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# EVERY $C_{00}$ CONTRACTION WITH HILBERT-SCHMIDT DEFECT OPERATOR IS OF CLASS $C_0$

KATSUTOSHI TAKAHASHI and MITSURU UCHIYAMA

## 1. INTRODUCTION

According to the theory of Sz.-Nagy and Foiaş [5], the most natural generalization of the theory of the minimal polynomial for a matrix of finite order is developed for a contraction $T$ of class $C_0$; there exists a non-zero analytic function $u(\lambda)$ in the Hardy space $H^\infty$ such that $u(T) = 0$ and the spectrum of $T$ in the open unit disc coincides with the set of zeros of the function $u(\lambda)$. Each contraction $T$ of class $C_0$ is of class $C_{00}$, that is, $T^n \to 0$ and $T^{*n} \to 0$ strongly as $n \to \infty$.

An important example of a contraction of class $C_0$ is any contraction $T$ of class $C_{00}$, which is a *weak contraction* in the sense that it satisfies the following two conditions (see [5, Chapter VIII]):

(i) the spectrum of $T$ does not fill the unit disc, and

(ii) the *defect operator* $D_T = (I - T^*T)^{1/2}$ is of Hilbert-Schmidt class, i.e. $\mathrm{tr}(I - T^*T) < \infty$.

The main result of the present paper is that, for a contraction of class $C_{00}$, (i) is a consequence of (ii). In this connection we remark that $\mathrm{tr}(I - T^*T) < \infty$ can not be replaced by $\mathrm{tr}((I - T^*T)^p) < \infty$ for any $p > 1$. Then we apply the result to obtain various characterizations for a contraction $T$, for which $T^{*n} \to 0$ and $\mathrm{tr}(I - T^*T) < \infty$, to be of class $C_0$. As another application, we present a characterization for a hyponormal (i.e. $T^*T \geqslant TT^*$) contraction to have no non-trivial normal direct summand.

## 2. CONTRACTION OF CLASS $C_{00}$

A contraction $T$, i.e. $\|T\| \leqslant 1$, on a separable Hilbert space is said to be *completely non-unitary* if it has no non-trivial unitary direct summand. Sz.-Nagy and Foiaş [5] developed a $H^\infty$-functional calculus for each completely non-unitary contraction, that is, there is a weak*-weak continuous (multiplicative) homomorphism, $u(\lambda) \to u(T)$, from the Hardy space $H^\infty$ on the open disc $\mathbf{D} = \{\lambda : |\lambda| < 1\}$ to the weakly closed algebra generated by $T$ that is an extension of the usual functional calculus with polynomials. Then $T$ is said to be of class $C_0$, in short $T \in C_0$, if the homomorphism is not injective, that is, $u(T) = 0$ for some non-zero function $u(\lambda)$.

To each completely non-unitary contraction $T$ there corresponds its characteristic function $\Theta_T(\lambda)$; it is an operator-valued $H^\infty$-function whose values are contractions from $\mathscr{D}_T$, the closure of the range of the defect operator $D_T$ to $\mathscr{D}_{T^*}$, the closure of the range of $D_{T^*}$. We refer the detailed theory to the monograph [5], but let us cite here only the following important result [5, Chapter VI, Theorem 5.1]: for a contraction $T$ of class $C_{00}$, $u(T) = 0$ for some non-zero $H^\infty$-function $u(\lambda)$ if and only if $u(\lambda)$ is a *scalar multiple* of $\Theta_T(\lambda)$ in the sense that there exists an operator-valued $H^\infty$-function $A(\lambda)$ whose values are bounded linear operators from $\mathscr{D}_{T^*}$ to $\mathscr{D}_T$ such that

$$A(\lambda)\Theta_T(\lambda) = u(\lambda)I_{\mathscr{D}_T} \quad \text{and} \quad \Theta_T(\lambda)A(\lambda) = u(\lambda)I_{\mathscr{D}_{T^*}} \quad \text{for } \lambda \in \mathbf{D},$$

where $I_{\mathscr{D}_T}$ and $I_{\mathscr{D}_{T^*}}$ are the identity operators on $\mathscr{D}_T$ and $\mathscr{D}_{T^*}$ respectively.

A contraction $T$ is said to be of class $C_{0\cdot}$, in short $T \in C_{0\cdot}$, if $T^n \to 0$ strongly as $n \to \infty$. $T$ is of class $C_{1\cdot}$, in short $T \in C_{1\cdot}$, if $\inf \|T^n h\| > 0$ for all non-zero $h$. The class $C_{\cdot 0}$ and $C_{\cdot 1}$ are defined by using $T^*$ instead of $T$. For $\alpha, \beta = 0, 1$, $T \in C_{\alpha\beta}$ means that $T \in C_{\alpha\cdot}$ and $T \in C_{\cdot\beta}$ simultaneously.

For other general definitions and results of operator theory we refer to the monograph [5].

THEOREM 1. *If a contraction $T$ is of class $C_{00}$ and the defect operator $D_T$ is of Hilbert-Schmidt class, then $T$ is of class $C_0$.*

*Proof.* Let us consider first the case when the point spectrum $\sigma_p(T)$ of $T$ does not fill the open disc $\mathbf{D}$. A proof for this case can be derived from a general result of the previous paper [6] of one of the authors, but let us present here a simpler version adapted to the present case. Since the Möbius transform $(T - \alpha I)(I - \bar{\alpha} T)^{-1}$ for any fixed $\alpha \in \mathbf{D}$ does not change any situations (see [5, p. 240]), we may assume $0 \notin \sigma_p(T)$. Since $D_T$ is of Hilbert-Schmidt class, the identity

$$I + (T^* | \mathscr{D}_{T^*})\, \Theta_T(\lambda) = D_T^2 + \lambda D_T T^*(I - \lambda T^*)^{-1} D_T$$

shows that $I + (T^* | \mathscr{D}_{T^*})\Theta_T(\lambda)$ is of trace class for $\lambda \in \mathbf{D}$, and we can define the determinant

$$\delta(\lambda) = \det((-T^* | \mathscr{D}_{T^*})\Theta_T(\lambda))$$

as a function in $H^\infty$. Since $0 \notin \sigma_p(T)$ and $\operatorname{tr}(I - T^*T) < \infty$ imply that $T^*T$ is invertible,

$$\delta(0) = \det(T^*T | \mathscr{D}_T) \neq 0,$$

so that $\delta(\lambda)$ is non-zero. According to Bercovici and Voiculescu [1], there exists an operator-valued $H^\infty$-function $A(\lambda)$ whose values are contractions on $\mathscr{D}_T$ such that

$$A(\lambda)\cdot(T^*\mid\mathscr{D}_T)\Theta_T(\lambda) = \delta(\lambda)I_{\mathscr{D}_T} \quad \text{for } \lambda\in\mathbf{D}.$$

Since $\Theta_T(e^{it})$ is unitary for almost all $t$ because of $T\in C_{00}$ (see [5, Chapter VI, Proposition 3.5]), we have also

$$\Theta_T(\lambda)\cdot A(\lambda)\,(T^*\mid\mathscr{D}_{T^*} = \delta(\lambda)I_{\mathscr{D}_{T^*}} \quad \text{for } \lambda\in\mathbf{D}.$$

Therefore $\delta(\lambda)$ is a scalar multiple of $\Theta_T(\lambda)$, which is equivalent to $T\in C_0$.

Let us prove next, by contradiction, that the assumption of the above case is really satisfied. Suppose that $\sigma_p(T)$ fills $\mathbf{D}$, and let $\mathscr{M}$ denote the closed linear span of $\{\ker(T-\lambda I): 0\neq\lambda\in\mathbf{D}\}$. Then $\mathscr{M}$ is an invariant subspace of $T$, different from $\{0\}$. Let $T_1$ be the restriction of $T$ to $\mathscr{M}$. First, since the restriction to an invariant subspace does not affect the $C_{00}$-property, $T_1$ is again of class $C_{00}$. Next, $I_1-T_1^*T_1$ is of trace class, $I_1$ being the identity operator on $\mathscr{M}$, because

$$\mathrm{tr}(I_1-T_1^*T_1)\leqslant \mathrm{tr}(I-T^*T)<\infty.$$

Since

$$h=\lambda^{-1}Th=\lambda^{-1}T_1h \quad \text{for } 0\neq\lambda\in\mathbf{D} \text{ and } h\in\ker(T-\lambda I),$$

$T_1$ has dense range by the definition of $\mathscr{M}$, so that $0\notin\sigma_p(T_1^*)$. Now the identity

$$(I_1-T_1T_1^*)T_1 = T_1(I_1-T_1^*T_1)$$

implies that the selfadjoint operators $I_1-T_1T_1^*$ and $(I_1-T_1^*T_1)\mid(\ker T_1)^\perp$ are unitarily equivalent (e.g. [2, p. 82]), hence $I_1-T_1T_1^*$ is of trace class. Then it follows from the already proved case that $T_1^*$ is of class $C_0$. Therefore the spectrum of $T_1^*$ in $\mathbf{D}$ is discrete, which contradicts $\lambda\in\sigma_p(T_1)$ for all non-zero $\lambda\in\mathbf{D}$. This contradiction proves that the point spectrum $\sigma_p(T)$ does not fill $\mathbf{D}$. ▨

Let us show, by an example, that the restriction $\mathrm{tr}(I-T^*T)<\infty$ in Theorem 1 can not be replaced by $\mathrm{tr}((I-T^*T)^p)<\infty$ for any $p>1$.

Consider an orthonormal basis $\{e_n\}_{n=0}^\infty$ of a Hilbert space, and let $T$ be a unilateral weighted shift defined by

$$Te_{n-1}=\alpha_n e_n, \quad \text{where } \alpha_n = 1-(n+1)^{-1},\ n=1,2,\ldots.$$

Obviously $T$ is a contraction of class $C_{\cdot 0}$. Since $\sum_{n=0}^{\infty}(1-\alpha_n)=\infty$, $\prod_{k=1}^{n}\alpha_k$ converges to 0 as $n\to\infty$, which implies $T^n\to 0$, hence $T$ is of class $C_{00}$. Finally since

$\{e_n\}$ is the complete system of eigenvectors of $I - T^*T$ with the system of eigenvalues $\{1-\alpha_n^2\}$, for any $p > 1$

$$\mathrm{tr}((I - T^*T)^p) \leqslant 2^p \sum_{n=1}^{\infty} n^{-p} < \infty.$$

But it is known (e.g. [2, p. 158]) that the spectrum of $T$ fills the whole unit disc, hence $T$ is not of class $C_0$.

## 3. CONTRACTION OF CLASS $C_{.0}$

Here we present a reformulation of Theorem 1 in the form of equivalence of various conditions for a contraction of class $C_{.0}$.

THEOREM 2. *Let $T$ be a contraction of class $C_{.0}$ such that the defect operator $D_T$ is of Hilbert-Schmidt class. Then the point spectrum of $T$ does not fill the open unit disc, and the following conditions are mutually equivalent.*

(1) *$T$ is of class $C_{0.}$.*
(2) *$T$ is of class $C_0$.*
(3) $\mathrm{ind}(T) = 0$.
(4) *$T = U + K$ for a unitary operator $U$ and an operator of trace class $K$.*
(5) *The planar Lebesgue measure of the spectrum of $T$ is zero.*
(6) *$T$ is a weak contraction.*

*Proof*. Let $T = \begin{pmatrix} T_1 & T_{12} \\ 0 & T_2 \end{pmatrix}$ be the triangulation of $T$ of type $\begin{pmatrix} C_{00} & * \\ 0 & C_{10} \end{pmatrix}$ (see [5, p. 73]). Then since the defect operator $D_{T_1}$ is of Hilbert-Schmidt class, $T_1$ is of class $C_0$ by Theorem 1. Since the point spectrum $\sigma_p(T_2)$ is empty, $\sigma_p(T)$ is included in $\sigma(T_1)$. This shows that $\sigma_p(T)$ is at most countable. (For a completely non-unitary contraction $T$, $\sigma_p(T)$ does not intersect with the unit circle.)

We now show the equivalence of the above six conditions. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 1 and the properties of operators of class $C_0$ mentioned in Introduction. (2) $\Rightarrow$ (3) follows from index theory of Fredholm operators (see [3, Chapter 5]). To see (3) $\Rightarrow$ (4), let $T = V|T|$ be the polar decomposition of $T$, that is, $|T| = (T^*T)^{1/2}$ and $V$ is the partial isometry with initial space range $(T^*)$ and final space range$(T)$ ($T$ being Fredholm). Then (3) implies that $V$ can be replaced by a unitary operator $U$. Since the assumption $\mathrm{tr}(I - T^*T) < \infty$ implies $\mathrm{tr}(I - |T|) < \infty$, the operator $K = -U(I - |T|)$ meets the requirement of (4). Since $\sigma_p(T)$ is at most countable, (4) $\Rightarrow$ (5) is clear. And (5) $\Rightarrow$ (6) is trivial. Finally (6) $\Rightarrow$ (2) follows from the $C_0 - C_{11}$ decomposition for weak contractions (see [5, p. 331]). □

COROLLARY 3. *Let $U$ be a unitary operator and $K$ an operator of trace class. If $T = U + K$ is a contraction, then $T^n \to 0$ is equivalent to $T^{*n} \to 0$.*

## 4. HYPONORMAL CONTRACTION

Recall that a *hyponormal* operator $T$, i.e. $T^*T \geqslant TT^*$, is said to be *completely non-normal* if $T$ has no non-trivial normal direct summand. Putnam [4] proved that every completely non-normal, hyponormal contraction is of class $C_{\cdot 0}$. We shall show that a better description of completely non-normality can be given in case the defect operator $D_T$ is of Hilbert-Schmidt class.

THEOREM 4. *Let $T$ be a hyponormal contraction such that the defect operator $D_T$ is of Hilbert-Schmidt class. Then $T$ is completely non-normal if and only if $T$ is of class $C_{10}$.*

*Proof.* Since, for a normal contraction $N$, the conditions $N \in C_{\cdot 0}$ and $N \in C_{0 \cdot}$ are equivalent, the assumption $T \in C_{10}$ excludes the existence of any non-trivial normal direct summand.

Suppose conversely that $T$ is completely non-normal. Then by the theorem of Putnam mentioned above, $T$ is of class $C_{\cdot 0}$. Let $T_1$ be the restriction of $T$ to the invariant subspace $\mathcal{M} := \{h: T^n h \to 0\}$. Then $T_1$ is a hyponormal contraction of class $C_{00}$, and $I_{\mathcal{M}} - T_1^* T_1$ is of trace class. Then according to Theorem 2, the planar Lebesgue measure of the spectrum of $T$ is zero. On the other hand, according to the area theorem of Putnam (see [2, p. 294]) the planar Lebesgue measure of any non-normal hyponormal operator is positive. This leads to the conclusion that $\mathcal{M} = \{0\}$, or equivalently $T$ is of class $C_{10}$. ▨

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*KATSUTOSHI TAKAHASHI*
*Division of Applied Mathematics,*
*Research Institute of Applied Electricity,*
*Hokkaido University,*
*Sapporo 060,*
*Japan.*

*MITSURU UCHIYAMA*
*Department of Mathematics,*
*Fukuoka University of Education,*
*Munakata, Fukuoka 811—41,*
*Japan.*

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