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APPLICATIONS OF THE KREĬN RESOLVENT FORMULA TO THE THEORY OF SELF-ADJOINT EXTENSIONS OF POSITIVE SYMMETRIC OPERATORS

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1. INTRODUCTION

Let T be the closure in $L^2([0, \infty))$ of $-\mathrm{d}^2/\mathrm{d}x^2$ defined on $C_0^\infty((0, \infty))$. T has deficiency indices (1,1) and its self-adjoint extensions T_α are indexed by the boundary conditions

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)\,|_{x=0} = \alpha f(0), \quad \alpha \in (-\infty, \infty]. \tag{1.1}$$

The Friedrichs extension T_{F} corresponds to $\alpha = \infty$ (more properly speaking $f(0) = 0$) and at the formal level is obtained by taking the limit $\alpha \to \infty$ in (1.1). Indeed one can verify that

$$\lim_{\alpha \to \infty} \|(T_\alpha + 1)^{-1} - (T_{\mathrm{F}} + 1)^{-1}\| = 0. \tag{1.2}$$

Again, at the formal level, one can see that T_{F} is also obtained by taking the limit $\alpha \to -\infty$ in (1.1) and indeed, one can verify with some work that

$$\lim_{\alpha \to -\infty} \|(T_\alpha + 1)^{-1} - (T_{\mathrm{F}} + 1)^{-1}\| = 0. \tag{1.3}$$

Now for $\alpha < 0$, $\sigma(T_\alpha) = \{-\alpha^2\} \cup [0, \infty)$, so in the sense of (1.3) T_{F} is the limit of self-adjoint extensions of T, which are not uniformly bounded from below.

The same phenomenon has been observed recently in the study of regularisations of the one-dimensional Schrödinger operator [5,8]. The initial motivation of this paper was to see whether this phenomenon (the fact that the Friedrichs extension of a positive symmetric operator is in some sense the limit of some sequences of self-adjoint extensions which are not bounded from below) is a generic one or is related to the structure of the above examples. Our results below prove the genericity of this phenomenon. Our main result, contained in Theorem 2 below, gives a "parametrisation" of classes of not bounded from below self-adjoint extensions, which in some sense made precise in Theorem 3, are "neighbourhoods" of the Friedrichs extension. Theorem 4 concerns the finite deficiency indices. An application of Theorem 4 to the problem discussed at the beginning of the Introduction is given in [15]. Some basic results of the Kreĭn-Birman theory are also recovered.

After the first version of this paper was finished we became aware of the fact that a particular case of our results was known. More exactly, the fact that the von Neumann extension corresponding to a point λ [1,2] converges in the strong resolvent sense to the Friedrichs extension as $\lambda \to -\infty$ (see Corollary 1 (i) below), has appeared in [2] with a completely different proof.

Our main tool is the Kreĭn resolvent formula [9, 16] combined with a distinguished property of the Friedrichs extension described in Lemma 3 below (for a related result see Proposition 4.a) in [14] and § 1 in [12]). Recently, the theory of generalised resolvents for nondensely defined contractions in Hilbert spaces has been thoroughly developed [12, 14]. The basic result of the Kreĭn-Birman theory in the form given in Theorem 5 below has appeared in [13] and is also implied, via the Cayley transform by the results in [12, 14].

2. THE KREĬN RESOLVENT FORMULA

As has already been said in the Introduction our main tool is the following resolvent formula given by Kreĭn (see [9] and for the case of infinite deficiency indices [11, 16]).

THEOREM 1. *Let A be a closed symmetric (densely defined) operator in a separable Hilbert space $\mathscr{H}$, with domain $\mathscr{D}(A)$ and equal (finite or infinite) deficiency indices. Let A_0 be a fixed self-adjoint extension of A, $\lambda_0 \in \rho(A_0)$ (the resolvent set of A_0) and let $P(\lambda_0)$ be the orthogonal projection on $((A-\lambda_0)\mathscr{D}(A))^{\perp}$. Then defining*

$$E(\lambda_1, \lambda_2) = (A_0 - \lambda_2)(A_0 - \lambda_1)^{-1}, \quad \lambda_1, \lambda_2 \in \rho(A_0); \tag{2.1}$$

$$F(\lambda_1, \lambda_2) = (\lambda_1 - \lambda_2)E(\lambda_1, \bar{\lambda}_2) + (\lambda_2 - \bar{\lambda}_2)/2 \tag{2.2}$$

the following assertion holds.

The formula

$$R_\lambda = (A_0 - \lambda)^{-1} - E(\lambda, \bar{\lambda}_0)Q[n + QF(\lambda, \lambda_0)Q]^{-1}QE(\lambda, \lambda_0) \tag{2.3}$$

gives a one-to-one correspondence between all the self-adjoint extensions of A different from A_0 and all the pairs (Q, n) where Q is a nontrivial orthogonal projection smaller than $P(\lambda_0)$ and n is a self-adjoint operator in $Q\mathscr{H}$.

3. APPLICATIONS TO THE THEORY OF SELF-ADJOINT EXTENSIONS OF POSITIVE SYMMETRIC OPERATORS

We shall start by listing some observations of a technically preparatory character.

1°. *Let A_n be a sequence of bounded self-adjoint operators satisfying the conditions:*

(i) $$0 \leqslant A_{n+1} \leqslant A_n, \quad n = 1,2, \dots ;$$

(ii) $$\operatorname*{s-lim}_{n\to\infty} A_n = 0;$$

(iii) $$\sigma_{\text{ess}}(A_n) \subset [0,\delta_n], \quad \lim_{n\to\infty} \delta_n = \delta.$$

Then $\lim\limits_{n\to\infty} \|A_n\| \leqslant \delta$.

Proof. Suppose that for some $\varepsilon > 0$

$$\lim_{n\to\infty} \|A_n\| \geqslant \delta + 2\varepsilon. \tag{3.1}$$

By (iii) it may be assumed that

$$\delta + \varepsilon \geqslant \delta_n, \quad n = 1,2, \dots . \tag{3.2}$$

Let P_n be the spectral projection of A_n corresponding to $[\delta + \varepsilon, \infty)$. Then from (iii) and (3.2) it follows that each P_n is a nonzero projection of finite rank. Let x_n be a normalised eigenvector of A_n corresponding to the eigenvalue $\|A_n\|$.
Then it follows from (3.1) that, if $k \leqslant n$

$$\delta + 2\varepsilon \leqslant \| A_n \| = (A_n x_n, x_n) \leqslant (A_k x_n, x_n) \leqslant$$
$$\leqslant \| A_k \| \, \|P_k x_n\|^2 + (\delta + \varepsilon) \{1 - \| P_k x_n \|^2\}$$

and hence that

$$\| P_k x_n \|^2 \geqslant \varepsilon(\| A_k \| - \delta - \varepsilon)^{-1} \geqslant \varepsilon(\|A_1\| - \delta - \varepsilon)^{-1} \equiv \gamma > 0. \tag{3.3}$$

The bounded sequence $\{x_n\}$ can be assumed to converge weakly to a vector x_∞. Since each P_k is of finite rank, (3.3) implies that

$$\| P_k x \quad \|^2 = \lim_{n\to\infty} \|P_k x_n\|^2 \geqslant \gamma \tag{3.4}$$

which yields

$$(A_k x_\infty, x_\infty) \geqslant (\delta + \varepsilon) \| P_k x_\infty \|^2 \geqslant (\delta + \varepsilon)\gamma. \tag{3.5}$$

But (ii) implies

$$\lim_{k\to\infty} (A_k x_\infty, x_\infty) = 0$$

a contradiction.

2°. *Let N be a self-adjoint operator bounded from below (respectively above) and $F(\lambda)$ a bounded self-adjoint operator valued function norm continuous for $\lambda \in [a, b] \subset \mathbf{R}$. Then $\inf \sigma(N + F(\lambda))$ (respectively $\sup \sigma(N + F(\lambda))$) is a continuous function of λ on $[a, b]$.*

3°. *Let B be a bounded positive operator with bounded inverse and P an orthogonal projection. Then the operator PBP considered as an operator in $P\mathscr{H}$ has a bounded inverse, $(PBP)^{-1}$, and*

$$(PBP)^{-1} \oplus 0 \leqslant B^{-1} \tag{3.6}$$

where the orthogonal sum is taken according to $\mathscr{H} = P\mathscr{H} \oplus (1 - P)\mathscr{H}$.

Proof. The inequality (3.6) is equivalent to

$$B^{1/2}P(PB^{1/2}B^{1/2}P)^{-1}PB^{1/2} \leqslant 1. \tag{3.7}$$

If $B^{1/2}P = V|B^{1/2}P|$ is the polar decomposition of $B^{1/2}P$ then the l.h.s. of (3.7) is just VV^*.

From now on A will be a closed densely defined symmetric operator satisfying

$$m(A) = \inf_{f\in\mathscr{D}(A)} (f, Af)/\|f\|^2 = 1.$$

In what follows, it is understood that A is not self-adjoint and that A_F denotes the Friedrichs extension of A [4, § 14], P denotes the orthogonal projection on $(A\mathscr{D}(A))^\perp$ and Q an orthogonal projection smaller than P.

Taking $A_0 = A_F$, $\lambda_0 = 0$, the formula (2.3) may be written as

$$(A_\alpha - \lambda)^{-1} = (A_F - \lambda)^{-1} - E(\lambda, 0)\, Q_\alpha [n_\alpha + f_{Q_\alpha}(\lambda)]^{-1} Q_\alpha E(\lambda, 0) \tag{3.8}$$

where

$$f_{Q_\alpha}(\lambda) = \lambda Q_\alpha A_F (A_F - \lambda)^{-1} \upharpoonright Q_\alpha\mathscr{H}. \tag{3.9}$$

LEMMA 1. *Let $\lambda_1 \in \mathbf{R} \cap \rho(A_F)$. Then $\lambda_1 \in \rho(A_\alpha)$ if and only if $0 \in \rho(n_\alpha + f_{Q_\alpha}(\lambda_1))$.*

Proof. The result is a direct consequence of (3.8).

To obtain further results we need some properties of the analytic operator valued function $f_{Q_\alpha}(\lambda): Q_\alpha\mathscr{H} \to Q_\alpha\mathscr{H}$.

LEMMA 2. *For* $\lambda \in (-\infty, 1)$ *and* $Q_\alpha \leqslant P$, $f_{Q_\alpha}(\lambda)$ *is a bounded self-adjoint operator, and*

(i) $\sigma(f_{Q_\alpha}(\lambda)) \subset [\lambda, \lambda/(1-\lambda)]$;

(ii) $f_{Q_\alpha}(\lambda)$ *is strictly increasing,*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{Q_\alpha}(\lambda) \geqslant \min\{1, 1/(1-\lambda)^2\}.$$

Proof. This is a direct consequence of the definition.

LEMMA 3. *For all* $Q \leqslant P$

(i) $\text{s-}\lim\limits_{\lambda\to-\infty} f_Q^{-1}(\lambda) = 0$;

(ii) *If in addition* A_F^{-1} *is compact, then*

$$\lim_{\lambda\to-\infty} \| f_Q^{-1}(\lambda) \| = 0.$$

Proof. (i) If G_μ is the spectral measure of A_F, then

$$(f, aA_F(A_F + a)^{-1}f) = \int_1^\infty \frac{a\mu}{a+\mu}\,\mathrm{d}(G_\mu f, f), \quad a > 0$$

and the monotone convergence theorem implies that

$$\lim_{a\to\infty} (f, aA_F(A_F + a)^{-1}f) < \infty$$

is equivalent to $f \in \mathscr{D}(A_F^{1/2})$. On the other hand [4; Proposition 15.2] $\mathscr{D}(A_F^{1/2}) \cap P\mathscr{H} = \{0\}$, from which, for $f \in Q\mathscr{H}$

$$\lim_{a\to\infty} (f, aA_F(A_F + a)^{-1}f) = \infty. \tag{3.10}$$

The first point of the lemma follows from (3.10) by standard arguments.

(ii) If $(A_F - \lambda)^{-1}$ is compact, it follows that $f_Q(\lambda) - \lambda = \lambda^2 Q(A_F - \lambda)^{-1}Q$ is compact and then $\lambda^{-1} - f_Q^{-1}(\lambda) = f_Q^{-1}(\lambda)(f_Q(\lambda) - \lambda)/\lambda$ is compact. Then $\sigma(_{\text{ess}} - f_Q^{-1}(\lambda)) = \{-1/\lambda\}$ and one can apply 1°.

Now let $a \in (-\infty, 1)$ and $A_N(a)$ be the self-adjoint extension of A corresponding to the pair $(P, -f_P(a))$. It is easy to see that $A_N(a)$ is nothing but the von Neumann extension [1; § 107,2] corresponding to the point a.

COROLLARY 1.

(i) $\text{s-}\lim_{a\to-\infty} (A_N^{-1}(a) - A_F^{-1}) = 0.$

(ii) *If one of the following is true*

(a) A_F^{-1} *is compact;*

(b) $\dim P < \infty$*;*

then

$$\lim_{a\to-\infty} \| A_N^{-1}(a) - A_F^{-1}\| = 0.$$

Proof. Let $a < 0$. From (3.8)

$$A_N^{-1}(a) - A_F^{-1} = Pf_P^{-1}(a)P$$

and Lemma 3 concludes the proof.

LEMMA 4. *Let* $\lambda_1 < 1$. *Suppose that*

(a) $\rho(A_\alpha) \supset (\lambda_1, 1)$*;*

(b) *for some* $\lambda_2 \in (\lambda_1, 1)$

$$(A_\alpha - \lambda_2)^{-1} - (A_F - \lambda)^{-1} \leqslant 0.$$

Then for all $\lambda \in (\lambda_1, 1)$

$$(A_\alpha - \lambda)^{-1} - (A_F - \lambda)^{-1} \leqslant 0.$$

Proof. For $\lambda \in (-\infty, 1)$, let

$$m_\alpha(\lambda) = n_\alpha + f_{Q_\alpha}(\lambda)$$

and

$$p(\lambda) = \inf \sigma(m_\alpha(\lambda)).$$

Since $p(\lambda_2) \geqslant 0$ it follows that $p(\lambda) > -\infty$ for all $\lambda \in (-\infty, 1)$. Then $p(\lambda) \in \sigma(m_\alpha(\lambda))$ and, by 2°, is a continuous function of λ. From Lemma 1 it follows that $p(\lambda) \neq 0$ for $\lambda \in (\lambda_1, 1)$, which implies that $p(\lambda) > 0$ for $\lambda \in (\lambda_1, 1)$. Then $m_\alpha(\lambda) > 0$ for $\lambda \in (\lambda_1, 1)$ which together with (3.8) proves the lemma.

PROPOSITION 1. *Let* $\lambda_1 \in (-\infty, 1)$. *Then*

$$m_\alpha(\lambda_1) \equiv n_\alpha + f_{Q_\alpha}(\lambda_1) \geqslant 0 \tag{3.11}$$

if and only if

$$\rho(A_\alpha) \supset (\lambda_1, 1) \tag{3.12}$$

and for some $\lambda_2 \in (\lambda_1, 1)$

$$(A_\alpha - \lambda_2)^{-1} - (A_F - \lambda_2)^{-1} \leqslant 0. \tag{3.13}$$

Proof. Suppose $m_\alpha(\lambda_1) \geqslant 0$. Then from Lemma 2 (ii) for all $\lambda \in (\lambda_1, 1)$

$$m_\alpha(\lambda) \geqslant \int_{\lambda_1}^{\lambda} \min\{1, 1/(1-t)^2\}\, dt$$

and from Lemma 1, $\lambda \in \rho(A_\alpha)$. The inequality (3.13) follows from (3.8). Suppose now (3.12), (3.13) to the true. Then by Lemma 2, Lemma 4 and (3.8) $m_\alpha(\lambda) > 0$ for all $\lambda \in (\lambda_1, 1)$. This together with 2°, finishes the proof of the proposition.

Let $\lambda \in (-\infty, 1)\setminus\{0\}$ and $\mathscr{C}(\lambda)$ be the class of self-adjoint operators in $P\mathscr{H}$ given by

$$\mathscr{C}(\lambda) = \left\{ C : P\mathscr{H} \to P\mathscr{H} \;\middle|\; \begin{matrix} 0 \leqslant C \leqslant f_{\bar{P}}^{-1}(\lambda) & \text{for } \lambda > 0 \\ 0 \geqslant C \geqslant f_{\bar{P}}^{-1}(\lambda) & \text{for } \lambda < 0 \end{matrix} \right\}.$$

By $\mathscr{C}(1)$, $\mathscr{C}(-\infty)$ we shall denote

$$\mathscr{C}(1) = \bigcap_{\lambda \in (0,1)} \mathscr{C}(\lambda)$$

$$\mathscr{C}(-\infty) = \bigcap_{\lambda \in (-\infty,0)} \mathscr{C}(\lambda).$$

By Lemma 2, $f_{\bar{P}}^{-1}(\lambda)$ is a decreasing function, positive for $\lambda > 0$ and negative for $\lambda < 0$. Thus [4; Proposition 7.8]

$$\mathscr{C}(1) = \{C : P\mathscr{H} \to P\mathscr{H} \mid 0 \leqslant C \leqslant \operatorname{s-lim}_{\lambda \to 1} f_{\bar{P}}^{-1}(\lambda)\}$$

$$\mathscr{C}(-\infty) = \{C : P\mathscr{H} \to P\mathscr{H} \mid 0 \geqslant C \geqslant \operatorname{s-lim}_{\lambda \to -\infty} f_{\bar{P}}^{-1}(\lambda)\}.$$

COROLLARY 2. $\mathscr{C}(-\infty) = \{0\}$.

Proof. This is an immediate consequence of Lemma 3.

With each $C_\alpha \in \mathscr{C}(\lambda)$ we shall associate the pair (Q_α, n_α), and then a self-adjoint extension A_α, taking

$$Q_\alpha = P - \mathrm{N}(C_\alpha)$$

where $\mathrm{N}(C_\alpha)$ is the orthogonal projection on the null subspace of C_α and

$$n_\alpha = -(Q_\alpha C_\alpha \restriction_{Q_\alpha \mathscr{H}})^{-1}.$$

Note that $C_1 = C_2$ if and only if $Q_1 = Q_2$ and $n_1 = n_2$.

THEOREM 2. *Let* $\lambda_1 \in (-\infty, 0)$. *Then* $C_\alpha \in \mathscr{C}(\lambda_1)$ *if and only if* $\rho(A_\alpha) \supset (\lambda_1, 1)$ *and for some* $\lambda_2 \in (\lambda_1, 1)$

$$(A_\alpha - \lambda_2)^{-1} - (A_{\mathrm{F}} - \lambda_2)^{-1} \leqslant 0.$$

If $C_{\alpha_1}, C_{\alpha_2} \in \mathscr{C}(\lambda_1)$ *then* $C_{\alpha_1} \geqslant C_{\alpha_2}$ *is equivalent to*

$$(A_{\alpha_1} - \lambda)^{-1} - (A_{\alpha_2} - \lambda)^{-1} \geqslant 0$$

for all $\lambda \in (\lambda_1, 1)$.

Proof. Suppose $C_\alpha \in \mathscr{C}(\lambda_1)$. Then for all $\varepsilon > 0$ [4; Proposition 6.1]

$$(-C_\alpha + \varepsilon)^{-1} \geqslant (-f_P^{-1}(\lambda_1) + \varepsilon)^{-1}.$$

This implies that for all $g \in \mathscr{D}(n_\alpha^{1/2}) \subset Q_\alpha \mathscr{H}$

$$(g, (n_\alpha^{-1} + \varepsilon)^{-1} g) \geqslant (g, (-f_P^{-1}(\lambda_1) + \varepsilon)^{-1} g). \tag{3.14}$$

Taking the limit $\varepsilon \to 0$ in (3.14) one obtains $n_\alpha \geqslant -f_P(\lambda_1)$ and $\rho(A_\alpha) \supset (\lambda_1, 1)$ by Proposition 1. Conversely, suppose that $\rho(A_\alpha) \supset (\lambda_1, 1)$ and (3.13) is true. Then by Proposition 1, $n_\alpha \geqslant -f_{Q_\alpha}(\lambda_1)$ and hence $0 \leqslant n_\alpha^{-1} \leqslant -f_{Q_\alpha}^{-1}(\lambda_1)$. Using the definition of C_α and 3° one obtains $0 \leqslant -C_\alpha \leqslant -f_P^{-1}(\lambda_1)$. The order relation follows from

$$A_\alpha^{-1} - A_{\mathrm{F}}^{-1} = P C_\alpha P$$

and the proof is completed.

THEOREM 3. (i) *Let* A_q *be a sequence of self-adjoint extensions of* A *satisfying:*

$$\rho(A_q) \supset (-a_q, 1), \quad \lim_{q\to\infty} a_q = \infty$$

$$A_q^{-1} - A_{\mathrm{F}}^{-1} \leqslant 0.$$

Then

$$\text{s-}\lim_{q\to\infty} (A_q^{-1} - A_{\mathrm{F}}^{-1}) = 0.$$

(ii) *If in addition, one of the following are true*

(a) A_{F}^{-1} *is compact;*

(b) $\dim P < \infty$;

then

$$\lim_{q\to\infty} \| A_q^{-1} - A_{\mathrm{F}}^{-1} \| = 0.$$

Proof. Now

$$A_q^{-1} - A_{\mathrm{F}}^{-1} = P C_q P$$

and since by Theorem 2, $C_q \in \mathscr{C}(-a_q)$, the use of Lemma 3 finishes the proof.

THEOREM 4. *Suppose* $\dim P = m < \infty$, *and let* A_q *be a sequence of self-adjoint extensions of* A *with the property that there exist* $\{a_q\}_1^\infty$, $a_q > 1$, $\lim_{q\to\infty} a_q = \infty$ *such that* A_q *has* m *eigenvalues (counting multiplicities) in* $(-\infty, -a_q)$. *Then*

$$\lim_{q\to\infty} \|A_q^{-1} - A_F^{-1}\| = 0.$$

Proof. Since A_q can have at most m eigenvalues in $(-\infty, 1)$, [1; § 107], it follows that $0 \in \rho(A_q)$. Let $\mathcal{M}_q$ be the m-dimensional subspace spanned by the eigenvectors of A_q corresponding to the eigenvalues $\lambda_i^q \in (-\infty, -a_q)$, $i = 1, \ldots, m$. Since $A_F^{-1} > 0$ it follows that

$$(x, (A_F^{-1} - A_q^{-1})x) \geqslant \min_i\{|\lambda_i^q|^{-1}\}\|x\|^2, \quad x \in \mathcal{M}_q$$

from which we see that $A_q^{-1} - A_F^{-1}$ has at least m negative eigenvalues. From

$$A_q^{-1} - A_F^{-1} = -Q_q n_q^{-1} Q_q$$

it follows that $A_q^{-1} - A_F^{-1}$ is of finite rank, at most m, i.e. has at most m eigenvalues different from zero. It follows that $A_q^{-1} - A_F^{-1} \leqslant 0$ and one can apply Theorem 3 (ii).

Repeating (with the obvious changes) the proofs of Proposition 1 and Theorem 2 one obtains the following basic facts of the Kreĭn-Birman theory [3, 10].

PROPOSITION 2 [3]. *Let* $\lambda_1 \in (-\infty, 1)$. *Then* $m_\alpha(\lambda_1) \leqslant 0$ *if and only if* $\rho(A_\alpha) \supset \supset (-\infty, \lambda_1)$.

THEOREM 5. (i) *Let* $\lambda_1 \in (0,1]$. *Then* $C_\alpha \in \mathscr{C}(\lambda_1)$ *if and only if* $\rho(A_\alpha) \supset (-\infty, \lambda_1)$. *If* $C_{\alpha_1}, C_{\alpha_2} \in \mathscr{C}(\lambda_1)$ *then* $C_{\alpha_1} \geqslant C_{\alpha_2}$ *is equivalent to*

$$A_{\alpha_1} \leqslant A_{\alpha_2}.$$

(ii) *A has an unique self-adjoint extension with the spectrum contained in* $[1, \infty)$, *if and only if*

$$\text{s-}\lim_{\lambda\to 1} f_P^{-1}(\lambda) = 0.$$

The following result announced in [6] has also a simple proof.

PROPOSITION 3. *Suppose* A_F^{-1} *is compact. Then* A_α *is bounded from below if and only if* n_α *is bounded from above.*

Proof. Suppose n_α is bounded from above. From Lemma 3 (ii), $f_{Q_\alpha}(\lambda) \leqslant -\beta(\lambda)$, $\lim_{\lambda\to-\infty} \beta(\lambda) = \infty$ so that for $\lambda \to -\infty$, $m_\alpha(\lambda) < 0$. Then Lemma 1 implies $\lambda \in \rho(A_\alpha)$. Conversely if $\rho(A_\alpha) \supset (-\infty, \lambda_1)$ by Proposition 1, $n_\alpha \leqslant -f_{Q_\alpha}(\lambda_1)$ and the proof is finished.

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