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CONTROL SUBSPACES OF MINIMAL DIMENSION, UNITARY AND MODEL OPERATORS

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Let us consider a linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t \geqslant 0,$$

where $A: X \to X$, $B: U \to X$ are bounded linear operators, X (the state space) and U (the control or input space) are some normed linear spaces. The general problem of control is to describe operator pairs (A, B) admitting an appropriate input signal $u(\cdot)$ such that the system starting from a fixed initial state $x(0)$ eventually reaches a prescribed neighbourhood of x, $x \in X$. If the pair (A, B) has this property one says that the system is controllable. It is well known (see e.g. [1], [2]) that for $x(0) = \mathbf{0}$ the system is controllable iff the subspace BU is cyclic for A, i.e.

$$X = \operatorname{span}\{A^k BU : k \geqslant 0\}$$

where span $\{\ldots\}$ is the closed linear hull of the set $\{\ldots\}$.

In the paper [3] (see also [4—6]) the following characteristic of an operator A was introduced

$$\operatorname{disc} A \overset{\text{def}}{=} \sup_{R \in \operatorname{Cyc} A} \min\{\dim R' : R' \subset R,\ R' \in \operatorname{Cyc} A\}$$

where $\operatorname{Cyc} A$ is the family of all finite dimensional cyclic subspaces, i.e. of all such R's, $R \subset X$, $\dim R < \infty$, that $E_R = X$, where

$$E_R = E_R^A \overset{\text{def}}{=} \operatorname{span}\{A^n R : n \geqslant 0\}.$$

(Here "disc" stands for "Dimension of the Input Subspace of Control".) For a transfer operator A of a controllable system the quantity $\operatorname{disc} A$ shows to what extent it is possible to minimize the dimension of the control subspace of our system without loss of controllability. The reader can find in [3,4] some discussions of properties of disc A and of its connection with the spectral multiplicity μ_A

$$\mu_A \stackrel{\text{def}}{=} \min\{\dim R : R \in \operatorname{Cyc} A\}.$$

The knowledge of Lat A (= the lattice of invariant subspaces of A) in some cases enables us to compute disc A, this quantity depending on Lat A only. This paper yields a collection of such cases: unitary and semiunitary operators, C_0-contractions and sums of these operators.

1. MAIN RESULT. Let U be a unitary operator in a separable Hilbert space H_U. We can think that

$$H_U = \int_{\mathbf{T}}^{\oplus} H(t)\, d\nu(t),$$

$$U : \int_{\mathbf{T}}^{\oplus} f(t)\, d\nu(t) \to \int_{\mathbf{T}}^{\oplus} t f(t)\, d\nu(t).$$

where ν is the scalar spectral measure of U on the unit circle $\mathbf{T} = \{\zeta : |\zeta| = 1\}$. Let $\nu_a + \nu_s$ be the standard Lebesgue decomposition of ν. Since the measure ν can be replaced by an equivalent one, we shall suppose that ν_a is a part of the Lebesgue measure m. The ν-a.e. defined function $r(t) \stackrel{\text{def}}{=} \dim H(t)$ is the local multiplicity of the spectrum of U. The numbers

$$\mu_s = \nu_s\text{-ess}\sup r(t),$$

$$\mu_a = \nu_a\text{-ess}\sup r(t)$$

are the spectral multiplicities of U_s (the singular part of U) and of U_a (the absolutely continuous part of U) respectively. If $\nu_a = m$ and

$$\boldsymbol{\mu} = m\text{-ess}\inf r(t),$$

the bilateral shift $\mathscr{S}_{\boldsymbol{\mu}}$ of multiplicity $\boldsymbol{\mu}$ is contained in U. So we can decompose

$$U = U_s \oplus U_{ar} \oplus \mathscr{S}_{\boldsymbol{\mu}}.$$

Here U_{ar} is the maximal reductive part of U_a.

We shall use the expression "for almost all n-dimensional subspaces of R" meaning the invariant measure τ_n on the Grassmann manifold $G_n(R)$ of all n-dimensional subspaces of R. We shall say that the operator A has many cyclic subspaces if for any $R \in \operatorname{Cyc} A$ almost all subspaces $R' \subset R$, $\dim R' = \operatorname{disc} A$, are cyclic.

Now we can state the main result.

THEOREM. *Let U be a unitary operator, T a C_0-contraction, S_n the unilateral shift of multiplicity n and let A be the following direct sum:*

$$A = S_k^* \dotplus U \dotplus S_n \dotplus T. \tag{1}$$

Then

$$\operatorname{disc} A = \max\{\mu_s, n + \max(1, \mu_T, k + \mu_a + \boldsymbol{\mu})\}$$

and A has many cyclic subspaces.

To understand better the statement let us consider the contribution of each summand into the disc of the sum. The following heuristic explanations constitute in a sense the plan of the proof (see also Section 3 below). But the proof depends on too many technical details to permit a short pithy description. Apparently it will be useful to return to these explanations when reading the proof of the theorem.

Note at first that disc (as well as the spectral multiplicity) is invariant under similarity, so we can assume without loss of generality the sum (1) to be orthogonalt

If operators A and B have many invariant subspaces and if the lattice of the invariant subspaces of the sum $A \oplus B$ splits up into the sum of lattices

$$\operatorname{Lat}(A \oplus B) = \operatorname{Lat} A \oplus \operatorname{Lat} B$$

(i.e. $E \in \operatorname{Lat}(A \oplus B) \Rightarrow E = E_1 \oplus E_2$, $E_1 \in \operatorname{Lat} A$, $E_2 \in \operatorname{Lat} B$) then

$$\operatorname{disc}(A \oplus B) = \max\{\operatorname{disc} A, \operatorname{disc} B\}.$$

Due to this property we can separate the singular part of the unitary operator. As to the other summands the operator S_n increases the whole disc by n. For S_n itself $\operatorname{disc} S_n = n + 1$, but this "superfluous" unit disappears being summed with other operators. A unitary operator with the absolutely continuous spectrum can be thought of as

$$U_a = \mathscr{S}_{\boldsymbol{\mu}} \oplus U_{ar}.$$

Remind that $\mathscr{S}_{\boldsymbol{\mu}}$ is the bilateral shift of multiplicity $\boldsymbol{\mu}$ and is the maximal reductive (i.e. $\operatorname{Lat} U_{ar} \subset \operatorname{Lat} U_{ar}^*$) part of U_a. The spectral measure of U_{ar} does not contain the Lebesgue measure. The formula

$$\operatorname{disc} U_a = \mu_a + \boldsymbol{\mu} = (\mu_a - \boldsymbol{\mu}) + 2\boldsymbol{\mu}$$

can be interpreted as follows:

$$\operatorname{disc} U_a = \operatorname{disc} U_{ar} + \operatorname{disc} \mathscr{S}_{\boldsymbol{\mu}},$$

i.e. the contribution of the reductive part is equal to its spectral multiplicity

$$\operatorname{disc} U_{ar} = \mu_{U_{ar}} = \mu_a - \boldsymbol{\mu}$$

and the contribution of the bilateral shift is twice as big as its spectral multiplicity

$$\operatorname{disc} \mathscr{S}_{\boldsymbol{\mu}} = 2\boldsymbol{\mu}.$$

For the backward shift we have

$$\operatorname{disc} S_k^* = k = \dim \operatorname{Ker} S_k^*,$$

and for a C_0-contraction

$$\operatorname{disc} T = \mu_T .$$

The first of these operators adds its disc to the disc of the unitary operator, but the second one makes its contribution to the whole disc only if its disc is bigger than disc $(S_k^* \oplus U_a)$. Here we have the situation analogous to the singular unitary operator. Though now the lattice of the invariant subspaces does not split up into the sum of lattices, the functional calculus for our operators is rich enough to separate the thin spectrum of C_0-contraction from the spectra of U and S_k^*.

Now we shall try to explain (not leaving the intuitive level) when the disc of a direct sum is the sum of the discs and when it is their maximum. Roughly speaking, if the spectra of operators overlap we have the first case otherwise the second possibility occurs. But "the overlapping" does not mean here the mere intersection of sets, it rather means the relative "density" of spectra. So the spectrum of U_s is very thin and chips off from everything else. The spectrum of T is thin enough also, it "filters" through the spectra of the unitary operator and of the backward shift, but the spectrum of the shift operator (the "densest" one) is impervious to the spectrum of a C_0-contraction. The spectrum of S_k^* is "friable", it consists of eigenvalues, however due to its mass (it fills the whole unit disc $\mathbf{D}$) it makes the same contribution as the absolutely continuous spectrum of the unitary operator, which is "denser", but lies on the unit circle only. (The "friability" is exhibited by the following fact also: adding S^* to itself we get cyclic operator again in spite of increase of the multiplicity of eigenvalues (i.e. the dimension of the eigensubspaces).)

2. NOTATION AND SOME AUXILIARY PROPOSITIONS. The symbol $L(X, Y)$ will be used to denote the set of all linear bounded operators from X into Y, $L(X) \overset{\text{def}}{=} L(X, X)$. A *subspace* always means a *closed* subspace. Let E be a subspace of a Hilbert space H, then $E^{\perp} = H \ominus E$ denotes the orthogonal subspace and P_E denotes the orthogonal projection of H onto E.

We shall deal with operators in *separable* Hilbert spaces only. Now we restate for this case some simple properties of $\operatorname{disc} A$, that can be found (with other introductory material) in [3].

2.1. $\operatorname{disc} A \geqslant \mu_A \geqslant \dim \operatorname{Ker} A^*$.

2.2. $\operatorname{disc}(A \oplus B) \geqslant \max(\operatorname{disc} A, \operatorname{disc} B)$.

2.3. $[E \in \operatorname{Lat} A,\ \mu_{A|E} < \infty] \Rightarrow \operatorname{disc}(P_{E^\perp} A | E^\perp) \leqslant \operatorname{disc} A$.

2.4. $[E \in \operatorname{Lat} A,\ R \in \operatorname{Cyc} A] \Rightarrow P_{E^\perp} R \in \operatorname{Cyc}(P_{E^\perp} A | E^\perp)$.

2.5. Let $\mathfrak{M}$ be a set of m-dimensional subspaces of a finite dimensional space R (i.e. $\mathfrak{M} \subset G_m(R)$) and suppose $\mathfrak{M}$ has the full measure (i.e. $\tau_m(\mathfrak{M}) = 1$). If for every $M, M \in \mathfrak{M}$, a family $\mathfrak{N}_M \subset G_n(R)$ is given such that $\tau_n(\mathfrak{N}_M) = 1$ then the set

$$G_{n+m}(R) \cap \{\operatorname{span}(N, M) \ : \ N \in \mathfrak{N}_M,\ M \in \mathfrak{M}\}$$

has the full measure in $G_{n+m}(R)$.

2.6. If $\mathfrak{M} \subset G_m(R')$, $\tau_m(\mathfrak{M}) = 1$ and $R' \subset R$ then

$$\tau_m\{M \ : \ M \in G_m(R),\ P_{R'} M \in \mathfrak{M}\} = 1.$$

2.7. If operators A and B have many invariant subspaces and

$$\operatorname{Lat}(A \oplus B) = \operatorname{Lat} A \oplus \operatorname{Lat} B$$

then

$$\operatorname{disc}(A \oplus B) = \max\{\operatorname{disc} A, \operatorname{disc} B\}$$

and the operator $A \oplus B$ has many invariant subspaces.

2.8. A complete information concerning C_0-contractions needed below can be found in the books [7,8] and in the paper [9].

We would only remind that every C_0-contraction is unitarily equivalent to the projection of the shift operator

$$S : f \to zf, \quad f \in H^2(E)$$

onto the subspace

$$K_\Theta \stackrel{\text{def}}{=} H^2(E) \ominus \Theta H^2(E)$$

for a suitable choice of the auxiliary Hilbert space E and of the inner function Θ. ($H^2(E)$ is the Hardy space of E-valued functions.) The orthogonal projection in $H^2(E)$ onto K_Θ is

$$P_\Theta = \Theta P_- \Theta^* = I - \Theta P_+ \Theta^*$$

where $P_- = I - P_+$ and P_+ is the Riesz projection of L^2 onto H^2. So we have

$$T = P_\Theta S \mid K_\Theta$$

or a unitarily equivalent representation

$$T = S^* \mid K_{\tilde{\Theta}}$$

where $\tilde{\Theta}(\zeta) = \Theta(\bar{\zeta})^*$, $\zeta \in \mathbf{D}$. The symbol $H^\infty(E_1, E_2)$ will be used to denote the set of all bounded analytic functions from $\mathbf{D}$ into $L(E_1, E_2)$; $H^\infty(E) \stackrel{\text{def}}{=} H^\infty(E, E)$.

3. PLAN OF THE PROOF. Beginning the proof of the theorem we separate at first the singular part of the unitary operator (Lemma 4) and later on we deal with the unitary operator with the absolutely continuous spectral measure. Lemma 5 is an auxiliary one, it is essentially contained in [3]. We state it here for the sake of completeness.

Lower estimates of the disc are given in Lemmata 6—8. In these lemmata we construct some of "the worst" cyclic subspaces. These examples sum up in Lemma 9 containing the following inequality

$$\operatorname{disc}(S_k^* \oplus U \oplus S_n \oplus T) \geqslant n + \max\{1, \mu_T, k + \mu_U + \boldsymbol{\mu}\}.$$

In the rest of the paper we prove the opposite inequality. Since the lattice of the investigated sum does not split up into the sum of lattices, we have to consider not each summand separately, but the sum as a whole. We do this in the concluding Lemma 23, but at first we consider some more simple parts of the sum $S_k^* \oplus U \oplus \oplus S_n \oplus T$. In Lemmata 10—12 C_0-contractions are investigated and the total is expressed by Lemma 13: $\operatorname{disc} T = \mu_T$. Further in Lemmata 14—16 we deal with the shift operator S_n. Lemma 17 unites the obtained results with the information concerning C_0-contractions. The results are stated in Corollaries 18—19: $\operatorname{disc} S_n = n + 1$, $\operatorname{disc}(S_n \oplus T) = n + \mu_T$. In Lemma 20 the backward shift S_k^* is investigated and the total is stated in Corollary 21: $\operatorname{disc} S_k^* = k$, and in Corollary 22: $\operatorname{disc}(S_k^* \oplus T) = \max\{k, \mu_T\}$.

In conclusion, as it was already said, all obtained results will be summarized in Lemma 23 to compute $\operatorname{disc}(S_k^* \oplus U \oplus S_n \oplus T)$. Let d be equal to $n + \max\{1, \mu_T, k + \mu_U + \boldsymbol{\mu}\}$. It is necessary to choose, in an arbitrary finite dimensional cyclic subspace R, d vectors whose span is cyclic. We choose at first $(n + \mu_U)$ vectors such that the direct sum operator in question induces in the invariant subspace they generate an operator containing $U_{ar} \oplus S_{n+\boldsymbol{\mu}}$. The operator U behaves like the sum $U_{ar} \oplus S_{\boldsymbol{\mu}} \oplus S_{\boldsymbol{\mu}}^*$, so in the orthogonal complement of the obtained subspace we have, roughly speaking, the operator $S_{k+\boldsymbol{\mu}}^* \oplus T$. Now we find $(k + \boldsymbol{\mu})$ more vectors that generate the whole space where the operator $S_{k+\boldsymbol{\mu}}^*$ is defined. Doing this we impose on the chosen vectors some additional restrictions ensuring the possibility of the simultaneous approximation every vector from the domain of T.

Now we turn to the realization of the sketched program.

4. LEMMA. *Let U_s be a unitary operator whose spectral measure is singular. Let A be a contraction, whose unitary dilation has an absolutely continuous spectral*

measure [1]. *If A has many cyclic subspaces then*

$$\operatorname{disc}(U_s \oplus A) = \max\{\mu_{U_s}, \operatorname{disc} A\}.$$

Proof. According to Proposition 2.7 it is sufficient to show that U_s has many cyclic subspaces and

$$\operatorname{Lat}(U_s \oplus A) = \operatorname{Lat} U_s \oplus \operatorname{Lat} A.$$

The last can be easy obtained from the F. and M. Riesz' theorem on analytic measures (see e.g. [8]). Indeed, let ν_s, ν_a be the respective scalar spectral measures for U_s and for the minimal unitary dilation of A. So if

$$U_s^n x_1 \oplus A^n x_2 \perp y_1 \oplus y_2, \quad n \geqslant 0,$$

then

$$\int_{\mathbf{T}} \zeta^n (x_1(\zeta), y_1(\zeta))\, d\nu_s(\zeta) + \int_{\mathbf{T}} \zeta^n (x_2(\zeta), y_2(\zeta))\, d\nu_a(\zeta) = 0, \quad n \geqslant 0,$$

and the F. and M. Riesz' theorem implies the vanishing of both integrals (for all $n \geqslant 0$). This just means that every subspace from $\operatorname{Lat}(U_s \oplus A)$, is the orthogonal sum of two subspaces from $\operatorname{Lat} U_s$, and from $\operatorname{Lat} A$.

The fact that U_s has many cyclic subspaces is a corollary of the following lemma.

5. LEMMA. *Let R be a finite dimensional subspace of the space H,*

$$H = \int_{\mathbf{T}}^{\oplus} H(t)\, d\nu(t),$$

$$\mu \stackrel{\text{def}}{=} \nu\text{-ess}\sup \dim H(t) < \infty.$$

Let $\{j(t)\}$ be a family of evaluation mappings, $j(t)\colon H \to H(t)$, such that

$$j(t)h = h(t) \quad \nu\text{-a.e.}, \quad \forall\, h \in R.$$

If

$$j(t)R = H(t) \quad \nu\text{-a.e.}$$

then for almost all μ-dimensional subspaces $R_\mu \subset R$ the following equality holds:

$$j(t)R_\mu = H(t) \quad \nu\text{-a.e.}\,.$$

[1] In other words, A is the sum of a unitary operator whose spectral measure is absolutely continuous and of a completely nonunitary contraction, see [7].

Proof. Let $h_1, h_2 \in R$. We may assume the equalities

$$h_i(t) = j(t)h_i$$

hold for every $t \in \mathbf{T}$. Put

$$\sigma_\alpha = \{t : h_1(t) + \alpha h_2(t) = 0,\ h_1(t) \neq 0,\ h_2(t) \neq 0\}, \quad \alpha \in \mathbf{C}.$$

Since $\sigma_\alpha \cap \sigma_\beta = \emptyset$ $(\alpha \neq \beta)$, we have $\nu(\sigma_\alpha) = 0$ for almost all α, $\alpha \in \mathbf{C}$. Taking into account the equality $j(t)R = H(t)$, we get

$$j(t)h = h(t) \neq 0 \quad \nu\text{-a.e.} \tag{2}$$

for almost all vectors $h \in R$. So lemma is proved for $\mu = 1$. Assuming the lemma is valid for some μ we prove it for the next one.

Fix a vector h satisfying (2) and put

$$H'(t) = H(t) \ominus h(t) \cdot \mathbf{C},$$

$$H' = \int^{\oplus} H'(t)\, d\nu(t), \quad R' = P_{H'}R.$$

Since $j(t)R' = H'(t)$, by the induction hypothesis we have $j(t)R'_\mu = H'(t)$ for almost all $R'_\mu \in G_\mu(R')$. Put $R_{\mu+1} = \operatorname{span}\{h, R''_\mu\}$, where R''_μ is an arbitrary μ-dimensional subspace of R such that $P_{H'}R''_\mu = R'_\mu$. Since h varies in a set of almost all vectors from R and R''_μ varies in a set of almost all subspaces from $G_\mu(R)$ (Proposition 2.6), $R_{\mu+1}$ is "almost every" element of $G_{\mu+1}(R)$ (Proposition 2.5). □

6. LEMMA. *Let T be a contraction whose minimal unitary dilation has absolutely continuous spectral measure. Then*

$$\operatorname{disc}(S_n \oplus T) \geqslant n + \operatorname{disc} T.$$

Proof. (Induction on n). Let $n = 1$, $R_T \in \operatorname{Cyc} T$, $\dim R_T = \operatorname{disc} T$ and suppose R_T does not contain proper cyclic subspaces. Such R_T exists by the definition of disc. Put

$$R = \operatorname{span}\{\mathbf{1} \oplus \mathbf{O},\ \mathbf{O} \oplus R_T\}.$$

It is clear that $R \in \operatorname{Cyc}(S \oplus T)$. Let $R' \subset R$ be a cyclic subspace for the operator $S \oplus T$. Choose a basis $\{x_i\}$ in R' where x_i are vectors of the form:

$$x_1 = \mathbf{1} \oplus h_1, \quad x_i = \mathbf{O} \oplus h_i, \quad 2 \leqslant i \leqslant \dim R'.$$

Letting

$$L = \operatorname{span}\{T^k h_i : k \geqslant 0,\quad 2 \leqslant i \leqslant \dim R'\}$$

we verify that L is the whole space, where T is defined. Indeed, let $f \perp L$. Since $R' \in \mathrm{Cyc}(S \oplus T)$, there exists a family of polynomials $\{p_{ik} : 1 \leqslant i \leqslant \dim R', k \geqslant 1\}$ such that

$$\mathbf{O} \oplus f = \lim_k \sum_{i=1}^{\dim R'} p_{ik}(S \oplus T)\, x_i =$$

$$= (\lim_k p_{1k}) \oplus \left(\lim_k \sum_{i=1}^{\dim R'} p_{ik}(T)\, h_i\right).$$

Hence

$$\lim_k p_{1k} = \mathbf{O}, \tag{3}$$

$$\lim_k \sum_{i=1}^{\dim R'} p_{ik}(T)\, h_i = f. \tag{4}$$

Assuming that the minimal unitary dilation of T is the operator of multiplication by $z : h \to zh$ in $\int^{\oplus} H(z)\, \mathrm{d}\nu(z)$, we can choose a bounded outer function φ satisfying $\overline{\varphi} f \in L^\infty$. Multiplying (4) by the vector $\varphi(T)^* f$ and using (3) and the fact that $f \perp L$, we get

$$(f, \varphi(T)^* f) = \lim_k (\varphi(T)\, p_{1k}(T)\, h_1, f) =$$

$$= \lim_k \int_{\mathbf{T}} p_{1k}(\zeta)\, \varphi(\zeta)(h_1(\zeta), f(\zeta))_{H(\zeta)}\, \mathrm{d}\nu(\zeta) = 0.$$

Clearly there exists a sequence $\{\varphi_n\}$ such that $|\varphi_n| \leqslant 1$ and $\varphi_n(\zeta) \to 1$ a.e. on $\mathbf{T}$, i.e. $\lim \varphi_n(T)^* f = f$. Hence $f = \mathbf{O}$, i.e.

$$\mathrm{span}\{h_i : 2 \leqslant i \leqslant \dim R'\} = R_T$$

and therefore $\dim R' = 1 + \dim R_T = \dim R$, i.e. $R' = R$. Hence R does not contain proper cyclic subspaces, and $\mathrm{disc}(S \oplus T) \geqslant \dim R = 1 + \mathrm{disc}\, T$.

It is now clear that the induction goes. The spectral measure of the minimal unitary dilation of the operator $S_n \oplus T$ is absolutely continuous. Hence the inequality

$$\mathrm{disc}(S_{n-1} \oplus T) \geqslant n - 1 + \mathrm{disc}\, T$$

yields

$$\mathrm{disc}(S_n \oplus T) = \mathrm{disc}[S \oplus (S_{n-1} \oplus T)] \geqslant$$

$$\geqslant 1 + \mathrm{disc}(S_{n-1} \oplus T) \geqslant n + \mathrm{disc}\, T. \qquad ▩$$

7. Lemma. $\mathrm{disc}\, S_n \geqslant n + 1$.

Proof. The equality $\operatorname{disc} S = 2$ has been proved in [3]. (To prove $\operatorname{disc} S \geqslant 2$ it is sufficient to check that the two dimensional cyclic subspace $R = \operatorname{span}\{z, z^2 + 1/2\}$ contains no cyclic vectors.) Then Lemma 6 implies

$$\operatorname{disc} S_n = \operatorname{disc}(S_{n-1} \oplus S) \geqslant n - 1 + \operatorname{disc} S = n + 1. \qquad \square$$

8. LEMMA. *Let U be a unitary operator with an absolutely continuous spectral measure. Then*

$$\operatorname{disc}(U \oplus S_k^*) \geqslant \mu_U + \boldsymbol{\mu} + k.$$

Proof. Put $\mu = \mu_U$. We can assume that U is the orthogonal sum of μ operators without multiplicity:

$$U = \sum_{i=1}^{\mu}{}^{\oplus} A_i, \quad A_i \in L(H_i),$$

$$H_i = L^2(E_i), \quad \mathbf{T} = E_1 = \ldots = E_{\boldsymbol{\mu}} \supsetneq E_{\boldsymbol{\mu}+1} \supset \ldots \supset E_\mu,$$

$$(A_i f_i)(\zeta) = \bar{\zeta} f_i(\zeta), \quad \zeta \in \mathbf{T},\ f_i \in H_i,\ 1 \leqslant i \leqslant \mu.$$

Then $U \oplus S_k^*$ can be rewritten in the following form

$$U \oplus S_k^* = \sum_{i=1}^{\mu+k}{}^{\oplus} A_i \stackrel{\text{def}}{=} A, \quad A_i \in L(H_i),$$

$$H_i = H^2, \quad \mu < i \leqslant \mu + k,$$

$$(A_i f)(\zeta) = \zeta^{-1}[f(\zeta) - f(0)], \quad \zeta \in \mathbf{T},\ f \in H^2,\ \mu < i \leqslant \mu + k.$$

Let χ be the characteristic function of the set E_μ, or of an arbitrary set of positive but not of full measure in the case $\mu = \boldsymbol{\mu}$. Define the following family of vectors $x_i, y_i \in H_i$:

$$x_i = \chi, \quad y_i = \mathbf{1} - \chi, \quad 1 \leqslant i \leqslant \boldsymbol{\mu};$$

$$x_i = \mathbf{1} \mid E_i, \quad \boldsymbol{\mu} < i \leqslant \mu;$$

$$x_i = P_+\chi, \quad \mu < i \leqslant \mu + k;$$

and put

$$R = \operatorname{span}\{x_i, y_j\colon\ 1 \leqslant i \leqslant \mu + k,\ 1 \leqslant j \leqslant \boldsymbol{\mu}\}.$$

It is clear that $R \in \operatorname{Cyc} A$, because $\operatorname{span}\{\chi, \mathbf{1} - \chi\} \in \operatorname{Cyc} S$, $\mathbf{1}$ is a cyclic vector of a reductive cyclic unitary operator and because $P_+\chi \in \operatorname{Cyc} S^*$ (cf. [8]).

Let us verify that $R' \notin \operatorname{Cyc} A$ if $R' \subsetneq R$. Consider a non zero vector $f \in R \ominus R'$. The vector f belonging to R is of the form

$$f = \sum_{i=1}^{\boldsymbol{\mu}}{}^{\oplus} (\alpha_i x_i + \beta_i y_i) \oplus \sum_{i=\boldsymbol{\mu}+1}^{\mu+k}{}^{\oplus} \alpha_i x_i.$$

Now we shall check that the A-invariant subspace spanned by R' is orthogonal to the following non zero vector g:

$$g = \left[\zeta \sum_{i=1}^{\boldsymbol{\mu}}{}^{\oplus}(\alpha_i\|x_i\|^2 x_i - \beta_i\|y_i\|^2 y_i)\right] \oplus \left[\sum_{i=\boldsymbol{\mu}+1}^{\mu}{}^{\oplus} \alpha_i\|x_i\|^2 \chi\right] \oplus$$

$$\oplus \left[\sum_{i=\mu+1}^{\mu+k}{}^{\oplus} \alpha_i\|x_i\|^2 \mathbf{1}\right].$$

To do this it is sufficient to prove that $R' \perp gH^2$. Letting $h \in R'$ we have

$$h = \sum_{i=1}^{\boldsymbol{\mu}}{}^{\oplus}(\alpha_i' x_i + \beta_i' y_i) \oplus \sum_{i=\boldsymbol{\mu}+1}^{\mu+k}{}^{\oplus} \alpha_i' x_i$$

and

$$0 = (h, f) = \sum_{i=1}^{\mu+k} \alpha_i'\bar{\alpha}_i\|x_i\|^2 + \sum_{i=1}^{\boldsymbol{\mu}} \beta_i'\bar{\beta}_i\|y_i\|^2.$$

Hence

$$(h, \zeta^n g) = \sum_{i=1}^{\boldsymbol{\mu}} [\alpha_i'\bar{\alpha}_i\|x_i\|^2 (x_i, \zeta^{n+1} x_i) - \beta_i'\bar{\beta}_i\|y_i\|^2 (y_i, \zeta^{n+1} y_i)] +$$

$$+ \sum_{i=\boldsymbol{\mu}+1}^{\mu} \alpha_i'\bar{\alpha}_i\|x_i\|^2 (x_i, \zeta^{n+1}\chi) + \sum_{i=\mu+1}^{\mu+k} \alpha_i'\bar{\alpha}_i\|x_i\|^2 (x_i, \zeta^{n+1}) =$$

$$= \left[\sum_{i=1}^{\mu+k} \alpha_i'\bar{\alpha}_i\|x_i\|^2 + \sum_{i=1}^{\boldsymbol{\mu}} \beta_i'\bar{\beta}_i\|y_i\|^2\right](\chi, \zeta^{n+1}) -$$

$$- \sum_{i=1}^{\boldsymbol{\mu}} \beta_i'\bar{\beta}_i\|y_i\|^2 (\mathbf{1}, \zeta^{n+1}) = 0.$$

So the cyclic subspace R contains no cyclic subspaces. Therefore

$$\operatorname{disc}(U \oplus S_k^*) \geqslant \dim R = \mu + \boldsymbol{\mu} + k.$$ □

9. Corollary. *If U is a unitary operator and $T \in C_0$ then*

$$\operatorname{disc}(S_k^* \oplus U \oplus S_n \oplus T) \geqslant n + \max\{1,\ \mu_T,\ k + \mu_a + \boldsymbol{\mu}\}.$$

Proof. Due to Proposition 2.2 we may assume without loss of generality U has an absolutely continuous spectral measure. Then

$$\operatorname{disc}(S_k^* \oplus U \oplus S_n \oplus T) \geqslant \qquad \text{(Lemma 6)}$$

$$\geqslant n + \operatorname{disc}(S_k^* \oplus U \oplus T) \geqslant \qquad \text{(Proposition 2.2}$$

$$\geqslant n + \max\{\operatorname{disc}(S_k^* \oplus U),\ \operatorname{disc} T\} \geqslant \qquad \text{(Lemma 8, Proposition 2.1)}$$

$$\geqslant n + \max\{k + \mu_a + \mu,\ \mu_T\}.$$

The case without operators S_k^*, U and T was considered in Lemma 7. □

Now we turn to the proof of the main part of the theorem, which gives upper bound for the disc. Let us begin with the consideration of C_0-contractions.

10. LEMMA. *Let T be a C_0-contraction, $\Theta_J = \operatorname{diag}\{m_1, m_2, \dots\}$ be the characteristic function of its Jordan model*[1]. *Let $R \in \operatorname{Lat} T$, $\mu \stackrel{\text{def}}{=} \mu_{T|R} < \infty$, and $m \in H^\infty$. If $m(P_{R^\perp} T | R^\perp) = \mathbf{O}$ then $m \in m_{\mu+1} H^\infty$.*

Proof. Let H be the space where T is defined, and let $H_m = \operatorname{clos} m(T)H$. Clearly $H_m \subset R$, and therefore $\mu_{T|H_m} \leqslant \mu$, i.e. there exists a subspace $L \in \operatorname{Cyc}(T|H_m)$ such that $\dim L \leqslant \mu$.

Let T_J be the Jordan model of T in the space H_J, $X \in L(H, H_J)$ be a quasiaffine transform intertwining T_J and T: $T_J X = XT$. We are going to show that the subspace XL is cyclic for M, $M \stackrel{\text{def}}{=} T_J | \operatorname{clos} m(T_J) H_J$. Firstly, since $L \subset H_m$,

$$XL \subset XH_m \subset \operatorname{clos} Xm(T)H = \operatorname{clos} m(T_J)XH = \operatorname{clos} m(T_J)H_J.$$

On the other hand

$$\operatorname{span}\{M^k XL : k \geqslant 0\} = \operatorname{span}\{T_J^k XL : k \geqslant 0\} =$$

$$= \operatorname{span}\{XT^k L : k \geqslant 0\} = \operatorname{clos} XH_m = \operatorname{clos} m(T_J)H_J.$$

So $XL \in \operatorname{Cyc} M$, hence, $\mu_M \leqslant \dim XL = \dim L \leqslant \mu$. However the Jordan operator M has the spectral multiplicity $\mu_M = \sup\{j : m \notin m_j H^\infty\}$, because $m(T) = \mathbf{O}$ for every contraction T whose minimal function divides m. Therefore, $m \in m_{\mu+1} H^\infty$. □

11. LEMMA. *Let $T \in L(H)$ be a C_0-contraction, R be a finite dimensional subspace of H. Then the equality $m_x = m_R$ holds for almost all $x \in R$.* (Recall that m_R is the minimal function of the operator $T | E_R$, $E_R = \operatorname{clos}\{T^n R : n \geqslant 0\}$.)

Proof. (Induction on $\dim R$). For $\dim R = 1$ the assertion is obvious. Assuming its validity for $\dim R \leqslant n$ let us prove it for $\dim R = n + 1$.

Let us fix a hyperplane $R' \subset R$, $\dim R' = n$, and a vector $x \in R \setminus R'$. Since $E_R = \operatorname{span}\{E_{R'}, E_x\}$, the equality

$$m_R = \operatorname{LCM}\{m_{R'}, m_x\} = \operatorname{LCM}\{m_{x'}, m_x\}$$

---

[1] The Jordan model of a C_0-contraction T is the unique operator of the form $P_\Theta S | K_\Theta$ with $\Theta = \operatorname{diag}\{m_1, m_2, \dots\}$, $m_i / m_{i+1} \in H^\infty$, which is quasisimilar to the contraction T, see [10], [8].

(where LCM means "Least Common Multiple") holds for almost all $x' \in R'$ (by the induction hypothesis).

It remains to prove that

$$m_{x'+\alpha x} = \mathrm{LCM}\{m_{x'}, m_x\}$$

for almost all α in $\mathbf{C}$. Let

$$m_\alpha = \frac{\mathrm{LCM}\{m_{x'}, m_x\}}{m_{x'+\alpha x}}.$$

These functions are mutually disjoint, because

$$\mathrm{GCD}\{m_\alpha, m_\beta\} = \frac{\mathrm{LCM}\{m_{x'}, m_x\}}{\mathrm{LCM}\{m_{x'+\alpha x}, m_{x'+\beta x}\}}$$

(where GCD means "Greatest Common Divisor") but $\mathrm{LCM}\{m_{x'+\alpha x}, m_{x'+\beta x}\} = m_{\mathrm{span}\{x, x'\}} = \mathrm{LCM}\{m_x, m_{x'}\}$ if $\alpha \neq \beta$. Therefore $m_\alpha = 1$ for all α except an at most countable set. ▨

12. Lemma. *Let T be a C_0-contraction, $\mu =: \mu_T < \infty$, $R \in \mathrm{Cyc}\, T$. Then almost all subspaces $R_\mu \in G_\mu(R)$ are cyclic.*

Proof. Let Θ denote the characteristic function of T, $T = P_\Theta S \mid K_\Theta$, $K_\Theta = H^2(E) \ominus \Theta H^2(E)$, and let $\Theta_J = \mathrm{diag}\{m_1, \ldots, m_\mu\}$ be the characteristic function of the Jordan model of T. Since $\mu < \infty$, T is a weak contraction (see [9]), hence there exists $\det \Theta$ and the following equality

$$\det \Theta = \det \Theta_J = \prod_{i=1}^{\mu} m_i$$

holds.

By Lemma 11 for almost all one dimensional subspaces $R_1 \in G_1(R)$ we have

$$m_{R_1} = m_R = m_1.$$

Now we fix such a subspace R_1 and define the functions Θ_1 and θ_1 by the equalities

$$E_{R_1} = \Theta_1 H^2(E) \ominus \Theta H^2(E), \quad \theta_1 = \Theta_1^* \Theta.$$

Then θ_1 is the characteristic function of the operator $T \mid E_{R_1}$. (More exactly, the pure part of θ_1 is the characteristic function of $T \mid E_{R_1}$, but since this difference has no influence on $\det \theta_1$, it does not matter for us (see [7], Chapter VII).) Since it is an operator without multiplicity, the determinant of its characteristic function coincides with its minimal function, i.e.

$$\det \theta_1 = m_{R_1} = m_1.$$

The function Θ_1 is the characteristic function of the operator T_1:

$$T_1 \overset{\text{def}}{=} P_{\Theta_1} T | K_{\Theta_1}, \quad K_{\Theta_1} = E_{R_1}^{\perp} = H^2(E) \ominus \Theta_1 H^2(E).$$

Since $\Theta = \Theta_1 \theta_1$, we have

$$\det \Theta_1 = \frac{\det \Theta}{\det \theta_1} = \frac{\prod_{i=1}^{\mu} m_i}{m_1} = \prod_{i=2}^{\mu} m_i.$$

Now, letting $R_s \in G_s(R)$, $E_{R_s} = \Theta_s H^2(E) \ominus \Theta H^2(E)$, we shall prove that the inclusion

$$\prod_{i=s+1}^{\mu} m_i \in (\det \Theta_s) H^\infty \tag{5}$$

is valid for almost all $R_s \in G_s(R)$. We have proved this inclusion for $s = 1$. Assuming this inclusion for almost all $R_s \in G_s(R)$ let us prove it for $(s+1)$-dimensional subspaces R_{s+1}.

Fixing a subspace R_s satisfying (5), put

$$T_s = P_{\Theta_s} T | K_s, \quad K_s = E_{R_s}^{\perp} = H^2(E) \ominus \Theta_s H^2(E).$$

By Proposition 2.4 we have $P_{\Theta_s} R \in \operatorname{Cyc} T_s$ and so by Lemma 11 the equality $m_x = m_{T_s}$ holds for almost all vectors $x \in P_{\Theta_s} R$. Noting that the set of $y \in R$ such that $P_{\Theta_s} y = x$ and $m_x = m_{T_s}$ has the full measure in $G_1(R)$ (Proposition 2.6), we choose any such vector y and put $R_{s+1} = \operatorname{span}\{R_s, y\}$. In this way we obtain almost all subspaces from $G_{s+1}(R)$ (Proposition 2.5). Now it is sufficient to show that the inclusion (5) is valid for the constructed R_{s+1}.

We shall verify, at first, that the following equality

$$E_x^{T_s} = \Theta_{s+1} H^2(E) \ominus \Theta_s H^2(E)$$

holds. Indeed, we have

$$E_{R_{s+1}}^T = \operatorname{span}\left\{E_{R_s}^T, E_y^T\right\} = E_{R_s}^T \oplus E_x^{T_s}$$

hence

$$E_x^{T_s} = E_{R_{s+1}}^T \ominus E_{R_s}^T = [\Theta_{s+1} H^2(E) \ominus \Theta H^2(E)] \ominus$$

$$\ominus [\Theta_s H^2(E) \ominus \Theta H^2(E)] = \Theta_{s+1} H^2(E) \ominus \Theta_s H^2(E);$$

since the function $\theta_{s+1} \overset{\text{def}}{=} \Theta_{s+1}^* \Theta_s$ is the characteristic function of the operator $T_s | E_x^{T_s}$ without multiplicity. Therefore

$$\det \Theta_s = \det \theta_{s+1} \det \Theta_{s+1} = m_{T_s} \det \Theta_{s+1}.$$

Since the multiplicity of $T|E^T_{R_s}$ does not exceed s, we have by Lemma 10

$$m_{T_s} \in m_{s+1}H^\infty.$$

And using the induction hypothesis we obtain

$$\prod_{i=s+2}^{\mu} m_i \in \left(\frac{\det \Theta_s}{m_{s+1}}\right) H^\infty = \left(\frac{m_{T_s}}{m_{s+1}} \det \Theta_{s+1}\right) H^\infty \subset \det \Theta_{s+1} H^\infty.$$

Thus $\mathbf{1} \in \det \Theta_\mu H^\infty$, hence $\det \Theta_\mu = \mathbf{1}$, i.e. $\Theta_\mu = I$, therefore the equality

$$E_{R_\mu} = H^2(E) \ominus \Theta H^2(E)$$

holds for almost all $R_\mu \in G_\mu(R)$. ▨

13. Corollary. *Every C_0-contraction T has many cyclic subspaces and* $\operatorname{disc} T = \mu_T$. ▨

For operators without multiplicity this assertion was noted by D. Rastović [11]. In this case it follows immediately from Lemma 11.

Now we turn to the proof of auxiliary propositions that will be needed for the examination of the shift operator.

14. Lemma. *If a family of functions $\{f_s\}_{s=1}^k, f_s \in H^2$, has no common inner divisor, then the inner part of the functions $\sum_{s=1}^k c_s f_s$ is mutually disjoint with any fixed inner function for almost all vectors $c = \{c_s\}_{s=1}^k \in \mathbf{C}^k$.*

Proof. (Induction on k.) If $k = 1$ the function f is outer by the assumption, therefore it is mutually disjoint with any inner function φ. Let lemma be valid for $k \leqslant n$ and consider the case $k = n + 1$.

Put $\psi = \mathrm{GCD}\{\varphi, f_s : 1 \leqslant s \leqslant n\}$, then the induction hypothesis implies

$$\mathrm{GCD}\left\{\varphi, \sum_{s=1}^{n} c_s f_s\right\} = \psi \tag{6}$$

for almost all $c = \{c_s\}_{s=1}^n \in \mathbf{C}^n$. Fix a vector $c \in \mathbf{C}^n$ satisfying (6) and put

$$\varphi_\alpha = \mathrm{GCD}\left\{\varphi, \sum_{s=1}^{n} c_s f_s + \alpha f_{n+1}\right\}.$$

Then

$$\mathrm{GCD}\{\varphi_\alpha, \varphi_\beta\} = \mathrm{GCD}\left\{\varphi, \sum_{s=1}^{n} c_s f_s, f_{n+1}\right\} =$$

$$= \mathrm{GCD}\{\psi, f_{n+1}\} = \mathrm{GCD}\{\varphi, f_s : 1 \leqslant s \leqslant n + 1\} = \mathbf{1},$$

i.e. the functions φ_α are mutually disjoint divisors of the function φ, and hence $\varphi_\alpha = 1$ for almost all α i.e.

$$\mathrm{GCD}\left\{\varphi, \sum_{s=1}^{n+1} c_s f_s\right\} = 1$$

for almost all $c \in \mathbf{C}^{n+1}$.

15. Lemma. *Let $F = \{F_{ij} : 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant N\} \in L(\mathbf{C}^n, H^2_n)$ be a strong outer H^2-function* [1]. *Then for almost all $R \in G_n(\mathbf{C}^N)$ the function* [2] $\det(F \mid R)$ *is mutually disjoint with any fixed inner function φ.*

Proof. Let $\{e^j\}_{j=1}^N$ be a basis in $(\mathbf{C}^N)^*$ and $\{h^s\}_{s=1}^r$ be a basis in the space $(\Lambda^n(\mathbf{C}^N))^*$ of n-forms over $\mathbf{C}^N$, $r = \dim \Lambda^n(\mathbf{C}^N) = \binom{N}{n}$. Let F be the following n-form

$$F = \bigwedge_{i=1}^{n}\left(\sum_{j=1}^{N} F_{ij} e^j\right) = \sum_{s=1}^{r} f_s h^s.$$

By the theorem on outer functions (see [8]) F is outer iff $\mathrm{GCD}\{f_s : 1 \leqslant s \leqslant r\} = \mathbf{1}$. By Lemma 14 for almost all n-vectors $G \in \Lambda^n(\mathbf{C}^N)$ the function

$$\langle F, G\rangle = \sum_{s=1}^{r} f_s \langle h^s, G\rangle$$

is mutually disjoint with φ. This implies the wanted assertion because there exists a measure preserving bijection, mapping n-dimensional subspaces onto the set of decomposable norm one n-vectors

$$R = \mathrm{span}\{g_1, \ldots, g_n\} \Leftrightarrow G = g_1 \wedge \ldots \wedge g_n$$

and $\det(F \mid R) = \langle F, G\rangle$.

16. Lemma. *Let $R \in \mathrm{Cyc}\, S_n$ and φ be a scalar inner function. Then*

a) *for almost all $R' \in G_n(R)$ the subspace $E^{S_n}_{R'}$, has the form ΘH^2_n, where Θ is a two-sided inner function, $\mu_\Theta = 1$* [3] *and* $\mathrm{GCD}\{\det\Theta, \varphi\} = \mathbf{1}$;

b) $R' \in \mathrm{Cyc}\, S_n$ *for almost all* $R' \in G_{n+1}(R)$ [4].

---

[1] A linear continuous transformation $F : E_1 \to H^2(E_2)$ is called a strong H^2-function. It is called outer if $\mathrm{span}\,\{z^n F E_1 : n \geqslant 0\} = H^2(E_2)$. The theorem on outer functions from [8] (p. 38) asserts that a strong H^2-function F is an outer one iff $\dim E_1 \geqslant \dim E_2$ and the principal minors of the matrix F_{ij} (that defines the function F in some bases of E_1 and E_2) have no common inner divisor.

[2] i.e. the inner part of the function.

[3] μ_Θ is by definition the spectral multiplicity of the operator $T_\Theta = P_\Theta S\, K_\Theta$. Recall (see e.g. [9]) that under condition $(I - \Theta) \in \mathfrak{S}_1$ the equality $\mu_\Theta = 1$ holds iff the set of all minors of Θ of corank one has no common inner divisor.

[4] This expression can be incorrect if $\dim R < n + 1$. In such case (here and below) writing that something is valid for almost all $R' \in G_m(R)$, we suppose that the assertion holds for R itself, if m turns out to exceed $\dim R$.

Proof. The proposition a)$_1$ (i.e. a) for $n = 1$) was proved in Lemma 14.

Let us demonstrate now that a)$_n$ $\Rightarrow$ b)$_n$. By Proposition 2.5 it is enough to verify that the subspace ΘH_n^2 (which is defined in the condition a)) together with almost any vector from R generate the whole space H_n^2. Since $\mu_\Theta = 1$, Lemma 12 implies that almost any vector from $P_\Theta R$ is cyclic for the operator $T = P_\Theta S_n | K_\Theta$, as $P_\Theta R \in \operatorname{Cyc} T$. Hence the projections onto K_Θ of almost all vectors $x \in R$ are T-cyclic, and therefore

$$\operatorname{span}\{\Theta H_n^2,\ S_n^k x\ :\ k \geqslant 0\} =$$

$$= \Theta H_n^2 \oplus \operatorname{span}\{T^k P_\Theta x\ :\ k \geqslant 0\} = H_n^2.$$

Now to complete the proof by induction, we have only to verify that the condition b)$_{n-1}$ implies a)$_n$ for all $n \geqslant 2$.

Let π_{n-1} be the orthogonal projection of $H_n^2 = H_{n-1}^2 \oplus H^2$ onto H_{n-1}^2. If $R \in \operatorname{Cyc} S_n$ then $\pi_{n-1} R \in \operatorname{Cyc} S_{n-1}$ and by condition b)$_{n-1}$ almost all n-dimensional subspaces of $\pi_{n-1}R$ are S_{n-1}-cyclic, hence the inclusion $\pi_{n-1}R' \in \operatorname{Cyc} S_{n-1}$ is valid for almost all $R' \in G_n(R)$ or, in other words, for Θ defined by the equality

$$E_{R'}^{S_n} = \Theta H_l^2 \tag{7}$$

the function $\pi_{n-1}\Theta$ is outer. On the other hand by Lemma 5 rank $\Theta = n$ for almost all n-dimensional subspaces $R' \subset R$. So in (7) $l = n$ for almost all $R' \in G_n(R)$, i.e. Θ is a two-sided inner function. Moreover $\mu_\Theta = 1$, because the fact that $\pi_{n-1}\Theta$ is outer implies that($\pi_{n-1}\Theta$)'s minors of rank $n - 1$ have no common inner divisor a fortiori it is valid for Θ's minors of rank $n - 1$.

The assertion that $\operatorname{GCD}\{\det \Theta, \varphi\} = 1$ for almost all R' is contained in Lemma 15. Indeed, let$\{F_j\}_{j=1}^N$ be a basis in R ($N = \dim R$) and F_{ij} be the coordinate functions of F_j, $F_j \in H_n^2$. The cyclicity of R implies that the strong H^2-function $F = \{F_{ij}\}$ is outer. Since the function Θ from (7) is the inner part of $F|R'$, we have

$$\operatorname{GCD}\{\det \Theta, \varphi\} = \operatorname{GCD}\{\det(F|R')\,\varphi\} = 1$$

for almost all $R' \in G_n(R)$. ▨

17. LEMMA. *Let $T \in L(H)$ be a C_0-contraction, $\mu_T < \infty$ and $R \in \operatorname{Cyc}(S_n \oplus T)$. Then almost all subspaces $R' \subset R$ with* $\dim R = n + \max\{1, \mu_T\}$ *are cyclic for $S_n \oplus T$.*

Proof. Since $P_{H_n^2} R \in \operatorname{Cyc} S_n$, Lemma 16 implies that almost all $(n + 1)$-dimensional subspaces of $P_{H_n^2} R$ are S_n-cyclic. Therefore we have the equality

$$\operatorname{span}\{S_n^k P_{H_n^2} R'\ :\ k \geqslant 0\} = H_n^2 \tag{8}$$

for almost all $R' \in G_{n+1}(R)$. Let $A = S_n \oplus T$ and m_T be the minimal function of T. If for R' (8) is valid, we have the following inclusion

$$\operatorname{span}\{A^k R' : k \geqslant 0\} \supset m_T(A) \operatorname{span}\{A^k R' : k \geqslant 0\} =$$

$$= [m_T(S_n) \operatorname{span}\{S_n^k (P_{H_k^2} R' : k \geqslant 0\}] \oplus \{\mathbf{O}\} = m_T H_n^2 \oplus \{\mathbf{O}\}.$$

Let $H' := (H_n^2 \ominus m_T H_n^2) \oplus H$ and $A' = P_{H'} A_{|} H'$. The operator A' is a C_0-contraction and its Jordan model is the orthogonal sum of n copies of the contraction with scalar characteristic function m_T and of the Jordan model for T, therefore $\mu_{A'} = n + \mu_T$. Since $P_{H'} R \in \operatorname{Cyc} A'$, Lemma 12 implies that

$$\operatorname{span}\{A'^k P_{H'} R' \ : \ k \geqslant 0\} = H' \tag{9}$$

for almost all $R' \subset R$, $\dim R' = n + \mu_T$ (see Proposition 2.6). So for almost all $R' \subset R$ with $\dim R' = n + \max\{1, \mu_T\}$ we have both (8) and (9). As it was shown (8) yields the inclusion

$$E_{R'}^A \supset m_T H_n^2 \oplus \{\mathbf{O}\} = (H')^{\perp}$$

therefore (9) implies the equality

$$E_{R'}^A = (H')^{\perp} \oplus \operatorname{span}\{A'^k P_{H'} R' \ : \ k \geqslant 0\} = H_n^2 \oplus H. \qquad \square$$

18. COROLLARY. *The operator* S_n *has many cyclic subspaces and* $\operatorname{disc} S_n = n + 1$. $\square$

19. COROLLARY. *If T is a C_0-contraction, the operator $S_n \oplus T$ has many cyclic subspaces and* $\operatorname{disc}(S_n \oplus T) = n + \mu_T$. $\square$

20. LEMMA. *Let $\Theta \in H^\infty(\mathbf{C}^n, \mathbf{C}^p)$ be an inner function, $\Theta = \Theta_*^e \Theta_*^i$ be its $*$-canonical factorization (for definition see [7]), $K \overset{\text{def}}{=} H_p^2 \ominus \Theta H_n^2$, $T \overset{\text{def}}{=} S_p^* | K$. Then $R \in \operatorname{Cyc} T$ implies the inclusion*

$$E_{R'}^T \supset H_p^2 \ominus \Theta_*^e H_n^2$$

for almost all $R' \in G_{p-n}(R)$.

Proof. (Induction on p, $p \geqslant n$). For $p = n$ the assertion of the lemma is trivial. In this case Θ is a two-sided inner function, i.e. $\Theta_*^e = I$, and both sides of the inclusion are zero spaces.

If $p > n$ then the subspace of vectors y, $y \in R$, such that $E_y^T = H_p^2 \ominus \Theta_y H_p^2$ (i.e. Θ_y is a two-sided inner function) has non zero codimension in R. (Such vector functions y are called pseudocontinuable: for every coordinate function y_i there exists a pair u_i, v_i of analytic functions bounded in $\hat{\mathbf{C}} \setminus \operatorname{clos} \mathbf{D}$ such that $u_i(\zeta) = y_i(\zeta) \times$

$\times v_i(\zeta)$ for almost all $\zeta \in \mathbf{T}$.) Indeed, K is the span of a finite set of subspaces E_T^y (e.g. y runs over a basis in R), but the span of a finite set of subspaces of the form $H_p^2 \ominus \Theta H_p^2$ is of the same form and cannot coincide with K.

So for almost all $y \in R$ we have $E_y = H_p^2 \ominus \Theta_y H_r^2$, where $r < p$. Since $E_y \subset K$, the inclusion $\Theta_y H_r^2 \supset \Theta H_n^2$ is valid, hence there exists an inner function θ such that $\Theta = \Theta_y \theta$. The following decomposition

$$K = E_y \oplus E_y^\perp = (H_p^2 \ominus \Theta_y H_r^2) \oplus \Theta_y (H_r^2 \ominus \theta H_n^2)$$

shows that the factor-operator $T' = P_{E_y^\perp} T | E_y^\perp$ is unitarily equivalent to $S_r^* | (H_r^2 \ominus \ominus \theta H_n^2)$. Now we use the induction hypothesis, namely, that lemma holds for every $r < p$, i.e. for almost all $(r-n)$-dimensional subspaces $R'' \subset P_{E_y^\perp} R$ the inclusion

$$E_{R''}^{T'} \supset \Theta_y (H_r^2 \ominus \theta_*^e H_n^2)$$

is valid. Therefore (see Propositions 2.5 and 2.6) for almost all $(r+1-n)$-dimensional subspaces $R' \subset R$ we have

$$E_{R'}^T \supset E_y^T \oplus \Theta_y (H_r^2 \ominus \theta_*^e H_n^2) = H_p^2 \ominus \Theta_y \theta_*^e H_n^2.$$

Since θ_*^i is a right-hand divisor of Θ_*^i, Θ_*^e is a left-hand divisor of $\Theta_y \theta_*^e$, i.e. $\Theta_y \theta_*^e H_n^2 \subset \Theta_*^e H_n^2$, hence

$$E_{R'}^T \supset H_p^2 \ominus \Theta_*^e H_n^2$$

for almost all $(r+1-n)$-dimensional subspaces $R' \subset R$ and consequently for almost all $(p-n)$-dimensional subspaces $R' \subset R$. ▨

21. Corollary. *The operator S_p^* has many cyclic subspaces and* $\operatorname{disc} S_p^* = P$.

Proof. Put $n = 0$ in Lemma 20 and use Lemma 8 with $\mu_U = \boldsymbol{\mu} = 0$. ▨

22. Corollary. *Let $T \in L(H)$ be a C_0-contraction. Then the operator $S_k^* \oplus T$ has many cyclic subspaces and* $\operatorname{disc}(S_k^* \oplus T) = \max\{k, \mu_T\}$.

Proof. Let $R \in \operatorname{Cyc}(S_k^* \oplus T)$. By the preceeding corollary the following equality

$$\operatorname{span}\{S_k^{*n} P_{H_k^2} R' : n \geqslant 0\} = H_k^2 \tag{10}$$

holds for almost all $R' \in G_k(R)$. Therefore for the same R' we have

$$E_{R'}^A \supset m_T(A)\, E_{R'}^A = m_T(S_k^*) H_k^2 \oplus \{\mathbf{O}\} =$$
$$= P_+ m_T(\bar{z}) H_k^2 \oplus \{\mathbf{O}\} = H_k^2 \oplus \{\mathbf{O}\},$$

where $A = S_k^* \oplus T$ and m_T is the minimal function of T.

By Lemma 12 for almost all $R' \in G_{\mu_T}(R)$ the equality

$$\text{(11)} \qquad \operatorname{span}\{T^n P_H R' : n \geqslant 0\} = H$$

holds. Hence for almost all $R' \in G_l(R)$, where $l = \max\{k, \mu_T\}$, we have both (10) and (11) simultaneously, i.e. $R' \subset \operatorname{Cyc} A$, therefore $\operatorname{disc} A \leqslant \max\{k, \mu_T\}$. The opposite inequality is trivial (see Proposition 2.2).

The following lemma completes the proof of the theorem.

23. LEMMA. *Let U be a unitary operator with an absolutely continuous spectral measure, T be a C_0-contraction, $A = S_k^* \times U \times S_n \times T$* [1]*, $d = n + \max\{1, \mu_T k + \mu_U + \mu\}$, $R \in \operatorname{Cyc} A$. Then almost all $R' \in G_d(R)$ are A-cyclic.*

Proof. We shall suppose that A is defined in the following Hilbert space $\mathscr{H}$:

$$\mathscr{H} = H^2_{-k} \times H_U \times H^2_n \times H_T$$

$U \in L(H_U)$, $T \in L(H_T)$, the backward shift S_k^* is defined in $H^2_{-k} \overset{\text{def}}{=} L^2_k \ominus H^2_k$ as follows $S_k^* = P_- \mathscr{S}_k | H^2_{-k}$. We can assume

$$\mathscr{H} \subset L^2_k \times H_U \times L^2_n \times H_T$$

and

$$L^2_k \times H_U \times L^2_n = \int_{\mathbf{T}}^{\oplus} (\mathbf{C}^k \times H(t) \times \mathbf{C}^n) \, d\nu(t),$$

$\dim H(t) = r(t)$ ν-a.e., $r(t)$ being local spectral multiplicity of U. Let us note that if $k + n > 0$ or if U is nonreductive then ν is the Lebesgue measure.

Let $j(t)$ be (so as in Lemma 5) the family of evaluation mappings from $L^2_k \times \times H_U \times L^2_n$ into $\mathbf{C}^k \times H(t) \times \mathbf{C}^n : j(t)h = h(t)$ a.e. $\forall h$. If $R \in \operatorname{Cyc} A$ then $j(t) P_{H_U \times H^2_n} R = H(t) \times \mathbf{C}^n$ ν-a.e. and by Lemma 5 for almost all $R_1 \in G_{n+\mu_U}(R)$ the equality

$$\text{(12)} \qquad j(t) P_{H_U \times H^2_n} R_1 = H(t) \times \mathbf{C}^n$$

holds ν-a.e. . Fixing such a subspace R_1 put

$$L = \operatorname{clos} m_T(A) \operatorname{span}\{A^s R_1 : s \geqslant 0\}.$$

For the sake of convenience we write sometimes, for example, $H_U \times H^2_n$ instead of $\{\mathbf{O}\} \times H_U \times H^2_n$, $H(t) \times \mathbf{C}^n$ instead of $\{\mathbf{O}\} \times H(t) \times \mathbf{C}^n$ and so on.

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[1] We write here and below the direct product instead of the orthogonal sum to distinguish it from orthogonal decompositions occuring in the proof.

Let us introduce the following auxiliary space

$$\mathscr{H}' = L_k^2 \times H_U \times H_n^2$$

and the operator

$$A' = \mathscr{S}_k \times U \times S_n$$

further put

$$L' = m_T(A')\mathrm{span}\{A' P_{\mathscr{H}'} R_1 : s \geqslant 0\}.$$

We shall denote, when it will be convenient, the elements of a direct product as a column. For example we can write the following evident equality

$$L \oplus \begin{pmatrix} H_k^2 \\ \mathbf{O} \\ \mathbf{O} \\ \mathbf{O} \end{pmatrix} = \mathrm{span}\left\{ L', \begin{pmatrix} H_k^2 \\ \mathbf{O} \\ \mathbf{O} \end{pmatrix} \right\} \times \{\mathbf{O}\}_{H_T} \overset{\mathrm{def}}{=} L'' \times \{\mathbf{O}\}_{H_T}.$$

Since L'' is invariant under multiplication by the independent variable, it has the form

$$L'' = \Theta H_l^2 \oplus M_\rho \tag{13}$$

where Θ is a measurable operator-valued function isometrical almost everywhere on $\mathbf{T}$, M_ρ is a reducing subspace of the multiplication operator; $\rho(t)$ is the local spectral multiplicity of the unitary operator being a part of multiplication operator in M_ρ.

The equality (12) yields

$$j(t)\, L'' = \mathbf{C}^k \times H(t) \times \mathbf{C}^n.$$

On the other hand

$$j(t)\, L'' = j(t)\, \Theta H_l^2 \oplus j(t)\, M_\rho$$

hence

$$\dim j(t)\, L'' = k + r(t) + n = l + \rho(t) \quad \nu\text{-a.e.}$$

and therefore

$$\mathscr{H}' \oplus \begin{pmatrix} \mathbf{O} \\ \mathbf{O} \\ H_{-n}^2 \end{pmatrix} = \Theta L_l^2 \oplus M_\rho.$$

Since $\bigcap z^n \mathscr{H}' = L_k^2 \times H_U$, we have $M_\rho \subset L_k^2 \times H_U$. This inclusion implies that $L_n^2 \subset \Theta L_l^2$, i.e. Θ^* is isometrical on L_n^2. Putting $\theta = P_{L_n^2}\Theta$ we can write

$$\Theta\theta^* = \begin{pmatrix} \mathbf{O} \\ \mathbf{O} \\ I \end{pmatrix}$$

and

$$\mathscr{H}'\ominus L''=\Theta H^2_{-l}\ominus\begin{pmatrix}\mathbf{O}\\ \mathbf{O}\\ H^2_{-n}\end{pmatrix}=\Theta[H^2_{-l}\ominus\theta^* H^2_{-n}]\overset{\text{def}}{=}\Theta K.$$

Note that $\theta H^2_l\subset H^2_n$ because $\Theta H^2_l\subset\mathscr{H}'$, hence $\theta\in H^\infty(\mathbf{C}^l,\mathbf{C}^n)$ i.e. θ is $*$-inner.

Let us return now to the operator A and show that the operator

$$A_1=P_{L^\perp}A\,|\,L^\perp$$

is unitarily equivalent to $(S_l^*|K_{\tilde\theta})\times T$, where $\tilde\theta$ is the inner function defined by the formula $\tilde\theta(z)=\theta(\bar z)^*$. For the orthogonal complement $L^\perp$ we have:

$$L^\perp=\mathscr{H}\ominus L=\begin{pmatrix}L^2_k\\ H_U\\ H^2_n\\ H_T\end{pmatrix}\ominus\left[L\oplus\begin{pmatrix}H^2_k\\ \mathbf{O}\\ \mathbf{O}\\ \mathbf{O}\end{pmatrix}\right]=$$

$$=[\mathscr{H}'\ominus L'']\times H_T=\Theta K\times H_T.$$

Letting now

$$\widetilde K\overset{\text{def}}{=}K_{\tilde\theta}=H^2_l\ominus\tilde\theta H^2_n$$

$$A_2\overset{\text{def}}{=}A_1\,|\,\Theta K$$

$$J:L^2(E)\to L^2(E),\quad (Jf)(\zeta)\overset{\text{def}}{=}\bar\zeta f(\bar\zeta),$$

we obtain for $f\in\Theta K$

$$A_2f=P_{\Theta K}A'f=\Theta(P_K\mathscr{S}_l|K)\,\Theta^*f=$$

$$=\Theta J(P_{\widetilde K}S_l^*\,|\,\widetilde K)\,J\Theta^*f=\Theta J(S_l^*\,|\,\widetilde K)J\Theta^*f.$$

Since $P_{\Theta K}R\in\operatorname{Cyc}A_2$, by Lemma 20 we have for almost all $R_2\in G_{l-n}(R)$ the following inclusion

$$\operatorname{span}\{A_2^sP_{\Theta K}R_2:s\geqslant 0\}\supset\Theta J[H^2_l\ominus(\tilde\theta)^e_*\,H^2_n]=$$

$$=\Theta J[H^2_l\ominus\tilde\theta^e H^2_n]=\Theta[H^2_{-l}\ominus(\theta^e)^*\,H^2_{-n}]=$$

$$=\Theta K\ominus\Theta[(\theta^e)^*\,H^2_{-n}\ominus\theta^*H^2_{-n}]=\Theta K\ominus\Theta\theta^*[\theta^iH^2_{-n}\ominus H^2_{-n}]=$$

$$=\Theta K\ominus\begin{pmatrix}\mathbf{O}\\ \mathbf{O}\\ I\end{pmatrix}[H^2_n\ominus\theta^iH^2_n]=\Theta K\ominus\begin{pmatrix}\mathbf{O}\\ \mathbf{O}\\ K_{\theta^i}\end{pmatrix}.$$

Now we put $R'=\operatorname{span}\{R_1,R_2\}$. Since R_1 varies in a set of full measure in $G_{n+\mu_U}(R)$ and R_2 varies in a set of full measure in $G_{l-n}(R)$, by Proposition 2.5 R'

fills a set of full measure in $G_{l+\mu_U}(R)$. Let us show that all these subspaces have the following property:

(14) $$E^A_{R'} \supset H^2_{-k} \times H_U.$$

It is sufficient to verify that

$$\operatorname{clos} m_T(A)E^A_{R'} \supset H^2_{-k} \times H_U$$

or an equivalent inclusion

$$\operatorname{span}\left\{\begin{pmatrix} H^2_k \\ \mathbf{O} \\ \mathbf{O}\end{pmatrix},\ m_T(A')A'^s P_{\mathscr{H}'}R' : s \geqslant 0\right\} \supset \begin{pmatrix} L^2_k \\ H_U \\ \mathbf{O}\end{pmatrix}.$$

However the left side of this inclusion contains the subspace

$$\operatorname{span}\left\{\begin{pmatrix} m_T H^2_k \\ \mathbf{O} \\ \mathbf{O}\end{pmatrix},\ m_T(A')A'^s P_{\mathscr{H}'}R' : s \geqslant 0\right\} =$$

$$= m_T(A') \operatorname{span}\left\{\begin{pmatrix} H^2_k \\ \mathbf{O} \\ \mathbf{O}\end{pmatrix},\ A'^s P_{\mathscr{H}'}R' : s \geqslant 0\right\} \supset$$

$$\supset m_T(A') \operatorname{span}\left\{\begin{pmatrix} H^2_k \\ \mathbf{O} \\ \mathbf{O}\end{pmatrix},\ L',\ A'^s P_{\mathscr{H}'}R_2 : s \geqslant 0\right\} =$$

$$= m_T(A')\,[L'' \oplus \operatorname{span}\{A_2^s P_{\Theta K} R_2\ :\ s \geqslant 0\}] \supset$$

$$\supset m_T(A')\left[L'' \oplus \Theta K \ominus \begin{pmatrix} \mathbf{O} \\ \mathbf{O} \\ K_{\theta^{\mathrm{i}}}\end{pmatrix}\right] =$$

$$= m_T(A')\begin{pmatrix} L^2_k \\ H_U \\ \theta^{\mathrm{i}} H^2_n\end{pmatrix} = \begin{pmatrix} L^2_k \\ H_U \\ m_T\theta^{\mathrm{i}} H^2_n\end{pmatrix} \supset \begin{pmatrix} L^2_k \\ H_U \\ \mathbf{O}\end{pmatrix}.$$

Thus (14) is valid for almost all $R' \in G_{l+\mu_U}(R)$ and therefore it is valid for almost all $R' \in G_d(R)$, because $l = \operatorname{rank}\Theta(t) \leqslant k + r(t) + n$ ν-a.e., i.e. $l \leqslant k + \boldsymbol{\mu} + n$ and $l + \mu_U \leqslant k + \boldsymbol{\mu} + n + \mu_U \leqslant d$.

By Lemma 17 for almost all subspaces $R' \subset R$, $\dim R' = n + \max\{1, \mu_T\}$, the following equality

$$\operatorname{span}\{(S_n \times T)^s P_{H_n^2 \times H_T} R' : s \geqslant 0\} = H_n^2 \times H_T \tag{15}$$

is valid. Consequently, it is valid for almost all $R' \in G_d(R)$ because $d \geqslant n + \max\{1, \mu_T\}$. Therefore for almost all $R' \in G_d(R)$ we have simultaneously the inclusion (14) and the equality (15), that is for these subspaces $E_{R'}^A = \mathscr{H}$. $\square$

24. CONCLUDING REMARKS. We have considered only the orthogonal sum of operators. It was mentioned, however, that orthogonality does not matter, because similarities preserve the lattice of invariant subspaces. It is unknown (see[3]) whether disc is invariant under quasi-similarities. This is anyway true for quasi--similarities preserving the lattice of invariant subspaces. This seems to be the case for quasi-similarities of weak contractions and their Jordan models (see [12]). Jordan operators are a particular case of operators considered above. We are going to return to this question elsewhere.

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