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ANALYTIC OPERATOR FUNCTIONS WITH COMPACT SPECTRUM. III. HILBERT SPACE CASE: INVERSE PROBLEM AND APPLICATIONS

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0. INTRODUCTION

In the earlier papers [29, 30] spectral linearizations and spectral pairs have been introduced to deal with classification and factorization problems for analytic operator functions with a compact spectrum. To explain the main results of the present paper we first recall the basic definitions of [29, 30].

Let Ω be an open set in $\mathbf{C}$, and let $W\colon \Omega \to \mathscr{L}(H)$ be an analytic operator function whose values are bounded linear operators on the (complex) Hilbert space H. A (bounded linear) operator $A\colon G \to G$ acting on a Hilbert space G is called a *spectral linearization* of W on Ω if the spectrum $\sigma(A)$ of A is contained in Ω and the operator functions $W(\lambda) \oplus I_G$ and $(\lambda - A) \oplus I_H$ are analytically equivalent (cf. [10, 11]) on Ω, which means that the following identities hold:

$$(0.1)\qquad E(\lambda)\begin{bmatrix} \lambda - A & 0 \\ 0 & I_H \end{bmatrix} F(\lambda) = \begin{bmatrix} W(\lambda) & 0 \\ 0 & I_G \end{bmatrix}, \quad \lambda \in \Omega,$$

where $E(\lambda)\colon G \oplus H \to H \oplus G$ and $F(\lambda)\colon H \oplus G \to G \oplus H$ are some invertible operators depending analytically on $\lambda \in \Omega$. Here the symbol I_Z stands for the identity operator on the space Z. If a spectral linearization exists, then the set

$$(0.2)\qquad \Sigma(W) = \{\lambda \in \Omega \mid W(\lambda) \text{ is not invertible}\}$$

is compact in Ω (in fact, as (0.1) shows, $\Sigma(W) = \sigma(A)$). The set $\Sigma(W)$ we call the *spectrum* of W. Conversely, if $\Sigma(W)$ is compact in Ω (in other words, if $W(\lambda)$ is invertible for λ near the boundary of Ω), then a spectral linearization of W exists and is unique up to similarity (see [29], where this is proved in a Banach space setting).

So the class of operator functions considered here (as well as in [29, 30]) consists of analytic operator functions with a compact spectrum.

A pair of Hilbert space operators (C, A) is called a *(right) spectral pair* of W on Ω if $A: G \to G$ is a spectral linearization of W on Ω and (assuming (0.1) holds) the operator $C: G \to H$ is given by

$$C = \frac{1}{2\pi i} \int_{\Gamma} \pi F(\lambda)^{-1} \tau(\lambda - A)^{-1} d\lambda.$$

Here $\pi: H \oplus G \to H$ is the canonical projection onto H, the map $\tau: G \to G \oplus H$ is the canonical embedding of G and Γ is a suitable curve in Ω around the spectrum $\Sigma(W)$. In the next section we shall give another description of spectral pairs, which will clarify the connections with earlier definitions of spectral pairs for matrix and operator polynomials and analytic matrix functions (cf. [22, 37, 17, 35]). Spectral pairs can be used to describe factorizations of analytic operator functions in terms of restrictions to invariant subspaces (see [30] and the next section for more details).

In the present paper we prove that a pair $C: G \to H$, $A: H \to H$ of Hilbert space operators can appear as a right spectral pair of an analytic operator function with a compact spectrum if and only if for some positive integer m the operator

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} : G \to H^m \tag{0.3}$$

is left invertible. Furthermore, if this condition is fulfilled and $|\lambda - z_0| < \delta$ is an open disc disjoint with $\sigma(A)$, then the operator function

$$W(\lambda) = I + CF^{-1}[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1}C^*, \tag{0.4}$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A^* - \bar{z}_0)^{-n-1}C^*C(A - z_0)^{-n-1}, \tag{0.5}$$

is a solution of the corresponding inverse problem. This means that the operator function W given by (0.4) is analytic on $\Omega = \{\lambda \in \mathbf{C} \mid |\lambda - z_0| > \delta\}$ and (C, A) is a right spectral pair for W on Ω. This is the first main result of the present paper.

On the basis of the solution of the inverse problem for spectral pairs we show that given an arbitrary invariant subspace of a spectral linearization there exists an analytic divisor with a compact spectrum of the corresponding analytic operator function, and an explicit formula for the divisor can be given. This fact allows us to construct explicitly spectral factorizations when the spectrum of the operator function decomposes into two disjoint compact parts. The existence of such factorizations has been proved earlier in [21] by using certain theorems on the triviality of analytic cocycles (see also [30], Section 5).

Further, in this way we are able to prove by construction the existence of a greatest common divisor for a finite family of analytic operator functions with compact spectra and to show the existence of a least common multiple for a finite family of analytic operator functions whose spectra are mutually disjoint compact sets.

There is a close connection between the inverse problem for spectral pairs and the spectrum displacement problem. The latter problem may be stated as follows. Given bounded linear Hilbert (or Banach) space operators $A: G \to G$ and $C: G \to H$, construct, if possible, a bounded linear operator $B: H \to G$ such that the spectrum of $A-BC$ lies in a prescribed open set Ω. This problem is of interest in Mathematical Systems Theory (especially the case when Ω is the open left half-plane) and in the finite dimensional case it is related to the well-known pole shifting theorem (see, e.g., [38]).

It turns out that the solution of the inverse problem for spectral pairs allows us to solve the spectrum displacement problem for Hilbert space operators in the following way. Let $A : G \to G$ and $C: G \to H$ be bounded linear operators acting between Hilbert spaces. Then there exists a bounded linear operator $B: H \to G$ such that the spectra $\sigma(A)$ and $\sigma(A-BC)$ are disjoint if and only if for some integer m the operator (0.3) is left invertible, and in that case the operator $B = F^{-1} (A^* - \bar{z}_0)^{-1}C^*$: $: H \to G$, where F is as in (0.5), has the property that

$$\sigma(A-BC) \subset \{\lambda \mid |\lambda - z_0| < \delta\}.$$

Recently, for separable Hilbert spaces a stronger version of the spectrum displacement theorem has been proved by Eckstein in [9], but the method used in [9] does not yield formulas for B.

Conversely, solutions of the spectrum displacement theorem can be used to provide solutions of the inverse problem for spectral pairs. For example, in this paper we use Eckstein's spectrum displacement theorem [9] to solve the inverse problem for spectral pairs of operator polynomials whose coefficients act on a separable Hilbert space.

This paper consists of 8 sections. In the first section we recall some definitions and results developed in [29, 30]. In Section 2 we solve the inverse problem for spectral pairs, and in Section 3 its solution is applied to establish the correspondence between factorizations and invariant subspaces of a spectral linearization. The spectrum displacement problem is dealt with in Section 4. In Sections 5 and 6 we study greatest common divisors and least common multiples, respectively. For operator polynomials (acting on separable Hilbert spaces) the inverse problem is solved in Section 7, and subsequently its solution is employed to solve the extension problem for monic operator polynomials (see [24] for the finite dimensional formulation and solution of this problem). In Section 8 we establish a connection between the theory of Sz.-Nagy—Foiaş characteristic operator functions (cf. [33]) and the approach developed in [29, 30] and the present paper. It turns out that, roughly speaking, a Hilbert space contraction is a spectral linearization of the corresponding Sz.-Nagy—Foiaş characteristic operator function. Finally, in an appendix for the infinite dimensional case we connect the left invertibility of the column (0.3) to Hautus' test for exact observability (see [27]).

Almost all results of this paper are stated and proved in a Hilbert space framework. We have no solution of the general inverse problem for spectral pairs in a general Banach space setting.

1. PRELIMINARIES

The notion of a spectral linearization is linked with the following notion of a spectral node. Let Ω be an open set in $\mathbf{C}$, and let $W\colon \Omega \to \mathscr{L}(H)$ be an analytic operator function with a compact spectrum $\Sigma(W)$. Here $\mathscr{L}(H)$ denotes the Banach algebra of all operators on the Hilbert space H. Throughout this paper all spaces are assumed to be complex and all operators are assumed to be bounded and linear. A quintet $\theta = (A, B, C; G, H)$ is called a *spectral node* for W on Ω if G is a Hilbert space,

$$A\colon G \to G, \quad B\colon H \to G, \quad C\colon G \to H$$

are operators and the following conditions are satisfied:

(P_1) $\sigma(A) \subset \Omega$;

(P_2) $W(\lambda)^{-1} - C(\lambda I - A)^{-1}B$ has an analytic extension on Ω;

(P_3) $W(\lambda)C(\lambda I - A)^{-1}$ has an analytic extension on Ω;

(P_4) $\bigcap_{j=0}^{\infty} \operatorname{Ker} CA^j = (0)$.

The operator A is called the *main operator* of the spectral node θ. The main operator A is a spectral linearization of W on Ω (see [29], Corollary 4.2) and, conversely, if A is a spectral linearization of W on Ω, then A is the main operator of some spectral node for W on Ω (see [29], Theorem 5.1). The connection between linearization (cf. [4, 32, 11]) and spectral nodes is explained in [29].

A spectral node for W on Ω exists and an explicit construction is given below. First we introduce some further terminology. By a bounded Cauchy domain Δ we mean a bounded open set in $\mathbf{C}$ whose boundary $\partial\Delta$ consists of a finite number of disjoint, closed, rectifiable and positively oriented Jordan curves. If $\Omega \subset \mathbf{C}$ is an open set and σ is a compact set in Ω, then there always exists a bounded Cauchy domain Δ such that $\sigma \subset \Delta \subset \bar{\Delta} \subset \Omega$ (see [34], Section 148). For a bounded Cauchy domain Δ we denote by $L_2(\partial\Delta, H)$ the Hilbert space of all strongly measurable H-valued L_2-functions on the boundary $\partial\Delta$ of Δ (see [28]).

THEOREM 1.1. *Let $W: \Omega \to \mathscr{L}(H)$ be an analytic operator function with compact spectrum $\Sigma(W)$, where $\Omega \subset \mathbf{C}$ is an open set containing zero. Suppose that Δ is a bounded Cauchy domain containing 0 such that $\Sigma(W) \subset \Delta \subset \bar{\Delta} \subset \Omega$, and let M be the set of all functions $f \in L_2(\partial\Delta, H)$ which admit an analytic continuation to an H-valued function on $\mathbf{C}_\infty \setminus \Sigma(W)$ vanishing at infinity, while $W(\lambda)f(\lambda)$ has an analytic continuation to Ω. The set M endowed with the L_2-norm is a Hilbert space. Put*

$$V: M \to M, \quad (Vf)(z) = zf(z) - (2\pi i)^{-1}\int_{\partial\Delta} f(w)\mathrm{d}w;$$

(1.1)

$$R: H \to M, \quad (Ry)(z) = (2\pi i)^{-1}\int_{\Gamma} (W(w))^{-1}(z-w)^{-1}y\mathrm{d}w;$$

(here Γ is the boundary of a bounded Cauchy domain Δ' such that $\Sigma(W) \subset \Delta' \subset \bar{\Delta}' \subset \Delta$);

$$Q: M \to H, \quad Qf = (2\pi i)^{-1}\int_{\partial\Delta} f(w)\mathrm{d}w. \tag{1.2}$$

Then $(V, R, Q; M, H)$ is a spectral node for W on Ω.

This result is a Hilbert space version of Theorem 3.1 in [29]. It follows from Theorem 1.1 that if H is a separable Hilbert space, then the Hilbert space on which the main operator V acts is separable too.

Any other spectral node for W on Ω is similar to the spectral node $(V, R, Q; M, H)$ defined in Theorem 1.1 (cf. [29], Theorem 1.2). Here similarity means the following. Two spectral nodes $\theta_i = (A_i, B_i, C_i; G_i, H)$, $i = 1,2$, are *similar* if there exists an invertible operator $S: G_1 \to G_2$ such that $A_1 = S^{-1}A_2S$, $B_1 = S^{-1}B_2$ and $C_1 = C_2S$.

A pair (C, A) (resp. (A, B)) of operators $C: G \to H$ and $A: G \to G$ (resp. $A: G \to G$ and $B: H \to G$) is called a *right* (resp. *left*) *spectral pair* for W on Ω if there exists an operator $B: H \to G$ (resp. $C: G \to H$) such that $\theta = (A, B, C; G, H)$ is a spectral node for W on Ω. The definition of a right spectral pair given here coincides with the one employed in Introduction. Since spectral nodes for a given function are unique up to similarity, the same is true for right and left spectral pairs. Spectral pairs can be defined intrinsically in terms of the operator function W itself, without reference to a spectral node (see [30], Section 3).

The notion of a spectral pair allows us to describe divisibility of analytic operator functions in terms of restrictions of spectral pairs. Let us give the necessary definitions. For $i = 1,2$ let (C_i, A_i) be a pair of Hilbert space operators $C_i: G_i \to \to H$ and $A_i: G_i \to G_i$. The pair (C_2, A_2) is called a *right restriction* of (C_1, A_1) if there exists a left invertible operator $S: G_2 \to G_1$ such that

$$C_1 S = C_2, \quad A_1 S = S A_2.$$

An analytic operator function $W_1: \Omega \to \mathscr{L}(H)$ with compact spectrum is called a *right divisor* on Ω of the analytic operator function $W: \Omega \to \mathscr{L}(H)$ with compact spectrum if $W(\lambda) = Q(\lambda) W_1(\lambda)$, $\lambda \in \Omega$, for some analytic function $Q: \Omega \to \mathscr{L}(H)$. Note that Q necessarily has a compact spectrum. The following result appeared in [30] as Theorem 2.1.

THEOREM 1.2. *For $i = 1,2$ let $W_i: \Omega \to \mathscr{L}(H)$ be an analytic operator function with compact spectrum, where $\Omega \subset \mathbf{C}$ is an open set and H is a Hilbert space. For $i = 1,2$ let (C_i, A_i) be a right spectral pair of W_i on Ω. Then the pair (C_2, A_2) is a right restriction of (C_1, A_1) if and only if W_2 is a right divisor of W_1 (on Ω).*

A dual result holds for left divisibility of analytic operator functions with compact spectrum. Here left restrictions of left spectral pairs will be involved.

The notion of a spectral node was introduced in [29] and studied in [29, 30]. For monic matrix and operator polynomials W and $\Omega = \mathbf{C}$ the analogous notion of a spectral triple has been studied in [17, 18, 19] (see also [20]). For monic operator polynomials and Ω the interior domain of a simple contour in $\mathbf{C}$, the notion of a spectral pair was introduced and employed in [22, 37].

Throughout the paper the following notation is used: $\mathbf{C}_\infty$ stands for the. Riemann sphere $\mathbf{C} \cup \{\infty\}$, the boundary of a set $\Delta \subset \mathbf{C}$ is denoted by $\partial\Delta$. Hilbert (resp. Banach) spaces are described by the letters G, H (resp. X, Y), possibly with subscripts. The finite block operator column (0.3) will often be denoted by $K_m(C, A)$.

2. INVERSE THEOREMS

In this section we give a full description of all pairs (C, A) of Hilbert space operators that can appear as a right spectral pair for some analytic operator function with compact spectrum. Further, we solve the corresponding inverse problem, that is, given such (C, A) we construct an analytic operator function for which (C, A) is a right spectral pair. Throughout this section Ω is an open set in $\mathbf{C}$ such that $\overline{\Omega} \neq \mathbf{C}$, i.e., the complement of Ω in $\mathbf{C}$ has a non-empty interior.

THEOREM 2.1. *Let Ω be an open set in* $\mathbf{C}$ *with* $\bar{\Omega} \neq \mathbf{C}$. *A pair* (C, A) *of Hilbert space operators* $A\colon G \to G$ *and* $C\colon G \to H$ *is a right spectral pair for some analytic operator function* $W\colon \Omega \to \mathscr{L}(H)$ *with compact spectrum if and only if the following conditions hold:*

(i) $\sigma(A) \subset \Omega$;

(ii) *the operator*

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} : G \to H^m \tag{2.1}$$

is left invertible for some positive integer m. Furthermore, if (i) *and* (ii) *hold true and* $\Omega \cap \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\}$ *is empty, then the operator function*

$$W(\lambda) = I + CF^{-1}[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1}C^*, \tag{2.2}$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A^* - \bar{z}_0)^{-n-1}C^*C(A - z_0)^{-n-1}, \tag{2.3}$$

is analytic on Ω and has compact spectrum, and (C, A) is a right spectral pair for W on Ω.

Proof. First assume that (C, A) is a right spectral pair for some analytic operator function $W\colon \Omega \to \mathscr{L}(H)$ with compact spectrum. Then, by definition, Condition (i) holds true. The left invertibility of the finite column (2.1) has already been proved in [30], Section 6, for any analytic operator function with compact spectrum, defined on any open set in $\mathbf{C}$ and with values acting on arbitrary Banach spaces. But the proof given in [30] is not straightforward and is based on factorization results that are proved using theorems on the triviality of certain cocycles (see [21]). For the case considered here a direct proof can be given, which goes as follows.

Choose $r > 0$ such that $\sigma(A) \subset \{\lambda \in \mathbf{C} \mid |\lambda| < r\}$. We first show that the operator

$$K_\infty = \begin{bmatrix} C \\ C(r^{-1}A) \\ C(r^{-1}A)^2 \\ \vdots \end{bmatrix} : G \to \ell_2(H) \tag{2.4}$$

is left invertible. Since the spectral radius of A is strictly less than r, the operator K_∞ is a well-defined bounded linear operator. Further, Ker $K_\infty = \{0\}$ in view of the condition that $\bigcap_{i \geq 0} \operatorname{Ker} CA^i = \{0\}$, which is satisfied by each right spectral pair. Since $\ell_2(H)$ is a Hilbert space, we have to show that there exists $\gamma > 0$ such that $\|K_\infty x\| \geq \gamma \|x\|$ for all $x \in G$. Suppose not. Then there exists a sequence $x_n \in G$, $\|x_n\| = 1$, $n = 1, 2, \ldots$, such that $K_\infty x_n \to 0$ if $n \to \infty$. In particular,

$$\lim_{n \to \infty} CA^j x_n = 0 \quad (j = 0, 1, 2, \ldots). \tag{2.5}$$

Let $\langle H \rangle = \ell_\infty(H)/c_0(H)$ and $\langle G \rangle = \ell_\infty(G)/c_0(G)$ (cf. the paragraph preceding Theorem 2.7 in [29]), and put

$$\langle y \rangle = \langle (x_1, x_2, \ldots) \rangle \in \langle G \rangle.$$

Denote by $\langle A \rangle : \langle G \rangle \to \langle G \rangle$ and $\langle C \rangle : \langle G \rangle \to \langle H \rangle$ the operators induced by A and C, respectively. From (2.5) we conclude that

$$\langle y \rangle \in \bigcap_{j \geq 0} \operatorname{Ker} \langle C \rangle \langle A \rangle^j. \tag{2.6}$$

According to Theorem 2.7 in [29] the pair ($\langle C \rangle$, $\langle A \rangle$) is a right spectral pair for some analytic $\mathscr{L}(\langle H \rangle)$-valued function with compact spectrum. So the right hand side of (2.6) consists of the zero element only. So $\langle y \rangle = 0$, which contradicts the fact that $\|x_n\| = 1$ for $n \geq 0$. Hence, the operator K_∞ is left invertible.

Since $\sigma(A) \subset \{\lambda \mid |\lambda| < r\}$, the spectral radius of $r^{-1}A$ is less than 1. But then we can use the left invertibility of K_∞ and the fact that the set of all left invertible operators is open to show that for m sufficiently large the operator $K_m(C, r^{-1}A)$ is left invertible. This proves Condition (ii).

Now conversely, assume that for the pair (C, A) Conditions (i) and (ii) hold true. First, let us prove that the operator F defined by (2.3) is invertible. Since $\Omega \cap \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\} = \emptyset$ and $\sigma(A) \subset \Omega$, the operator $A - z_0$ is invertible and the spectral radius of $(A - z_0)^{-1}$ is less than δ^{-1}. It follows that the series at the right hand side of (2.3) converges in the operator norm, and hence F is well-defined. Observe that

$$\langle Fx, x \rangle = \sum_{n=0}^{\infty} \delta^{2n} \|C(A - z_0)^{-n-1} x\|^2.$$

Since the finite column $K_m(C, A)$ is left invertible for some m, the same is true for $K_m(C, (A - z_0)^{-1})$. To see this, observe that

$$\begin{bmatrix} I & 0 & \dots\dots 0 \\ \binom{1}{0} z_0 I & \binom{1}{1} I & 0 \\ \vdots & \vdots & \ddots \quad \vdots \\ \binom{m-1}{0} z_0^{m-1} I & \binom{m-1}{1} z_0^{m-2} I \dots & \dots I \end{bmatrix} \begin{bmatrix} C \\ C(A - z_0) \\ \vdots \\ C(A - z_0)^{m-1} \end{bmatrix} =$$

(2.7)

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix}.$$

So there exists $\gamma > 0$ such that

$$\sum_{n=0}^{m-1} \| C(A - z_0)^{-n} x \|^2 \geqslant \gamma \|x\|^2, \qquad x \in G.$$

But then F must be strictly positive, and hence F is invertible. Note that F is the (unique) solution of the equation

$$\delta^2 F - (A^* - \bar{z}_0) F (A - z_0) = -C^* C. \tag{2.8}$$

To analyse the function W defined by (2.2) we introduce the following auxiliary operator

$$B = F^{-1}(A^* - \bar{z}_0)^{-1} C^* \colon H \to G. \tag{2.9}$$

From (2.8) it is clear that

$$A - BC = z_0 + \delta^2 F^{-1}(A^* - \bar{z}_0)^{-1} F,$$

and so

$$\lambda - (A - BC) = F^{-1} [\lambda - z_0 - \delta^2 (A^* - \bar{z}_0)^{-1}] F.$$

Since the spectral radius of $(A^* - \bar{z}_0)^{-1}$ is less than δ^{-1}, the operator $\lambda - z_0 - \delta^2(A^* - \bar{z}_0)^{-1}$ is invertible whenever $|\lambda - z_0| \geq \delta$. It follows that

$$\sigma(A - BC) \subset \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\}. \tag{2.10}$$

In particular, for $\lambda \in \Omega$ the operator $\lambda - (A - BC)$ is invertible and

$$I - C[\lambda - (A - BC)]^{-1}B = I - CF^{-1}[\lambda - z_0 - \delta^2(A^* - \bar{z}_0)^{-1}]^{-1}FB =$$
$$= I + CF^{-1}[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1}C^*.$$

It follows that the operator function W defined by (2.2) can be rewritten as

$$W(\lambda) = I - C[\lambda - (A - BC)]^{-1}B, \quad \lambda \in \Omega. \tag{2.11}$$

From this representation it is clear that W is analytic on Ω. Further, for $\lambda \in \Omega\setminus\sigma(A)$ the operator $W(\lambda)$ is invertible and its inverse is given by (cf. [3], Section 1.1)

$$W(\lambda)^{-1} = I + C(\lambda - A)^{-1}B. \tag{2.12}$$

Next we prove that $(A, B, C; G, H)$ is a spectral node for W on Ω. By Condition (i), Property (P_1) for a spectral node is fulfilled. From (2.12) it is clear that $W(\lambda)^{-1} - C(\lambda - A)^{-1}B$ has an analytic continuation on Ω, and so (P_2) holds true. To prove (P_3) we use (2.11). So for $\lambda \in \Omega\setminus\sigma(A)$ we have

$$W(\lambda)C(\lambda - A)^{-1} = C(\lambda - A)^{-1} - C[\lambda - (A - BC)]^{-1}BC(\lambda - A)^{-1}.$$

Now write $BC = \lambda - (A - BC) - (\lambda - A)$. It follows that

$$W(\lambda)C(\lambda - A)^{-1} = C[\lambda - (A - BC)]^{-1}.$$

Since the spectrum of $A-BC$ is disjoint with Ω, it is clear that $W(\lambda)C(\lambda - A)^{-1}$ has an analytic continuation on Ω, which establishes (P_3). Finally, Property (P_4) holds true because of the left invertibility of $K_m(C, A)$. So $(A, B, C; G, H)$ is a spectral node for W on Ω. But then (C, A) is a right spectral pair for W on Ω. □

We note that the function W given by (2.2) is of a rather special type. For example, W admits an analytic continuation to a function which is analytic on the Riemann sphere outside the disc $|\lambda - z_0| < \delta$ and whose value at infinity is equal to the identity operator on H. Further, the function $W(\cdot)^{-1}$ has an analytic continuation outside $\sigma(A)$ including the point infinity. These extra properties one reads off directly from the formulas (2.10), (2.11) and (2.12). We shall come back to this in Section 4.

In Section 7 we shall see that in the case when H and G are separable Hilbert spaces one can take Ω to be any open set in $\mathbf{C}$. In fact, in that case one can prove that (C, A) is a right spectral pair for an operator polynomial, but we do not have an explicit formula for the polynomial.

Using the Hilbert space version of Theorem 2.8 in [29] we immediately obtain the following result, which is the dual of Theorem 2.1.

THEOREM 2.2. *Let Ω be an open set in* **C** *with $\bar{\Omega} \neq$* **C**. *A pair (A, B) of Hilbert space operators $A: G \to G$ and $B: H \to G$ is a left spectral pair for some analytic operator function $W: \Omega \to \mathscr{L}(H)$ with compact spectrum if and only if the followng conditions hold:*

(i) $\sigma(A) \subset \Omega$;

(ii) *the operator $[B\ AB\ \ldots\ A^{m-1}B]: H^m \to G$ is right invertible for some positive integer m.*

Furthermore, if (i) *and* (ii) *hold true and $\Omega \cap \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\}$ is empty, then the operator function*

$$W(\lambda) = I + B^*[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1}F^{-1}B,$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A - z_0)^{-n-1}BB^*(A^* - \bar{z}_0)^{-n-1},$$

is analytic on Ω and has compact spectrum, and (A, B) is a left spectral pair for W on Ω.

Let $A: G \to G$, $B: H \to G$ and $C: G \to H$ be given Hilbert space operators, and let $\Omega \subset \mathbf{C}$ be an open set with $\bar{\Omega} \neq \mathbf{C}$. In order that $(A, B, C; G, H)$ is a spectral node for an analytic operator function on Ω with compact spectrum, it is necessary that $\sigma(A) \subset \Omega$ and for some positive integer m the operators $K_m(C, A)$ and $[B\ AB\ \ldots\ A^{m-1}B]$ are left invertible and right invertible, respectively. This is clear from the previous theorems. In the finite dimensional case (i.e., whenever $\dim H < < \infty$) these conditions are sufficient; this can be shown using the results from [25]. If $\dim H = +\infty$, these conditions are not sufficient as the following counterexample shows.

EXAMPLE 2.3. Let $W_1, W_2: \Omega \to \mathscr{L}(H)$ be analytic operator functions with compact spectrum such that their H-extensions $W_1(\cdot) \oplus I_H$ and $W_2(\cdot) \oplus I_H$ are analytically equivalent on Ω, while W_1 and W_2 are not analytically equivalent on Ω. If $\dim H = +\infty$, such a pair of functions exists indeed (see [11]). For instance, take $H = \ell_2$, $\Omega = \{z \mid |z| < 1\}$, and let W_1 and W_2 be defined by

$$W_1(\lambda)(x_n)_{n=0}^{+\infty} = \left(\left(\lambda - \frac{1}{2}\right)x_n\right)_{n=0}^{+\infty},$$

(2.13)

$$W_2(\lambda)(x_n)_{n=0}^{+\infty} = \left(x_0, \left(\lambda - \frac{1}{2}\right)x_1, \left(\lambda - \frac{1}{2}\right)x_2, \ldots\right).$$

Since the H-extensions of W_1 and W_2 are analytically equivalent on Ω, by Theorem 5.3 of [29] there exist spectral nodes for W_1 and W_2 on Ω of the form

$$\theta_1 = (A, B_1, C_1; G, H), \quad \theta_2 = (A, B_2, C_2; G, H),$$

respectively. For example, if W_1 and W_2 are as in (2.13), take $H = G = \ell_2$, $A = \frac{1}{2} I_{\ell_2}$, $B_1 = C_1 = I_{\ell_2}$ and define $B_2\colon \ell_2 \to \ell_2$ and $C_2\colon \ell_2 \to \ell_2$ by

$$B_2(x_n)_{n=0}^{+\infty} = (x_1, x_2, x_3, \ldots),$$

$$C_2(x_n)_{n=0}^{+\infty} = (0, x_0, x_1, \ldots).$$

Observe that the quintet $\theta_3 = (A, B_2, C_1; G, H)$ has the following properties:

(i) $\sigma(A) \subset \Omega$;

(ii) for some positive integer m the operator $K_m(C_1, A)$ is left invertible and the operator $[B_2 \; AB_2 \; \ldots \; A^{m-1}B_2]$ is right invertible.

However, θ_3 is not a spectral node for some analytic operator function on Ω.

To prove this, let us suppose that θ_3 is a spectral node on Ω for the analytic operator function W_3 with compact spectrum, and let us enforce a contradiction. By Corollary 2.3 of [30] there exist invertible operators $E(\lambda)$ and $F(\lambda)$, depending analytically on λ in Ω, such that

$$W_3(\lambda) = E(\lambda)W_1(\lambda), \quad W_2(\lambda) = W_3(\lambda)F(\lambda); \quad \lambda \in \Omega.$$

But then $W_2(\lambda) = E(\lambda)W_1(\lambda)F(\lambda)$, $\lambda \in \Omega$, and thus W_1 and W_2 are analytically equivalent on Ω. Contradiction. Hence, θ_3 is not a spectral node on Ω for some analytic operator function with compact spectrum. □

3. FIRST APPLICATION TO DIVISIBILITY

Let $W\colon \Omega \to \mathscr{L}(H)$ be an analytic operator function with a compact spectrum, where $\Omega \subset \mathbf{C}$ is open. Let (C, A) be a right spectral pair for W on Ω. The space on which A acts is denoted by G. Recall from [30], Section 4, that a closed subspace N of G is called a *supporting subspace* of the pair (C, A) if N is invariant under A and the pair of restricted operators $(C|N, A|N)$ is a right spectral pair for some analytic operator function $W_1\colon \Omega \to \mathscr{L}(H)$ with compact spectrum. In that case W_1 is a right divisor of W on Ω (Theorem 1.2). The converse statement is also true, i.e., if $W_1\colon \Omega \to \mathscr{L}(H)$ is a right divisor of W on Ω, then there exists a (unique) supporting subspace N of (C, A) such that the pair $(C|N, A|N)$ is a right spectral pair for W_1 on Ω (see [30], Proposition 4.1). As an immediate corollary of Theorem 2.1 we have the following proposition.

PROPOSITION 3.1. *Let Ω be an open set with $\bar{\Omega} \neq \mathbf{C}$, and let (C, A), where $A: G \to G$ and $C: G \to H$, be a right spectral pair for W on Ω. A closed subspace N of G is a supporting subspace of the pair (C, A) if and only if N is A-invariant and $\sigma(A|N) \subset \Omega$.*

Proof. If N is a supporting subspace of (C, A), then by definition N is A-invariant and because of the properties of a spectral pair on Ω one has $\sigma(A|N) \subset \Omega$.

Conversely, now assume that N is A-invariant and $\sigma(A|N) \subset \Omega$. By Theorem 2.1 the operator $K_m(C, A)$ is left invertible for some m. Hence, the same is true for $K_m(C|N, A|N)$. Again we apply Theorem 2.1. So there exists an analytic operator function $W_1: \Omega \to \mathscr{L}(H)$ with compact spectrum and right spectral pair $(C|N, A|N)$. So N is a supporting subspace. ▨

From the previous proposition it is clear that a spectral subspace of A (i.e., the image of a Riesz projection $(2\pi i)^{-1}\int_\Gamma (\lambda - A)^{-1} d\lambda$ for a suitable contour Γ) is a supporting subspace of (C, A) (cf. [30], Corollary 5.3, where this was proved in the Banach space setting on the basis of theorems concerning the triviality of certain analytic cocycles). The fact that any spectral subspace is supporting has the following interesting consequence.

Let Ω be an open set with $\bar{\Omega} \neq \mathbf{C}$, and let $W: \Omega \to \mathscr{L}(H)$ be an analytic operator function. Now assume that the spectrum $\Sigma(W)$ of W decomposes into two disjoint compact sets σ_1 and σ_2. Then W admits a factorization $W(\lambda) = W_1(\lambda) \cdot W_2(\lambda)$, where $W_i: \Omega \to \mathscr{L}(H)$ is analytic and its spectrum is equal to σ_i $(i = 1,2)$. This fact is known and has been proved by using theorems on the triviality of certain analytic cocycles (see [21]; also [8]). However, the fact that any spectral subspace is supporting allows us to give a simple operator theoretical proof of this result and to give formulas for the factors.

THEOREM 3.2. *Let $\Omega \subset \mathbf{C}$ be an open set with $\bar{\Omega} \neq \mathbf{C}$. Let $W: \Omega \to \mathscr{L}(H)$ be an analytic operator function with right spectral pair (C, A), and assume that $\Sigma(W) = \sigma(A)$ decomposes into two disjoint compact sets σ_1 and σ_2. Let P be the orthogonal projection onto the spectral subspace N of A corresponding to σ_2, and define $E: N \to N$ by*

$$Ex = \sum_{n=0}^{\infty} \delta^{2n} P(A^* - \bar{z}_0)^{-n-1} C^* C (A - z_0)^{-n-1} x, \quad x \in N,$$

where $z_0 \in \mathbf{C}$ and $\delta > 0$ are such that $\Omega \cap \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\} = \emptyset$. For $\lambda \in \Omega$ put

$$W_1(\lambda) = W(\lambda) + W(\lambda) C(\lambda - A)^{-1} E^{-1} P(A^* - \bar{z}_0)^{-1} C^*, \tag{3.1}$$

$$W_2(\lambda) = I + CE^{-1}P[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1} C^*. \tag{3.2}$$

Then $W_1, W_2: \Omega \to \mathscr{L}(H)$ *are analytic,* $\Sigma(W_1) = \sigma_1$, $\Sigma(W_2) = \sigma_2$ *and* $W(\lambda) = W_1(\lambda) \cdot W_2(\lambda)$, $\lambda \in \Omega$.

Proof. Let $W_2: \Omega \to \mathscr{L}(H)$ be a right divisor of W corresponding to the supporting subspace N of (C, A); so $(C|N, A|N)$ is a right spectral pair for W_2 on Ω. One such W_2 is given by formula (3.2), which is derived from (2.2). Now by formula (2.12) the quotient $W_1(\lambda) = W(\lambda)W_2(\lambda)^{-1}$ is equal to

$$W(\lambda) + W(\lambda)(C|N)(\lambda - A|N)^{-1}B, \tag{3.3}$$

where $B = E^{-1}[(A|N)^* - \bar{z}_0]^{-1}(C|N)^*$ (cf. formula (2.9)). It is easily seen that (3.3) coincides with (3.1). Finally, $\Sigma(W_2) = \sigma_2$ and the spectrum of W_1 is equal to σ_1 (cf. [30], Theorem 2.4). □

If the set Ω in Proposition 3.1 is simply connected (or, more generally, if the bounded components of $\mathbf{C}\setminus\sigma(A)$ do not intersect the bounded components of $\mathbf{C}\setminus\bar{\Omega}$), then the condition $\sigma(A|N) \subset \Omega$ is satisfied automatically for every A-invariant subspace N. In this case Proposition 3.1 gives a one-to-one correspondence between the invariant subspaces of the spectral linearization of W and equivalence classes of right divisors of W (two right divisors are called equivalent if $W_1W_2^{-1}$ and $W_2W_1^{-1}$ are analytic on Ω). A correspondence of this type has been observed for matrix polynomials in [17, 12, 23] and for analytic matrix functions in [25, 36] (cf., the divisibility theory for characteristic operator functions [33, 5]; also [3], Chapter 1).

4. SPECTRUM DISPLACEMENT THEOREMS

In this section we show that the solution of the inverse problem given in Section 2 also provides a solution of the following problem: Given Hilbert space operators $A: G \to G$ and $C: G \to H$, construct, if possible, an operator $B: H \to G$ such that the spectrum of the operator $A-BC$ does not intersect with a prescribed open set Δ in the complex plane. As we mentioned in the introduction, this problem is of interest in Mathematical Systems Theory and is related to the pole-shifting theorem (see [38]).

THEOREM 4.1. *Let* $A: G \to G$ *and* $C: G \to H$ *be Hilbert space operators. Then there exists an operator* $B: H \to G$ *such that* $\sigma(A) \cap \sigma(A - BC) = \emptyset$ *if and only if for some positive integer* m *the operator*

$$K_m(C, A) = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{pmatrix} : G \to H^m \tag{4.1}$$

is left invertible. Moreover, in that case given an open disc $|\lambda - z_0| < \delta$ disjoint with $\sigma(A)$ the operator B defined by

$$B = F^{-1}(A^* - \bar{z}_0)^{-1}C^* : H \to G, \tag{4.2}$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A^* - \bar{z}_0)^{-n-1}C^*C(A - z_0)^{-n-1}, \tag{4.3}$$

has the property that

$$\sigma(A - BC) \subset \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\}.$$

Proof. Assume there exists an operator $B: H \to G$ such that $\sigma(A - BC) \cap \cap\, \sigma(A)$ is empty. Let Ω be an open set in $\mathbf{C}$ containing $\sigma(A)$ and disjoint with $\sigma(A - BC)$. Put

$$W(\lambda) = I - C[\lambda - (A - BC)]^{-1}B, \quad \lambda \in \Omega.$$

Obviously, W is analytic on Ω. Note that for each $\lambda \in \Omega$ the operator

$$\begin{pmatrix} A - \lambda & B \\ C & I_H \end{pmatrix} : G \oplus H \to G \oplus H$$

is invertible and its inverse is equal to

$$\begin{pmatrix} (A - BC - \lambda)^{-1} & -(A - BC - \lambda)^{-1}B \\ -C(A - BC - \lambda)^{-1} & W(\lambda) \end{pmatrix}. \tag{4.4}$$

Since the operator (4.4) depends analytically on $\lambda \in \Omega$, we can apply Theorem 4.3 in [29] to show that W has a compact spectrum in Ω and $(A, B, C; G, H)$ is a spectral node for W on Ω. In particular, (C, A) is a right spectral pair for W on Ω. Without loss of generality we may assume that $\bar{\Omega} \neq \mathbf{C}$. So we can apply Theorem 2.1 to show that for some positive integer m the operator $K_m(C, A)$ is left invertible.

Conversely, assume that for some positive integer m the operator (4.1) is left invertible. Note that formulas (4.3) and (2.3) are identical, as well as formulas (4.2) and (2.9). This theorem is now immediate from Theorem 2.1. ▨

COROLLARY 4.2. *If for some positive integer m the operator $K_m(C, A)$ is left invertible and $\Delta \neq \emptyset$ is any open set in* $\mathbf{C}$, *then there exists an operator $B: H \to G$ such that $\sigma(A - BC) \subset \Delta$.*

Proof. Without loss of generality we may assume that $\mathbf{C} \setminus (\Delta \cup \sigma(A))$ has a non-empty interior. Choose an open disc $|\lambda - z_0| < \delta$ which is disjoint with Δ and $\sigma(A)$. Since the operator $K_m(C, A)$ is left invertible for some positive integer m

we can apply the second part of Theorem 4.1 to show that there exists $B_1: H \to G$ such that

$$\sigma(A - B_1C) \subset \{\lambda \mid |\lambda - z_0| < \delta\}.$$

Now observe that $\sigma(A - B_1C)$ is disjoint with Δ. Further, from our hypothesis on $K_m(C, A)$ it follows that the operator $K_m(C, A - B_1C)$ is also left invertible for some m, in view of the formula

$$\begin{pmatrix} I & 0 & 0 & \dots & 0 \\ CB_1 & I & 0 & \dots & 0 \\ CAB_1 & CB_1 & I & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ CA^{m-1}B_1 & CA^{m-2}B_1 & CA^{m-3}B_1 & \dots & I \end{pmatrix} \begin{pmatrix} C \\ CA^x \\ C(A^x)^2 \\ \dots \\ C(A^x)^{m-1} \end{pmatrix} = $$

$$= \begin{pmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{m-1} \end{pmatrix}$$

where A^x stands for $A - B_1C$ (cf. Section IV.3 in [3]). Now apply the second part of Theorem 4.1 to the pair $(C, A - B_1C)$. So there exists an operator $B_2: H \to G$ such that $\sigma(A - B_1C - B_2C) \subset \Delta$. Hence, the operator $B = B_1 + B_2$ has the desired properties. □

When H and G are finite dimensional, Theorem 4.1 and Corollary 4.2 are well-known and the operator B can be obtained, for example, from Brunovsky's canonical form (see [7, 16]). In fact, in the finite dimensional case a stronger result holds true, namely, given a finite set $K \subset \mathbf{C}$ with at most k different points ($k = \dim G$), there exists $B: H \to G$ such that $\sigma(A - BC) = K$, and to a certain extent one can prescribe the multiplicities of the eigenvalues of $A - BC$. Here, of course, we assume that $K_m(C, A)$ is left invertible for some positive integer m, which in this case amounts to the requirement that $\bigcap_{i=0}^{k-1} CA^i = \{0\}$, where $k = \dim G$.

For infinite dimensional separable Hilbert spaces H and G the analogue of the stronger finite dimensional result has been proved recently by Eckstein in [9]; however, an explicit formula for B is not provided there. In [9] the connection between the left invertibility of $K_m(C, A)$ and the existence of an analytic operator function with (C, A) as a right spectral pair does not appear. The following version of Eckstein's result will be used later on.

THEOREM 4.3. *Let $A: G \to G$ and $C: G \to H$ be operators acting between separable Hilbert spaces, and assume that the operator $K_m(C, A)$ is left invertible. Then there exists $F: H \to G$ such that $(A-FC)^{3m-2} = 0$.*

In the first part of the proof of Theorem 4.1 the fact that H and G are Hilbert spaces is not used, and hence the arguments used there can be employed for Banach spaces too. This leads to the following Banach space version of the spectrum displacement theorem.

THEOREM 4.4. *Let $A: X \to X$ and $C: X \to Y$ be operators acting between Banach spaces, and let $\Omega \subset \mathbf{C}$ be an open set containing $\sigma(A)$. Then there exists an operator $B: Y \to X$ such that $\sigma(A - BC) \cap \Omega = \emptyset$ if and only if (C, A) is a right spectral pair for an analytic operator function $W: \Omega \to \mathscr{L}(Y)$ with the property that $W(\cdot)^{-1}$ has an analytic continuation to an analytic operator function on $\mathbf{C}_\infty \setminus \sigma(A)$ which is invertible at ∞.*

Proof. The "only if" part is covered by the first part of the proof of Theorem 4.1. To prove the "if" part, assume that (C, A) is a right spectral pair for an analytic operator function $W: \Omega \to \mathscr{L}(Y)$ such that $W(\cdot)^{-1}$ has the desired analytic continuation, U say. Let $D = U(\infty)$, and choose $F: Y \to X$ such that $(A, F, C; X, Y)$ is aspectral node for W on Ω. According to Theorem 4.3 in [29] the operator

$$\begin{pmatrix} A-\lambda & F \\ C & D \end{pmatrix}: X \oplus Y \to X \oplus Y$$

is invertible for $\lambda \in \Omega$. Since D is invertible, it follows that $A - FD^{-1}C - \lambda$ is invertible for all $\lambda \in \Omega$. Now put $B = FD^{-1}$. Then $\sigma(A - BC) \cap \Omega = \emptyset$. ∎

It is not known whether Theorem 4.1 holds true for Banach space operators C and A.

5. GREATEST COMMON DIVISORS OF ANALYTIC OPERATOR FUNCTIONS

Let $W_1, \ldots, W_r: \Omega \to \mathscr{L}(H)$ be analytic operator functions with compact spectra. A right divisor W of each W_i, $i = 1, \ldots, r$, on Ω is called a *greatest common (right)* divisor of $W_1, \ldots, W_r$ if every right divisor of $W_1, \ldots, W_r$ on Ω also is a right divisor of W on Ω. A greatest common divisor (if it exists) is uniquely determined up to multiplication from the left by an analytic operator function $E: \Omega \to \mathscr{L}(H)$ such that $E(\lambda)$ is invertible for all $\lambda \in \Omega$. The following theorem is the main result of this section.

THEOREM 5.1. *Let $W_1, \ldots, W_r: \Omega \to \mathscr{L}(H)$ be analytic operator functions with compact spectra, and assume that Ω is an open and simply connected set with $\bar{\Omega} \neq \mathbf{C}$. Then there exists a greatest common right divisor of $W_1, \ldots, W_r$ on Ω.*

Proof. Without loss of generality we assume that zero is a point of Ω (otherwise we replace $W_i(\lambda)$ by $W_i(\lambda+\alpha)$ for a suitable $\alpha \in \mathbf{C}$). Choose a bounded Cauchy domain Δ containing zero such that $\bigcup_{i=1}^{r} \Sigma(W_i) \subset \Delta \subset \bar{\Delta} \subset \Omega$, and let $\tilde{V}: L_2(\partial\Delta, H) \to L_2(\partial\Delta, H)$ and $\tilde{Q}: L_2(\partial\Delta, H) \to H$ be the (bounded linear) operators defined by the following formulas:

$$\tilde{V}f(z) = zf(z) - (2\pi i)^{-1} \int_{\partial\Delta} f(w)\, dw, \quad z \in \partial\Delta; \tag{5.1}$$

$$\tilde{Q}f = (2\pi i)^{-1} \int_{\partial\Delta} f(w)\, dw. \tag{5.2}$$

For $i = 1, \ldots, r$ let M_i be the subspace of $L_2(\partial\Delta, H)$ consisting of all functions $f \in L_2(\partial\Delta, H)$ that admit an analytic continuation to $\mathbf{C}_\infty \setminus \Sigma(W_i)$ vanishing at infinity, while $W_i(\lambda)f(\lambda)$ has an analytic continuation to Ω. By Theorem 1.1, the restriction $(\tilde{Q}|M_i, \tilde{V}|M_i)$ of $(\tilde{Q}, \tilde{V})$ is a right spectral pair for W_i on Ω $(i = 1, \ldots, r)$. In particular, the operator $K_m(\tilde{Q}|M_i, \tilde{V}|M_i)$ is left invertible for some m (not depending on $i = 1, \ldots, r$). Put $M = \bigcap_{i=1}^{r} M_i$; then also $K_m(\tilde{Q}|M, \tilde{V}|M)$ is left invertible for some m.

As Ω is simply connected, one has $\sigma(\tilde{V}|M) \subset \Omega$. By Theorem 2.1 there exists an analytic operator function $W: \Omega \to \mathcal{L}(H)$ with compact spectrum, for which $(\tilde{Q}|M, \tilde{V}|M)$ is a right spectral pair on Ω. By the divisibility theorem (i.e., Theorem 1.2), the function W is a right divisor of each W_i on Ω. The same Theorem 1.2 also ensures that W is a greatest common divisor of $W_1, \ldots, W_r$ on Ω. □

The proof of Theorem 5.1, together with Theorem 2.1, yields the following formula for a greatest common divisor of $W_1, \ldots, W_r$ on Ω (assuming zero belongs to Ω). Let Δ be a bounded Cauchy domain such that $\{0\} \cup \bigcup_{i=1}^{r} \Sigma(W_i) \subset \Delta \subset \bar{\Delta} \subset \Omega$, and let M be the set of all $f \in L_2(\partial\Delta, H)$ such that f is analytic in $\mathbf{C}_\infty \setminus \bigcup_{i=1}^{r} \Sigma(W_i)$, $f(\infty) = 0$ and for $i = 1, \ldots, r$ the function $W_i f$ is analytic in Ω. Let $A = \tilde{V}|M$ and $C = \tilde{Q}|M$, where $\tilde{V}$ and $\tilde{Q}$ are given by (5.1) and (5.2), respectively. Then

$$W(\lambda) = I + C \left\{ \sum_{n=0}^{\infty} \delta^{2n} (A^* - \bar{z}_0)^{-n-1} C^* C (A - z_0)^{-n-1} \right\}^{-1} \cdot$$
$$\cdot [\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1} C^*, \quad \lambda \in \Omega,$$

is a greatest common divisor of $W_1, \ldots, W_r$ on Ω, where $z_0 \in \mathbf{C}$ and $\delta > 0$ are chosen such that $\{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\} \cap \Omega = \emptyset$.

In case $r = 2$, $W_1(\lambda) = \lambda I - T_1$ and $W_2(\lambda) = \lambda I - T_2$, Theorem 5.1 implies the following statement.

COROLLARY 5.2. *Let $T_1, T_2 \in \mathscr{L}(H)$ be Hilbert space operators. Then a greatest common divisor of $\lambda I - T_1$ and $\lambda I - T_2$ (on $\Omega \supset \sigma(T_1) \cup \sigma(T_2)$) is given by the formula $I - P_1 + (\lambda I - T_1)P_1$, where P_1 is the orthogonal projection onto the maximal T_1-invariant subspace N such that $T_1|N = T_2|N$.*

For matrix polynomials common divisors and greatest common divisors were studied in [12] in terms of spectral pairs, and in [13] in terms of generalized resultant matrices.

6. LEAST COMMON MULTIPLES OF ANALYTIC OPERATOR FUNCTIONS

Let Ω be an open set in $\mathbf{C}$, and let $W_i: \Omega \to \mathscr{L}(H)$, $i = 1,2, \ldots, r$, be analytic operator functions with compact spectrum. We say that $W: \Omega \to \mathscr{L}(H)$ is a *(left) common multiple* on Ω of the functions $W_1, \ldots, W_r$ if W is analytic, has compact spectrum and

$$W(\lambda) = U_1(\lambda)W_1(\lambda) = \ldots = U_r(\lambda)W_r(\lambda), \quad \lambda \in \Omega,$$

for some analytic operator functions $U_1, \ldots, U_r$. A right common multiple is defined in an analogous way. We shall consider left common multiples only (the corresponding results for right common multiples can be obtained by taking adjoints).

A common multiple W on Ω of $W_1, \ldots, W_r$ is called a *least common multiple*, if every other common multiple $\tilde{W}$ of $W_1, \ldots, W_r$ on Ω is divisible by W on the right, i.e., the function $\tilde{W}(\cdot)W(\cdot)^{-1}$ is analytic on Ω. Clearly, if a least common multiple exists, it is determined uniquely up to multiplication from the left by an analytic (on Ω) $\mathscr{L}(H)$-valued function whose values are invertible operators.

In the finite dimensional case ($\dim H < \infty$) there always exists a least common multiple for analytic operator functions with compact spectrum (see [36]; for matrix polynomials see [14, 13, 12]). In the infinite dimensional case this is not true: common multiples (let alone least common multiples) do not always exist (see the example given below). A necessary condition is given by the following proposition.

PROPOSITION 6.1. *Let $W_1, \ldots, W_r: \Omega \to \mathscr{L}(H)$ be analytic functions with compact spectra and right spectral pairs $(C_1, A_1), \ldots, (C_r, A_r)$, respectively, where $\Omega \subset \mathbf{C}$ is an open set. Assume that there is a common multiple W of $W_1, \ldots, W_r$*

on Ω. Then

$$\text{Ker}\, K_l([C_1C_2 \dots C_r],\ A_1 \oplus \dots \oplus A_r) = \tag{6.1}$$
$$= \bigcap_{i=0}^{\infty} \text{Ker}\,[C_1A_1^i\ C_2A_2^i \dots C_rA_r^i]$$

for some integer l.

Proof. Let (C, A) be a right spectral pair for W on Ω; so $K_l(C, A)$ is left invertible for some l. By Theorem 7.1 of [15] there is a Hilbert space G_0 and a pair of operators $A_0\colon G_0 \to G_0$, $C_0\colon G_0 \to H$ such that (C, A) is a right restriction of (C_0, A_0) and the operator $K_l(C_0, A_0)$ is invertible. Then

$$C_0A_0^l\,[K_l(C_0, A_0)]^{-1}\, K_l(C, A) = CA^l. \tag{6.2}$$

By Theorem 1.2, for $i = 1, \dots, r$ the pair (C_i, A_i) is a right restriction of (C, A). Now (6.2) implies that

$$C_0A_0^l\,[K_l(C_0, A_0)]^{-1}\, K_l([C_1 \dots C_r],\ A_1 \oplus \dots \oplus A_r) =$$
$$= [C_1A_1^l \dots C_rA_r^l],$$

and (6.1) follows. □

Proposition 6.1 allows us to produce examples of analytic operator functions without a common multiple. For example (cf. Example 2.1 in [15]), define

$$T_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad S_k := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & & 1 & 0 \end{pmatrix}$$

as operators from $\mathbf{C}^k \to \mathbf{C}^k$. Let $H := \mathbf{C}^1 \oplus \mathbf{C}^2 \oplus \mathbf{C}^3 \oplus \dots$, and put $T := T_1 \oplus T_2 \oplus T_3 \oplus \dots$; $S := S_1 \oplus S_2 \oplus S_3 \oplus \dots$. Then for every open set Ω in $\mathbf{C}$ containing $\sigma(T) \cup \sigma(S)$, the analytic operator functions $\lambda I - T$, $\lambda I - S\colon \Omega \to \mathscr{L}(H)$ do not have a common multiple (Condition (6.1) is violated).

The following result reduces the existence problem for a least common multiple to the existence problem of a common multiple (assuming Ω is simply connected).

THEOREM 6.2. *Let Ω be a simply connected open set in $\mathbf{C}$ with $\bar{\Omega} \neq \mathbf{C}$, and let $W_1, \dots, W_r\colon \Omega \to \mathscr{L}(H)$ be analytic operator functions with compact spectra. Then there exists a least common multiple of $W_1, \dots, W_r$ on Ω if and only if there exists a common multiple of $W_1, \dots, W_r$ on Ω.*

Proof. Assume W_0 is a common multiple of $W_1, \ldots, W_r$ on Ω. Without loss of generality we may assume that $0 \in \Omega$. Choose a bounded Cauchy domain Δ containing zero such that $\bigcup_{i=0}^{r} \Sigma(W_i) \subset \Delta \subset \bar{\Delta} \subset \Omega$. Let M_i be the set of all $f \in L_2(\partial\Delta, H)$ that are analytic on $\mathbf{C}_\infty \setminus \Sigma(W_i)$, are zero at ∞ such that $W_i f$ is analytic in Ω $(i = 1, \ldots, r)$. By Theorem 1.1, the restriction $(\tilde{Q}|M_i, \tilde{V}|M_i)$ is a right spectral pair for W_i on Ω, where $\tilde{V}$ and $\tilde{Q}$ are given by (5.1) and (5.2), respectively. Since W_0 is a common multiple of $W_1, \ldots, W_r$, we have $M_0 \supset M_1 + \ldots + M_r$. Put $N = \overline{M_1 + \ldots + M_r}$. Clearly, N is a $\tilde{V}$-invariant subspace and $N \subset M_0$. Further, by Theorem 2.1, $K_m(\tilde{Q}|M_0, \tilde{V}|M_0)$ is left invertible for some m; so $K_m(\tilde{Q}|N, \tilde{V}|N)$ is left invertible as well. Since $\sigma(\tilde{V}|M_0) \subset \Omega$ and Ω is simply connected, also $\sigma(\tilde{V}|N) \subset \Omega$. By Theorem 2.1 there exists an analytic operator function $\tilde{W}: \Omega \to \mathscr{L}(H)$ with right spectral pair $(\tilde{Q}|N, \tilde{V}|N)$. Using (divisibility) Theorem 1.2 it is easily seen that $\tilde{W}$ is a least common multiple of $W_1, \ldots, W_r$ on Ω. ▨

We point out one case when the existence of a least common multiple is ensured.

THEOREM 6.3. *Let $\Omega \subset \mathbf{C}$ be an open set such that $\bar{\Omega} \neq \mathbf{C}$. Let $W_1, \ldots, W_r$ be $\mathscr{L}(H)$-valued analytic (in Ω) functions with compact spectra such that $\Sigma(W_i) \cap \Sigma(W_j) = \emptyset$ for $i \neq j$. Then there exists a least common multiple W of $W_1, \ldots, W_r$ on Ω, and $\Sigma(W) = \bigcup_{i=1}^{r} \Sigma(W_i)$.*

Proof. We shall consider the case $r = 2$ (the general case can be obtained easily by induction on r). For $i = 1,2$ let (C_i, A_i) be a right spectral pair for W_i on Ω, where $A_i: G_i \to G_i$, $C_i: G_i \to H$ (G_i is a Hilbert space). Without loss of generality we may assume that the spectral radius of A_i is less than 1, $i = 1,2$ (otherwise replace $W(\lambda)$ by $W(\alpha\lambda)$ for a suitable fixed positive number α). Consider

$$M = \{(x_1, x_2) \in G_1 \oplus G_2 \mid C_1 A_1^j x_1 + C_2 A_2^j x_2 = 0;\ j = 0, 1, \ldots\}.$$

Note that M is a closed subspace of $G_1 \oplus G_2$. We shall prove that M is, in fact, the zero subspace. Define

$$N_1 = \{x_1 \in G_1 \mid \exists\, x_2 \in G_2: (x_1, x_2) \in M\};$$

$$N_2 = \{x_2 \in G_2 \mid \exists\, x_1 \in G_1: (x_1, x_2) \in M\}.$$

Clearly, N_1 and N_2 are linear sets. Given $x_1 \in N_1$, there is a unique $x_2 \in G_2$ such that $(x_1, x_2) \in M$ (this follows from the fact that $\operatorname{Ker} K_m(C_2, A_2) = \{0\}$ for some m); write

$x_2 = S_1 x_1$. Thus $S_1: N_1 \to N_2$ is a linear map. Similarly, given $x_2 \in N_2$, there is a unique $x_1 \in G_1$ such that $(x_1, x_2) \in M$; write $x_1 = S_2 x_2$. So $S_2: N_2 \to N_1$ is a linear map and $S_2 = S_1^{-1}$.

Choose an integer $m > 0$ such that $K_m(C_2, A_2)$ and $K_m(C_1, A_1)$ are left invertible. Let

$$\Lambda_1 = -[K_m(C_2, A_2)]^+ K_m(C_1, A_1): G_1 \to G_2;$$

$$\Lambda_2 = -[K_m(C_1, A_1)]^+ K_m(C_2, A_2): G_2 \to G_1,$$

where the superscript + denotes a left inverse. Then $S_i = \Lambda_i | N_i$, $i = 1,2$. As for $i = 1,2$ the map S_i is a restriction of a (bounded linear) operator Λ_i, and M is closed, it is easily seen that N_1 and N_2 are closed as well. Observe that N_i is A_i-invariant, $i = 1,2$, and the restrictions $A_1 | N_1$ and $A_2 | N_2$ are similar. So for the boundaries of their spectra we have $\partial\sigma(A_1 | N_1) = \partial\sigma(A_2 | N_2)$, which in view of $\partial\sigma(A_i | N_i) \subset \sigma(A_i)$, $i = 1,2$, and $\sigma(A_1) \cap \sigma(A_2) = \emptyset$ leads to a contradiction, unless $N_i = \{0\}$, $i = 1,2$. So $M = \{0\}$. Put

$$K_\infty = \begin{pmatrix} C_1 & C_2 \\ C_1 A_1 & C_2 A_2 \\ C_1 A_1^2 & C_2 A_2^2 \\ \vdots & \vdots \end{pmatrix} : G_1 \oplus G_2 \to \ell_2(H). \tag{6.3}$$

Since $\sigma(A_1) \cup \sigma(A_2) \subset \{\lambda \mid |\lambda| < 1\}$, formula (6.3) defines a (bounded linear) operator K_∞. We have shown above that $\operatorname{Ker} K_\infty = \{0\}$. It turns out that K_∞ is moreover, left invertible. To see this, let $\langle G_i \rangle = \ell_\infty(G_i)/c_0(G_i)$, $i = 1,2$, and $\langle H \rangle = \ell_\infty(H)/c_0(H)$. Now C_i and A_i induce operators $\langle A_i \rangle: \langle G_i \rangle \to \langle G_i \rangle$ and $\langle C_i \rangle: \langle G_i \rangle \to \langle H \rangle$. Using the property $\sigma(A_i) = \sigma(\langle A_i \rangle)$, $i = 1,2$, we see that K_∞ induces an operator $\langle K_\infty \rangle: \langle G_1 \rangle \oplus \langle G_2 \rangle \to \ell_2(\langle H \rangle)$. As above we prove that $\operatorname{Ker} \langle K_\infty \rangle = \{0\}$, which means that there exists $\gamma > 0$ such that $\|K_\infty x\| \geq \gamma \|x\|$ for all $x \in G_1 \oplus G_2$ (cf. the proof of Theorem 2.1). Thus K is left invertible. Consequently, a finite column $K_\infty([C_1\ C_2], A_1 \oplus A_2)$ is left invertible for some integer m. By Theorem 2.1, there is an analytic function $W: \Omega \to \mathscr{L}(H)$ with compact spectrum whose right spectral pair is $([C_1\ C_2], A_1 \oplus A_2)$. Clearly, W is a common multiple of W_1 and W_2. It is not difficult to check (using, for instance, the particular spectral pairs described in Theorem 1.1) that W is, in fact, a least common multiple of W_1 and W_2. Since $A_1 \oplus A_2$ is a spectral linearization of W on Ω, clearly $\Sigma(W) = \Sigma(W_1) \cup \Sigma(W_2)$. □

In the framework of monic operator polynomials acting in Banach spaces, Theorem 6.3 has been proved in [36].

7. AN INVERSE THEOREM FOR OPERATOR POLYNOMIALS

In this section we assume the Hilbert spaces H and G to be separable. Using Theorem 4.3 (due to Eckstein) we shall prove the existence of an operator polynomial with given right spectral pair (C, A), provided $K_m(C, A)$ is left invertible for some m. In this way we shall derive a sharpened version of the inverse Theorem 2.1. More exactly, the following result holds true.

THEOREM 7.1. *Let $C\colon G \to H, A\colon G \to G$ (resp. $A\colon G \to G,\ B\colon H \to G$) be operators acting between separable Hilbert spaces, and assume that the operator $K_m(C, A)\colon G \to H^m$ (resp. $[B\ AB\ \dots\ A^{m-1}B]\colon H^m \to G$) is left (resp. right) invertible for some $m \in \mathbf{N}$. Then there exists an operator polynomial $W(\lambda)$ with degree at most $3m-2$ such that (C, A) (resp. (A, B)) is its right (resp. left) spectral pair on $\mathbf{C}$.*

Proof. We shall prove the part of Theorem 7.1 concerning the pair (C, A) only. Without loss of generality we assume that A is invertible (otherwise replace A by $A-\lambda_0$ for some $\lambda_0 \notin \sigma(A)$). Since $K_m(C, A)$ is left invertible, so is $K_m(CA^{-1}, A^{-1})$. By Theorem 4.3 there exists $B\colon H \to G$ such that $(A^{-1} - BCA^{-1})^{3m-2} = 0$. Put

$$W(\lambda) = I - CA^{-1}(\lambda^{-1} - (A^{-1} - BCA^{-1}))^{-1}B, \quad \lambda \in \mathbf{C}\setminus\{0\}.$$

Clearly, $W(\lambda)$ is a comonic operator polynomial in λ (i.e., a polynomial with constant term I) and its degree does not exceed $3m-2$. We shall check that $(A, -AB, C;\ G, H)$ is a spectral node for $W(\lambda)$ on $\mathbf{C}$, thereby proving the theorem. Define

$$\tilde{W}(\lambda) = I - CA^{-1}(\lambda - A^{-1} + BCA^{-1})^{-1}B, \quad \lambda \in \mathbf{C}\setminus\{0\}.$$

As in the proof of Theorem 3.1, one verifies that $(A^{-1}, B, CA^{-1};\ G, H)$ is a spectral node for $\tilde{W}$ on $\mathbf{C}\setminus\{0\}$. Since $W(\lambda) = \tilde{W}(\lambda^{-1})$ $(0 \neq \lambda \in \mathbf{C})$, in view of Theorem 2.6 in [29] we conclude that $(A, -AB, C;\ G, H)$ is a spectral node for W on $\mathbf{C}\setminus\{0\}$ (this fact is easily verified directly too). As $W(0) = I$ is invertible, $(A, -AB, C;\ G, H)$ is a spectral node for W on the whole complex plane. ∎

As a corollary of Theorem 7.1 we obtain the following factorization theorem (which is a stronger result than the factorization result mentioned in Section 3; here, however, we need the separability condition on the Hilbert spaces H and G for its proof).

THEOREM 7.2. *Let H be a separable Hilbert space, and let $\Omega \subset \mathbf{C}$ be an open set and $W\colon \Omega \to \mathcal{L}(H)$ an analytic operator function with compact spectrum. Assume that $\Sigma(W) = \sigma_1 \cup \sigma_2$, where σ_1 and σ_2 are disjoint compact sets. Then W admits a factorization*

$$W(\lambda) = R_1(\lambda)\, E(\lambda)\, R_2(\lambda), \quad \lambda \in \Omega, \tag{7.1}$$

where for $i = 1,2$ the function $R_i: \mathbf{C} \to \mathscr{L}(H)$ is an operator polynomial with $\Sigma(R_i) = \sigma_i$, and $E: \Omega \to \mathscr{L}(H)$ is an analytic operator function with invertible values.

Proof. Let (C, A) be a right spectral pair for W on Ω. Note that we may assume that the operators C and A act on separable Hilbert spaces. By Theorem 7.1, there is an operator polynomial $R_2(\lambda)$ with right spectral pair $(C|N, A|N)$, where N is the spectral subspace of A corresponding to σ_2. Let $\tilde{W}(\lambda) = W(\lambda)R_2(\lambda)^{-1}$, and let $R_1(\lambda)$ be an operator polynomial whose left spectral pair on $\mathbf{C}$ coincides with the one for $\tilde{W}$ on Ω (such a polynomial exists by virtue of Theorem 7.1). With these polynomials R_1 and R_2, the factorization (7.1) follows. □

Let $A: G \to G$ and $C: G \to H$ be a given pair of operators. It is known that (C, A) is a right spectral pair on $\mathbf{C}$ for a monic (i.e., leading coefficient equal to the identity I) operator polynomial of degree k if and only if $K_k(C, A)$ is invertible (see [19], where this is proved in a Banach space setting). If $K_k(C, A)$ is only left invertible, then one can construct a monic operator polynomial of degree k with spectral pair $(\tilde{C}, \tilde{A})$ on $\mathbf{C}$ such that (C, A) is a right restriction of the pair $(\tilde{C}, \tilde{A})$ (see [19, 15, 24]). In the finite dimensional case $\tilde{A}$ can always be chosen in such a way that $\tilde{A}$ has at most one eigenvalue more that A (see [24], Theorem 0.1), but in general the existing knowledge about the spectrum of $\tilde{A}$ is not very extensive. Therefore the following theorem is of interest.

THEOREM 7.3. *Let $C: G \to H$, $A: G \to G$ be separable Hilbert space operators such that $K_m(C, A)$ is left invertible. Assume $\lambda_0 \notin \sigma(A)$. Then there exist separable Hilbert space operators $C_0: G_0 \to H$ and $A_0: G_0 \to G_0$ such that $\lambda_0 - A_0$ is nilpotent and the block operator*

$$\begin{pmatrix} C & C_0 \\ CA & C_0A_0 \\ \vdots & \vdots \\ CA^{3m-3} & C_0A_0^{3m-3} \end{pmatrix}: G \oplus G_0 \to H^{3m-2} \tag{7.2}$$

is invertible, i.e., $([C \;\; C_0], A \oplus A_0)$ is a right spectral pair on $\mathbf{C}$ for a monic operator polynomial of degree $3m-2$.

Proof. With no loss of generality (see formula (2.7)) we assume $\lambda_0 = 0$. So A is invertible. Let $W(\lambda) = I + \sum_{j=1}^{3m-2} \lambda^j W_j$ be the operator polynomial with right spectral pair (C, A^{-1}). According to Theorem 7.1 such a polynomial exists indeed. Put

$$\tilde{W}(\lambda) = \lambda^{3m-2} W(\lambda^{-1}). \tag{7.3}$$

Then $\tilde{W}(\lambda)$ is a monic operator polynomial of degree $3m-2$. It is easily seen that (C, A) is a right spectral pair for $\tilde{W}$ on $\mathbf{C}\setminus\{0\}$ (cf. Theorem 2.6 in [29]), and consequently $\sigma(A) \subset \Sigma(W) \subset (\sigma(A) \cup \{0\})$. Assume $\Sigma(\tilde{W}) = \sigma(A) \cup \{0\}$ (otherwise $K_{3m-2}(C, A)$ is invertible and the theorem holds true for $G_0 = \{0\}$), and let (C_0, T_0), $T_0: G_0 \to G_0$, $C_0: G_0 \to H$, be a right spectral pair for $\tilde{W}$ on a neighbourhood of zero which does not intersect $\sigma(A)$. Put $\tilde{C} = [C\ C_0]$ and $\tilde{A} = A \oplus T_0$. Using the definition of a spectral pair (Section 1), it is not difficult to check that $(\tilde{C}, \tilde{A})$ is a right spectral pair for $\tilde{W}$ on $\mathbf{C}$ (cf. the proof of Theorem 6.3). Hence, the operator (7.2) with A_0 replaced by T_0 is invertible.

Since $\sigma(T_0) = \{0\}$, the operator T_0 is a limit in the uniform topology of nilpotent operators (see [2]; also [26]). So for some nilpotent operator A_0 sufficiently close to T_0 the operator (7.2) is invertible. ▨

Let $W: \Omega \to \mathscr{L}(H)$ be an analytic operator function with compact spectrum. Assume $\bar{\Omega} \neq \mathbf{C}$. Then Theorem 7.3 implies that there exists a monic operator polynomial $L(\lambda)$ such that $L(\lambda)W(\lambda)^{-1}$ and $W(\lambda)L(\lambda)^{-1}$ are both analytic on Ω.

Using Theorem 7.3 one can show that in case of a separable Hilbert space H the condition $\bar{\Omega} \neq \mathbf{C}$ can be omitted in the statements of Theorems 6.2, 6.3 and 5.1.

8. A CONNECTION WITH CHARACTERISTIC OPERATOR FUNCTIONS

Let $T: H \to H$ be a Hilbert space contraction, i.e., $\|T\| \leqslant 1$, and let $\theta_T(\lambda)$ be its Sz.-Nagy—Foiaş characteristic operator function (see the definition given below). In general, T is not a spectral linearization of $\theta_T(\lambda)$ on the open unit disc Δ (because T may have spectrum on the boundary of Δ). Nevertheless, there is a relationship between T and $\theta_T(\lambda)$ which is much like the definition of a spectral linearization (see formula (8.1) below). This formula will be used to compare spectral properties of T and $\theta_T(\lambda)$.

First we recall some definitions. For a Hilbert space contraction $T: H \to H$, define as usual:

$$D_T = (I - T^*T)^{1/2}, \quad D_{T^*} = (I - TT^*)^{1/2}$$

$$\mathscr{D}_T = \overline{\operatorname{Im} D_T}, \quad \mathscr{D}_{T^*} = \overline{\operatorname{Im} D_{T^*}}.$$

By definition the Sz.-Nagy—Foiaş characteristic operator function of T is given by

$$\theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] \mid \mathscr{D}_T : \mathscr{D}_T \to \mathscr{D}_{T^*}$$

(cf. [33]). Note that θ_T is defined and analytic on the set of all λ such that $I - \lambda T^*$ is invertible. In particular, θ_T is analytic on Δ.

THEOREM 8.1. *For a Hilbert space contraction T the following equality holds:*

$$E(\lambda)\begin{bmatrix} I_H & 0 \\ 0 & \theta_T(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda - T & 0 \\ 0 & I_{\mathscr{D}_T} \end{bmatrix} F(\lambda), \quad \lambda \in \Delta,$$

where $E(\lambda)$ and $F(\lambda)$ are invertible operators depending analytically on $\lambda \in \Delta$, given by the following formulas:

$$E(\lambda) = \begin{bmatrix} T - \lambda & D_{T^*} \\ D_T & -T^* \end{bmatrix} : H \oplus \mathscr{D}_{T^*} \to H \oplus \mathscr{D}_T;$$

$$F(\lambda) = \begin{bmatrix} -I_H & (I - \lambda T^*)^{-1} D_T \\ D_T & I - D_T(I - \lambda T^*)^{-1} D_T \end{bmatrix} : H \oplus \mathscr{D}_T \to H \oplus \mathscr{D}_T;$$

$$E(\lambda)^{-1} = \begin{bmatrix} T^*(I - \lambda T^*)^{-1} & (I - \lambda T^*)^{-1} D_T \\ D_{T^*}(I - \lambda T^*)^{-1} & \theta_T(\lambda) \end{bmatrix};$$

$$F(\lambda)^{-1} = \begin{bmatrix} T^*(I - \lambda T^*)^{-1}(\lambda - T) & (I - \lambda T^*)^{-1} D_T \\ D_T & I_{\mathscr{D}_T} \end{bmatrix}.$$

Proof. By direct verification, using the following relations (see [33]):

$$TD_T = D_{T^*}T, \quad T^*D_{T^*} = D_T T^*;$$

$$\theta_T(\lambda) D_T = D_{T^*}(I - \lambda T^*)^{-1}(\lambda - T).$$ □

As a corollary of Theorem 8.1 we compute spectral nodes for contractions when we do not have spectrum on the unit circle. Observe that for such a contraction T the characteristic operator function $\theta_T(\lambda)$ is unitary on the unit circle (this follows from the formula

$$\theta_T(\lambda)^{-1} = [-T^* + D_T(\lambda - T)^{-1} D_{T^*}]|\mathscr{D}_{T^*} : \mathscr{D}_{T^*} \to \mathscr{D}_T, \quad \lambda \in \bar{\Delta} \setminus \sigma(T)).$$

In particular, the spaces $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ are isomorphic.

COROLLARY 8.2. *Assume T is a contraction with spectral radius $r(T) < 1$. Then for any unitary operator $U: \mathscr{D}_{T^*} \to \mathscr{D}_T$ the quintet*

$$(T, D_{T^*}U^*, D_T; H, \mathscr{D}_T)$$

is a spectral node for $U\theta_T$ on the open unit disc Δ.

Proof. Apply Theorem 4.1 in [29] to (8.1), taking into account the formulas for $E(\lambda)$ and $F(\lambda)^{-1}$. □

As another application of formula (8.1) we have the following result (cf. [6]; see also [31]).

COROLLARY 8.3. *Let T be a Hilbert space contraction. Then for every $|\lambda_0| < 1$ and every $n = 1,2, \dots$ the subspaces* $\operatorname{Ker}(\lambda_0 - T)^n$ *and*

$$\operatorname{Ker}\begin{bmatrix} \theta_T(\lambda_0) & 0 & \dots 0 \\ \frac{1}{1!}\theta_T^{(1)}(\lambda_0) & \theta_T(\lambda_0) & \dots 0 \\ \vdots & \vdots & \vdots \\ \frac{1}{(n-1)!}\theta_T^{(n-1)}(\lambda_0) & \frac{1}{(n-2)!}\theta_T^{(n-2)}(\lambda_0) & \dots \theta_T(\lambda_0) \end{bmatrix}$$

are isomorphic.

Proof. Given an analytic $\mathscr{L}(Y)$-valued function W in a neighbourhood of $\lambda_0 \in \mathbf{C}$ (here Y is a Banach space), denote

$$\Phi_n(W;\lambda_0) = \begin{bmatrix} W(\lambda_0) & 0 & \dots 0 \\ \frac{1}{1!}W^{(1)}(\lambda_0) & W(\lambda_0) & \dots 0 \\ \vdots & \vdots & \vdots \\ \frac{1}{(n-1)!}W^{(n-1)}(\lambda_0) & \frac{1}{(n-2)!}W^{(n-2)}(\lambda_0) & \dots W(\lambda_0) \end{bmatrix},$$

$n = 1,2,3, \dots$. So $\Phi_n(W;\lambda_0)$ is an operator acting on Y^n. The following properties of the operators $\Phi_n(W;\lambda_0)$ are easily verified:

(i) $\Phi_n(W;\lambda_0)$ is invertible if and only if $W(\lambda_0)$ is invertible;

(ii) $\Phi_n(W_1W_2;\lambda_0) = \Phi_n(W_1;\lambda_0)\Phi_n(W_2;\lambda_0)$;

(iii) the Banach spaces $\operatorname{Ker}\Phi_n(W;\lambda_0)$ and $\operatorname{Ker}\Phi_n(W \oplus I_Z;\lambda_0)$ are isomorphic for any Banach space Z.

Corollary 8.3 follows immediately from formula (8.1) and properties (i) — (iii). ▨

APPENDIX

In Theorem 2.1 we have seen that for Hilbert space operators $A: G \to G$ and $C: G \to H$ the left invertibility of

$$K_m(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} : \ G \to H^m$$

for some integer $m > 0$ is equivalent to the condition that (C, A) is a right spectral pair for an analytic operator function with compact spectrum on some open set $\Omega \supset \sigma(A)$. It is well-known that in the finite dimensional case (i.e., $\dim G < \infty$, $\dim H < < \infty$) the column $K_m(C, A)$ is left invertible for some integer $m > 0$ if and only if the matrix

$$\begin{bmatrix} \lambda - A \\ C \end{bmatrix} \tag{A.1}$$

has full rank for each λ in $\mathbf{C}$. In Mathematical Systems Theory the latter statement is known as Hautus' test for observability (cf. [27]). Note that the matrix (A.1) has full rank if and only if it is left invertible. With this terminology the equivalence result mentioned above extends to the infinite dimensional case, as follows.

THEOREM A.1. *Let $A : X \to X$ and $C : X \to Y$ be bounded linear operators acting between complex Banach spaces. Then the operator*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} : X \to Y^m \tag{A.2}$$

is left invertible for some integer $m > 0$ if and only if the operator

$$\begin{bmatrix} \lambda - A \\ C \end{bmatrix} : X \to X \oplus Y \tag{A.3}$$

is left invertible for each $\lambda \in \mathbf{C}$.

Proof. Let $Z_0, \ldots, Z_{m-1}$ be bounded linear operators from Y into X, and assume that $[Z_0\ Z_1 \ldots Z_m]$ is a left inverse of the operator (A.2). Put $R(\lambda) = \sum_{j=0}^{m-1} \lambda^j Z_j$, and let $Q(\lambda) = \sum_{j=0}^{m-2} \lambda^j Q_j$, where

$$Q_j = -(Z_{j+1}C + Z_{j+2}CA + \ldots + Z_{m-1}CA^{m-j-2}), \quad j = 0, \ldots, m-2.$$

From $\sum_{j=0}^{m-1} Z_j CA^j = I$, one easily deduces that

$$Q(\lambda) + R(\lambda)C(\lambda^{-1}I + \lambda^{-2}A + \lambda^{-3}A^2 + \ldots) = \\ = \lambda^{-1}I + \lambda^{-2}A + \lambda^{-3}A^2 + \ldots$$

for $|\lambda| > \|A\|$. In other words $Q(\lambda) + R(\lambda)C(\lambda - A)^{-1} = (\lambda - A)^{-1}$ for $|\lambda| > \|A\|$. But then

$$Q(\lambda)(\lambda - A) + R(\lambda)C = I_X \tag{A.4}$$

for $|\lambda| > \|A\|$. Since both sides of (A.4) are operator polynomials, this equality holds for each $\lambda \in \mathbf{C}$. This implies that for each λ in $\mathbf{C}$ the operator $[Q(\lambda)\ R(\lambda)]$ is a left inverse of the operator (A.3).

To prove the converse, assume that for each $\lambda \in \mathbf{C}$ the operator (A.3) has a left inverse. Since the operator (A.3) depends analytically on λ in $\mathbf{C}$, one can apply a result of G.R. Allan (Corollary of Theorem 1 in [1]) to show that there exist entire operator functions $V(\lambda)$ and $U(\lambda)$ such that

$$V(\lambda)(\lambda - A) + U(\lambda)C = I, \quad \lambda \in \mathbf{C}.$$

In particular, we have

$$V(\lambda) + U(\lambda)C(\lambda - A)^{-1} = (\lambda - A)^{-1}, \quad |\lambda| > \|A\|. \tag{A.5}$$

Write $U(\lambda) = \sum_{j=0}^{\infty} \lambda^j U_j$. Let Γ be a positively oriented circle centered at zero with radius $r > \|A\|$. By integrating both the left- and right-hand side of (A.5) over Γ we obtain

$$I = \frac{1}{2\pi i}\int_\Gamma (\lambda - A)^{-1} d\lambda = \frac{1}{2\pi i}\int_\Gamma U(\lambda)C(\lambda - A)^{-1}\, d\lambda =$$

$$= \sum_{j=0}^{\infty} U_j C A^j.$$

It follows that for m sufficiently large the operator $\sum_{j=0}^{m-1} U_j C A^j$ is invertible. But then we may conclude that for some integer $m > 0$ the operator (A.2) is left invertible. ▨

Note that the operator (A.3) is left invertible for all λ not in the spectrum of A. In that case a left inverse of (A.3) is given by $[(A - \lambda)^{-1}\ \ 0]$.

THEOREM A.2. *Let $A: X \to X$ and $B: Y \to X$ be bounded linear operators acting between complex Banach spaces. Then the operator*

$$[B\ \ AB\ \ldots\ A^{m-1}B]: Y^m \to X$$

is right invertible for some $m > 0$ if and only if the operator

$$[\lambda - A\ \ B]: X \oplus Y \to X$$

is right invertible for each λ in $\mathbf{C}$.

Theorem A.2 may be viewed as the dual statement of Theorem A.1. Its proof follows the same line of arguments as in the proof of Theorem A.1 and, therefore, is omitted.

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