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POSITIVE SEMIGROUPS AND RESOLVENTS

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1. INTRODUCTION

The general theory of C_0-semigroups $S = \{S_t : t \geqslant 0\}$ of operators on Banach spaces is well established by now (see [5, 10, 13]), but in most applications there is some additional structure present, which often allows some reformulation and simplification of their properties. It has recently been proved [1] that the fundamental condition

$$\|(\lambda - Z)^{-n}\| \leqslant M(\lambda - \alpha)^{-n}$$

for the operator Z on a real Banach space $\mathscr{B}$ to be the generator of a C_0-semigroup can be replaced by positivity of the resolvent if $\mathscr{B}$ is ordered by a normal generating positive cone $\mathscr{B}_+$, assuming that $\mathscr{B}_+$ has non-empty interior (interior points of $\mathscr{B}_+$ are the order-units for $\mathscr{B}$). It is shown in Theorem 1 below that this last assumption cannot be omitted, and in Theorems 2 and 3 that other conjectures made in [4] are false.

For positive C_0-semigroups S on several different types of ordered Banach space, it has been established that the limit

$$\omega_s = \lim_{t\to\infty} t^{-1} \log\|S_t\|$$

belongs to the spectrum of the generator. In Theorem 4, this will be shown to be the case if $\mathscr{B}_+$ is also "α-directed", for example if $\mathscr{B}$ is the self-adjoint part of any C^*-algebra. Finally in Theorem 5 it is shown that ω_s may be determined by inspection of a single interior point of $\mathscr{B}_+$ and extremal functionals in $\mathscr{B}_+^*$.

2. GENERATORS AND POSITIVE RESOLVENTS

Throughout the paper, $\mathscr{B}$ will denote a real Banach space ordered by a normal generating cone $\mathscr{B}_+$. It was shown in [1] (see also [4]) that if $\mathscr{B}_+$ has non-empty interior, then a densely-defined linear operator Z on $\mathscr{B}$ is the generator of a positive C_0-semigroup if and only if the resolvents $R(\lambda, Z) =: (\lambda I - Z)^{-1}$ exist and are positive operators on $\mathscr{B}$ for all sufficiently large real λ. It was conjectured in [4] that this result remains true even if $\mathscr{B}_+$ has no interior, but the following theorem shows that the result is in fact false for $\mathscr{B} = C_0(\mathbf{R})$ (the space of continuous real functions on $\mathbf{R}$ vanishing at infinity).

THEOREM 1. *There is a densely defined (closed) linear operator Z on $C_0(\mathbf{R})$ whose resolvents $R(\lambda, Z)$ exist and are positive for $\lambda > 0$, but which is not the generator of a C_0-semigroup.*

Proof. Suppose that $v : \mathbf{R} \to (0, \infty)$ is continuous and bounded away from zero, $v \in L^1(\mathbf{R}) + L^\infty(\mathbf{R})$. Define Z by

$$Zf = -\frac{(fv)'}{v}$$

whenever the right-hand side exists in $C_0(\mathbf{R})$. If g is a C^1-function of compact support then $Z(g/v)$ exists; so Z is densely defined. If $(\lambda I - Z)f = 0$ for some $\lambda > 0$, then $\lambda(fv) + (fv)' = 0$, so $f(x) = ce^{-\lambda x}/v(x)$. Since $\liminf_{x\to-\infty} v(x) < \infty$ and $\lim_{x\to-\infty} f(x) = 0$, it follows that $f = 0$.

For g in $C_0(\mathbf{R})$ and $\lambda > 0$, the function $k_\lambda := (e_\lambda * (gv))/v$ lies in $C_0(\mathbf{R})$, where $*$ denotes convolution and

$$e_\lambda(x) = \begin{cases} e^{-\lambda x} & (x \geqslant 0) \\ 0 & (x < 0). \end{cases}$$

Furthermore Zk_λ exists and equals $\lambda k_\lambda - g$. Thus $\lambda I - Z$ is invertible, and $R(\lambda, Z)g = k_\lambda$. In particular, $R(\lambda, Z)$ is positive.

If f is a C^1-function of compact support and

$$g_t(x) = \frac{f(x-t)}{v(x)}$$

then $\frac{d}{dt}(g_t) = Zg_t$. Hence, if Z is a generator, $g_t = e^{tZ}(f/v)$ [5, Theorem 1.7]. It follows from the continuity of e^{tZ} that

$$(e^{tZ}g)(x) = \frac{v(x-t)\,g(x-t)}{v(x)} \tag{1}$$

for all g in $C_0(\mathbf{R})$. But (1) defines a strongly continuous semigroup (if and) only if

$$\limsup_{t\downarrow 0} \sup_x \frac{v(x-t)}{v(x)} < \infty.$$

One may certainly arrange that this fails (without violating the other conditions on v), for example by taking $v(x) = 1 + \max_{n \in \mathbf{N}} (n - n^4|x - n|)$.

In Theorem 1 it is possible to arrange that the resolvents $R(\lambda, Z)$ are strictly positive in the sense that they map $C_0(\mathbf{R})_+ \setminus (0)$ into its quasi-interior, the strictly positive functions in $C_0(\mathbf{R})$ (see [4, Proposition 2.5]). If Z_1 is defined by

$$Z_1 f = \frac{(fv)''}{2v}$$

where v satisfies the same conditions as above, then $R(\lambda, Z_1)g = (h_\lambda * (gv))/v$, where $h_\lambda(x) = (2\lambda)^{-1/2} \exp(-(2\lambda)^{1/2}|x|)$. But Z_1 is a generator (if and) only if

$$\limsup_{t \downarrow 0} \sup_x (r_t * v)(x)/v(x) < \infty$$

where $r_t(x) = (2\pi t)^{-1/2} \exp(-x^2/2t)$. In this case,

$$(\mathrm{e}^{tZ_1} f)(x) = (r_t * (fv))/v. \tag{2}$$

It was conjectured in [4] that, if $\mathscr{B}$ is the self-adjoint part of a C^*-algebra, then any positive semigroup S on $\mathscr{B}$ satisfies a bound of the form $\|S_t\| \leqslant \mathrm{e}^{\omega t}$ $(t \geqslant 0)$ for some real ω. If $\mathscr{B}$ has a unit 1, then $\|S_t\| = \|S_t 1\| \to 1$ as $t \downarrow 0$. If 1 belongs to the domain of the generator Z, then $\|S_t\| \leqslant \mathrm{e}^{\|Z1\|t}$ [4]. The next two results show that more than this cannot be said. The constructions are similar to that in Theorem 1 except that the perturbations of standard semigroups are less severe.

THEOREM 2. *There is a positive C_0-semigroup S on $C(\mathbf{T})$ such that $t^{-1}\log\|S_t\| \to \infty$ as $t \downarrow 0$.*

Proof. Let v be any strictly positive function in $C(\mathbf{T})$. Consider $\mathbf{T}$ as the interval $[0,2\pi]$ with endpoints identified, and define $S_t = \mathrm{e}^{tZ}$ by (1). Then $\|S_t\| = \sup_x v(x - t)/v(x)$, and it is easy to arrange that $t^{-1}\log\|S_t\| \to \infty$, for example by taking $v(x) = 1 + |\sin x|^{1/2}$.

THEOREM 3. *There is a positive C_0-semigroup S on $C_0(\mathbf{R})$ such that $\limsup_{t \downarrow 0}\|S_t\| > 1$. Such a semigroup cannot be extended to a positive C_0-semigroup on $C(\overline{\mathbf{R}})$, where $\overline{\mathbf{R}}$ is the one-point compactification of $\mathbf{R}$.*

Proof. Let v be any bounded continuous real function on $\mathbf{R}$, which is bounded away from zero, and define $S_t = \mathrm{e}^{tZ}$ by (1). Then $\|S_t\| = \sup_x v(x - t)/v(x)$, so one may arrange that $\limsup_{t \downarrow 0}\|S_t\| > 1$, for example by taking $v(x) = 2 + \sin x^2$.

The final statement follows from the fact that if S extends to a positive C_0-semigroup $\bar{S}$ on $C(\bar{\mathbf{R}})$, then

$$\|S_t\| \leqslant \|\bar{S}_t\| = \|\bar{S}_t 1\| \to 1 \quad \text{as } t \downarrow 0.$$

The semigroups in Theorems 2 and 3 can be taken to be strictly positive, for example by using (2) (with appropriate choice of v) as a definition.

3. THE PERIPHERAL SPECTRUM

It is a well-known property of C_0-semigroups $S_t = e^{tZ}$ that the limit $\omega_s = \lim_{t\to\infty} t^{-1} \log\|S_t\|$ exists in $[-\infty, \infty)$, the spectral radius of S_t is $\exp(\omega_s t)$, and $e^{\lambda t}$ belongs to the spectrum $\operatorname{Sp} S_t$ whenever λ belongs to $\operatorname{Sp} Z$ [5, Theorems 1.22, 2.16]. In particular,

(3) $$\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp} Z\} \leqslant \omega_s .$$

For positive semigroups, equality often holds in (3). For example there is equality if S is a positive semigroup and either

(a) $\mathscr{B} = L^1(\mu)$ [6],

(b) $\mathscr{B} = L^2(\mu)$ [11], or

(c) $\mathscr{B}$ has an order-unit ([9], where the result is stated for unital C^*-algebras, but the proof may easily be modified).

On the other hand, equality may fail in (3) if $\mathscr{B}$ is a Banach lattice [7]; it is unknown whether it holds if $\mathscr{B} = L^p(\mu)$ $(1 < p < \infty;\ p \neq 2)$.

Recall from [2] that $\mathscr{B}$ is *α-directed* ($\alpha \geqslant 1$) if, for any finite number of elements $b_1, \ldots, b_n$ in the closed unit ball of $\mathscr{B}$, there is some b in $\mathscr{B}$ with $\|b\| \leqslant \alpha$, $b_i \leqslant b$ $(i = 1, \ldots, n)$. If u is an order-unit of norm 1, then $\mathscr{B}$ is δ-directed where δ is the distance from u to the complement of $\mathscr{B}_+$; so Theorem 4 below extends case (c) mentioned above. The self-adjoint part of any C^*-algebra is α-directed for any $\alpha > 1$ [12, Theorem 1.4.2].

THEOREM 4. *Suppose that $\mathscr{B}$ is α-directed for some finite α. Let S be a positive C_0-semigroup on $\mathscr{B}$ with generator Z and with $\omega_s > -\infty$. Then ω_s belongs to* $\operatorname{Sp} Z$.

Proof. Replacing S_t by $\exp(-\omega_s t)S_t$, there is no loss of generality in assuming that $\omega_s = 0$. The proof that follows will be similar to that given in [6] for AL-spaces, but using the dual semigroup S^*.

Suppose that $0 \notin \operatorname{Sp} Z$. It follows from [5, Theorem 2.16, Lemma 2.11] that $\operatorname{Sp} \mathrm{R}(1, Z) \subset \{\lambda : |\lambda - 1/2| \leqslant 1/2\} \setminus \{1\}$, so that the spectral radius ρ of $\mathrm{R}(1, Z)$ is strictly less than 1. Standard resolvent theory (see [5, p. 35]) shows that if $|1 - \mu| < \rho^{-1}$, then $\mu \notin \operatorname{Sp} Z$ and

$$\mathrm{R}(\mu, Z) = \sum_{n=0}^{\infty} (1 - \mu)^n\, \mathrm{R}(1, Z)^{n+1}$$

the sum being absolutely convergent. Fix μ with $1-\rho^{-1}<\mu<0$. For b in $\mathscr{B}_+$ and φ in $\mathscr{B}_+^*$,

$$\langle R(1, Z)^{n+1}b, \varphi\rangle = \int_0^\infty \langle S_t b, \varphi\rangle \frac{t^n}{n!} e^{-t}\, dt,$$

the integral being absolutely convergent. It follows from the Monotone Convergence Theorem that

$$\langle R(\mu, Z)b, \varphi\rangle = \int_0^\infty \langle S_t b, \varphi\rangle \sum_{n=0}^{\infty} \frac{(1-\mu)^n t^n}{n!} e^{-t}\, dt = \int_0^\infty \langle S_t b, \varphi\rangle e^{-\mu t}\, dt.$$

Since $\mathscr{B}_+$ is generating, this formula is valid for all b in $\mathscr{B}$. The norm is α-additive on $\mathscr{B}_+^*$ [2, Theorem 2.3.5] in the sense that $\|\varphi_1\| + \ldots + \|\varphi_n\| \leqslant \alpha(\|\varphi_1 + \ldots + \varphi_n\|)$ $(\varphi_i \in \mathscr{B}_+^*)$, so the use of approximating Riemann sums shows that

$$\int_0^\infty e^{-\mu t} \|S_t^*\varphi\|\, dt \leqslant \alpha \left\| \int_0^\infty e^{-\mu t} S_t^* \varphi\, dt \right\| = \alpha \|R(\mu, Z)^*\varphi\| \leqslant \alpha \|R(\mu, Z)^*\|\, \|\varphi\| < \infty$$

at least for φ in the set

$$\mathscr{S} = \{\psi \in \mathscr{B}_+^* : \lim_{t\to 0} \|S_t^*\psi - \psi\| = 0\} = \{\psi \in \mathscr{B}_+^* : t \to S_t^*\psi \text{ is norm continuous}\}.$$

But the proof of [5, Theorem 1.36] shows that every $\varphi \in \mathscr{B}_+^*$ is the weak*-limit of a sequence $\varphi_n \in \mathscr{S}$, so

$$\int_0^\infty e^{-\mu t} \|S_t^*\varphi\|\, dt \leqslant \liminf_{n\to\infty} \int_0^\infty e^{-\mu t} \|S_t^*\varphi_n\|\, dt \leqslant \tag{4}$$
$$\leqslant \alpha \|R(\mu, Z)^*\| \liminf_{n\to\infty} \|\varphi_n\| < \infty$$

by the Uniform Boundedness Theorem.

Since $\mu < 0 = \omega_s$, $\|e^{-\mu m} S_m^*\| \to \infty$ as $m \to \infty$. Since $\mathscr{B}_+^*$ is generating, the Uniform Boundedness Theorem implies that there is a φ in $\mathscr{B}_+^*$ and a sequence

$m_n \to \infty$ such that $e^{-\mu m_n} \|S_{m_n}^* \varphi\| \geq 1$. If $M := \sup \{e^{-\mu t} \|S_t\| : 0 \leq t \leq 1\}$, then

$$e^{-\mu t} \|S_t^* \varphi\| \geq \frac{1}{M} \quad (m_n - 1 \leq t \leq m_n).$$

This contradicts (4).

4. EXTREMAL POSITIVE FUNCTIONALS

Suppose that $\mathscr{B}$ has an order-unit u, and let $\mathscr{B}$ be a positive C_0-semigroup on $\mathscr{B}$. The Kreĭn-Rutman Theorem shows that for each $t > 0$, there is a non-zero φ_t in $\mathscr{B}_+^*$ such that $S_t^* \varphi_t = e^{\omega_s t} \varphi_t$ (see [8, 2.1]). Thus, if for each non-zero φ in $\mathscr{B}_+^*$,

(5) $$\inf_{t > 0} \langle S_t u, \varphi \rangle < \langle u, \varphi \rangle$$

then $\omega_s < 0$ (and conversely). The next result, the discrete version of which was proved in [3], shows that it is sufficient that (5) should hold for a restricted class of functionals.

THEOREM 5. *Suppose that $\mathscr{B}$ has an order-unit u, and that S is a positive C_0-semigroup on $\mathscr{B}$ such that* (5) *holds whenever φ is a non-zero weak*-limit of extrema functionals in the cone $\mathscr{B}_+^*$. Then $\|S_t\| < 1$ for some t, and hence $\omega_s < 0$.*

Proof. Let $K = \{\varphi \in \mathscr{B}_+^* : \varphi(u) = 1\}$. Then $\mathscr{B}$ is linearly homeomorphic to the space $A(K)$ of weak*-continuous affine real functions on K. It follows from compactness of the weak* closure $\overline{\partial K}$ of the extreme boundary of K and strong continuity of S that there are rational numbers $t_1, \ldots, t_k$ such that

$$\inf_{1 \leq i \leq k} \langle S_{t_i} u, \varphi \rangle < 1 \quad (\varphi \in \overline{\partial K}).$$

So there is a positive number τ such that

$$\inf_{n \in \mathbf{N}} \langle S_{n\tau} u, \varphi \rangle < 1 \quad (\varphi \in \overline{\partial K}).$$

It follows from [3, Theorem 3.2] that $\|S_{n\tau} u\| \to 0$ as $n \to \infty$.

Similarly one may prove that if $\sup_{t > 0} \langle S_t u, \varphi \rangle > 1$ whenever φ is a non-zero weak*-limit of extremal positive functionals, then $\omega_s > 0$. However in Theorem 5 it is not possible to assume that (5) is valid only for extremal positive functionals φ, as may be seen by embedding certain of the operators considered in [3, Example 3.1] in suitable semigroups.

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