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NUCLEAR OPERATORS IN NEST ALGEBRAS

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1. INTRODUCTION

The main result shows that each nuclear operator T in a nest algebra $\mathrm{Alg}\,\mathscr{E}$ admits a representation

$$T = \int_{\mathscr{E}} T_E \mathrm{d}\tau(E),$$

where τ is a finite positive Borel measure on the nest and $T \to T_E$ is a nuclear operator valued function on $\mathscr{E}$ such that $T_E = ET_E(I - E_-)$ almost everywhere. This representation leads to conditions under which T can be decomposed into an *exact* sum of rank one operators in $\mathrm{Alg}\,\mathscr{E}$ in the following sense:

$$T = \sum_{i=1}^{\infty} R_i, \quad \|T\|_1 = \sum_{i=1}^{\infty} \|R_i\|_1$$

with $R_1, R_2, \ldots$ rank one operators in $\mathrm{Alg}\,\mathscr{E}$. We call this property exact decomposability and it is shown, in particular, that T is exactly decomposable if $\mathscr{E}$ is countable or if T is dissipative.

A basic result required in the analysis is a construction of Lance, Lemmas 3.2, 3.3 of [11], which splits an upper triangular 2×2 operator matrix into a sum of two operators of the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$. An indication of some of the power of this decomposition is given in the fact that it leads naturally to a useful result of Parrott [14]. In [11] it is used to derive Arveson's distance formula [1], to which Parrott's result is closely related [15].

In Section 3 we make inductive use of the lemma, and an inherent left continuity, in order to associate with each positive operator C and nest $\mathscr{E}$ a positive operator valued Borel measure $C(\Delta)$ on $\mathscr{E}$. If this construction is applied to the positive part C of an operator $T = UC$ in $\mathrm{Alg}\,\mathscr{E}$ then the operator measure $UC(\Delta)$ on $\mathscr{E}$ provides the appropriate generalisation of Lance's construction, and in case $\mathscr{E}$

has three elements coincides with this construction. In Section 4 we give a Radon-Nikodym theorem for nuclear operator valued measures. For a nuclear operator T this allows us to form the derivative T_E of the measure $UC(\Delta)$ with respect to the scalar measure $\tau(\Delta) =:$ trace $C(\Delta)$ and thereby obtain the main result. The relationship between C and $C(\Delta)$ seems to be worthy of further analysis.

In Section 5 we complete the proof of the main result and give various applications. A natural corollary, of wider interest, is discussed more fully in [16]. This is Lidskii's theorem that the trace of a nuclear operator is the sum of its eigenvalues (counted with their algebraic multiplicity).

NOTATION. We fix a separable complex Hilbert space H. The term *subspace* means *closed* linear subspace. We let $\mathscr{E}$ denote a complete *nest* of self-adjoint projections on H. Thus $\mathscr{E}$ is a totally ordered (under range inclusion) family which contains the projections 0 and I, and which is closed in the strong operator topology. If $E \in \mathscr{E}$ and $E \neq 0$ (resp. $E \neq I$) then we define E_- (resp. E_+) as the supremum (resp. infimum) of the collection of F in $\mathscr{E}$ with $F < E$ (resp. $F > E$). The algebra of all bounded linear operators on H is denoted by $B(H)$, and $B_1(H)$ denotes the class of nuclear operators (trace class operators). The nuclear operators form a Banach space under the norm

$$\|T\|_1 = \operatorname{tr}((T^*T)^{1/2})$$

where tr denotes the trace on $B_1(H)$.

The nest algebra Alg $\mathscr{E}$ associated with a nest $\mathscr{E}$ is the algebra of all operators T such that $(I - E)TE = 0$ for all $E \in \mathscr{E}$. We denote the family of nuclear operators in Alg $\mathscr{E}$ by $\mathrm{Alg}_1\,\mathscr{E}$. The rank one operator $x \to (u, x)v$ is denoted $u \otimes v$.

2. A LEMMA OF E. C. LANCE

Our starting point is the following fundamental lemma of [11], reformulated in a manner appropriate for later induction.

LEMMA 2.1. *Let C be a positive operator which has an operator matrix $\begin{bmatrix} A & B \\ B^* & D \end{bmatrix}$ with respect to a given decomposition of H. Then the limit, as $n \to \infty$, of the sequence $B^*(A + n^{-1}I)^{-1}B$ exists in the strong operator topology. If D_1 denotes this limit then the following hold.*

(i) $D_1 \leqslant D$.

(ii) *The operator* $C_1 = \begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$ *is positive.*

(iii) *If U is an operator on H and UC has the form* $\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, *then* UC_1 *and* $U(C - C_1)$ *have, respectively, the forms* $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$.

COROLLARY 2.2. *If* $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}$ *is a nuclear operator then there exist* T_2' *and* T_2'' *so that if* $R = \begin{bmatrix} T_1 & T_2' \\ 0 & 0 \end{bmatrix}$ *and* $S = \begin{bmatrix} 0 & T_2'' \\ 0 & T_3 \end{bmatrix}$ *then* $T = R + S$ *and* $\|T\|_1 = \|R\|_1 + \|S\|_1$.

Proof. The corollary follows immediately from an application of the lemma to the polar decomposition $T = UC$. Note that

$$\|T\|_1 = \mathrm{tr}(C) = \mathrm{tr}(C_1) + \mathrm{tr}(C - C_1) = \|UC_1\|_1 + \|U(C - C_1)\|_1,$$

so we may take $R = UC_1$ and $S = U(C - C_1)$.

The corollary may be used now to obtain a useful result of Parrott (see [14] and its footnote for partial anticipations). The proof below makes free use of the $B_1(H)$, $B(H)$ duality and is closely related to the discussions of the distance formula in [11] and [12].

COROLLARY 2.3.

$$\inf_X \left\| \begin{bmatrix} X & A \\ C & B \end{bmatrix} \right\| = \max\left\{ \left\| \begin{bmatrix} 0 & A \\ 0 & B \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 & 0 \\ C & B \end{bmatrix} \right\| \right\}.$$

Proof. Let us suppose that the operator matrices are relative to an orthogonal decomposition $H = H_1 \oplus H_2$. If $Z \in B(H)$ then write Z_r for the functional on the annihilator of $B(H_1)$ which is induced by Z. That is, Z determines a functional on $B_1(H)$ and Z_r is the restriction of Z to the annihilator mentioned. This annihilator is simply the collection of nuclear operators whose first operator matrix entry is zero. If $Z = \begin{bmatrix} 0 & A \\ C & B \end{bmatrix}$

$$\|Z_r\| \leqslant \inf_X \left\| \begin{bmatrix} X & A \\ C & B \end{bmatrix} \right\|, \tag{2.1}$$

since operators X in $B(H_1)$ induce the zero functional on the annihilator. On the other hand, by the Hahn-Banach theorem, Z_r has a norm maintaining extension, and so equality occurs in (2.1). But, using Corollary 2.2, we see that

$$\|Z_r\| = \sup_{\left\|\begin{bmatrix} 0 & U \\ V & W\end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ V & W\end{bmatrix} Z\right)\right| =$$

$$= \sup_{\left\|\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix}\right\|_1 + \left\|\begin{bmatrix} 0 & U \\ 0 & W_2\end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix} Z\right) + \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ 0 & W_2\end{bmatrix} Z\right)\right| =$$

$$= \sup_{\left\|\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix}\right\|_1 + \left\|\begin{bmatrix} 0 & U \\ 0 & W_2\end{bmatrix}\right\|_1 = 1} \left| \operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix}\begin{bmatrix} 0 & A \\ 0 & B\end{bmatrix}\right) + \operatorname{tr}\left(\begin{bmatrix} 0 & U \\ 0 & W_2\end{bmatrix}\begin{bmatrix} 0 & 0 \\ C & B\end{bmatrix}\right)\right| =$$

$$= \max\left\{\left\|\begin{bmatrix} 0 & A \\ 0 & B\end{bmatrix}\right\|, \left\|\begin{bmatrix} 0 & 0 \\ C & B\end{bmatrix}\right\|\right\}.$$

The last equality follows because the supremum of $\operatorname{tr}\left(\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix}\begin{bmatrix} 0 & A \\ 0 & B\end{bmatrix}\right)$ as $\begin{bmatrix} 0 & 0 \\ V & W_1\end{bmatrix}$ varies in the unit ball of $B_1(H)$, is the operator norm of $\begin{bmatrix} 0 & A \\ 0 & B\end{bmatrix}$. The corollary is now proven.

A well known result of Ringrose (see Erdos [5]) asserts that each operator T in Alg $\mathcal{E}$ with finite rank n may be written as a sum of n rank one operators in Alg $\mathcal{E}$. Lemma 2.1 provides an alternative proof of this with the strengthening of the conclusion to an *exact* sum, as we now show. Moreover the method provides a constructive rather than existential approach and so may be of added interest. Extensions of Ringrose's result have been made by various authors to reflexive algebras Alg $\mathcal{L}$ for certain commutative subspace lattices $\mathcal{L}$. We refer the reader to Hopenwasser and Moore [10] for a good discussion of this and for the following two results:

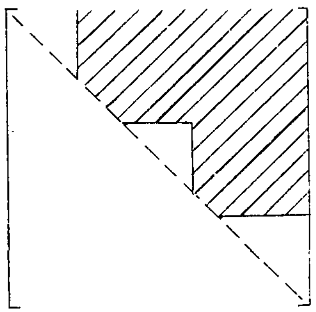
(i) decomposition into rank ones is possible if $\mathcal{L}$ has finite width (although the length of the sum may have to be greater than the rank),

(ii) there is a totally atomic $\mathcal{L}$ and a rank two operator in Alg $\mathcal{L}$ that cannot be written as a sum of rank one operators in Alg $\mathcal{L}$.

Before proceeding it is convenient to introduce the following concept.

DEFINITION 2.4. An operator T in Alg $\mathcal{E}$ is said to be *suspended* by a set $\mathcal{G} \subseteq \mathcal{E}$ if $(E - F)T(E - F) = 0$ whenever the interval $(F, E]$ is disjoint from $\mathcal{G}$.

If T is suspended by two disjoint intervals then T looks like this



One can verify that T is suspended by a singleton $E \neq 0$ if and only if $ET(I - E_-) = T$. Each rank one operator in Alg $\mathscr{E}$ is thus suspended by a singleton since, as is well known, it may be expressed as $e \otimes f$ with $f \in E$ and $e \in I - E_-$, for some $E \neq 0$. If $T \in \mathrm{Alg}_1 \mathscr{E}$ is suspended by the singleton E then it is easy to obtain an exact decomposition of T. Let $C = \sum_{i=1}^{\infty} C_i$ be *any* decomposition of C into positive rank one operators where $T = UC$ is the polar decomposition of T. Then $T = \sum_{i=1}^{\infty} UC_i$ is an exact sum. Also

$$\sum_{i=1}^{\infty} UC_i = T = ET(I - E_-) = \sum_{i=1}^{\infty} EUC_i(I - E_-),$$

and so $\|EUC_i(I - E_-)\|_1 = \|UC_i\|_1$, $i = 1, 2, \ldots$, and hence each summand UC_i belongs to Alg $\mathscr{E}$ and is suspended by E.

It can be shown that every exact sum $X = \sum_{i=1}^{\infty} X_i$, with each X_i of rank one, must arise through a rank one positive decomposition of the positive part of the nuclear operator X. One often takes a spectral decomposition for the positive part, giving a Schmidt expansion for X, but in our context this takes no account of the invariant subspaces of X and need not correspond to the internal exact decomposition for Alg $\mathscr{E}$ obtained below.

COROLLARY 2.5. *Let $T \in \mathrm{Alg}_1 \mathscr{E}$ be a finite rank operator of rank n. Then there are rank one operators $R_1, R_2, \ldots, R_n$ in* Alg $\mathscr{E}$ *with $T = R_1 + R_2 + \ldots + R_n$ and $\|T\|_1 = \|R_1\|_1 + \|R_2\|_1 + \ldots + \|R_n\|_1$.*

Proof. We use the notation of Lemma 2.1. Let $T = UC$ be the polar decomposition, let $E \in \mathscr{E}$, $E \neq 0$, I and let C_1 be constructed from C, as in Lemma 2.1, relative to the decomposition induced by E. Let $C_2 = C - C_1$. We first show that $\mathrm{rank}\, C_1 + \mathrm{rank}\, C_2 = \mathrm{rank}\, C$.

Let P denote the range projection of A. Then the positivity of C_1 shows that $B^*P = B^*$ (see Lance's proof). Thus

$$D_1 = \lim_n B^*(A + n^{-1})^{-1}B = \lim_n B^*P(A + n^{-1})^{-1}PB = B^*(PAP)^{-1}B$$

where $(PAP)^{-1}$ denotes, informally, the operator which is 0 on $(I - P)H$ and the inverse of PAP on PH. Let S be the invertible operator

$$S = \begin{bmatrix} I & 0 \\ B^*(PAP)^{-1} & I \end{bmatrix}.$$

Then since $B^*(PAP)^{-1}A = B^*P = B^*$, and $B^*(PAP)^{-1}B = D_1$ we have

$$C_1 = S\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

Also

$$C_2 = SC_2 = S\begin{bmatrix} 0 & 0 \\ 0 & D - D_1 \end{bmatrix}. \tag{2.3}$$

Since $B^*P = B^*$ we have

$$\ker \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}^* \supseteq \ker \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

and thus

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} = \operatorname{rank} A.$$

Hence

$$\operatorname{rank} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} + \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & D - D_1 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A & B \\ 0 & D - D_1 \end{bmatrix}.$$

Now 'apply' S^{-1} to this last equation and use (2.2), (2.3) to see that $\operatorname{rank} C_1 + \operatorname{rank} C_2 = \operatorname{rank} C$, as desired.

To complete the proof the above is used inductively until we obtain $C = K_1 + K_2 + \ldots + K_m$ relative to $0 = E_0 < E_1 < \ldots < E_{k-1} < E_k = I$ with the following properties:

(i) $\operatorname{rank} K_i > 0$;

(ii) $\operatorname{rank} C = \sum_i \operatorname{rank} K_i$;

(iii) UK_i is suspended by $[E_{i-1}, E_i)$, $i = 1, 2, \ldots, k$;

(iv) K_i cannot be further decomposed with non zero summands relative to any projection in $[E_{i-1}, E_i)$.

Plainly, (iii) and (iv) show that UK_i is in fact suspended by a single projection. The proof is now completed.

REMARK. As observed in [11] there is a version of Corollary 2.2 for upper triangular operator matrices relative to decompositions of both domain space and range space. For example suppose P, Q are self-adjoint projections with $Q < P$ and that T has the form

$$T = \begin{array}{c} \\ Q \end{array}\overset{P}{\begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix}}.$$

We construct an associated operator $\widetilde{T}$, so that $\widetilde{T}$ is upper triangular and

$$\widetilde{T} = \begin{bmatrix} 0 & 0 & 0 \\ T_1 & T_2 & 0 \\ 0 & T_3 & 0 \end{bmatrix}.$$

It can be checked that the Lance decomposition of $\widetilde{T}$ provides an associated decomposition of T.

With the ideas above one can obtain a version of Corollary 2.5 for finite rank operators in a weakly closed operator module of Alg $\mathscr{E}$, and hence a strengthening of Lemma 2.1 of [8].

3. OPERATOR VALUED MEASURES

We now make inductive use of Lemma 2.1 to associate with each positive operator C in $B(H)$ a positive operator valued measure. This association will depend only upon the fixed nest $\mathscr{E}$. The construction of Lemma 2.1 has an inherent left continuity property with respect to the weak operator topology. This is expressed in Lemma 3.2 and provides just the continuity property required for extending finitely additive measures to measures.

Let $\mathscr{F}$ be the finite subnest $0 = E_0 < E_1 < \ldots < E_n = I$ of $\mathscr{E}$. Let C be a fixed positive operator on H and decompose C as in Lemma 2.1 with respect to E_1 to obtain $C = C_1 + C_2'$. Next decompose C_2' with respect to E_2 to obtain $C_2' = C_2 + C_3'$, and so on, until we have the following decomposition

$$C = C_1 + C_2 + \ldots + C_n \tag{3.1}$$

associated with $\mathscr{F}$. (Here C_1 has the form $\begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$ and $C_2' = C - C_1$, and so on.) We now define $C_{\mathscr{F}}[E_{i-1}, E_i) = C_i$, $i = 1, 2, \ldots, n$. The next lemma shows that $C_{\mathscr{F}}[E, F)$ is independent of the subnest $\mathscr{F}$, and so we shall denote the common value by $C[E, F)$.

Let $\mathscr{R}(\mathscr{E})$ be the ring of subsets of $\mathscr{E}$ generated by the collection of semi-intervals $[E, F)$ with $E, F \in \mathscr{E}$, $E < F$.

LEMMA 3.1 (i) *The operator* $C_{\mathscr{F}}[E_{i-1}, E_i)$ *is independent of* $\mathscr{F}$, *the finite nest containing* E_{i-1} *and* E_i.

(ii) *The correspondence* $[E, F) \to C[E, F)$ *extends to a finitely additive positive operator valued function on* $\mathscr{R}(\mathscr{E})$.

Proof. We first claim that the decomposition (3.1) arises independently of the *order* of successive applications of Lemma 2.1. More specifically consider a quadruple subnest $0 = E_0 < E_1 < E_2 < E_3 = I$. Use Lemma 2.1 to decompose C as $C' + C_3'$ relative to E_2. Next decompose C' relative to E_1 as $C' = C_1' + C_2'$. We show that, with the notation used earlier, $C_1' = C_1$, $C_2' = C_2$ and $C_3' = C_3$. That $C_1' = C_1$ should be clear. Since $C_1 + C_2$ is positive and $(C_1 + C_2)E_2 = C'E_2$ it follows, by the minimality property of Lemma 2.1(i), that $C' \leqslant C_1 + C_2$. Hence $C_1' + C_2' \leqslant C_1 + C_2$ and $C_2' \leqslant C_2$. But $C_2'E_2 = C_2E_2$, and so, by minimality again, $C_2 \leqslant C_2'$. Thus $C_2 = C_2'$ and $C_3 = C_3'$. Our original claim now follows easily by induction with the quadruple case.

The proof of (i) is now immediate, because if two finite subnests $\mathscr{F}_1$ and $\mathscr{F}_2$ determine $C_{\mathscr{F}_1}[E, F)$ and $C_{\mathscr{F}_2}[E, F)$ then, from the above, $C_{\mathscr{F}_1}[E, F) = C_{\mathscr{F}_1 \cup \mathscr{F}_2}[E, F) = C_{\mathscr{F}_2}[E, F)$.

To establish (ii) we need only verify that if $E < F < G$ belong to $\mathscr{E}$ then $C[E, G) = C[E, F) + C[F, G)$. This too is an immediate consequence of the claim and its proof.

LEMMA 3.2. *If* $E \in \mathscr{E}$ *and* $E_- = E$ *then* $C[F, E)$ *converges to zero in the weak operator topology as* F *increases to* E *with* $F < E$.

Proof. Note that, with respect to the Hilbert space decomposition induced by E, $C[0, E)$ has the form $\begin{bmatrix} A & B \\ B^* & D_1 \end{bmatrix}$, as in Lemma 2.1. Also with respect to the decomposition induced by F ($F < E$), $C[0, F)$ has the form $\begin{bmatrix} A' & B' \\ B'^* & D_1' \end{bmatrix}$. Moreover, since $E_- = E$, we have $A' \to A$, $B' \to B$ in the weak operator topology as $F \to E$, $F < E$. Thus the monotone increasing net $C[0, F)$ converges in the weak operator topology to an operator $X \leqslant C[0, E)$ which has the form $\begin{bmatrix} A & B \\ B^* & Z \end{bmatrix}$ with respect to

E. But, by the minimality of D_1, $C[0, E) \leqslant X$. Hence $X = C[0, E)$ and the lemma follows.

From the last two lemmas and the basic theory of positive operator valued measures, [2, p. 15], there is a unique positive operator valued set function $C(\Delta)$ defined on the Borel subsets Δ of $\mathscr{E}$ ($\mathscr{E}$ is metrized by the strong operator topology), which coincides with $C[E, F)$ on $\mathscr{R}(\mathscr{E})$, and is such that

$$C(\Delta) = \sum_{i=1}^{\infty} C(\Delta_i) \tag{3.2}$$

whenever Δ is a disjoint union on Borel subsets Δ_i, and convergence is with respect to weak operator topology.

It follows from Lemma 2.1 (iii) and the constructions above that if $UC \in \operatorname{Alg}\mathscr{E}$ then $UC[E, F) \in \operatorname{Alg}\mathscr{E}$ and is suspended by $[E, F)$ for each $E, F \in \mathscr{E}$, $E < F$.

4. A RADON-NIKODYM THEOREM

We now establish some integration theory for nuclear operator valued functions sufficient for our application. No attempt is made at generality.

Let (Ω, Σ, μ) be a sigma finite measure space. A function $f\colon \Omega \to B_1(H)$ is said to be *measurable* if the function $t \to (f(t)x, y)$, $t \in \Omega$, is measurable for every pair of vectors x, y in H. In view of our separability assumption on H it would suffice here to require measurability for x, y in a dense subset. If f is such a measurable function then, again by separability, $t \to \|f(t)\|_1$ is measurable. The function f is said to be *integrable* if $t \to \|f(t)\|_1$ is integrable. Simple applications of Lebesgue's dominated convergence theorem reveal that for an integrable function f the sesquilinear form $[\,,]$ defined by

$$[x, y] = \int_\Omega (f(t)x, y)\, \mathrm{d}\mu(t)$$

satisfies

$$\sum_{n=1}^{\infty} |[x_n, y_n]| \leqslant \int_\Omega \|f(t)\|_1\, \mathrm{d}\mu(t)$$

for every pair of orthonormal sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$. Hence there exists a nuclear operator T such that $[x, y] = (Tx, y)$ for $x, y \in H$. The operator T is called the integral of f and we write $T = \int_\Omega f\, \mathrm{d}\mu$.

THEOREM 4.1. *Let (Ω, Σ, μ) be a sigma finite measure space and let $C(\Delta)$ be an operator valued measure on Σ such that $C(\Omega)$ is nuclear and $C(\Delta) = 0$ whenever $\Delta \in \Sigma$*

and $\mu(\Delta) = 0$. *Then there exists a positive integrable nuclear operator valued function* $D(t)$ *such that* $C(\Delta) = \int_\Delta D(t)\mathrm{d}\mu(t)$ *for all* $\Delta \in \Sigma$.

Proof. Let Q denote a subset of H consisting of all linear combinations, with coefficients in $\mathbf{Q} + i\mathbf{Q}$ of a fixed orthonormal basis $e_1, e_2, \ldots$. For x, y in H let $\mu_{x,y}$ denote the scalar complex measure on Σ defined by $\mu_{x,y}(\Delta) = (C(\Delta)x, y)$. By the Radon-Nikodym theorem there exists a measurable integrable function $D_{x,y}$ such that $\mu_{x,y}(\Delta) = \int_\Delta D_{x,y}(t)\mathrm{d}\mu(t)$. The derivative $D_{x,y}(t)$ is determined almost everywhere. Thus it is possible to choose a null set N so that for all $t \notin N$ the mapping $x, y \to D_{x,y}(t)$ is a finite and sesquilinear form, over $\mathbf{Q} + i\mathbf{Q}$ on the vector pairs x, y in Q. Also, by the monotone convergence theorem,

$$
\begin{aligned}
\int_\Omega \left(\sum_n D_{e_n, e_n}(t)\right) \mathrm{d}\mu(t) &= \sum_n \int_\Omega D_{e_n, e_n}(t)\, \mathrm{d}\mu(t) = \\
&= \sum_n \mu_{e_n, e_n}(\Omega) = \sum_n (C(\Omega)e_n, e_n) = \mathrm{tr}(C(\Omega)).
\end{aligned}
\tag{4.1}
$$

Hence we can arrange N so that $\sum_n D_{e_n, e_n}(t)$ is finite for all $t \notin N$. It follows by standard arguments that for each $t \notin N$ there exist a positive nuclear operator $D(t)$ such that $D_{x,y}(t) = (D(t)x, y)$ for all $x, y \in Q$. Set $D(t) = 0$ for $t \in N$. Since Q is dense it follows that $D(t)$ is measurable and, by (4.1), integrable. Since

$$
\begin{aligned}
(C(\Delta)x, y) = \mu_{x,y}(\Delta) &= \int_\Delta D_{x,y}(t)\mathrm{d}\mu(t) = \\
= \int_\Delta (D(t)x, y)\mathrm{d}\mu(t) &= \left(\int_\Delta D(t)\mathrm{d}\mu(t)x, y\right) \quad \text{for } x, y \in Q,
\end{aligned}
$$

the theorem follows.

The integral of an integrable function has been defined in a weak sense and such a description could be used to integrate suitable $B(H)$ valued functions. For $B_1(H)$ valued functions however the integral exists in the following, much stronger, sense, and this will be useful.

THEOREM 4.2. *Let* (Ω, Σ, μ) *be a sigma finite measure space and let* $D(t)$ *be an integrable nuclear operator valued function on* Ω. *Then for each* $\varepsilon > 0$ *there exists a measurable partition* $\Delta_1, \Delta_2, \ldots, \Delta_r$ *of* Ω *and* $t_i \in \Delta_i$ *for* $i = 1, 2, \ldots r$ *such that*

$$
\left\| \int_\Omega D(t)\mathrm{d}\mu(t) - \sum_{i=1}^r D(t_i)\mu(\Delta_i) \right\|_1 < \varepsilon.
$$

Proof. We make the simplifying assumption that $\mu(\Omega) = 1$ and that $\|D(t)\|_1 \leqslant M$ almost everywhere since the theorem follows easily from this special case Let P_n, $n = 1, 2, \ldots$ be finite rank projections such that P_n tends strongly to the identity. If $X \in B_1(H)$ then $P_n X P_n \to X$ in $B_1(H)$. Thus $P_n D(t) P_n \to D(t)$ in $B_1(H)$ for almost every t. In particular there is a measurable set K with $\mu(K) < \frac{\varepsilon}{5M}$ and an integer N_0 such that $\|P_n D(t) P_n - D(t)\|_1 < \frac{\varepsilon}{5}$ for all $n > N_0$ and $t \notin K$. Also there exists an $N > N_0$ such that

$$\left\| \int P_N D(t) P_N \mathrm{d}\mu(t) - \int D(t) \mathrm{d}\mu \right\|_1 < \frac{\varepsilon}{5}.$$

Since $P_N D(t) P_N$ is an integrable operator valued function with values in $B(\mathbf{C}^n)$ it follows from the integration theory for scalar functions that there exists a partition $\Delta_1, \Delta_2, \ldots, \Delta_r$ of Ω such that

$$\left\| \int P_N D(t) P_N \mathrm{d}\mu(t) - \sum_{i=1}^{r} P_N D(t_i) P_N \mu(\Delta_i) \right\|_1 < \frac{\varepsilon}{5}$$

for almost every choice of $t_i \in \Delta_i$, $i = 1, 2, \ldots, r$. We can also assume that $K = \bigcup_{i=1}^{s} \Delta_i$ for some $s \leqslant r$. It follows that

$$\left\| \sum_{i=1}^{r} P_N D(t_i) P_N \mu(\Delta_i) - \sum_{i=s+1}^{r} P_N D(t_i) P_N \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5},$$

$$\left\| \sum_{i=s+1}^{r} P_N D(t_i) P_N \mu(\Delta_i) - \sum_{i=s+1}^{r} D(t_i) \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5}$$

and

$$\left\| \sum_{i=s+1}^{r} D(t_i) \mu(\Delta_i) - \sum_{i=1}^{r} D(t_i) \mu(\Delta_i) \right\|_1 \leqslant \frac{\varepsilon}{5}.$$

Combine the displayed inequalities above and the theorem follows.

5 MAIN RESULT AND APPLICATIONS

THEOREM 5.1. *Let $T \in \mathrm{Alg}_1 \mathscr{E}$. Then there exists a finite positive Borel measure τ on $\mathscr{E}$ and an integrable nuclear operator valued function $E \to T_E$ on $\mathscr{E}$ such that*

(i) $$T = \int_{\mathscr{E}} T_E \mathrm{d}\tau(E),$$

(ii) $$\|T\|_1 = \int_{\mathscr{E}} \|T_E\|_1 \mathrm{d}\tau(E),$$

(iii) $$T_E = ET(I - E_-) \textit{ almost everywhere.}$$

Proof. Let $T = UC$ be a polar decomposition of T with U an isometry and C a positive operator. By the construction of Section 3 there is a nuclear operator valued measure $C(\Delta)$ defined on the Borel algebra of $\mathscr{E}$, such that $UC[E, F)$ is suspended by $[E, F)$ whenever $E, F \in \mathscr{E}$, $E < F$. Let τ be the scalar Borel measure on $\mathscr{E}$ defined by $\tau(\Delta) = \mathrm{tr}(C(\Delta))$. Plainly $C(\Delta)$ is absolutely continuous with respect to τ and so, by Theorem 4.1, there exists a positive integrable $B_1(H)$ valued derivative $E \to D_E$ such that $C(\Delta) = \int_{\Delta} D_E \, \mathrm{d}\tau(E)$. Define $T_E = UD_E$. Then $E \to T_E$ is integrable and (i) and (ii) follow.

Let $\mathscr{G}$ be a countable order dense subset of $\mathscr{E}$ and let $\mathscr{I}$ be the collection of intervals $\Delta = (F, G]$ whose endpoints belong to $\mathscr{G}$. To establish (iii) it will be sufficient, in view of the remarks following Definition 2.4, to show that for almost every E we have $\Delta T_E \Delta = 0$ for every projection $\Delta = G - F$ with $\Delta \in \mathscr{I}$ and $E \notin \Delta$. (The notational economy here should cause no confusion.)

Fix M, N in $\mathscr{G}$ with $M < N$ and consider a scalar step function $\varphi(E)$ on $[M, N)$ on the form $\varphi(E) = \sum_{k=1}^{n} a_k \chi_{\Delta_k}(E)$, where $\Delta_k = [E_{k-1}, E_k)$ and $M = E_0 < E_1 < \ldots < E_n = N$ is a finite measurable partition. Since $\int_{\Delta_k} T_E \, \mathrm{d}\tau = UC(\Delta_k)$ is suspended by Δ_k it follows that $\int_{[M,N)} \varphi(E) T_E \mathrm{d}\tau$ is suspended by $[M, N)$ and thus that

$$\int_{[M,N)} \varphi(E) \Delta T_E \Delta \mathrm{d}\tau = \Delta \int_{[M,N)} \varphi(E) T_E \mathrm{d}\tau \, \Delta = 0$$

for every $\Delta \in \mathscr{I}$ which is disjoint from $[M, N)$. Since φ is arbitrary it follows that there is a null set $\Delta_{M,N}$ such that $\Delta T_E \Delta = 0$ for all $E \in [M, N) \setminus \Delta_{M,N}$ and all Δ disjoint from $[M, N)$. Let Δ^* be the union of all the sets $\Delta_{M,N}$ with M, N in $\mathscr{G}$. Then it follows that if $E \notin \Delta^*$ then $\Delta T_E \Delta = 0$ for all $\Delta \in \mathscr{I}$ with $E \notin \Delta$. Thus (iii) is proven, since $\tau(\Delta^*) = 0$.

Recall that an operator $T \in \mathrm{Alg}_1 \mathscr{E}$ is said to be *exactly decomposable* if there exist rank one operators $R_1, R_2, \ldots$ in $\mathrm{Alg}\,\mathscr{E}$ such that $\|T\|_1 = \sum_{i=1}^{\infty} \|R_i\|_1$ and $T = \sum_{i=1}^{\infty} R_i$.

COROLLARY 5.2. (i) *If $\mathscr{E}$ is countable then each T in $\mathrm{Alg}_1\,\mathscr{E}$ is exactly decomposable.*

(ii) *Let $T \in \mathrm{Alg}_1\,\mathscr{E}$ and let $\varepsilon > 0$. Then there exist rank one operators $R_1, R_2, \ldots$ in $\mathrm{Alg}\,\mathscr{E}$ such that $T = \sum_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} \|R_i\|_1 < \|T\|_1 + \varepsilon$.*

Proof. (i) Theorem 5.1 shows that $T = \sum_{E \in \mathscr{E}} \tau(\{E\})T_E$ and that this sum is exact. Since T_E is nuclear and suspended by a singleton, our remarks following Definition 2.4 show that each T_E is exactly decomposable. This proves (i).

(ii) Note first that if $S \in \mathrm{Alg}_1\,\mathscr{E}$ is suspended by a finite number of points then S is exactly decomposable. This is a consequence of Theorem 5.1 but follows from Corollary 2.2 more directly. Theorems 5.1 and 4.2 show that there is an approximating sum S_1, which is suspended by a finite number of points, such that $\|T - S_1\|_1 < \varepsilon/2$. Similarly obtain $S_2, S_3, \ldots$ each suspended by a finite number of points, such that

$$\|T - (S_1 + \ldots + S_n)\|_1 < \frac{\varepsilon}{2^n}, \quad n = 1, 2, \ldots$$

$$\|S_n\|_1 < \frac{\varepsilon}{2^{n+1}}, \quad n = 2, 3, \ldots .$$

Write each S_j as an exact decomposition $S_j = \sum_{i=1}^{\infty} R_i^{(j)}$. Then $T = \sum_{i,j} R_i^{(j)}$ and $\sum_{i,j} \|R_i^{(j)}\|_1 < \|T\|_1 + \varepsilon$.

REMARK. The second part of the corollary shows that every nuclear operator is approximately decomposable, and shows that in the unit ball of $\mathrm{Alg}_1\,\mathscr{E}$ the finite rank operators are dense. This could also be obtained as a consequence of Erdos' density theorem: In the unit ball of $\mathrm{Alg}\,\mathscr{E}$ the finite rank operators are dense in the weak operator topology [5]. This useful result (e.g. see [6], [8]) is usually applied in the equivalent form: there is a net F_α of finite rank operators in $\mathrm{Alg}\,\mathscr{E}$ with $\|F_\alpha\| \leqslant 1$ and $F_\alpha \to I$ in the weak topology. This looks like a bounded approximate identity for the weak operator topology, and in fact provides a (norm) bounded approximate identity for the Banach algebra $(\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K}$ with the operator norm ($\mathscr{K}$ = the compact operators). In particular factorisation is possible (by means of Cohen's factorisation theorem [3, p. 61]). This algebra is rather interesting, being radical if $\mathscr{E}$ is continuous. All closed ideals can be described by using the methods of [8]. Each closed ideal J of $(\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K}$ is of the form

$$J = \{X \in (\mathrm{Alg}\,\mathscr{E}) \cap \mathscr{K} \mid (I - \widetilde{E})XE = 0, \text{ all } E \in \mathscr{E}\}$$

where $E \to \widetilde{E}$ is a left continuous order homomorphism of $\mathscr{E}$, with $\widetilde{E} \leqslant E$ for all

E in $\mathscr{E}$. A similar description holds for the closed ideals of the Banach algebra $(\mathrm{Alg}_1 \mathscr{E}, \|\cdot\|_1)$.

REMARK. It also follows from Corollary 5.2 (ii) that the upper triangular integral (in the usual sense [7]) of an operator $T \in \mathrm{Alg}_1 \mathscr{E}$ converges to T in the nuclear norm. That is, if $\mathscr{U}_{\mathscr{F}}(T) = \sum E_i T(E_i - E_{i-1})$ is the upper triangular sum associated with a finite subset $\mathscr{F} = \{E_0 < E_1 < \ldots < E_n\}$ then $\mathscr{U}_{\mathscr{F}}(T)$ converges $\|\ \|_1$ to the operator T as $\mathscr{F}$ runs through the directed set of all finite subsets.

This contrasts sharply with the well known fact that $\mathscr{U}_{\mathscr{F}}(X)$ need not converge $\|\ \|_1$ for $X \in B_1(H)$ (although it does converge $\|\ \|_p$, $1 < p < \infty$). Indeed the canonical projection from $B_1(H)$ to $\mathrm{Alg}_1 \mathscr{E}$ is not $\|\ \|_1$ bounded if $\mathscr{E}$ is infinite. Let us digress a moment to indicate that $\mathrm{Alg}_1 \mathscr{E}$ has *no* complement in $B_1(H)$. The proof is modeled on Newman's proof that H^1 has no complement in L^1 [9]. Specifically we show that if there is a continuous projection $\pi : B_1(H) \to \mathrm{Alg}_1 \mathscr{E}$ then, by averaging, we can deduce the uniform boundedness of certain canonical projections on $\mathrm{Alg}\,\mathscr{F}$, $\mathscr{F}$ a finite subnest of $\mathscr{E}$, and thus obtain a contradiction. Indeed for a given finite subnest $\mathscr{F}$ let $G_{\mathscr{F}}$ denote the unitary group in $\mathscr{F}''$ (the double commutant) with Haar measure $\mathrm{d}U$. Define

$$\pi_{\mathscr{F}}(X) = \int_{G_{\mathscr{F}}} \int_{G_{\mathscr{F}}} U^* \pi(UXV^*) V \,\mathrm{d}U \,\mathrm{d}V.$$

This exists as a Riemann integral of $\|\ \|_1$ continuous $B_1(H)$ valued functions on $G_{\mathscr{F}} \times G_{\mathscr{F}}$. We have $\|\pi_{\mathscr{F}}\| \leqslant \|\pi\|$, for the operator norms of these mappings, and, since $G_{\mathscr{F}}^* \mathrm{Alg}_1 \mathscr{E} = (\mathrm{Alg}_1 \mathscr{E}) G_{\mathscr{F}} = \mathrm{Alg}_1 \mathscr{E}$ it follows that $\pi_{\mathscr{F}}$ is a projection. Since $\pi_{\mathscr{F}}(WXY) = W\pi_{\mathscr{F}}(X)Y$ for $W, Y \in G_{\mathscr{F}}$ it follows that $\pi_{\mathscr{F}}(SXT) = S\pi_{\mathscr{F}}(X)T$ for $S, T \in \mathscr{F}''$. In particular

$$(E_j - E_{j-1})\, \pi_{\mathscr{F}}(X)\, (E_k - E_{k-1}) = 0$$

for $j > k$. If $\tilde{\pi}_{\mathscr{F}}$ denotes the restriction to operators X with $0 = (E_j - E_{j-1})X(E_j - E_{j-1})$ then it follows that $\tilde{\pi}_{\mathscr{F}}$ is the canonical projection into $\mathrm{Alg}\,\mathscr{F}$. Now we have $\|\tilde{\pi}_{\mathscr{F}}\| \leqslant \|\pi\|$ for all $\mathscr{F}$, which is a contradiction.

THEOREM 5.3. *Let $T \in \mathrm{Alg}_1 \mathscr{E}$. If T is dissipative then T is exactly decomposable.*

Proof. Recall that an operator is dissipative if $\mathrm{i}(T^* - T) \geqslant 0$. Let $T = \int_{\mathscr{E}} T_E \,\mathrm{d}\tau$ be the decomposition of Theorem 5.1. Since $\mathrm{tr}(T_E) = 0$ when $E_- = E$ we have,

$$\mathrm{tr}(\mathrm{i}(T^* - T)) = \int_{\mathscr{E}} \mathrm{tr}(\mathrm{i}(T_E^* - T_E))\,\mathrm{d}\tau =$$

$$= \int_{\mathscr{D}} \mathrm{i}\,\mathrm{tr}(T_E^* - T_E)\,\mathrm{d}\tau$$

where $\mathscr{D} = \{E : E_- < E\}$. This shows that $\mathscr{D}$ is non void if $T^* - T \neq 0$. Let H_0 be the closed span of $\{(E - E_-)H : E \in \mathscr{D}\}$. This is the subspace on which $\mathscr{E}$ is totally atomic. More precisely, if P is the orthogonal projection onto H_0 then P commutes with $\mathscr{E}$ and if $P \neq 0$ then $\mathscr{E}_0 = \{EP : E \in \mathscr{E}\}$ is a totally atomic nest on H_0. Moreover $\mathscr{E}_1 = \{E(I - P) : E \in \mathscr{E}\}$ is a continuous nest on $H_1 = (I - P)H$ if $P \neq I$. Let us write

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

relative to the decomposition $H_0 \oplus H_1$. Since T_4 is also dissipative and belongs to the continuous nest algebra Alg $\mathscr{E}_1$, by our initial observation T_4 is self-adjoint. Hence $T_4 = 0$. But since T is dissipative this now implies that $T_2 = T_3^*$. Thus $T_2^* T_2 = T_3 T_2 = (I - P)TPT(I - P)$ is a compact self-adjoint operator in a continuous nest algebra, and so $T_2 = 0$. By Corollary 5.3(i) T_1 is exactly decomposable relative to $\mathscr{E}_0$, and this provides an exact decomposition relative to $\mathscr{E}$.

REMARK. The first part of this proof shows that a non zero dissipative nuclear operator cannot possess a continuous nest of invariant subspaces. In fact it is a theorem of Lidskii that the closed range of T is the closed linear span of the principal vectors of T. This is a simple consequence (see [17, p. 149]) of another well known theorem of his, namely that the trace of a nuclear operator is the sum of the eigenvalues counted with their algebraic multiplicity [13], [4, p. 1104], [17, p. 139], [18, Chapter 3], [6]. It is shown in [16] how the formula $\mathrm{tr}(T) = \int \mathrm{tr}(T_E)\, d\tau$ also leads to this result, thereby providing a triangularisation proof. (The triangularisation proof of [6] uses Erdos' density theorem.)

REMARK. If $T \in \mathrm{Alg}\,\mathscr{E}$ and $C(\Delta)$ is the operator measure for $C = |T|$ then it may happen that $\tau(\Delta) = \mathrm{tr}(C(\Delta))$ is a *locally finite* measure in the sense that $\tau((E,F)) < +\infty$ for all $E > 0$ and $F < I$ and $0_+ = 0$ and $I_- = I$. In this case we could refer to T as a *locally nuclear* operator. Such an operator admits a representation $T = \int T_E\, d\tau$ which exists, for example, as a weak integral. One can obtain a mild generalisation of Lidskii's trace theorem: If T is locally nuclear with eigenvalues $\lambda_1(T), \lambda_2(T), \ldots$ counted with their algebraic multiplicity, such that $\sum_{i=1}^{\infty} |\lambda_i(T)| < +\infty$ then

$$\sum_{i=1}^{\infty} \lambda_i(T) = \lim \mathrm{tr}((F - E)T(F - E)) \quad \text{as } E \downarrow 0,\ F \uparrow I.$$

It may be of interest to obtain external characterisations of locally nuclear operators and of the *sigma nuclear* operators, where sigma nuclear means $\tau(\Delta)$ is sigma finite.

REMARK. We do not have an example of a nuclear operator T which is not exactly decomposable.

If the measure τ of Theorem 5.1 is discrete then, as in the proof of Corollary 5.2(i), T is exactly decomposable. However there are exactly decomposable operators for which τ is continuous.

REFERENCES

1. ARVESON, W. B., Interpolation problems in nest algebras, *J. Functional Analysis*, **20**(1975), 208–233.
2. BERBERIAN, S. K., *Notes on spectral theory*, van Nostrand Math. Studies No. 5, 1966.
3. BONSALL, F. F.; DUNCAN, J., *Complete normed algebras*, Berlin–Heidelberg–New York, Springer, 1973.
4. DUNFORD, N.; SCHWARTZ, J. T., *Linear Operators. Part II*, Interscience, 1963.
5. ERDOS, J. A., Operators of finite rank in nest algebras, *J. London Math. Soc.*, **43**(1968), 391–397.
6. ERDOS, J. A., On the trace of a trace class operator, *Bull. London Math. Soc.*, **6**(1974), 47–50.
7. ERDOS, J. A., Triangular integration on symmetrically normed ideals, *Indiana Univ. Math. J.*, **27**(1978), 401–408.
8. ERDOS, J. A.; POWER, S. C., Weakly closed ideals of nest algebras, *J. Operator Theory*, **7**(1982), 219–235.
9. HOFFMAN, K., *Banach spaces of analytic functions*, Englewood Cliffs, Prentice Hall, 1962.
10. HOPENWASSER, A.; MOORE, R., Finite rank operators in reflexive operator algebras, preprint.
11. LANCE, E. C., Cohomology and perturbations of nest algebras, *Proc. London Math. Soc.*, **43**(1981), 334–356.
12. LARSON, D. R., Annihilators of operator algebras, preprint.
13. LIDSKII, V. B., Non-self-adjoint operators with a trace (Russian), *Dokl. Akad. Nauk S.S.S.R.*, **125**(1959), 485–487; English transl., *Amer. Math. Soc. Transl.*, **47**(1965), 43–46.
14. PARROTT, S., On a quotient norm and the Sz.-Nagy–Foiaş lifting theorem, *J. Functional Analysis*, **30**(1978), 311–328.
15. POWER, S. C., The distance to upper triangular operators, *Math. Proc. Cambridge Philos. Soc.*, **88**(1980), 327–329.
16. POWER, S. C., Another proof of Lidskii's theorem on the trace, *Bull. London Math. Soc.*, to appear.
17. RINGROSE, J. R., *Compact non-self-adjoint operators*, van Nostrand Math. Studies 35, 1971.
18. SIMON, B., *Trace ideals and their applications*, London Math. Soc. Lecture Notes 35, Cambridge Univ. Press, 1979.

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