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ECONOMICAL COMPACT PERTURBATIONS. I: ERASING NORMAL EIGENVALUES

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1. INTRODUCTION

Several recent works on compact perturbations of Hilbert space operators deal with problems of the following type:

Given an operator T acting on a complex separable infinite dimensional Hilbert space $\mathscr{H}$ and a certain property (P), there exists a compact operator K such that $T - K$ satisfies (P).

In certain cases (e.g., [1], [11], [14]) it is shown that, given $\varepsilon > 0$, K can be chosen so that $\|K\| < \varepsilon$. In other cases (e.g., [1], [13]), the authors obtain a mere existence result with either no control at all or only a very rough estimate on the value of $\|K\|$.

In the first part of this article it will be obtained the "most economical" value of $\|K\|$ for a particular perturbation problem, considered in [13].

We shall need some notation. In what follows $\mathscr{L}(\mathscr{H})$ will denote the algebra of all (bounded linear) operators acting on $\mathscr{H}$ and $\mathscr{K}(\mathscr{H})$ will denote the ideal of all compact operators. Given T in $\mathscr{L}(\mathscr{H})$, $\sigma(T)$ and $\sigma_e(T)$ will denote the *spectrum* and the *essential spectrum* of T (i.e., the spectrum of the canonical image $\tilde{T} = T + \mathscr{K}(\mathscr{H})$ of T in the quotient Calkin algebra $\mathscr{A}(\mathscr{H}) = \mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$, respectively).

An isolated point $\lambda \in \sigma(T)$ is a *normal eigenvalue* of T if the Riesz spectral (dempotent associated to the clopen subset $\{\lambda\}$ of σ(T) via functional calculus is a finite rank operator. (This is equivalent to saying that λ is an isolated point of $\sigma(T)\setminus\sigma_e(T)$.)

In [13, Lemma 6], J. G. Stampfli proved that given T in $\mathscr{L}(\mathscr{H})$ there exists K in $\mathscr{K}(\mathscr{H})$ such that $\sigma_0(T - K) = \emptyset$, where $\sigma_0(R)$ denotes the set of all normal eigenvalues of the operator R; furthermore, K can be chosen so that

$$\|K\| = \max\ \{\text{dist}\,[\lambda, \sigma_e(T)] : \lambda \in \sigma_0(T)\}$$

isee Proposition 2.6 below). But this estimate of the norm of the compact perturbation necessary to "erase" the normal eigenvalues of T is very crude in all cases.

The result of [4, Proposition 6.6] (see also [7, Section 2.4)] suggests that one-half of the above value "plus ε" always suffices and this is, indeed, the case.

In fact, the first result of the article provides an estimate for $\|K\|$ which is the best possible *general formula*. (The results of [7, Chapter II] and [8] indicate that a formula for the optimal value of $\|K\|$ will necessarily involve the particular structure of T; see also Example 2.3 below.)

In the second part of the article it is shown that the distance from a given operator T to the set of $\mathcal{N}(\mathcal{H})$ of all nilpotent operators is exactly equal to the maximum between these two quantities

$$\varkappa(T) = \operatorname{dist}[T, \mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]$$

$$\delta_0(A) = \inf\{\|A\| : A \in \mathcal{L}(\mathcal{H}),\ \sigma_0(T - A) = \emptyset\}.$$

(The precise value of $\varkappa(T)$ was obtained in [6]; see Theorem 3.1 below.)

Indeed, the same arguments can be applied to a very general kind of similar problems.

2. THE NORM OF A COMPACT PERTURBATION ERASING THE NORMAL EIGENVALUES

Given $T \in \mathcal{L}(\mathcal{H})$, we define

$$\mathrm{m}(T) = \min\{\lambda \in \sigma([T^* T]^{1/2})\} \quad (= \text{the } \textit{minimum modulus} \text{ of } T),$$

$$\mathrm{m_e}(T) = \min\{\lambda \in \sigma([\tilde{T}^* \tilde{T}]^{1/2})\} \quad (= \text{the } \textit{essential minimum modulus} \text{ of } T)$$

and

$$\Delta_\gamma(T) = \{\lambda \in \mathbf{C} : \mathrm{m_e}(\lambda - T) \leqslant \gamma\} \quad (\gamma \geqslant 0).$$

The most immediate properties of $\mathrm{m}(T)$ and $\mathrm{m_e}(T)$ have been analyzed in [2], [7, Proposition 6.10]. In particular

$$|m(\lambda - T) - \mathrm{m}(\mu - T)| \leqslant |\lambda - \mu| \quad \text{and} \quad |\mathrm{m_e}(\lambda - T) - \mathrm{m_e}(\mu - T)| \leqslant |\lambda - \mu|$$

(for all $\lambda, \mu \in \mathbf{C}$), the *left essential spectrum* $\sigma_{le}(T) = \{\lambda \in \mathbf{C} : (\lambda - \tilde{T})$ is not left invertible$\}$ coincides with $\Delta_0(T)$; $\Delta_\gamma(T)$ is a compact neighbourhood of $\Delta_{\gamma'}(T)$ for all $\gamma > \gamma' \geqslant 0$, $\bigcup_{\gamma \geqslant 0} \Delta_\gamma(T) = \mathbf{C}$ and each component of $\Delta_\gamma(T) \cup \sigma(T)$ meets $\sigma(T)$ (see [6]).

Given $\lambda \in \mathbf{C}$, define

$$\mathrm{m_e}(T; \lambda) = \min\{\gamma \geqslant 0 : \operatorname{dist}[\lambda, \Delta_\gamma(T)] \leqslant \gamma\}.$$

It is convenient to remark that $m_e(T;\lambda) \leqslant (1/2)\text{dist}[\lambda, \sigma_{1e}(T)] \leqslant (1/2)\text{dist}[\lambda, \partial\sigma_e(T)]$, where $\partial\Omega$ denotes the boundary of $\Omega \subset \mathbf{C}$. We have the following.

THEOREM 2.1. *Given $T \in \mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ such that*

$$\|K\| < \varepsilon + \max\{m_e(T;\lambda) : \lambda \in \sigma_0(T)\}$$

and

$$\sigma_0(T - K) = \emptyset.$$

Let $\rho_{s\text{-}F}(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is a semi-Fredholm operator}\}$ denote the semi-Fredholm domain of T. (The reader is referred to [10] for definition and properties of semi-Fredholm operators.) If $\lambda \in \rho_{s\text{-}F}(T)$, then following [1] we define

$$\min\text{ind}(\lambda - T) = \min\{\text{nul}(\lambda - T),\ \text{nul}(\lambda - T)^*\},$$

where $\text{nul}\,A = \dim\ker A$.

Combining Theorem 2.1 with the main result of [1] (and its proof), we obtain the following.

THEOREM 2.2. *Given $T \in \mathcal{L}(\mathcal{H})$ and $\varepsilon > 0$, there exists $K \in \mathcal{K}(\mathcal{H})$ such that*

$$\|K\| < \varepsilon + \max\{m_e(T;\lambda) : \lambda \in \sigma_0(T)\}$$

and

$$\min\text{ind}(\lambda - T) = 0 \quad \textit{for all } \lambda \in \rho_{s\text{-}F}(T).$$

A simple example will show that Theorem 2.1 is the best possible general result (i.e., without involving the particular structure of the operator). If $A_1 \in \mathcal{L}(\mathcal{H}_1)$ and $A_2 \in \mathcal{L}(\mathcal{H}_2)$, then $A_1 \oplus A_2$ will denote the direct sum of A_1 and A_2 acting in the usual fashion on the *orthogonal* direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$. Similar notation will be used for arbitrary finite or denumerable direct sums. If α is a cardinal number, $0 \leqslant \alpha \leqslant \infty$, then $A^{(\alpha)}$ will denote the direct sum of α copies of A acting on the orthogonal direct sum $\mathcal{H}^{(\alpha)}$ of α copies of $\mathcal{H}$.

EXAMPLE 2.3. Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator; then

$$\delta = m_e(T^{(\infty)};0) \geqslant \frac{1}{2} m_e(T^{(\infty)}) = \frac{1}{2}\|T^{-1}\|^{-1} > 0.$$

If $A \in \mathcal{L}(\mathcal{H}^{(\infty)} \oplus \mathbf{C}^1)$ and $\sigma_0(T^{(\infty)} \oplus 0_1 - A) = \emptyset$ (where 0_1 denotes the zero operator acting on $\mathbf{C}^1$), then $\|A\| \geqslant \delta$. Furthermore, if 0 belongs to the unbounded component of $\mathbf{C} \setminus \sigma(T)$ and A is compact, then $\|A\| > \delta$.

Proof. Assume that $B \in \mathcal{L}(\mathcal{H}^{(\infty)} \oplus \mathbf{C}^1)$ and $\|B\| < \delta$; then $(T^{(\infty)} \oplus 0_1 - B) - \lambda$ is invertible and uniformly bounded below by $\delta - \|B\|$ for all $\lambda \in \mathbf{C}$ such that $|\lambda| = \delta$.

It follows from the functional calculus (see, e.g., [7, Section 1.1] that $\sigma_0(T^{(\infty)} \oplus \oplus 0_1 - B) \cap \{\lambda : |\lambda| < \delta\} \neq \emptyset$. Therefore $\|A\|$ cannot be smaller than δ.

Moreover, proceeding as in the above reference we see that if $\|A\| = \delta$, then $\sigma(T^{(\infty)} \oplus 0_1 - A) \cap \{\lambda : |\lambda| \leqslant \delta\} \neq \emptyset$. Suppose that $\|A\| = \delta$, A is compact and 0 belongs to the unbounded component of $\mathbf{C} \setminus \sigma(T)$. If $\mu \in \sigma(T^{(\infty)} \oplus 0_1 - A) \cap \cap \{\lambda : |\lambda| \leqslant \delta\}$, then $(T^{(\infty)} \oplus 0_1 - A) - \mu$ is a Fredholm operator of index 0. It is completely apparent that μ also belongs to the unbounded component of $\mathbf{C} \setminus \sigma(T) = \rho_F(T^{(\infty)} \oplus 0_1 - A)$. Since $T^{(\infty)} \oplus 0_1 - A - \mu$ is not invertible, we conclude that μ is a normal eigenvalue for this operator, i.e., $\sigma_0(T^{(\infty)} \oplus 0_1 - A) \neq \emptyset$, contradicting our hypothesis. □

REMARK 2.4. The hypothesis "A is compact" is crucial: If $T = I \oplus 0_1$ and $A = (1/2)\, I \oplus (-1/2)_1$, then $\mathrm{m_e}(T; 0) = 1/2 = \|A\|$ and $T - A = (1/2)\, I \oplus (1/2)_1$, so that $\sigma_0(T - A) = \emptyset$.

EXAMPLE 2.5. Let U be a bilateral shift with respect to the ONB $\{e_n\}_{-\infty}^{\infty}$ of $\mathscr{H}$ (i.e., $Ue_n = e_{n+1}$, $n \in \mathbf{Z}$) and define $U_m \in L(H)$ by

$$U_m e_n = \begin{cases} e_{n+1}, & n \neq 0 \\ \dfrac{1}{m} e_1, & n = 0. \end{cases}$$

Then U_m is similar to U, $\sigma(U_m) = \sigma(U) = \sigma_e(U_m) = \sigma_e(U) = \partial D$ (where D denotes the open unit disk) and

$$\mathrm{m_e}(\lambda - U_m) = \mathrm{m_e}(\lambda - U) = 1 - |\lambda|, \quad \lambda \in D.$$

If $F \in \mathscr{L}(\mathbf{C}^d)$ is an arbitrary operator such that $\sigma(F) \subset D$, then there exist compact operators $K_m \in \mathscr{K}(\mathscr{H} \oplus \mathbf{C}^d)$, $\|K_m\| < 2/m$, such that

$$\sigma(U_m \oplus F - K_m) = \sigma_e(U_m \oplus F - K) = \partial D.$$

Indeed, if $K'_m \in \mathscr{K}(\mathscr{H})$ is the rank one operator defined by

$$K'_m e_n = \begin{cases} 0, & n \neq 0 \\ \dfrac{1}{m} e_1, & n = 0, \end{cases}$$

then $U_m - K'_m \simeq S \oplus S^*$, where S denotes a unilateral shift (and S^* is the adjoint of S). Thus, $\sigma(U_m - K'_m) = D^-$, $\sigma_e(U_m - K'_m) = \partial D$ and $U_m - K'_m - \lambda$ is a Fredholm operator of index 0 and nontrivial kernel for all $\lambda \in D$.

According to [1] (see also Theorem 2.2), there exists $K_m'' \in \mathcal{L}(\mathcal{H} \oplus \mathbf{C}^d)$, $\|K_m''\| < 1/m$, such that

$$\sigma((U_m - K_m') \oplus F - K_m'') = \sigma_e((U_m - K_m') \oplus F - K_m'') = \partial D.$$

Thus, $K_m = K_m' \oplus 0_d + K_m''$ satisfies our requirements. ▨

This simple example shows that, in general, Theorem 2.1 does not provide the best possible answer.

The following result has been taken from J. G. Stampfli's article [13]. Its proof will be included here because it will help to understand the proof of Theorem 2.1.

PROPOSITION 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$; then there exists a compact operator K such that $\sigma_0(T - K) = \emptyset$. Furthermore, K can be chosen so that*

$$\|K\| = \max\{\operatorname{dist}[\lambda, \sigma_e(T)] : \lambda \in \sigma_0(T)\}.$$

Proof. Let $\{\lambda_n\}_{1 \leq n < m}$ $(1 \leq m \leq \infty)$ be an enumeration of the points of $\sigma_0(T)$ and let $\mathcal{H}_n$ be the Riesz spectral subspace corresponding to the clopen subset $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of $\sigma(T)$. Define $\mathcal{H}_0 = \{0\}$ and $\mathcal{M}_n = \mathcal{H}_n \ominus \mathcal{H}_{n-1}$, $n = 1, 2, \ldots,$ ther

$$T = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & * & \\ & & \ddots & & & & \\ & & & \lambda_n & & & \\ & & & & \ddots & & \\ 0 & & & & & \ddots & \\ & & & & & & T_\infty \end{pmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \\ \mathcal{M}_n \\ \vdots \\ \mathcal{H}_\infty \end{matrix}$$

where $\mathcal{H}_\infty = \mathcal{H} \ominus (\bigoplus_{n=1}^{\infty} \mathcal{M}_n)$.

For each n, let $\mu_n \in \partial\sigma_e(T)$ be any point such that $|\lambda_n - \mu_n| = \operatorname{dist}[\lambda_n, \sigma_e(T)]$ and define

$$K = \begin{pmatrix} \lambda_1 - \mu_1 & & & & & & \\ & \lambda_2 - \mu_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & \lambda_n - \mu_n & & & \\ & & & & \ddots & & \\ 0 & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} \begin{matrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \\ \mathcal{M}_n \\ \vdots \\ \mathcal{H}_\infty. \end{matrix}$$

Since $|\lambda_n - \mu_n| = \text{dist}[\lambda_n, \sigma_e(T)] \to 0$ $(n \to \infty)$ and $\mathcal{M}_n$ is finite dimensional for all $n = 1, 2, \ldots$, it readily follows that $K \in \mathcal{K}(\mathcal{H})$. On the other hand, it is straightforward to check that $\sigma_0(T-K) = \emptyset$ and

$$\|K\| = \sup |\lambda_n - \mu_n| = \max \{\text{dist}[\lambda, \sigma_e(T)] : \lambda \in \sigma_0(T)\}]. \qquad \square$$

LEMMA 2.7. *Let γ be a Jordan arc joining* 0 *and μ in* **C** *and let $\varepsilon > 0$. Then there exists $d = d(\varepsilon, \gamma)$, a normal operator N and a nilpoint operator Q in $\mathcal{L}(\mathbf{C}^d)$ such that*

(i) $\mu \in \sigma(N) \subset \gamma$;

and

(ii) $\|N - Q\| < \varepsilon$.

Proof. By [5, Proposition 2.28] (see also [7, Section 2]) for each $d \geqslant 1$ there exist a normal operator M_d and a nilpotent Q_d in $\mathcal{L}(\mathbf{C}^d)$ such that $1 \in \sigma(M_d)$, $\|M_d\| = 1$ and $\|M_d - Q_d\| \to 0$ $(d \to \infty)$.

Let $\varphi : D^- \to \Omega^-$ be a conformal mapping from (some neighborhood of) D^-) onto a compact neighborhood Ω^- of $\gamma\setminus\{\mu\}$ such that $\varphi(D) = \Omega$ is open and simply connected, $\Omega \subset (\gamma)_{\varepsilon/2}$, $\gamma\setminus Q = \{\mu\}$, $\varphi(0) = 0$ and $\varphi(1) = \mu$ (where $X_\varepsilon \overset{(\text{def})}{=} \{\lambda \in \mathbf{C} : \text{dist}\ [\lambda, X] \leqslant \varepsilon\}$, $X \subset \mathbf{C}$). Then $\varphi(M_d)$ is normal, $\varphi(Q_d)$ is nilpotent, $\mu \in \sigma[\varphi(M_d)] \subset \Omega^- \subset (\gamma)_{\varepsilon/2}$ and (since φ is analytic in some neighborhood of D^{-1}) it is not difficult to check that $\|\varphi(M_d) - \varphi(Q_d)\| \to 0$ $(d \to \infty)$.

Define $d = d(\varepsilon, \gamma)$ as the first dimension such that $\|\varphi(M_d) - \varphi(Q_d)\| < \varepsilon/2$. Now define $Q = \varphi(Q_d)$. Since $\mu \in \sigma[\varphi(M_d)] \subset (\gamma)_{\varepsilon/2}$, there exists a normal operator N such that $\mu \in \sigma(N) \subset \gamma$ and $\|N - \varphi(M_d)\| < \varepsilon/2$. (Roughly speaking : write $\varphi(M_d)$ as a diagonal matrix with eigenvalues $\mu_1, \mu_2, \ldots, \mu_d$ with respect to a suitable ONB of $\mathbf{C}^d$ and define N by "pushing" $\mu_2, \mu_3, \ldots, \mu_d$ to suitable points of γ.)

Since $\|N - Q\| < \varepsilon$, we are done. $\square$

If $T - \lambda = V_\lambda H_\lambda$ (polar decomposition), then $\text{m}(\lambda - T) = \text{m}(H_\lambda) \in \sigma(H_\lambda)$ and therefore $\text{m}(\lambda - T)$ is an approximate eigenvalue of H_λ. Assume that $\text{m}(\lambda - T)$ is actually an eigenvalue of H_λ and let e be a unit vector such that $[H_\lambda - \text{m}(\lambda - T)]e = 0$. As usual, if $f, g \in \mathcal{H}$ then $f \otimes g \in \mathcal{L}(\mathcal{H})$ is the rank one operator defined by $f \otimes g(x) = \langle x, g \rangle f$. With this notation in mind, it is not difficult to check that

$$e \in \ker(T - \lambda - \text{m}(\lambda - T)V_\lambda e \otimes e).$$

LEMMA 2.8. *Let $T, \lambda, V_\lambda, H_\lambda$ and e be as above. If $\mu \neq \lambda$ and $(T - \text{m}(\lambda - T)V_\lambda e \otimes e - \mu)\ f = 0$ for some unit vector f, then $\text{m}(\mu - T) < \text{m}(\lambda - T)$.*

Proof. Since e and f are unit eigenvectors of $T - \text{m}(\lambda - T)V_\lambda e \otimes e$ with different eigenvalues, they must be linearly independent and therefore $|\langle e, f\rangle| < 1$.

Since $(T - \text{m}(\lambda - T)V_\lambda e \otimes e - \mu)f = 0$, it follows that

$$(T - \mu)f = \text{m}(\lambda - T)V_\lambda e \otimes e(f) = \text{m}(\lambda - T)\langle f, e\rangle V_\lambda e$$

and therefore

$$m(\mu - T) \leqslant \|(T - \mu)f\| = m(\lambda - T)|\langle f, e\rangle| < m(\lambda - T). \qquad ▨$$

The strict inequality cannot be improved in any sense. Namely, if U is a bilateral shift ($Ue_n = e_{n+1}$, $n \in \mathbf{Z}$, with respect to the ONB $\{e_n\}_{-\infty}^{+\infty}$) and $\lambda = 0$, then $U = U \cdot I$ (polar decomposition), $m(U) = 1$ and $e_0 \in \ker(U - Ue_0 \otimes e_0)$. But $U - Ue_0 \otimes e_0 \simeq S \oplus S^*$, where S is a unilateral shift with adjoint S^*, and therefore each point μ of the open disk D is an eigenvalue of $U - Ue_0 \otimes Ue_0$ and $m(\mu - U) = 1 - |\mu|$ ($\mu \in D$).

Hence, we cannot expect that $m(\lambda - T) - m(\mu - T)$ will be bounded below by some positive constant depending on T and λ!

Even the two dimensional case produces some surprises:

EXAMPLE 2.9. Let $E_r = \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \in \mathscr{L}(\mathbf{C}^2)$ $(r > 0)$; then $\varepsilon_r = m(E_r - 1) \downarrow 0$ as $r \uparrow \infty$, so that if $E_r - 1 = V_r H_r$, then H_r has two eigenvalues: ε_r, which is small for r large, and $r = \|E_r\|$.

Let e_r be a unit vector in the kernel of $H_r - \varepsilon_r$; then $1 \in \sigma(E_r - \varepsilon_r V_r e_r \otimes e_r)$ and

$$|\mathrm{trace}\,(E_r - \varepsilon_r V_r e_r \otimes e_r)| = \varepsilon_r |\langle V_r e_r, e_r\rangle| < \varepsilon_r,$$

so that the second eigenvalue of $E_r - \varepsilon_r V_r e_r \otimes e_r$ is a number $\gamma_r \in D$ such that $|1 + \gamma_r| < \varepsilon_r$. Hence γ_r is very close to (-1)!

It is not hard to check that $m(\lambda - E_r)$ only depends on $|\lambda|$ and $m(\gamma_r - E_r)$ is very close to ε_r. (In the sense that $\varepsilon_r - m(\gamma_r - E_r) = o(\varepsilon_r)$.)

The following lemma is the key result of the first part of the article. Example 2.3 shows that this is the best possible result along these lines.

LEMMA 2.10. *Let $A \in \mathscr{L}(\mathscr{H})$ be an operator such that $A \simeq A^{(\infty)}$ and*

$$\ker([(A - \lambda_n)^* (A - \lambda_n)]^{1/2} - m(\lambda_n - A)) \neq \{0\}$$

for a dense subset $\{\lambda_n\}_{n=1}^{\infty}$ of points of the complex plane such that $\{\lambda_n\} \cap \sigma(A)$ is dense in $\sigma(A)$.

Assume that $\lambda_0 \notin \sigma(A)$ and let $T = \lambda_0 \oplus A \in \mathscr{L}(\mathbf{C}^1 \oplus \mathscr{H})$. Given $\varepsilon > 0$ there exists $C \in \mathscr{K}(\mathbf{C}^1 \oplus \mathscr{H})$ such that

$$\|C\| < \varepsilon + m_e(T; \lambda_0) \quad \textit{and} \quad \sigma(T - C) = \sigma_e(T) = \sigma(A).$$

Proof. Let $\gamma_0 = m_e(T; \lambda_0)$ and let $\mu_0' \in \partial\Delta_{\gamma_0}(T)$ be any point such that $|\lambda_0 - \mu_0'| = \gamma_0$.

Define $\gamma_0 > \gamma_1 > \gamma_2 > \dots > \gamma_{m-1} > \gamma_m = 0$ so that

$$d_H[\partial\Delta_{\gamma_j}(T), \partial\Delta_{\gamma_{j-1}}(T)] = \varepsilon/6, \quad j = 1, 2, \dots, m-1,$$

and

$$d_H[\partial\Delta_{\gamma_{m-1}}(T), \partial\sigma_{le}(T)] \leqslant \varepsilon/6,$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff distance. (Since $\Delta_{\gamma_0}(T)$ is compact, we cannot have infinitely many pairwise disjoint open disks of diameter $\varepsilon/6$ in $\Delta_{\gamma_0}(T)$. This guarantees the finiteness of the decreasing sequence $\gamma_0 > \gamma_1 > \gamma_2 > \dots > \gamma_m$.)

First Perturbation. Now we can find (inductively) $\mu_j' \in \partial\Delta_j(T)$ ($j = 0, 1, 2, \dots, m-1$) and $\mu_m' \in \partial\sigma_e(T)$ such that $|\mu_j' - \mu_{j-1}'| \leqslant \varepsilon/6$, $j = 1, 2, \dots, m$. Let μ_j be any point of $\{\lambda_n\} \cap [\Delta_{\gamma_j}(T) \setminus \Delta_{\gamma_{j-1}}(T)]$ such that $|\mu_j - \mu_j'| < \varepsilon/6$, $j = 0, 1, 2, \dots m$. Let γ be the polygonal line obtained as the union of the segments $[\mu_j, \mu_{j+1}]$, $j = 0, 1, 2, \dots m-1$. By Lemma 2.7 there exist a normal operator M_0 and a nilpotent operator Q_0 acting on a finite dimensional space such that $\mu_0 \in \sigma(M_0) \subset \gamma$ and $\|M_0 - (\mu_m + Q_0)\| < \varepsilon/6$. Clearly, we can find a normal operator N_0 acting on the same space as M_0 such that $\mu_0 \in \sigma(N_0) \subset \{\mu_j\}_{j=0}^m$, $\mathrm{nul}(\mu_0 - N_0) = 1$ and $\|N_0 - M_0\| < \varepsilon/2$. A fortiori, $\|N_0 - (\mu_m + Q_0)\| < 2\varepsilon/3$.

Let

$$N_0 = \begin{pmatrix} \mu_0 & & & & 0 \\ & \mu_1 & & & \\ & & \mu_2 & & \\ & 0 & & \ddots & \\ & & & & \mu_m \end{pmatrix} \begin{matrix} \mathcal{M}_0 \\ \mathcal{M}_1 \\ \mathcal{M}_2 \\ \vdots \\ \mathcal{M}_m, \end{matrix}$$

where $\dim \mathcal{M}_0 = 1$ and $\dim \mathcal{M}_j = r_j < \infty$ ($\mathcal{M}_j = \ker(\mu_j - N_0)$), $j = 1, 2, \dots m$.

Let $A - \mu_j = V_j H_j$ (polar decomposition) and let e_j be a unit vector in the kernel of $H_j - \mathrm{m}(\mu_j - A)$, $j = 1, 2, \dots m$; then

$$A - \mathrm{m}(\mu_j - A)\, V_j e_j \otimes e_j = \begin{pmatrix} \mu_j & B_j \\ 0 & A_j \end{pmatrix} \begin{matrix} \vee\{e_j\} \\ \mathscr{H} \ominus \vee\{e_j\} \end{matrix},$$

$$T \simeq \lambda_0 \oplus A^{(\infty)} \simeq \lambda_0 \oplus \left\{ \bigoplus_{j=1}^m A^{(r_j)} \right\} \oplus A^{(\infty)}$$

and

$$\lambda_0 \oplus \left\{ \bigoplus_{j=1}^m A^{(r_j)} \right\} \oplus A^{(\infty)} - (\lambda_0 - \mu_0) \oplus \left\{ \bigoplus_{j=1}^m [\mathrm{m}(\mu_j - A) V_j e_j \otimes e_j]^{(r_j)} \right\} \oplus 0^{(\infty)} =$$

$$= \mu_0 \oplus \left\{ \bigoplus_{j=1}^m \begin{pmatrix} \mu_j & B_j \\ 0 & A_j \end{pmatrix}^{(r_j)} \right\} \oplus A^{(\infty)} =$$

$$
= \left(\begin{array}{c|cccc|cccc}
\mu_0 & & & & & & & & \\
\hline
 & \mu_1^{(r_1)} & & & & B_1^{(r_1)} & & & \\
 & & \mu_2^{(r_2)} & & & & B_2^{(r_2)} & & \\
 & & & \ddots & & & & \ddots & \\
 & & & & \mu_m^{(r_m)} & & & & B_m^{(r_m)} \\
\hline
 & & & & & A_1^{(r_1)} & & & \\
 & & & & & & A_2^{(r_2)} & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & A_m^{(r_m)}
\end{array}\right) \oplus A^{(\infty)} \simeq
$$

$$
\simeq \begin{pmatrix} N_0 & B_0 \\ 0 & A_0 \end{pmatrix} \oplus A^{(\infty)},
$$

where $B_0 \simeq \begin{pmatrix} 0 \\ \bigoplus_{j=1}^{m} B_j^{(r_j)} \end{pmatrix}$ and $A_0 = \bigoplus_{j=1}^{m} A_j^{(r_j)}$.

It is easily seen that

$$
K_0 = (\lambda_0 - \mu_0) \oplus \left\{ \bigoplus_{j=1}^{m} [\mathrm{m}(\mu_j - A) V_j e_j \otimes e_j]^{(r_j)} \right\} \oplus 0
$$

is a finite rank operator such that $\|K_0\| = |\lambda_0 - \mu_0| = \gamma_0$.

SECOND PERTURBATION. It is clear that $\sigma_0(A_j) \subset \sigma_0(A - \mathrm{m}(\mu_j - A) V_j e_j \otimes \otimes e_j) \setminus \{\mu_j\}$; therefore (by Lemma 2.8) $\sigma_0(A_j) \subset \Delta_{\gamma_1}(T)$. A fortiori, $\sigma_0(A_0) \subset \Delta_{\gamma_1}(T)$. Since $\sigma_e(A_0) \subset \sigma(A)$, it follows that $\sigma_0(A_0) \setminus \Delta_{\gamma_2}(T)$ is a *finite* set.

Assume that the Riesz spectral subspace of A_0 corresponding to the clopen subset $\sigma_0(A_0) \setminus \Delta_{\gamma_2}(T)$ of $\sigma(A_0)$ has (necessarily finite) dimension d_1 and let λ_1 be any point of $\sigma_0(A_0) \setminus \Delta_{\gamma_2}(T)$; then

$$
A_0 = \begin{pmatrix} \lambda_1 & * \\ 0 & A_1' \end{pmatrix},
$$

where λ_1 acts on a one-dimensional subspace, $\sigma_0(A_0) \setminus \{\lambda_1\} \subset \sigma_0(A_1') \subset \sigma_0(A_0)$ and the Riesz spectral subspace of A_1' corresponding to $\sigma_0(A_1') \setminus \Delta_{\gamma_2}(T)$ has dimension $d_1 - 1$.

Thus

$$\lambda_0 \oplus A^{(\infty)} - K_0 \simeq \begin{pmatrix} N_0 & B_0 \\ 0 & A_0 \end{pmatrix} \oplus A^{(\infty)} \simeq$$

(1)

$$\simeq \begin{pmatrix} N_0 & B_0' & 0 & B_0'' \\ 0 & \begin{pmatrix} \lambda_1 & 0 \\ 0 & A^{(\infty)} \end{pmatrix} & & * \\ 0 & & & 0 \\ 0 & 0 & 0 & A_1' \end{pmatrix} \oplus A^{(\infty)},$$

where $(B_0' \ B_0'') = B_0$.

Now we can proceed exactly as for the First Perturbation and find a point $\nu_m \in \partial\sigma_{1c}(T)$ and a normal operator N_1 and a nilpotent operator Q_1 acting on the same finite dimensional space such that

$$N_1 = \begin{pmatrix} \lambda_1 & & & & & \\ & \nu_2 & & & & \\ & & \nu_3 & & 0 & \\ & & & \cdot & & \\ & & & & \cdot & \\ & 0 & & & \cdot & \\ & & & & & \nu_m \end{pmatrix} \begin{matrix} \mathcal{N}_1 \\ \mathcal{N}_2 \\ \mathcal{N}_3 \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{N}_m, \end{matrix}$$

$\dim \mathcal{N}_1 = 1$ and $\dim \mathcal{N}_j = s_j < \infty$ $(j = 2, 3, \ldots, m)$, $\nu_j \in \Delta_{\gamma_j}(T)$ $(j = 2, 3, \ldots \ldots, m)$ and $\|N_1 - (\nu_m + Q_1)\| < 2\varepsilon/3$.

As in the previous step, we can find a finite rank operator

$$K_1 \simeq 0 \oplus 0 \oplus \left(\left\{ \bigoplus_{j=2}^{m} [\mathrm{m}(\nu_j - A) W_j f_j \otimes f_j]^{(s_j)} \right\} \oplus 0^{(\infty)} \right) \oplus 0 \oplus 0^{(\infty)}$$

(with respect to the decomposition of (1)), where $A - \nu_j = W_j H_j$ (polar decomposition) and f_j is a unit vector in the kernel of $H_j - \mathrm{m}(\nu_j - A)$, such that

$$\lambda_0 \oplus A^{(\infty)} - (K_0 + K_1) \simeq \begin{pmatrix} N_0 & * & * \\ 0 & N_1 & * \\ 0 & 0 & A_1 \end{pmatrix} \oplus A^{(\infty)},$$

where $\sigma_e(A_1) \subset \sigma(A)$, $\sigma_0(A_1') \subset \sigma_0(A_1) \subset \Delta_{\gamma_1}(T)$ and $\sigma_0(A_1) \setminus \Delta_{\gamma_2}(T) = \sigma_0(A_1') \setminus \Delta_{\gamma_2}(T)$.

It is completely apparent that the action of K_1 only modifies a certain number of copies of A corresponding to a subspace contained in $\ker K_0 \cap \ker K_0^*$ and $K_0|(\ker K_1 \vee \ker K_1^*) = 0$. Hence $K_0 + K_1$ is a finite rank operator such that $\|K_1 + K_0\| = \max\{\|K_1\|, \|K_0\|\} = \gamma_0$.

If there is any point λ_2 in $\sigma_0(A_1)\setminus\Delta_{\gamma_2}(T)$, we repeat the argument with λ_2. Otherwise, we consider a point $\lambda_2 \in [\sigma_0(A_1) \cap \Delta_{\gamma_2}(T)]\setminus\Delta_{\gamma_3}(T)$, etc. After finitely many steps we shall obtain a finite rank operator $C_p = K_0 + K_1 + K_2 + \ldots + K_p$, such that $\|C_p\| = \gamma_0$ and

$$\lambda_0 \oplus A^{(\infty)} - C_p \simeq \begin{pmatrix} N & * \\ 0 & B' \end{pmatrix} \oplus A^{(\infty)},$$

where

$$N = \begin{pmatrix} N_0 & N_{01} & N_{02} & \cdots & N_{0p} \\ & N_1 & N_{12} & \cdots & N_{1p} \\ & & N_2 & \cdots & N_{2p} \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & & N_p \end{pmatrix}$$

acts on a finite dimensional space, $\sigma_e(B') \subset \sigma(A)$ and $\sigma_0(B') \subset \Delta_{\gamma_{m-1}}(T)$.

THIRD PERTURBATION. Furthermore, the above construction also yields an operator

$$F = (\mu_m + Q_0) \oplus (\nu_m + Q_1) \oplus \cdots \oplus (\tau_m + Q_p)$$

($Q_0, Q_1, \ldots, Q_p$ are nilpotent; Q_j acts on the space of N_j, $j = 0, 1, 2, \ldots, p$) such that $\sigma(F) \subset \partial\sigma(A)$ and $\left\| \bigoplus_{j=0}^{p} N_j - F \right\| < 2\varepsilon/3$.

Hence, there exists a finite rank operator $C_p' \simeq \left(\bigoplus_{j=0}^{p} N_j - F \right) \oplus 0 \oplus 0^{(\infty)}$ such that

(2) $$\lambda \oplus A^{(\infty)} - (C_p + C_p') \simeq \begin{pmatrix} G & * \\ 0 & B' \end{pmatrix} \oplus A^{(\infty)} \simeq \begin{pmatrix} G & * \\ 0 & B' \oplus A^{(\infty)} \end{pmatrix} \oplus A^{(\infty)}$$

where

$$G = \begin{pmatrix} \mu_m + Q_0 & N_{01} & N_{02} & \cdots & N_{0p} \\ & \nu_m + Q_1 & N_{12} & \cdots & N_{1p} \\ & & \pi_m + Q_2 & \cdots & N_{2p} \\ & & & \ddots & \vdots \\ & 0 & & & \vdots \\ & & & & \tau_m + Q_p \end{pmatrix}.$$

It is completely apparent that $\sigma(G) = \sigma(F) = \{\mu_m, \nu_m, \pi_m, \ldots, \tau_m\} \subset \partial\sigma(A)$ and $\|C_p'\| < 2\varepsilon/3$.

FOURTH PERTURBATION. Finally, since $\sigma_e(B' \oplus A^{(\infty)}) = \sigma(A)$ and $\sigma_0(B' \oplus \oplus A^{(\infty)}) \subset \Delta_{\gamma_{m-1}}(T) \subset \sigma(A)_{\varepsilon/6}$, it follows from Stampfli's construction (Proposition 2.6) and the main result of [1] that we can find a compact operator C_p'', $\|C_p''\| < \varepsilon/3$, such that

$$\lambda_0 \oplus A^{(\infty)} - (C_p + C_p' + C_p'') \simeq \begin{pmatrix} G & * \\ 0 & B \end{pmatrix} \oplus A^{(\infty)},$$

(where B is a small compact perturbation of $B' \oplus A^{(\infty)}$) satisfies $\sigma(B) = \sigma_e(B) = \sigma(A)$.

By hypothesis there exists a unitary mapping $U : \mathbf{C}^1 \oplus \mathscr{H} \to \mathbf{C}^1 \oplus \mathscr{H}^{(\infty)}$ such that $T = \lambda_0 \oplus A = U^*(\lambda_0 \oplus A^{(\infty)})\, U$. Define $C = U^*(C_p + C_p' + C_p'')U$; then $C \in \mathscr{K}(\mathbf{C}^1 \oplus \mathscr{H})$ is a compact operator, $\sigma(T - C) = \sigma(A) = \sigma_e(T)$ and

$$\|C\| \leqslant \|C_p\| + \|C_p'\| + \|C_p''\| < \gamma_0 + 2\varepsilon/3 + \varepsilon/3 = \varepsilon + \mathrm{m_e}(T\,;\lambda_0).$$

The proof of Lemma 2.10 is now complete. □

Now we are in a position to prove Theorem 2.1. Let $T \in \mathscr{L}(\mathscr{H})$ and let $\rho : C^*(\tilde{T}) \to \mathscr{L}(\mathscr{H}_\rho)$ be a faithful unital *-representation of the C^*-algebra generated by $\tilde{T}$ and $\tilde{1}$ in a separable infinite dimensional Hilbert space $\mathscr{H}_\rho$. By using the observations of [3] (see also [7, Section 4.4]), ρ can be chosen so that if $A' = \rho(\tilde{T})$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a dense subset of $\mathbf{C}$ such that $\{\lambda_n\} \cap \partial\sigma(A')$ is dense in $\partial\sigma(A') = \sigma_e(T)$, then $\ker([(A' - \lambda_n)^*(A' - \lambda_n)]^{1/2} - \mathrm{m}(\lambda_n - A')) \neq \{0\}$ for all $n = 1, 2, \ldots$. (It is easily seen that $\mathrm{m}(\lambda - A') = \mathrm{m_e}(\lambda - T)$, $\lambda \in \mathbf{C}$.)

Given $\varepsilon > 0$, it follows from Voiculescu's theorem [14] that there exists $K' \in K(\mathscr{H})$, $\|K'\| < \varepsilon/3$, such that $T - K' \simeq T \oplus A$, where $A \simeq A'^{(\infty)}$.

Let $\{\mu_1, \mu_2, \ldots, \mu_m\}$ be an enumeration of the (finitely many) eigenvalues of T contained in $\sigma_0(T) \setminus \sigma_e(T)_{\varepsilon/3}$ (each eigenvalue counted with its algebraic multiplicity); then

$$T = \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & * & \\ & & \ddots & & \\ 0 & & & \mu_m & \\ & & & & T_0 \end{pmatrix} \begin{matrix} \mathscr{M}_1 \\ \mathscr{M}_2 \\ \vdots \\ \mathscr{M}_m \\ \mathscr{M}_0\,, \end{matrix}$$

where $\dim \mathscr{M}_j = 1, j = 1, 2, \ldots, m$, and $\sigma_0(T_0) = \sigma_0(T) \cap \sigma_e(T)_{\varepsilon/3}$. By Proposition 2.6, we can find $K_m'' \in \mathscr{K}(\mathscr{M}_0)$, $\|K_m''\| \leqslant \varepsilon/3$, such that $\sigma_0(T_0 - K_m'') = \emptyset$.

Hence, there exists $K'' \in \mathscr{K}(\mathscr{H})$, $K'' \simeq K_m'' \oplus 0$, such that

$$T-(K'+K'')\simeq \begin{pmatrix} \mu_1 & & & & \\ & \mu_2 & & * & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \mu_m & \\ & & & & T_0 \end{pmatrix} \oplus A^{(\infty)} \simeq$$

$$\simeq \begin{pmatrix} \mu_1 \oplus A^{(\infty)} & & & & \\ & \mu_2 \oplus A^{(\infty)} & & * & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \mu_m \oplus A^{(\infty)} & \\ & & & & T_0 \end{pmatrix} \oplus A^{(\infty)}.$$

Applying Lemma 2.10 to $\mu_j \oplus A^{(\infty)}$, $j=1,2,\ldots,m$, we can construct a compact operator $K''' \simeq K_1''' \oplus K_2''' \oplus \ldots \oplus K_m''' \oplus 0 \oplus 0^{(\infty)}$ such that $\|K'''\| < \varepsilon/3 + \max\{\mathrm{m_e}(T;\mu_j): j=1,2,\ldots,m\}$ and $\sigma_0(T-K)=\emptyset$, where $K=K'+K''+K''' \in \mathscr{K}(\mathscr{H})$. It is apparent that

$$\|K\| \leqslant \|K'\| + \|K''\| + \|K'''\| < \varepsilon + \max\{\mathrm{m_e}(T;\lambda): \lambda \in \sigma_0(T)\}. \qquad ▨$$

Clearly, the same arguments can be used to "erase" the normal eigenvalues of T lying in a certain region, without modifying the remaining ones. For instance, in certain problems related with compact perturbations of operators in nest algebras it is necessary to remove the eigenvalues of T not contained in $\sigma_e(T)^\wedge=$ the polynomial hull of $\sigma_e(T)$ (i.e., the complement of unbounded component of $\mathbf{C}\setminus\sigma(_eT)$) [9]. Exactly the same argument as in Theorem 2.1 yields the following.

COROLLARY 2.11. *Given* $T \in \mathscr{L}(\mathscr{H})$ *and* $\varepsilon > 0$, *there exists* $K^\wedge \in \mathscr{K}(\mathscr{H})$ *such that*

$$\|K^\wedge\| < \varepsilon + \max\{\mathrm{m_e}(T;\lambda): \lambda \in \sigma_0(T)\setminus\sigma_e(T)^\wedge\}$$

and $\sigma_0(T-K)=\sigma_0(T)\cap\sigma_e(T)^\wedge$.

3. THE DISTANCE TO THE SET OF NILPOTENT OPERATORS

In [6] the author announced a general argument to compute the distance from a given operator to a subset $\mathscr{R}$ of $\mathscr{L}(\mathscr{H})$ invariant under similarities, provided $\mathscr{R}$ is also *invariant under compact perturbations* and satisfies certain "very general" conditions. The argument sketched in [6] can be modified to obtain an "acceptable" formula for the distance from a given A in $\mathcal{L}(\mathcal{H})$ to a similarity-invariant subset $\mathcal{R}$ satisfying those "general conditions", but *not invariant under compact perturbations.* The results will be developed for the case when $\mathcal{R} = \mathcal{N}(\mathcal{H})$ is the set of all nilpotent operators, but they can be easily translated, e.g., to the case when $\mathcal{R}$ is the set of all operators with spectrum equal to a fixed compact set or with spectra contained in a fixed set, etc.

Recall that $B \in$ (BQT) (= biquasitriangular operators) if and only if $\operatorname{ind}(\lambda - B) = 0$ for all $\lambda \in \rho_{s\text{-}F}(B)$. (This is not the original definition, but the one that will be used here. The reader is referred to [7, Chapter VI] for the original definition and properties of these operators. In particular, $[\mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]^- = \mathcal{N}(\mathcal{H})^- + \mathcal{K}(\mathcal{H}) \subset$ (BQT).)

THEOREM 3.1. *Let* $A \in \mathcal{L}(\mathcal{H})$; *then*

(i) dist $[A, \mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})] = \varkappa(A)$, *where* $\varkappa(A)$ *is the maximum among the four quantities*

$$\alpha = \max\{\mathrm{m}_e(\lambda - A) : \lambda \in \rho_{s\text{-}F}(A) \textit{ and } \operatorname{ind}(\lambda - A) < 0, \textit{ or } \lambda \in \partial\sigma_e(A)\},$$

$$\alpha^* = \max\{\mathrm{m}_e((\lambda - A)^*) : \lambda \in \rho_{s\text{-}F}(A) \textit{ and } \operatorname{ind}(\lambda - A) > 0, \textit{ or } \lambda \in \partial\sigma_e(A)\},$$

$$\beta = \min\{\gamma \geqslant 0 : \Delta_\gamma(A) \textit{ is connected}\}$$

and

$$\delta = \mathrm{m}_e(A).$$

(ii) *Furthermore, given* $\varepsilon > 0$ *there exists* $T \in [\mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]^-$ *such that* $\|A - T\| = \varkappa(A)$, $A - T \in$ (BQT), $\sigma_e(T) = \sigma_e(A) \cup \Delta_{\varkappa(A)}(A)$ *and* $\mathrm{d_H}[\sigma(T), \sigma(A) \cup \cup \sigma_e(T)] < \varepsilon$, *and* $\mathrm{m}_e(\lambda - T) = \max\{\mathrm{m}_e(\lambda - A) - \varkappa(A), 0\}$ *for all* $\lambda \in \mathbf{C}$.

(iii) dist $[A, \mathcal{N}(\mathcal{H})] = \max\{\varkappa(A), \delta_0(A)\}$, *where* $\delta_0(A) = \inf\{\|M\| : M \in \mathcal{L}(\mathcal{H}), \sigma_0(A - M) = \emptyset\}$.

(iv)

$$\max\{\varkappa(A), \beta_0(A)\} \leqslant \operatorname{dist}[A, \mathcal{N}(\mathcal{H})] \leqslant \max\{\varkappa(A), \max[\mathrm{m}_e(A\,;\lambda) : \lambda \in \sigma_0(A)]\},$$

where

$$\beta_0(A) = \min\{\gamma \geqslant 0 : \textit{the set } \{\lambda \in \mathbf{C} : \mathrm{m}(\lambda - A) \leqslant \gamma\} \textit{ is connected}\}.$$

We shall need an auxiliary result. Lemma 3.2 below has some interest in itself and raises a related problem.

LEMMA 3.2. *Let* $A, L \in \mathcal{L}(\mathcal{H})$ *and assume that* $\sigma_0(A - L) = \emptyset$. *Given* $\varepsilon > 0$ *there exists* $M \in$ (BQT) *such that*

(i) $\sigma_0(A - M) = \emptyset$,

(ii) $\|M\| < \|L\| + \varepsilon$ *and* $\|\tilde{M}\| = \|\tilde{L}\|$,

(iii) *each component of* $\sigma_e(A - M)$ *contains some component of* $\sigma_e(A)$, *and*

(iv) $\rho_{s\text{-}F}(A - M) \subset \rho_{s\text{-}F}(A)$ *and* $\operatorname{ind}(\lambda - (A - M)) = \operatorname{ind}(\lambda - A)$ *for all* $\lambda \in \rho_{s\text{-}F}(A - M)$.

Proof. Let $\rho : C^*(\tilde{A}, \tilde{L}) \to \mathscr{L}(\mathscr{H}_\rho)$ ($\mathscr{H}_\rho$ a separable space) be a faithful unital *-representation of the C^*-algebra $C^*(\tilde{A}, \tilde{L})$ generated by $\tilde{A}$, $\tilde{L}$ and $\tilde{1}$, and let $B = \rho(\tilde{A})$ and $S = \rho(\tilde{L})$. By Voiculescu's theorem [14], there exist K_ε, $C_\varepsilon \in \mathscr{K}(\mathscr{H})$, $\|K_\varepsilon\| < \varepsilon/2$, $\|C_\varepsilon\| < \varepsilon/2$, such that

$$A - K_\varepsilon = U(A \oplus (B^{(\infty)})^{(\infty)} \oplus (B^{(\infty)})^{(\infty)})\, U^*$$

and

$$L - C_\varepsilon = U(L \oplus (S^{(\infty)})^{(\infty)} \oplus (S^{(\infty)})^{(\infty)})\, U^*,$$

where U is a unitary mapping from $\mathscr{H} \oplus \left\{\bigoplus_{j=1}^{\infty} (\mathscr{H}_\rho^{(\infty)})_j^1\right\} \oplus \left\{\bigoplus_{j=1}^{\infty} (\mathscr{H}_\rho^{(\infty)})_j^2\right\}$ onto $\mathscr{H}$.

Let $\{r_j\}_{j=0}^{\infty}$ ($r_0 = 0$, $r_1 = 1$) be a denumerable dense subset of the interval $[0, 1]$ and let N be a normal operator such that $\sigma(N) = \{\lambda : |\lambda| \leqslant \|\tilde{L}\|\}$. Now we define

$$M = U\left(L \oplus \left\{\bigoplus_{j=1}^{\infty} r_j S^{(\infty)}\right\} \oplus \left\{\bigoplus_{j=1}^{\infty} r_j N^{(\infty)}\right\}\right) U^* + K_\varepsilon.$$

Then $A - M = U\left[(A - L) \oplus \left\{\bigoplus_{j=1}^{\infty} (B - r_j S)^{(\infty)}\right\} \oplus \left\{\bigoplus_{j=1}^{\infty} (B - r_j N)^{(\infty)}\right\}\right] U^*$, so that $\sigma_0(A - M) \subset \sigma_0(A - L) = \emptyset$. It is completely apparent that $\|M\| \leqslant \|L\| + \|K_\varepsilon\| + \|C_\varepsilon\| < \|L\| + \varepsilon$ and $\|\tilde{M}\| = \max\{\|\tilde{L}\|, \|S\|, \|N\|\} = \|\tilde{L}\|$. On the other hand, since $\sigma_e(M) = \sigma_e(L) \cup \sigma(S) \cup \sigma(N) = \sigma(N) = \{\lambda : |\lambda| \leqslant \|\tilde{M}\|\}$ and the normal operator N is a direct summand of M, it readily follows that $M \in$ (BQT).

Since $\{r_j\}^- = [0, 1]$, it is not difficult to check that

$$\sigma_e(A - M) = \bigcup_{0 \leqslant t \leqslant 1} \sigma((B - tS) \oplus (B - tN)) \supset \bigcup_{0 \leqslant t \leqslant 1} \sigma_e(A - tL).$$

By using this equality, we can easily check that each component of $\sigma_e(A - M)$ contains some component of $\sigma_e(A)$, the left essential spectrum $\sigma_{le}(A - M)$ of $A - M$ contains $\sigma_{le}(A)$ and the right essential spectrum $\sigma_{re}(A - M)$ of $A - M$ contains $\sigma_{re}(A)$. Thus, $\rho_{s\text{-}F}(A - M) \subset \rho_{s\text{-}F}(A)$.

Assume that $\lambda \in \rho_{s\text{-}F}(A - M)$, but $\operatorname{ind}(\lambda - A) \neq \operatorname{ind}(\lambda - (A - M))$; then the stability properties of the index imply that $\lambda \in \partial\sigma_e((B - \tau S) \oplus (B - \tau N))$ for some τ, $0 < \tau < 1$, whence we conclude that $\lambda - (A - M)$ cannot be semi-Fredholm, a contradiction.

Hence, $\operatorname{ind}(\lambda - (A - M)) = \operatorname{ind}(\lambda - A)$ for all $\lambda \in \rho_{s\text{-}F}(A - M)$. ∎

Lemma 3.2. and Theorem 2.1 suggest the following.

CONJECTURE 3.3. *Let A, $L \in \mathscr{L}(\mathscr{H})$ and assume that $\sigma_0(A - L) = 0$. Then given $\varepsilon > 0$ there exists $K \in \mathscr{K}(\mathscr{H})$ such that $\sigma_0(A - K) = 0$ and $\|K\| < \|L\| + \varepsilon$.*

Proof of Theorem 3.1. (i) *and* (ii). The formula $\operatorname{dist}[A, \mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})] = \varkappa(A)$ is the content of [6, Proposition 4.2]. In order to prove (ii) it suffices to modify the approximant T obtained there.

Let $\rho : C^*(\tilde{A}) \to \mathcal{L}(\mathcal{H}_\rho)$ ($\mathcal{H}_\rho$ separable) be a faithful unital *-representation and let $R = \rho(\tilde{A})$. By Voiculescu's theorem, given $\eta, 0 < \eta < \varepsilon$, there exists $K_\eta \in \mathcal{K}(\mathcal{H})$, $\|K_\eta\| < \eta$, such that $A - K_\eta \simeq A \oplus (R^{(\infty)})^{(\infty)} \oplus R$; furthermore (as in the proof of Theorem 2.1) we can also assume that $\operatorname{nul}([(\lambda_n - R)^* (\lambda_n - R)]^{1/2} - \operatorname{m_e}(\lambda_n - A)) = \infty$ for all λ_n in a given dense subset $\{\lambda_n\}_{n=1}^\infty$ of the complex plane.

If $R - \lambda_n = V_n H_n$ (polar decomposition) and $\{r_k\}_{k=1}^\infty$ is a dense subset of the interval [0, 1], then we define

$$T_1 = \bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} \{\lambda_n + V_n(H_n r_k \varkappa(A))\}$$

(acting in the obvious way in the space of $(R^{(\infty)})^{(\infty)}$, so that $\|(R^{(\infty)})^{(\infty)} - T_1\| = \sup_k r_k \varkappa(A) = \varkappa(A)$).

Let N be a normal operator such that $\sigma(N) = \{\lambda : |\lambda| \leqslant \varkappa(A)\}$ and define $T_2 = R - N$ (acting in the space of the last direct summand R).

Clearly, we can find $T \in \mathcal{L}(\mathcal{H})$ such that $T - K_\eta \simeq A \oplus T_1 \oplus T_2$ and

$$\|A - T\| = \max\{\|0\|, \|(R^{(\infty)})^{(\infty)} - T_1\|, \|R - T_2\|\} = \varkappa(A).$$

Applying the results of [6, Section 2] to T_1, we see that $A \oplus T_1 \in [\mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]^-$ and $\sigma_e(A \oplus T_1) = \sigma_e(A) \cup \Delta_{\varkappa(A)}(A)$; moreover, since $\sigma(T_2) \subset \sigma(R)_{\varkappa(A)} \subset \sigma(R) \cup \Delta_{\varkappa(A)}(A) = \sigma_e(A \oplus T_1)$, it readily follows that $T \simeq A \oplus T_1 \oplus T_2 +$ + compact also belongs to $[\mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]^-$. Thus, T is an *approximant* for A in $[\mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]^-$ and $\sigma_e(T) = \sigma_e(A \oplus T_1) \cup \sigma_e(T_2) = \sigma_e(A) \cup \Delta_{\varkappa(A)}(A)$.

Since $\|A - T\| = \varkappa(A)$, it is immediate that $\operatorname{m_e}(\lambda - T) \leqslant \max\{\operatorname{m_e}(\lambda - A) - \varkappa(A), 0\}$ for all $\lambda \in \mathbf{C}$. But the definition of T_1 makes it apparent that this inequality is actually an equality! (Recall that $\{\lambda_n\}^- = \mathbf{C}$ and $\{r_k\}^- = [0, 1]$.)

On the other hand, $A - T \simeq 0 \oplus \left\{\bigoplus_{n=1}^{\infty} \bigoplus_{k=1}^{\infty} S_{nk}\right\} \oplus N^{(\infty)}$, where $\|S_{nk}\| \leqslant \varkappa(A)$ for all n and k. Since N is normal and $\sigma(N) = \{\lambda : |\lambda| \leqslant \varkappa(A)\}$, it readily follows that $A - T \in$ (BQT).

Finally, since η can be chosen arbitrarily small, the upper semicontinuity of separate parts of the spectrum (see, e.g., [7, Section 1.1], [10]) implies that η can be chosen so that $\operatorname{d_H}([\sigma(T), \sigma(A) \cup \sigma_e(T)] < \varepsilon$.

(iii) If $\|A - B\| < \delta_0(A)$, then $\sigma_0(B) \neq 0$ and therefore B cannot be a limit of nilpotents [7, Theorem 5.1]. Hence, the distance from A to $\mathcal{N}(\mathcal{H})$ cannot be smaller than $\delta_0(A)$.

On the other hand, it is completely apparent that this distance cannot be smaller than $\operatorname{dist}[A, \mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})]$. Hence (by (i)),

$$\operatorname{dist}[A, \mathcal{N}(\mathcal{H})] \geqslant \max\{\varkappa(A), \delta_0(A)\}.$$

Assume that $\sigma_0(A-L)=\emptyset$, $\|L\|<\delta_0(A)+\varepsilon$, and construct M as in Lemma 3.2 such that $\sigma_0(A-M)=\emptyset$, $\|M\|<\delta_0(A)+2\varepsilon$, each component of $\sigma_e(A-M)$ contains some component of $\sigma_e(A)$, $\rho_{s\text{-}F}(A-M)\subset\rho_{s\text{-}F}(A)$ and $\operatorname{ind}(\lambda-(A-M))=$ $=\operatorname{ind}(\lambda-A)$ for all $\lambda\in\rho_{s\text{-}F}(A-M)$.

Define K_η $(0<\eta<\varepsilon)$, R, T_1 and T_2 as in the proof of (i) and (ii). Comparing that proof and the proof of Lemma 3.2 it is completely apparent that we can find an operator

$$T_0\simeq(A-M)\oplus T_1\oplus T_2+C_\eta,$$

where $C_\eta\simeq K_\eta\oplus 0\oplus 0$, such that $\sigma_0(T_0)=\emptyset$, $T_0\in(\mathrm{BQT})$, $\sigma_e(T_0)$ is a connected set containing the origin (so that $T_0\in\mathcal{N}(\mathcal{H})^-$ [7, Theorem 5.1]) and

$$A-T_0\simeq M\oplus\{(R^{(\infty)})^{(\infty)}-T_1\}\oplus(R-T_2),$$

so that

$$\|A-T_0\|=\max\{\|M\|,\|A-T\|\}=\max\{\delta_0(A)+2\varepsilon,\varkappa(A)\}.$$

Since $T_0\in\mathcal{N}(\mathcal{H})^-$ and ε can be chosen arbitrarily small, we conclude that

$$\operatorname{dist}[A,\mathcal{N}(\mathcal{H})]=\max\{\varkappa(A),\sigma_0(A)\}.$$

(iv) The inequality $\varkappa(A)\leqslant\operatorname{dist}[A,\mathcal{N}(\mathcal{H})]$ is a trivial consequence of (i) and the inequality $\operatorname{dist}[A,\mathcal{N}(\mathcal{H})]\leqslant\max\{\varkappa(A),\max[\mathrm{m}_e(A;\lambda):\lambda\in\sigma_0(A)]\}$ follows immediately from (iii) and Theorem 2.1.

On the other hand, if $\|A-B\|<\beta_0(A)$, then it follows from the upper semicontinuity of separate parts of the spectrum [7, Section 1.1] that $\sigma(B)$ is a disconnected set and therefore $B\notin\mathcal{N}(\mathcal{H})^-$ [7, Theorem 5.1].

Hence, $\operatorname{dist}[A,\mathcal{N}(\mathcal{H})]\geqslant\beta_0(A)$.

The proof of Theorem 3.1 is now complete. ▨

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