 0$ such that

$$\|(I + \alpha H)^{-n}\| \leqslant M\|(I + \alpha H)^{-n}\|_u \leqslant M(1 - \alpha\lambda)^{-n}.$$

Let $(\mathscr{B}, \mathscr{B}_+)$ be the dual of an ordered Banach space $(\mathscr{B}_*, \mathscr{B}_{*+})$, and assume that $\mathscr{B}_+$ has non-empty interior. In this case $\operatorname{int}\mathscr{B}_+ = \operatorname{quint}\mathscr{B}_+$. Proposition 2.3 established that a positive C_0-semigroup maps $\operatorname{int}\mathscr{B}_+$ into itself, but Example 3.1 and Theorem 3.2 show that this is not necessarily true for a C_0^*-semigroup. We can even show more, using the concept of N-interior points in $\mathscr{B}_+$. If $a \in \mathscr{B}_+$, then a is said to be an N-interior point, $a \in \operatorname{Nint}\mathscr{B}_+$, if $\omega(a) > 0$ for all $\omega \in \mathscr{B}_{*+}\setminus\{0\}$. One has the inclusions

$$\operatorname{int}\mathscr{B}_+ \subseteq \operatorname{Nint}\mathscr{B}_+ \subseteq \mathscr{B}_+$$

and all the inclusions may be proper. For example, if $\mathscr{B} = L^\infty[0, 1]$, then $f \in \operatorname{int}\mathscr{B}_+$ if, and only if, there exists an $\varepsilon > 0$ such that $f(s) \geqslant \varepsilon$ for almost all s, whilst $f \in \operatorname{Nint}\mathscr{B}_+$ if, and only if, $f(s) > 0$ for almost all s.

The mapping properties of C_0^*-semigroups and their resolvents on the N-interior can be quite complicated. It follows by the argument used to prove Proposition 2.3 that

$$(I + \alpha H)^{-1}(\operatorname{Nint}\mathscr{B}_+) \subseteq \operatorname{Nint}\mathscr{B}_+$$

but Example 3.1 demonstrates that neither of the relations

$$S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{Nint}\mathscr{B}_+$$

$$(I + \alpha H)^{-1}(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$$

is valid in general. It can however happen that

$$(*) \qquad S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$$

for small $t > 0$ implies

$$(**) \qquad S_t(\operatorname{Nint}\mathscr{B}_+) \subseteq \operatorname{Nint}\mathscr{B}_+$$

for all $t > 0$. For example if for each $u \in \operatorname{Nint}\mathscr{B}_+$ the set $\mathscr{B}_u = \{a\,;\, -\lambda u \leqslant a \leqslant \lambda u$ for some $\lambda > 0\}$ is weak* dense then this implication is valid. To prove it assume there exists a $u \in \operatorname{Nint}\mathscr{B}_+$ and a $t > 0$ such that $S_t u \notin \operatorname{Nint}\mathscr{B}_+$. But then there is a functional $\omega \in \mathscr{B}_{*+}\diagdown\{0\}$ such that $\omega(S_t u) = 0$ and hence $\omega(S_t a) = 0$ for all $a \in \mathscr{B}_u$. Since $\omega \circ S_t$ is a weak* continuous functional it follows that $\omega \circ S_t = 0$. But if $S_t(\operatorname{int}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+$ for small $t > 0$ it is true for all $t > 0$ and hence $(*)$ is not valid. In particular $(*)$ implies $(**)$ if $\mathscr{B}$ is a von Neumann algebra. The following example shows however that $(**)$ does not generally imply $(*)$.

EXAMPLE 3.6. Let $\mathscr{B} = L^\infty[0,1]$ be as in Example 3.1, and define

$$S_t f(x) = e^{-t/x} f(x), \quad x \in [0,1],\ t \geqslant 0.$$

Then S_t is a positive C_0^*-semigroup such that

$$S_t(\operatorname{Nint}\mathscr{B}_+) \subseteq \operatorname{Nint}\mathscr{B}_+ \quad \text{for all } t > 0$$

but

$$S_t(\mathscr{B}_+) \cap \operatorname{int}\mathscr{B}_+ = \emptyset \quad \text{for all } t > 0.$$

As for ergodicity of positive C_0^*-semigroups, the strongest form is still that S is strictly positive, i.e.,

$$S_t(\mathscr{B}_+\diagdown\{0\}) \subseteq \operatorname{int}\mathscr{B}_+$$

for $t > 0$. Candidates for weaker forms of ergodicity are

$$S_t(\mathscr{B}_+\diagdown\{0\}) \subseteq \operatorname{Nint}\mathscr{B}_+ \quad \text{for all } t > 0$$

and

$$S_t(\operatorname{Nint}\mathscr{B}_+) \subseteq \operatorname{int}\mathscr{B}_+ \quad \text{for all } t > 0.$$

We show in the next two examples that neither of these conditions imply strict positivity, and the last condition does not even imply a weak* form of irreducibility, i.e., S_t may leave a nontrivial weak*-closed hereditary subcone of $\mathscr{B}_+$ invariant.

EXAMPLE 3.7. Let $\mathscr{B} = L^\infty(\mathbf{R})$, and $\mathscr{B}_+$ the positive functions in $\mathscr{B}$. Define S_t as the diffusion semigroup with generator $H = -\frac{d^2}{dx^2}$, i.e.,

$$(S_t f)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} dy\, e^{-\frac{(x-y)^2}{2t}} f(y), \quad s > 0,\ f \in \mathscr{B}.$$

Then S is a positive C_0^*-semigroup, and $S_t(\mathscr{B}_+ \setminus \{0\}) \subseteq \mathrm{Nint}\,\mathscr{B}_+$. On the other hand S leaves the subspace $C_0(\mathbf{R}) = \{f \in C(\mathbf{R})\,;\ \lim_{|x|\to\infty} f(x) = 0\}$ invariant. But this space contains N-interior points in $\mathscr{B}_+$ because $f \in C_0(\mathbf{R}) \cap \mathrm{Nint}\,\mathscr{B}_+$ is equivalent to $f(x) > 0$ for almost all $x \in \mathbf{R}$. But the space does not contain interior points of $\mathscr{B}_+$ and hence S is not strictly positive.

EXAMPLE 3.8. Let X be the disjoint union of $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ with itself, $X = \mathbf{T} \cup \mathbf{T}$, and let

$$\mathscr{B} = L^\infty(X) = L^\infty(\mathbf{T}) \oplus L^\infty(\mathbf{T}).$$

Define

$$S_t(f_1 \oplus f_2) = S_t f_1 \oplus S_t f_2$$

where

$$(S_t f_i)(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} dy\, e^{-\frac{(x-y)^2}{2t}} f_i(y), \quad i = 1, 2$$

and this relation is understood modulo $\mathbf{Z}$ in x and y. Then $S_t(\mathrm{Nint}\,\mathscr{B}_+) \subseteq \mathrm{int}\,\mathscr{B}_+$ for $t > 0$, but if f is a positive L^∞-function supported on one of the components $\mathbf{T}$ in X, then $S_t f$ is supported on the same component, and hence

$$S_t(\mathscr{B}_+ \setminus \{0\}) \nsubseteq \mathrm{int}\,\mathscr{B}_+.$$

4. COMPLEMENTARY REMARKS ON OPERATOR ALGEBRAS

Much of the foregoing analysis was motivated by the theory of semigroups acting on operator algebras. If $\mathfrak{A}$ is a C^*-algebra with identity the positive elements $\mathfrak{A}_+$ form a normal cone with non-empty interior and the C_0-theory can be applied. If $\mathfrak{A}$ is a W^*-algebra it is the dual of the Banach space $\mathfrak{A}_*$ of normal forms, the positive cone $\mathfrak{A}_+$ is again normal with non-empty interior, and the C_0^*-theory applies. In these algebraic settings a number of the statements on ergodic properties can however be improved, e.g., one can deduce a C_0^*-version of Proposition 2.5. These improvements are mainly based upon a bipolar theorem for cones. Since it is unclear under what general assumptions such theorems are true we first isolate the requisite bipolar properties before discussing the algebraic systems.

Let $\mathscr{B}$, $\mathscr{F}$ be real Banach spaces in duality, i.e., $\mathscr{F} \subseteq \mathscr{B}^*$; $\mathscr{B} \subseteq \mathscr{F}^*$ and

$$\|a\| = \sup\{|\omega(a)| ; \omega \in \mathscr{F}, \|\omega\| \leqslant 1\}$$

for $a \in \mathscr{B}$ and

$$\|\omega\| = \sup\{|\omega(a)| ; a \in \mathscr{B}, \|a\| \leqslant 1\}$$

for $\omega \in \mathscr{F}$. Let $\mathscr{B}_+$, $\mathscr{F}_+$ be dual cones, i.e.,

$$\mathscr{B}_+ = \{a \in \mathscr{B} ; \omega(a) \geqslant 0 \text{ for all } \omega \in \mathscr{F}_+\}$$

$$\mathscr{F}_+ = \{\omega \in \mathscr{F} ; \omega(a) \geqslant 0 \text{ for all } a \in \mathscr{B}_+\}.$$

In particular this means that $\mathscr{B}_+$ (resp. $\mathscr{F}_+$) is closed in the $\sigma(\mathscr{B}, \mathscr{F})$-topology (resp. $\sigma(\mathscr{F}, \mathscr{B})$-topology).

If $\mathscr{C}$ is a subset of $\mathscr{B}_+$ (resp. $\mathscr{F}_+$) define

$$\mathscr{C}^\perp = \{\omega \in \mathscr{F}_+ ; \omega(a) = 0 \text{ for all } a \in \mathscr{C}\}$$

$$(\text{resp. } \mathscr{C}^\perp = \{a \in \mathscr{B}_+ ; \omega(a) = 0 \text{ for all } \omega \in \mathscr{C}\}).$$

Then $\mathscr{C}^\perp$ is an hereditary subcone of $\mathscr{F}_+$(resp. $\mathscr{B}_+$), closed in the appropriate topology.

We say that the quadruple $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ has the *weak positive bipolar property* if

$$\mathscr{C} = \mathscr{C}^{\perp\perp}$$

for each $\sigma(\mathscr{B}, \mathscr{F})$-closed hereditary subcone $\mathscr{C}$ in $\mathscr{B}_+$, and each $\sigma(\mathscr{F}, \mathscr{B})$-closed hereditary subcone $\mathscr{C}$ in $\mathscr{F}_+$. Furthermore, $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ is said to have the *$\mathscr{B}$-strong bipolar property* if it has the weak positive bipolar property and the $\sigma(\mathscr{B}, \mathscr{F})$-closure $\overline{\mathscr{C}}$ of each hereditary subcone $\mathscr{C}$ in $\mathscr{B}_+$ is hereditary, i.e., if

$$\overline{\mathscr{C}} = \mathscr{C}^{\perp\perp}$$

for each hereditary subcone $\mathscr{C}$ in $\mathscr{B}_+$.

We first discuss the implications of these concepts for various ergodicity properties of positive semigroups, and subsequently give examples of quadruples satisfying the positive bipolar properties.

Define

$$\operatorname{Nint} \mathscr{B}_+ = \{a \in \mathscr{B}_+ ; \quad \omega(a) > 0 \text{ for all } \omega \in \mathscr{F}_+ \setminus \{0\}\}$$

$$\operatorname{Nint} \mathscr{F}_+ = \{\omega \in \mathscr{F}_+ ; \quad \omega(a) > 0 \text{ for all } a \in \mathscr{B}_+ \setminus \{0\}\}.$$

Let S be a semigroup of bounded positive operators on $\mathscr{B}$ such that each S_t is $\sigma(\mathscr{B}, \mathscr{F})$-closed. Then there exists a dual semigroup S^* on $\mathscr{F}$ which also consists of bounded, positive, $\sigma(\mathscr{F}, \mathscr{B})$-closed operators.

We define S to be *strictly positive* if $S_t(\mathscr{B}_+\setminus\{0\}) \subseteq \operatorname{Nint}\mathscr{B}_+$ for all $t > 0$ (or, equivalently, for small $t > 0$), to be *irreducible* if the only $\sigma(\mathscr{B}, \mathscr{F})$-closed hereditary sub-cones of $\mathscr{B}_+$ invariant under S are $\{0\}$ and $\mathscr{B}_+$ itself, and to be *ergodic* if the hereditary cone generated by $\{S_t a\,;\ t \geqslant 0\}$ is $\sigma(\mathscr{B}, \mathscr{F})$-dense in $\mathscr{B}_+$ for all $a \in \mathscr{B}_+\setminus\{0\}$.

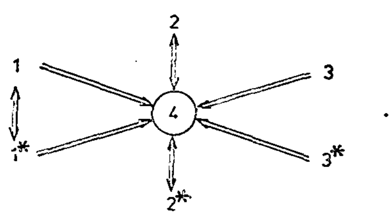
If S is a positive semigroup of $\sigma(\mathscr{B}, \mathscr{F})$-closed operators, consider the following properties:

1. S is strictly positive;
1*. S^* is strictly positive;
2. S is irreducible;
2*. S^* is irreducible;
3. S is ergodic;
3*. S^* is ergodic;

4. for each $a \in \mathscr{B}_+\setminus\{0\}$, $\omega \in \mathscr{F}_+\setminus\{0\}$ there exists a $t>0$ such that

$$\omega(S_t a) > 0.$$

If $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ satisfies the weak positive bipolar property the following implications are valid:



Moreover, if $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ satisfies the $\mathscr{B}$-strong positive bipolar property then one also has the additional implication $4 \Rightarrow 3$.

The proof of these statements is a straightforward extension of the proof of Proposition 2.5, together with the fact that the weak positive bipolar property implies that there is a one-one correspondence between closed hereditary cones $\mathscr{B}_+$ and $\mathscr{F}_+$ given by $\mathscr{C} \leftrightarrow \mathscr{C}^\perp$.

Assume for the moment that the $\sigma(\mathscr{B}, \mathscr{F})$-closed convex hull of every $\sigma(\mathscr{B}, \mathscr{F})$-compact set in $\mathscr{B}$ is $\sigma(\mathscr{B}, \mathscr{F})$-compact, and the same with $\mathscr{B}$ and $\mathscr{F}$ interchanged. This condition is fulfilled if $\mathscr{F} = \mathscr{B}^*$ or $\mathscr{B} = \mathscr{F}^*$ and allows one to express the resolvent $(I + \alpha H)^{-1}$ of S in terms of a Laplace transform

$$\omega((I + \alpha H)^{-1} a) = \int_0^\infty dt\ e^{-t}\ \omega(S_{\alpha t} a),$$

whenever $t \to \omega(S_t a)$ is continuous for all $a \in \mathscr{B}$, $\omega \in \mathscr{F}$. Under these circumstances Condition 4 is trivially equivalent to each of the conditions:

5. the operators $(I + \alpha H)^{-1}$ are strictly positive for all small $\alpha > 0$;
6. the operator $(I + \alpha H)^{-1}$ is strictly positive for some $\alpha > 0$;

and these are equivalent to the corresponding conditions 5* and 6* for S^*.

We now give some examples of quadruples satisfying the positive bipolar properties.

1. Let $(\mathscr{B}, \mathscr{B}_+)$ be a separable, countably order complete Banach lattice, or, more generally, an order complete Banach lattice with an order continuous norm, i.e., every order convergent net in $\mathscr{B}$ is norm convergent. Thus $\mathscr{B}$ could be L^p where $1 \leqslant p < \infty$ (but $\mathscr{B} = L^\infty$ must be treated separately, either as the dual of L^1 or as in Example 3 below). Then every norm-closed hereditary subcone of $\mathscr{B}_+$ is order-complete, [18], Theorem II.5.14 and Corollary.

Let $\mathscr{F} = \mathscr{B}^*$, $\mathscr{F}_+ = \mathscr{B}^*_+$. Then $\mathscr{F}$ is an order complete Banach lattice, [18], Proposition II.4.2, and the $\sigma(\mathscr{F}, \mathscr{B})$-closed hereditary subcones of $\mathscr{B}^*_+$ are the order-complete hereditary subcones of $\mathscr{B}^*_+$.

Now, if $\mathscr{C}$ is an order-complete hereditary subcone of an order-complete vector lattice $(\mathscr{E}, \mathscr{E}_+)$, define

$$\mathscr{C}_\perp = \{a \in \mathscr{E}_+ ;\ a \wedge b = 0 \quad \text{for all } b \in \mathscr{C}\}.$$

Then $\mathscr{E}_+$ is the direct sum of $\mathscr{C}$ and $\mathscr{C}_\perp$, [18], Theorem II.2.10. Thus if $(\mathscr{E}, \mathscr{E}_+) = (\mathscr{B}, \mathscr{B}_+)$, $\mathscr{C}^\perp$ is identified with the positive linear functionals on $\mathscr{C}_\perp - \mathscr{C}_\perp$. Thus the polar operation defines a one-one correspondence between the order-complete hereditary subcones of $\mathscr{B}_+$ and $\mathscr{F}_+$. But as the order complete hereditary subcones of $\mathscr{B}_+$, resp. $\mathscr{F}_+$, are just the hereditary subcones which are closed in the $\sigma(\mathscr{B}, \mathscr{F})$, resp. $\sigma(\mathscr{F}, \mathscr{B})$, topology it follows that the weak positive bipolar property is valid. It is easily shown that the norm closure of an hereditary cone in a Banach lattice is hereditary, and hence the $\mathscr{B}$-strong positive bipolar property holds. A compactness argument establishes that the weak*-closure of each hereditary cone in $\mathscr{F}_+$ is hereditary, and thus the $\mathscr{F}$-strong positive bipolar property also follows.

2. Let $\mathscr{B}$ be the self-adjoint part of a C^*-algebra, $\mathscr{B}_+$ the positive part, $\mathscr{F} = \mathscr{B}^*$, $\mathscr{F}_+ = \mathscr{B}^*_+$. Then $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ has the weak positive bipolar property by [13], Theorem 1.5.2 and Theorem 3.10.7. But if $\mathscr{C}$ is a hereditary cone in $\mathscr{B}_+$ then $\overline{\mathscr{C}}$, the norm, or weak, closure of $\mathscr{C}$, is a hereditary cone by the following reasoning:

If $u_1, \ldots, u_n \in \mathscr{C}$ and $n \in \mathbf{N}$, define

$$u_\alpha = \left(\sum_{k=1}^n u_k\right)\left(\frac{1}{n}\mathbf{I} + \sum_{k=1}^n u_k\right)^{-1}$$

where $\alpha = (u_1, \ldots, u_n)$. The set of α's is ordered by inclusion. Also

$$u_\alpha = \left(\sum_{k=1}^{n} u_k\right)^{1/2}\left(\frac{1}{n}\mathbf{I} + \sum_{k=1}^{n} u_k\right)^{-1}\left(\sum_{k=1}^{n} u_k\right)^{1/2} \leqslant \left\|\left(\frac{1}{n}\mathbf{I} + \sum_{k} u_k\right)^{-1}\right\|\left(\sum_{k=1}^{n} u_k\right)$$

and hence $u_\alpha \in \mathscr{C}$ by the hereditary property.

Furthermore,

$$a^{1/2} = \lim_\alpha a^{1/2}u_\alpha = \lim_\alpha u_\alpha a^{1/2}$$

for all $a \in \mathscr{C}$, by [3], proof of Proposition 2.2.18.

Define

$$\mathscr{C}_1 = \{a \in \mathscr{B}_+;\ a^{1/2} = \lim_\alpha a^{1/2}u_\alpha = \lim_\alpha u_\alpha a^{1/2}\}$$

$$\mathscr{C}_2 = \{a \in \mathscr{B}_+;\ a = \lim_\alpha u_\alpha a u_\alpha\}$$

where the limits are assumed to exist in norm. We now show that

$$\bar{\mathscr{C}} = \mathscr{C}_1 = \mathscr{C}_2.$$

a. $\bar{\mathscr{C}} \subseteq \mathscr{C}_1$. We already argued that $\mathscr{C} \subseteq \mathscr{C}_1$, but if a_n is a sequence of positive elements converging in norm to an element a, then it follows by polynomial approximation that $f(a_n) \to f(a)$ for all continuous functions f, and in particular $a_n^{1/2} \to a^{1/2}$. Therefore $\mathscr{C}_1$ is closed, and hence $\bar{\mathscr{C}} \subseteq \mathscr{C}_1$.

b. $\mathscr{C}_1 \subseteq \mathscr{C}_2$ is trivial.

c. $\mathscr{C}_2 \subseteq \bar{\mathscr{C}}$. If $a \in \mathscr{B}_+$ and $\lim_\alpha u_\alpha a u_\alpha$ exists, then $\lim_\alpha u_\alpha a u_\alpha \in \bar{\mathscr{C}}$ because

$$u_\alpha a u_\alpha \leqslant \|a\| u^2 \leqslant \|a\| u_\alpha$$

and hence $u_\alpha a u_\alpha \in \mathscr{C}$ by hereditarity. Thus $\mathscr{C}_2 \subseteq \bar{\mathscr{C}}$.

We now show that $\bar{\mathscr{C}}$ is hereditary. Assume $b \in \mathscr{B}_+$, $a \in \bar{\mathscr{C}}$ and

$$0 \leqslant b \leqslant a.$$

Realizing $\mathscr{B}$ on a Hilbert space $\mathscr{H}$, let P be the range projection of a, and use spectral theory to define $a^{-1/2}$ as an unbounded positive operator on P. Extend $a^{-1/2}$ to $\mathscr{H}$ by setting $a^{-1/2}(I - P) = (I - P)a^{-1/2} = 0$. Then the above relation implies

$$0 \leqslant a^{-1/2}ba^{-1/2} \leqslant P.$$

Put $c = a^{-1/2}ba^{-1/2}$. Then $c \geqslant 0$, $\|c\| \leqslant 1$ and

$$b = a^{1/2}ca^{1/2}.$$

But then

$$u_\alpha b u_\alpha = (u_\alpha a^{1/2})\, c(a^{1/2}u_\alpha) \underset{\alpha}{\rightarrow} a^{1/2}ca^{1/2} = b$$

and hence $b \in \mathscr{C}_2 = \bar{\mathscr{C}}$.

It follows that $\bar{\mathscr{C}}$ is hereditary, and $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ has the $\mathscr{B}$-strong positive bipolar property.

3. Let $\mathscr{B}$ be the self-adjoint part of a von Neumann algebra, $\mathscr{B}_+$ the positive part of $\mathscr{B}$ and $\mathscr{F}$ the hermitian functionals in the predual of $\mathscr{B}$. Then $\mathscr{F}_+$ consists of the normal positive functionals on $\mathscr{B}$. The quadruple $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ has the weak positive bipolar property by [13], Theorem 3.6.11. If $\mathscr{C}$ is a hereditary cone in $\mathscr{B}_+$, then $\bar{\mathscr{C}}$, the weak*-closure of $\mathscr{C}$, is a hereditary cone by a reasoning similar to the $\mathscr{C}^*$-case: if $u_1, \ldots, u_n \in \mathscr{C}$ one again defines

$$u_\alpha = \left(\sum_{k=1}^{n} u_k\right)\left(\frac{1}{n}\mathbf{I} + \sum_{k=1}^{n} u_k\right)^{-1}$$

and u_α is again an increasing net of positive elements in $\mathscr{B}_+$. It follows that u_α converges to a positive operator $p \in \mathscr{B}_+$ in the strong operator topology, [3] .It is not hard to show that p is a projection, and then

$$\bar{\mathscr{C}} = \{a \in \mathscr{B}_+ \,;\; a = \lim_\alpha u_\alpha a u_\alpha\} = p\mathscr{B}_+ p$$

where closures and limits now are in the weak*-topology. Thus $\bar{\mathscr{C}} = p\mathscr{B}_+p$ is a hereditary cone, and $(\mathscr{B}, \mathscr{B}_+, \mathscr{F}, \mathscr{F}_+)$ has the $\mathscr{B}$-strong positive bipolar property.

We conclude with a few remarks on the generator theorem for positive C_0-semigroups S on a C^*-algebra $\mathfrak{A}$. The cone $\mathfrak{A}_+$ of positive elements of $\mathfrak{A}$ has a non-empty interior if, and only if, $\mathfrak{A}$ contains an identity $\mathbf{I}$. In this case $\mathbf{I}$ is an interior element and if $\mathbf{I} \in D(H)$ it follows that the C_0-semigroup $S_t = \exp\{-tH\}$ has a bound of the form $\|S_t\| \leqslant \exp\{\omega t\}$. Thus the rescaled semigroup $S_t = \exp\{-t(H + \omega\mathbf{I})\}$ is contractive. It is unclear whether the condition $\mathbf{I} \in D(H)$ is necessary for this type of boundedness but it is easy to construct examples in which this condition is not satisfied. For example let $\mathfrak{A} = C(\mathbf{T})$ where $\mathbf{T}$ is the circle and

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V(t)$$

where the function is piecewise continuous but discontinuous. The operator H is defined on the continuously differentiable functions in $C(\mathbf{T})$ whose second derivatives make suitable jumps at the points of discontinuity of V. Hence H is densely defined and it is dispersive in the sense of [4], [9], [1], whenever $V \geqslant 0$. Hence H generates a positive C_0-semigroup of contractions on $C(\mathbf{T})$ but $\mathbf{I} \notin D(H)$. (The remark after 6.6 in [1] must be modified accordingly.)

It is of some interest to know whether C_0-semigroups on a C^*-algebra automatically satisfy a bound of the form $\|S_t\| \leqslant \exp\{\omega t\}$ because this could allow one to discuss generator theorems for algebras without identity by adjoining an identity $\mathbf{I}$ and setting $H\mathbf{I} = -\omega\mathbf{I}$. *)

APPENDIX

In this section we collect some results concerning linear maps between ordered Banach spaces; in particular we investigate the relationship between boundedness and order boundedness.

DEFINITION A1. Let $(\mathscr{B}_i, \mathscr{B}_{i+})$, $i = 1,2$, be ordered Banach spaces. The linear map $S: \mathscr{B}_1 \to \mathscr{B}_2$ is said to be *order bounded* if it carries order bounded sets into order bounded sets.

In particular, the difference of two positive maps is order bounded. Under certain conditions boundedness can be derived from order boundedness, and the following result is a generalization of a classic result which deals with positive linear maps [2].

THEOREM A2. *Let $(\mathscr{B}_i, \mathscr{B}_{i+})$, $i = 1, 2$, be ordered Banach spaces, and assume $\mathscr{B}_{1+}$ is generating and $\mathscr{B}_{2+}$ is normal. Then every order bounded linear map $S: \mathscr{B}_1 \to \mathscr{B}_2$ is bounded.*

Proof. If S were not bounded we cound find a sequence $\{b_n\}$ in $\mathscr{B}_1$ with $\|b_n\| = 1$ such that $\|Sb_n\| > 2^{2n}$. Since $\mathscr{B}_{1+}$ is generating, there is a constant $M > 0$ such that $b_n = a_n - c_n$ with $a_n, c_n \geqslant 0$ and $\|a_n\|, \|c_n\| \leqslant M$. Since $\|Sa_n - Sc_n\| > 2^{2n}$, there is, for each n, a positive b'_n such that $\|b'_n\| \leqslant M$ and $\|Sb'_n\| > \frac{2^{2n}}{2}$. Put $b' = \sum 2^{-n} b'_n$. Then $0 \leqslant 2^{-n}b'_n \leqslant b'$ for all n, and, since S is order bounded, there are $x, y \in \mathscr{B}_2$ such that $x \leqslant S(2^{-n}b'_n) \leqslant y$ for all n. By normality of $\mathscr{B}_{2+}$ we have $\|S(2^{-n}b'_n)\| \leqslant \frac{1}{\gamma} \cdot \max(\|x\|, \|y\|)$ for some constant $\gamma > 0$; but this is impossible since $\|S(2^{-n}b'_n)\| > \frac{2^n}{2}$ for all n. Hence S is bounded.

The next result goes in the opposite direction and supplies a criterion for boundedness to imply order boundedness. First recall from Section 1 that if $\mathrm{o}(\mathscr{B})$ (resp. $\mathrm{b}(\mathscr{B})$) denotes the set of order bounded (resp. bounded) subsets of an ordered Banach space, then normality of $\mathscr{B}_+$ implies $\mathrm{o}(\mathscr{B}) \subseteq \mathrm{b}(\mathscr{B})$ and $\mathrm{int}(\mathscr{B}_+) \neq \emptyset$ if, and only if, $\mathrm{b}(\mathscr{B}) \subseteq \mathrm{o}(\mathscr{B})$. This immediately gives:

PROPOSITION A3. *Let $(\mathscr{B}_i, \mathscr{B}_{i+})$, $i = 1, 2$, be ordered Banach spaces. Then if $\mathscr{B}_{1+}$ is normal and $\mathscr{B}_{2+}$ is generating, every bounded linear map $S: \mathscr{B}_1 \to \mathscr{B}_2$ is order bounded.*

*) See Notes added in proofs, page 399.

Proof. Under the stated conditions we have $S(\mathrm{o}(\mathscr{B}_1)) \subseteq S(\mathrm{b}(\mathscr{B}_1)) \subseteq \mathrm{b}(\mathscr{B}_2) \subseteq$ $\subseteq \mathrm{o}(\mathscr{B}_2)$, thus S is order bounded.

COROLLARY A4. *Let* $(\mathscr{B}, \mathscr{B}_+)$ *be an ordered Banach space.*

a. *If* $\mathscr{B}_+$ *is normal and* $\mathrm{int}\,\mathscr{B} \neq \emptyset$, *a linear map* $S\colon \mathscr{B} \to \mathscr{B}$ *is bounded if, and only if, it is order bounded.*

b. *If* $\mathscr{B}_+$ *is normal and generating, a linear functional on* $\mathscr{B}$ *is bounded if, and only if, it is order bounded.*

The following proposition gives sufficient conditions for the cone of positive operators in $\mathfrak{B}(\mathscr{B}_1, \mathscr{B}_2)$ to be normal.

PROPOSITION A5 (cf. [12]). *Let* $\mathfrak{B}_+$ *denote the cone of positive operators in* $\mathfrak{B}(\mathscr{B}_1, \mathscr{B}_2)$. *Then if* $\mathscr{B}_{1+}$ *is generating and* $\mathscr{B}_{2+}$ *is normal,* $\mathfrak{B}_+$ *is normal (in the operator norm topology of* $\mathfrak{B}(\mathscr{B}_1, \mathscr{B}_2)$*).*

Proof. For $S \in \mathfrak{B}_+$, define $\|S\|_+ = \sup\{\|Sa\|\,;\ a \in \mathscr{B}_{1+}\setminus\{0\},\ \|a\| \leqslant 1\}$. Since $\mathscr{B}_{1+}$ is generating, every $a \in \mathscr{B}$ can be written $a = a_1 - a_2$ with $a_1, a_2 \geqslant 0$ and $\|a_1\|, \|a_2\| \leqslant M\|a\|$ for some constant $M > 0$. Setting $a' = a_1 + a_2$, we have $-a' \leqslant$ $\leqslant a \leqslant a'$ and $-Sa' \leqslant Sa \leqslant Sa'$; thus, by normality of $\mathscr{B}_{2+}$, $\|Sa\| \leqslant \delta\|Sa'\|$ for some constant $\delta > 0$. Since $\|a'\| \leqslant 2M\|a\|$, this gives $\|Sa\|/\|a\| \leqslant 2M\delta$ for $a \neq 0$, i.e., $\|S\| \leqslant 2M\delta\|S\|_+$. Now, if $0 \leqslant S \leqslant T$, we have $0 \leqslant Sa \leqslant Ta$ for $a \geqslant 0$, and hence $\|Sa\| \leqslant \delta\|Ta\|$ by normality of $\mathscr{B}_{2+}$, i.e., $\|S\|_+ \leqslant \delta\|T\|_+$, and finally $\|S\| \leqslant$ $\leqslant 2M\delta\|S\|_+ \leqslant 2M\delta^2\|T\|_+ \leqslant 2M\delta^2\|T\|$, since obviously $\|T\|_+ \leqslant \|T\|$ for any positive T.

It would be of interest to derive general conditions which guarantee that $\mathfrak{B}_+$ is generating in $\mathfrak{B}(\mathscr{B}_1, \mathscr{B}_2)$. It should suffice that $\mathscr{B}_{1+}$ is normal and $\mathscr{B}_{2+}$ is generating and in this case one could then deduce that an operator is order bounded if, and only if, it is the difference of two positive operators.

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Notes added in proofs. After this paper was finished, Batty and Davies constructed examples showing that the questions on page 379 and 398 have negative answers, i. e. they construct first order differential operators H_1, H_2 on the Banach lattice $C_0(\mathbf{R})$ such that $(1+\alpha H_1)^{-1}$ exists as a positive operator for small $\alpha>0$ but H_1 is not a generator, and H_2 generates a positive C_0-semigroup, but no estimate of the form $\|e^{-tH_2}\|\leqslant e^{\omega t}$ is valid, [19]. The present results has been pushed further in [20].

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