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PRODUCT STATES OF THE GAUGE INVARIANT AND ROTATIONALLY INVARIANT CAR ALGEBRAS

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1. INTRODUCTION

In this paper we study the restriction of product states of the CAR algebra $\mathfrak{A}$ to the gauge invariant algebra $\mathfrak{A}^T$ and the rotationally invariant algebra $\mathfrak{A}^G \subset \subset \mathfrak{A}^T$. We obtain necessary and sufficient conditions for such states to be factor states, classify the factors according to type and obtain necessary and sufficient conditions for the quasi-equivalence of two such states. We also obtain results for the restriction of general factor states of $\mathfrak{A}$ to $\mathfrak{A}^T$ and $\mathfrak{A}^G$.

This paper is a continuation of [4] although it can be read with no knowledge of our earlier paper. The main new idea of this paper emerged while we were talking with Vaughan Jones and we wish to acknowledge our debt to him. The idea is this. Suppose π is a *-representation of $\mathfrak{A}$ induced by a factor state ω. Then if ω has certain properties, the weak closures $\pi(\mathfrak{A}^T)''$ and $\pi(\mathfrak{A}^G)''$ may be bigger than one would expect. In fact, one may have $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$ or $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^T)''$ (see Theorems 3.4, 3.6, 3.8 and 3.9). Then the analysis of representations of $\mathfrak{A}^G$ and $\mathfrak{A}^T$ reduces to analysis of representations of $\mathfrak{A}$, about which much more is known. This is a great simplification, which among other things allows us to classify with regards to type the restriction of product states to $\mathfrak{A}^T$ and $\mathfrak{A}^G$ (see Theorems 4.3 and 4.7).

2. NOTATION AND DEFINITIONS

Let $\mathscr{B}_0 = M(2, \mathbf{C})$ be the algebra of all complex (2×2)-matrices. Set $\mathscr{B}_p$ isomorphic to $\mathscr{B}_0$ for all $p \in \mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathfrak{A}_m = \bigotimes_{p=1}^{m} \mathscr{B}_p$ isomorphic to $M(2^m, \mathbf{C})$ for all $m \in \mathbf{N}$. We denote by $\mathfrak{A}$ the C^*-algebra obtained from the norm completion of the inductive limit of the $\mathfrak{A}_m$, i.e., $\mathfrak{A} = \overline{\bigcup_{m=1} \mathfrak{A}_m}$. This C^*-algebra, sometimes called the CAR algebra, is a UHF algebra of type 2^∞ (see [7]).

Let $G = \mathrm{SU}(2)$ be the group of all complex unitary (2×2)-matrices of determinant one and T be the subgroup of all diagonal unitary matrices of determinant one. We associate G with the group of rotations (SO(3)) and T with the group of rotations around the z-axis. We denote elements of G by letters g and h, and when we want to draw attention to the fact that they are matrices we will write U_g or U_h.

We define a product action of G (and, therefore, T) on $\mathfrak{A}$ as follows. Fixing matrix units E_{ij}, $e_{ij}^{(p)}$ for $\mathscr{B}_0$ and $\mathscr{B}_p$, respectively and isomorphisms $\gamma_p : \mathscr{B}_0 \ni E_{ij} \to e_{ij}^{(p)} \in \mathscr{B}_p$ we denote by $\alpha_g^{(p)}$ the *-automorphism of $\mathscr{B}_p$ given by

$$\alpha_g^{(p)}(A) = U_g^{(p)} A U_g^{(p)-1}$$

for all $A \in \mathscr{B}_p$ where $U_g^{(p)} = \gamma_p(U_g)$. We define the *-automorphism α_g of $\mathfrak{A}$ as follows. If $A = A_1 \otimes A_2 \otimes \ldots \otimes A_m \in \mathfrak{A}_m$ we set

$$\alpha_g(A) = \alpha_g^{(1)}(A_1) \otimes \alpha_g^{(2)}(A_2) \otimes \ldots \otimes \alpha_g^{(m)}(A_m).$$

By linearity α_g extends to $\mathfrak{A}_m$ and by norm continuity α_g has a unique extension to all of $\mathfrak{A}$.

We denote by $\mathfrak{A}^G$ and $\mathfrak{A}^T$ the subalgebras,

$$\mathfrak{A}^G = \{A \in \mathfrak{A};\ \alpha_g(A) = A \text{ for all } g \in G\}$$

$$\mathfrak{A}^T = \{A \in \mathfrak{A};\ \alpha_g(A) = A \text{ for all } g \in T\}.$$

Clearly, $\mathfrak{A} \supset \mathfrak{A}^T \supset \mathfrak{A}^G$. These algebras are AF-algebras generated by increasing sequences $\mathfrak{A}_m^T$ and $\mathfrak{A}_m^G$ of finite dimensional subalgebras (see [5] and [4] for further details). If ω is a state of $\mathfrak{A}$ we denote by ω^G and ω^T the restriction of ω to $\mathfrak{A}^G$ and $\mathfrak{A}^T$, respectively.

In this paper we will be primarily concerned with product states. Given states ω_p of $\mathscr{B}_p$ we may construct the product state $\omega = \bigotimes_{p=1}^{\infty} \omega_p$ of $\mathfrak{A}$ as follows. If $A = A_1 \otimes A_2 \otimes \ldots \otimes A_m \in \mathfrak{A}_m$ we define $\omega(A) = \omega_1(A_1)\omega_2(A_2) \ldots \omega_m(A_m)$. By linearity ω extends to a state of $\mathfrak{A}_m$ and by norm continuity to a unique state ω of $\mathfrak{A}$.

We will make use of the Pauli spin matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $a \in \mathbf{R}^3$ we denote by $a\cdot\sigma = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$. The association between elements $g \in G = \mathrm{SU}(2)$ and rotations R_g is given by

$$(R_g a)\cdot\sigma = U_g(a\cdot\sigma)U_g^{-1}$$

for all $a \in \mathbf{R}^3$. We will write $\sigma_{px}, \sigma_{py}, \sigma_{pz}$ for $\gamma_p(\sigma_x)$, $\gamma_p(\sigma_y)$, $\gamma_p(\sigma_z)$ and $\sigma_k \cdot \sigma_l = \sigma_{kx}\sigma_{lx} + \sigma_{ky}\sigma_{ly} + \sigma_{kz}\sigma_{lz}$. Note $\sigma_k \cdot \sigma_l$ is in $\mathfrak{A}^G$ and $\mathfrak{A}^G$ is generated by these elements (see [12]).

The Pauli spin matrices are useful for describing states ω_k of $\mathscr{B}_k$. A state ω_k of $\mathscr{B}_k$ may be characterized by a vector $a_k \in \mathbf{R}^3$ by the relation $(a_{kx}, a_{ky}, a_{kz}) = (\omega_k(\sigma_{kx}), \omega_k(\sigma_{ky}), \omega_k(\sigma_{kz}))$. The vector a_k must have length not exceeding one, $|a_k| \leqslant 1$, since ω_k is a state. Then to describe a product state $\omega = \bigotimes_{k=1}^{\infty} \omega_k$ of $\mathfrak{A}$ we need only give a sequence $\{a_k\}$ of three vectors of length not greater than one. To draw attention to these vectors a_k we will sometimes write $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$.

Several times in this paper we will be proving that two states ω_1 and ω_2 are quasi-equivalent, denoted $\omega_1 \underset{q}{\sim} \omega_2$. This means that the cyclic $*$-representations they induce are quasi-equivalent (see e.g. [6]). We will often make use of the fact that two $*$-representations π_1 and π_2 are quasi-equivalent if and only if the mapping φ defined by $\varphi(\pi_1(A)) = \pi_2(A)$ for all A in the algebra is σ-strongly bicontinuous (or σ-weakly bicontinuous). The extension of φ to the weak closure is a $*$-isomorphism of $\pi_1(\mathfrak{A})''$ with $\pi_2(\mathfrak{A})''$.

For the case of factor states (states which induce factor representations) the induced representations are either quasi-equivalent or disjoint. It follows that if ω_1 and ω_2 are factor states and $\|\omega_1 - \omega_2\| < 2$ then $\omega_1 \underset{q}{\sim} \omega_2$.

We will need some estimates on the norm differences of two products states of $\mathfrak{A}$. Each state ω of an $(n \times n)$-matrix algebra M corresponds to a density matrix $\Omega \in M$ via the relation $\omega(A) = \mathrm{tr}(A\Omega)$ for all $A \in M$, where tr is the normalized trace. If ω_1 and ω_2 are states of M with density matrices Ω_1 and Ω_2 then it follows from the Powers-Størmer inequality (see [10]) that

$$2(1 - \mathrm{tr}(\Omega_1^{1/2}\Omega_2^{1/2})) \leqslant \|\omega_1 - \omega_2\| \leqslant 2(1 - \mathrm{tr}(\Omega_1^{1/2}\Omega_2^{1/2})^2)^{1/2}. \tag{2.1}$$

If $\omega_1 = \bigotimes_{k=1}^{m} \omega_{a_k}$ and $\omega_2 = \bigotimes_{k=1}^{m} \omega_{b_k}$ are products states of $\mathfrak{A}_m$ one can use the above formula to estimate the norm difference of ω_1 and ω_2 using the facts,

$$\mathrm{tr}(\Omega_1^{1/2}\Omega_2^{1/2}) = \mathrm{tr}(\Omega_{a_1}^{1/2}\Omega_{b_1}^{1/2} \otimes \ldots \otimes \Omega_{a_m}^{1/2}\Omega_{b_m}^{1/2}) =$$

$$= \Gamma(a_1, b_1)\,\Gamma(a_2, b_2) \ldots \Gamma(a_m, b_m)$$

where

$$\Gamma(a, b) = \mathrm{tr}(\Omega_a^{1/2}\Omega_b^{1/2}) = \mathrm{tr}((I + a\cdot\sigma)^{1/2}(I + b\cdot\sigma)^{1/2}) =$$

$$= \frac{1}{4}(\sqrt{1+|a|} + \sqrt{1-|a|})(\sqrt{1+|b|} + \sqrt{1-|b|}) + \tag{2.2}$$

$$+ \frac{1}{4}(\sqrt{1+|a|} - \sqrt{1-|a|})(\sqrt{1+|b|} - \sqrt{1-|b|})\, a \cdot b/|a|\,|b|.$$

Since this formula holds for all m and the $\mathfrak{A}_m$ are dense in $\mathfrak{A}$ it follows for product states $\omega_1 = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ and $\omega_2 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ that we have

$$2(1 - s) \leqslant \|\omega_1 - \omega_2\| \leqslant 2\sqrt{1 - s^2} \tag{2.3}$$

where

$$s = \prod_{k=1}^{\infty} \Gamma(a_k, b_k).$$

It follows (see also [10]) that the product states ω_1 and ω_2 induce quasi-equivalent factor representations if and only if

$$\sum_{k=1}^{\infty} (1 - \Gamma(a_k, b_k)) < \infty.$$

We end this section with a computational lemma concerning $\Gamma(a, b)$.

LEMMA 2.1. *Suppose $a, b, c \in \mathbf{R}^3$ are vectors of length not greater than one and $a = b + c$ with b and c orthogonal, i.e. $b \cdot c = 0$. Then $1 - \Gamma(a, b) \leqslant |c|^2$.*

Proof. Assume a, b and c satisfy the hypothesis of the lemma. Rewriting the expression for $\Gamma(a, b)$ we have

$$\begin{aligned} 1 - \Gamma(a, b) = 1 - \frac{1}{2}\sqrt{1 + |a|}\sqrt{1 + |b|} - \frac{1}{2}\sqrt{1 - |a|}\sqrt{1 - |b|} + \\ + (|a|\,|b| - a \cdot b)\left(\frac{\sqrt{1 + |a|} - \sqrt{1 - |a|}}{2|a|}\right)\left(\frac{\sqrt{1 + |b|} - \sqrt{1 - |b|}}{2|b|}\right). \end{aligned} \tag{2.4}$$

Since $\sqrt{1 + |a|} - \sqrt{1 - |a|} \leqslant \sqrt{2}|a|$ (similarly with a replaced by b) and $a \cdot b = |b|^2$ we have

$$\begin{aligned} 1 - \Gamma(a, b) \leqslant 1 - \frac{1}{2}\sqrt{1 + |a|}\sqrt{1 + |b|} - \frac{1}{2}\sqrt{1 - |a|}\sqrt{1 - |b|} + \\ + \frac{1}{2}|b|\,(|a| - |b|). \end{aligned} \tag{2.5}$$

Since $\sqrt{1 - |a|}\sqrt{1 - |b|} \geqslant 1 - |a|$, $\sqrt{1 + |a|}\sqrt{1 + |b|} \geqslant 1 + |a| - (|a| - |b|)$ and $|a| - |b| \leqslant \frac{1}{2}|c|^2|b|^{-1}$ it follows that

$$1 - \Gamma(a, b) \leqslant \frac{1}{2}(|a| - |b|) + \frac{1}{2}|b|(|a| - |b|) \leqslant \frac{1}{4}(1 + |b|^{-1})\,|c|^2.$$

Hence for $|b| \geqslant 1/3$ we have

$$1 - \Gamma(a, b) \leqslant |c|^2. \tag{2.6}$$

Now we need an estimate for when $|b| \leqslant 1/3$. Let $s = \frac{1}{2}(|a| + |b|)$ and $d = \frac{1}{2}(|a| - |b|)$. We have

$$\sqrt{1+|a|}\sqrt{1+|b|} + \sqrt{1-|a|}\sqrt{1-|b|} = \sqrt{(1+s)^2 - d^2} + \sqrt{(1-s)^2 - d^2}.$$

Since for $0 \leqslant x \leqslant y$ we have $(y^2 - x^2)^{1/2} \geqslant y - x^2/y$ it follows from inequality (2.5) that

$$1 - \Gamma(a, b) \leqslant \frac{1}{2} d^2(1+s)^{-1} + \frac{1}{2} d^2(1-s)^{-1} + |b|d.$$

Since we assume $|b| \leqslant 1/3$ we have $0 \leqslant s \leqslant 2/3$ and

$$1 - \Gamma(a, b) \leqslant \frac{1}{2} d^2 + 3d^2/2 + \frac{1}{4}|c|^2 = 2d^2 + \frac{1}{4}|c|^2.$$

Since $d = \frac{1}{2}|c|^2(|a| + |b|)^{-1} \geqslant \frac{1}{2}|c|^2|a|^{-1} \geqslant \frac{1}{2}|c|$ we have

$$1 - \Gamma(a, b) \leqslant (3/4)|c|^2$$

for $|b| \leqslant 1/3$. Combining this inequality with (2.6) we have $1 - \Gamma(a, b) \leqslant |c|^2$. Done.

3. RELATION BETWEEN $\pi(\mathfrak{A}^T)''$, $\pi(\mathfrak{A}^G)''$ AND $\pi(\mathfrak{A})''$

In this section we will show that, under certain conditions, if π is a factor representation of $\mathfrak{A}$ then $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$ and $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A}^T)''$.

LEMMA 3.1. *Fix $k \in \mathbf{N}$. Then $\mathfrak{A}^T$ is generated as a C^*-algebra by σ_{kz} and $\mathfrak{A}^G$.*

Proof. Let $\mathscr{B}$ be the C^*-algebra generated by σ_{kz} and $\mathfrak{A}^G$. Since $\sigma_{kz} \in \mathfrak{A}^T$ and $\mathfrak{A}^G \subset \mathfrak{A}^T$ we have $\mathscr{B} \subset \mathfrak{A}^T$. Let $P_{rs} = \sum_{i,j=1}^{2} e_{ij}^{(r)} e_{ji}^{(s)}$. One can show (see [12]) that P_{rs} is an hermitian unitary element contained in $\mathfrak{A}^G$ which exchanges $\mathscr{B}_r$ and $\mathscr{B}_s$ for $r \neq s \in \mathbf{N}$. We will call the P_{rs} transposition elements. Since $\sigma_{kz} \in \mathscr{B}$ and $P_{rk} \in \mathscr{B}$ we have $\sigma_{rz} = P_{rk}\sigma_{kz}P_{rk} \in \mathscr{B}$ for all $r \in \mathbf{N}$. Since $\sigma_{rz} = e_{11}^{(r)} - e_{22}^{(r)}$ we have

$e_{ii}^{(r)} \in \mathscr{B}$ for all $r \in \mathbf{N}$ and $i = 1, 2$. Next we note that $e_{12}^{(r)} e_{21}^{(s)} = e_{11}^{(r)} e_{22}^{(s)} P_{rs} \in \mathscr{B}$. Now it follows from [5] that $\mathfrak{A}^T$ is generated by elements of the form

$$e_{i_1 j_1}^{(k_1)} e_{i_2 j_2}^{(k_2)} \ldots e_{i_m j_m}^{(k_m)}$$

where the k's are distinct and the sum of the i's equals the sum of the j's, i.e., $\sum_{p=1}^{m} i_p = \sum_{p=1}^{m} j_p$. Since such terms can be written as a product of terms $e_{ii}^{(p)}$ or $e_{12}^{(r)} e_{21}^{(s)} = e_{21}^{(s)} e_{12}^{(r)}$ it follows that such terms are in $\mathscr{B}$. Hence $\mathscr{B} \supset \mathfrak{A}^T$. Done.

LEMMA 3.2. *Fix $k \in \mathbf{N}$. Then σ_{kx} and $\mathfrak{A}^T$ generate $\mathfrak{A}$.*

Proof. Let $\mathscr{B}$ be the C^*-algebra generated by σ_{kx} and $\mathfrak{A}^T$. Arguing as in the last lemma we have $P_{rs} \in \mathfrak{A}^G \subset \mathfrak{A}^T \subset \mathscr{B}$ and, therefore, $\sigma_{rx} = P_{rk} \sigma_{kx} P_{rk} \in \mathscr{B}$ for all $r \in \mathbf{N}$. Since $\sigma_{rz} \in \mathscr{B}$ and σ_{rx} and σ_{rz} generate $\mathscr{B}_r$ we have $\mathscr{B}_r \subset \mathscr{B}$ for all $r \in \mathbf{N}$. Since the $\mathscr{B}_r$ generate $\mathfrak{A}$ we have $\mathscr{B} = \mathfrak{A}$. Done.

LEMMA 3.3. *Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$ and $(\pi, \mathscr{H}, f_0)$ is a cyclic $*$-representation induced by ω. Suppose $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 = \infty$. Then, $\pi(\sigma_{px}) \in \pi(\mathfrak{A}^T)''$ for each $p \in \mathbf{N}$.*

Proof. Suppose ω satisfies the hypothesis of the lemma. Let $Z_n = \sum_{k=2}^{n} a_{kx}^2 + a_{ky}^2$. We will show that $\pi(\sigma_{1x}) \in \pi(\mathfrak{A}^T)''$. Since $\pi(P_{rs}) \in \pi(\mathfrak{A}^T)$ for all $r, s \in \mathbf{N}$ it will follow by the argument of Lemma 3.1 that $\pi(\sigma_{px}) = \pi(P_{1p} \sigma_{1x} P_{1p}) \in \pi(\mathfrak{A}^T)''$ for all $p \in \mathbf{N}$.

We define $A_n \in \mathfrak{A}^T$ as follows:

$$A_n = 2Z_n^{-1} \sum_{k=2}^{n} (a_{kx} + \mathrm{i} a_{ky})\, e_{12}^{(1)} e_{21}^{(k)}$$

or

$$A_n = Z_n^{-1} \sum_{k=2}^{n} (a_{kx} + \mathrm{i} a_{ky})\, e_{12}^{(1)} (\sigma_{kx} - \mathrm{i} \sigma_{ky}).$$

Let $U_n(t) = \pi(\exp(\mathrm{i}t(A_n + A_n^*)))$. We will show

$$U_n(t) \to \pi(\exp(\mathrm{i}t\sigma_{1x})) = V(t)$$

strongly as $n \to \infty$, where the convergence is uniform for t in a bounded interval. To accomplish this we will show $\|V(t)f - U_n(t)f\| \to 0$ as $n \to \infty$ for unit vectors of the form $f = \pi(A_1 \otimes A_2 \otimes \ldots \otimes A_m) f_0$ with $A_p \in \mathscr{B}_p$. Since the linear span of such f are dense in $\mathscr{H}$ and the $U_n(t)$ and $V(t)$ are unitary it will follow that $U_n(t) \to$

$\to V(t)$ strongly. For such f the state $\omega_f(A) = (f, \pi(A)f)$ is again a product state $\omega_f = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ with $b_k = a_k$ for k sufficiently large ($k > m$). Now we have

$$\|V(t)f - U_n(t)f\|^2 = 2 - 2\,\mathrm{Re}(\omega_f(\exp(-\mathrm{i}t(A_n + A_n^*))\exp(\mathrm{i}t\sigma_{1x}))). \tag{3.1}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\exp(-\,\mathrm{i}t(A_n + A_n^*))\exp(\mathrm{i}t\sigma_{1x}) =$$

$$= -\,\mathrm{i}\exp(-\,\mathrm{i}t(A_n + A_n^*))\,(A_n + A_n^* - \sigma_{1x})\exp(\mathrm{i}t\sigma_{1x}).$$

Integrating we have

$$\exp(-\mathrm{i}t(A_n + A_n^*))\exp(\mathrm{i}t\sigma_{1x}) =$$

$$= I - \mathrm{i}\int_0^t \exp(-\mathrm{i}s(A_n + A_n^*))\,(A_n + A_n^* - \sigma_{1x})\exp(\mathrm{i}s\sigma_{1x})\,\mathrm{d}s.$$

Inserting this in equation (3.1) we have

$$\|V(t)f - U_n(t)f\|^2 =$$

$$= 2\,\mathrm{Re}\left(\mathrm{i}\int_0^t \omega_f(\exp(-\mathrm{i}s(A_n + A_n^*))\,(A_n + A_n^* - \sigma_{1x})\exp(\mathrm{i}s\sigma_{1x}))\,\mathrm{d}s\right).$$

Then we have

$$\|V(t)f - U_n(t)f\|^2 \leqslant 2\int_0^t \|\pi((A_n + A_n^* - \sigma_{1x})\exp(\mathrm{i}s\sigma_{1x}))f\|\,\mathrm{d}s. \tag{3.2}$$

Now the effect of the $\pi(\exp(\mathrm{i}s\sigma_{1x}))$ on f is just to change the state ω_f determined by f by rotating the first vector b_1 around the x-axis by $2s$ radians. Since this does not change the general form of f under consideration we will simply replace $\pi(\exp(\mathrm{i}s\sigma_{1x}))f$ by f in estimating the right hand side of inequality (3.2). We have

$$\|\pi(A_n + A_n^* - \sigma_{1x})f\| \leqslant \|\pi(A_n - e_{12}^{(1)})f\| + \|\pi(A_n^* - e_{21}^{(1)})f\|.$$

Estimating the first term we have

$$\|\pi(A_n - e_{12}^{(1)})f\|^2 = \omega_f((A_n^* - e_{21}^{(1)})\,(A_n - e_{12}^{(1)})) =$$

$$= Z_n^{-2}\sum_{k,l=2}^{n}(a_{kx} - \mathrm{i}a_{ky})\,(a_{lx} + \mathrm{i}a_{ly})\,\omega_f(e_{22}^{(1)}(\sigma_{kx} + \mathrm{i}\sigma_{ky})\,(\sigma_{lx} - \mathrm{i}\sigma_{ly})) -$$

$$- 2Z_n^{-1}\sum_{k=2}^{n}\mathrm{Re}((a_{kx} + \mathrm{i}a_{ky})\,\omega_f(e_{22}^{(1)}(\sigma_{kx} - \mathrm{i}\sigma_{ky}))) + \omega_f(e_{22}^{(1)}).$$

Using the product state property of $\omega_f = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ we have

$$\|\pi(A_n - e_{12}^{(1)})f\|^2 = \omega_f(e_{22}^{(1)})|\alpha_n - 1|^2 + \tag{3.3}$$
$$+ \omega_f(e_{22}^{(1)}) Z_n^{-2} \sum_{k=2}^{n} (a_{kx}^2 + a_{ky}^2)(2 + 2b_{kz} - b_{kx}^2 - b_{ky}^2)$$

where

$$\alpha_n = Z_n^{-1} \sum_{k=2}^{n} (a_{kx} + ia_{ky})(b_{kx} - ib_{ky}).$$

Since $b_k = a_k$ for $k > m$ we have for $n > m$

$$\alpha_n = 1 + Z_n^{-1} \sum_{k=2}^{m} (a_{kx} + ia_{ky})(b_{kx} - a_{kx} - i(b_{ky} - a_{ky})).$$

Since $Z_n \to \infty$ as $n \to \infty$ we have $\alpha_n \to 1$ as $n \to \infty$. Hence, the first term in equation (3.3) tends to zero as $n \to \infty$. Since $2 + 2b_{kz} - b_{kx}^2 - b_{ky}^2 \leq 4$ we have that the second term in equation (3.3) is not greater than $4\omega_f(e_{22}^{(1)}) Z_n^{-1}$. Since $Z_n \to \infty$ this second term tends to zero as $n \to \infty$. Hence, $\|\pi(A_n - e_{12}^{(1)})f\| \to 0$ as $n \to \infty$.

A similar calculation shows $\|\pi(A_n^* - e_{21}^{(1)})f\| \to 0$ as $n \to \infty$. Recalling we have replaced $\pi(\exp(is\sigma_{1x}))f$ by f we see our estimate of the convergence is uniform in s. Hence, from inequality (3.2), $|t|^{-1}\|U_n(t)f - V(t)f\| \to 0$ as $n \to \infty$, where the convergence is uniform in t. Hence $V(t) = \pi(\exp(it\sigma_{1x}))$ is in the strong closure of $\pi(\mathfrak{A}^T)$. Hence, $V(t)$ and, therefore, $\pi(\sigma_{1x})$ (its generator) are contained in $\pi(\mathfrak{A}^T)''$. And as we have seen this implies $\pi(\sigma_{px}) \in \pi(\mathfrak{A}^T)''$ for all $p \in \mathbf{N}$. Done.

THEOREM 3.4. *Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$ and $(\pi, \mathscr{H}, f_0)$ is a cyclic *-representation of $\mathfrak{A}$ induced by ω. Suppose $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 = \infty$. Then. $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$.*

Proof. Suppose the hypothesis of the theorem is satisfied. From Lemma 3.3 we have $\pi(\sigma_{1x}) \in \pi(\mathfrak{A}^T)''$ and from Lemma 3.2 we have that $\pi(\sigma_{1x})$ and $\pi(\mathfrak{A}^T)$ generate $\pi(\mathfrak{A})$ as a C^*-algebra. Hence, $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$. Done.

REMARK. Let $\widetilde{\mathfrak{A}}$ be the C^*-subalgebra of $\mathfrak{A}$ generated by the $\mathscr{B}_k$ with $k \geq 2$ and $\widetilde{\mathfrak{A}}^T$ and $\widetilde{\mathfrak{A}}^G$ be the group invariant C^*-subalgebras of $\mathfrak{A}$. Recalling the arguments of Lemmas 3.2, 3.3 and 3.4 we see that if $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 = \infty$ then $\pi(\sigma_{kx}) \in \pi(\widetilde{\mathfrak{A}}^T)''$ for $k \geq 2$ and $\pi(\widetilde{\mathfrak{A}}^T)'' = \pi(\widetilde{\mathfrak{A}})''$ (i.e., Theorem 3.4 holds for $\widetilde{\mathfrak{A}}^T$ and $\widetilde{\mathfrak{A}}$).

To study the relation between $\pi(\mathfrak{A}^G)''$ and $\pi(\mathfrak{A}^T)''$ we will need the following lemma.

LEMMA 3.5. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *and* $(\pi, \mathscr{H}, f_0)$ *is a cyclic *-representation induced by* ω. *Suppose* $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. *Then there is a subsequence* $a_{k(i)}$ *so that* $\sum_{i=1}^{\infty} |a_{k(i)}|^2 = \infty$ *and* $|a_{k(i)}|^{-1} a_{k(i)} \to s$ *as* $i \to \infty$. *Furthermore,* $\pi(s\cdot\sigma_p) \in \pi(\mathfrak{A}^G)''$ *for each* $p \in \mathbf{N}$.

Proof. Suppose the hypothesis of the lemma is satisfied. Since the surface S^2 of the unit sphere in $\mathbf{R}^3$ is compact it can be covered with a finite number n_1 o open discs $\{D_{1i},\ i = 1, \ldots, n_1\}$ of radius 1/2. Let Q_{1i} be the set of $k \in \mathbf{N}$ so that $a_k \neq 0$ and $|a_k|^{-1} a_k \in D_{1i}$. Clearly,

$$\sum_{k\in Q_{1i}} |a_k|^2 = \infty$$

for some $i = 1, \ldots, n_1$, say i_0. Begin forming the subsequence $i \to k(i)$ by choosing the first m_1 integers from Q_{1i_0} where m_1 is big enough to insure that the sum of the squares of the lengths of the chosen a_k exceed one.

Since the closure of D_{1i_0} is compact it can be covered with a finite number n_2 of open discs $\{D_{2i},\ i = 1, \ldots, n_2\}$ of radius 1/4. Let the sets Q_{2i} be defined as were the Q_{1i}. Then, we have

$$\sum_{k\in Q_{2i}} |a_k|^2 = \infty$$

for some $i = 1, \ldots, n_2$, say i_1. Pick the next m_2 integers for the sequence $i \to k(i)$ from Q_{2i_0} in the order they appear in Q_{2i_1} and choose them not repeating those previously chosen, where m_2 is sufficiently large to insure that the sum of the squares of these next choosen a_k's exceeds one.

Continuing the procedure we have begun in the obvious manner we obtain a subsequence $i \to k(i)$ so that $\sum_{i=1}^{\infty} |a_{k(i)}|^2 = \infty$ and $|a_{k(i)}|^{-1} a_{k(i)} \to s$ as $i \to \infty$.

Now we show $\pi(s\cdot\sigma_1) \in \pi(\mathfrak{A}^G)''$. Let

$$Z_n = \sum_{k=2}^{n} |a_{k(i)}|^2 \quad \text{and} \quad A_n = Z_n^{-1} \sum_{i=2}^{n} |a_{k(i)}| \sigma_1\cdot\sigma_{k(i)}$$

$$U_n(t) = \pi(\exp(itA_n)) \quad V(t) = \pi(\exp(its\cdot\sigma_1)).$$

We will show $U_n(t)$ converges strongly to $V(t)$ as $n \to \infty$. Arguing as in Lemma 3.3 it is sufficient to show $\|U_n(t)f - V(t)f\| \to 0$ for unit vectors f of the form $f = \pi(A_1 \otimes \otimes A_2 \otimes \ldots \otimes A_m)f_0$. For such vectors the state ω_f is again a product state $\omega_f =$

$= \bigotimes_{k=1}^{\infty} \omega_{b_k}$ with $b_k = a_k$ for $k > m$. Since the computations are almost identical to those in Lemma 3.3 we summarize them as follows,

$$\|U_n(t)f - V(t)f\|^2 \leqslant 2\int_0^t \|\pi((A_n - s\cdot\sigma_1)\exp(it's\cdot\sigma_1))f\|\,dt'.$$

And replacing $\pi(\exp(it's\cdot\sigma_1)f$ by f as we did in Lemma 3.3 we have

$$\|\pi(A_n - s\cdot\sigma_1)f\|^2 = \omega_f((A_n - s\cdot\sigma_1)^2) =$$

$$= |c_n - s|^2 + Z_n^{-2}\sum_{i=2}^{n} |a_{k(i)}|^2 (3 - 2b_1\cdot b_{k(i)} - |b_{k(i)}|^2) \leqslant |c_n - s|^2 + 3Z_n^{-1}$$

where

$$c_n = \sum_{i=2}^{n} |a_{k(i)}|\, b_{k(i)}.$$

We can express $c_n - s$ in the form

$$c_n - s = Z_n^{-1}\sum_{i=2}^{n} |a_{k(i)}|^2 (|a_{k(i)}|^{-1} b_{k(i)} - s).$$

Suppose $\varepsilon > 0$. Since $b_{k(i)} = a_{k(i)}$ for i sufficiently large and $|a_{k(i)}|^{-1}a_{k(i)} \to s$ as $i \to \infty$ there is an integer m so that $|\,|a_{k(i)}|^{-1}b_{k(i)} - s| < \varepsilon$ for $i > m$. Then it follows

$$|c_n - s| \leqslant \left(2Z_n^{-1}\sum_{i=2}^{m} |a_{k(i)}|\right) + \varepsilon.$$

Since $Z_n \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ is arbitrary we have $|c_n - s| \to 0$ as $n \to \infty$. Hence, $\|\pi(A_n - s\cdot\sigma_1)f\| \to 0$ and, thus, $U_n(t)$ converges strongly to $V(t)$ as $n \to \infty$. Hence $V(t)$ and its generator $\pi(s\cdot\sigma_1)$ is contained in $\pi(\mathfrak{A}^G)''$. By the argument of Lemma 3.1 $\pi(\mathfrak{A}^G)$ contains the transposition operators $\pi(P_{1p})$ and, hence, $\pi(s\cdot\sigma_p) = \pi(P_{1p}(s\cdot\sigma_1)P_{1p}) \in \pi(\mathfrak{A}^G)''$ for all $p \in \mathbf{N}$. Done.

THEOREM 3.6. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *and* $(\pi, \mathscr{H}, f_0)$ *is a cyclic *-representation of* $\mathfrak{A}$ *induced by* ω. *Suppose* $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. *Then there is a unit vector* $s \in \mathbf{R}^3$ *satisfying the conclusion of Lemma* 3.5, *and* $\pi(\mathfrak{A}^G)'' = \pi(\alpha_g(\mathfrak{A}^T))''$ *where* R_g *is a rotation so that* $R_g(0, 0, 1) = s$.

Proof. Assume the hypothesis of the theorem is satisfied. From Lemma 3.5 it follows that $\pi(s\cdot\sigma_1) \in \pi(\mathfrak{A}^G)''$. From Lemma 3.1 it follows that σ_{1z} and $\mathfrak{A}^G$ generate $\mathfrak{A}^T$ as a C^*-algebra. Since $\alpha_g(\sigma_{1z}) = s \cdot \sigma_1$ it follows that $s \cdot \sigma_1$ and $\mathfrak{A}^G$ generate $\alpha_g(\mathfrak{A}^T)$ as a C^*-algebra. Hence, $\pi(\mathfrak{A}^G)'' = \pi(\alpha_g(\mathfrak{A}^T))''$. Done.

REMARK. As in the remark following Theorem 3.4 we see that if $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ then there is a unit vector $s \in \mathbf{R}^3$ satisfying the conclusion of Lemma 3.5, and $\pi(\widetilde{\mathfrak{A}}^G)'' = \pi(\alpha_g(\widetilde{\mathfrak{A}}^T))''$ where $R_g(0, 0, 1) = s$ and $\widetilde{\mathfrak{A}}$ is the C^*-subalgebra of $\mathfrak{A}$ generated by the $\mathscr{B}_k$ with $k \geqslant 2$.

COROLLARY 3.7. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *and* $(\pi, \mathscr{H}, f_0)$ *is a cyclic *-representation of* $\mathfrak{A}$ *induced by* ω. *Suppose* $\sum_{k=1}^{\infty} |a_k|^2 - (a_k \cdot s)^2 = \infty$ *for all unit vectors* $s \in \mathbf{R}^3$. *Then* $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$.

Proof. Suppose the hypothesis of the corollary is satisfied. Then from Theorem 3.6 there is a unit vector s so that $\pi(\mathfrak{A}^G)'' = \pi(\alpha_g(\mathfrak{A}^T))''$ where $R_g(0, 0, 1) = s$. Since $\sum_{k=1}^{\infty} |a_k|^2 - (a_k \cdot s)^2 = \infty$ it follows from Theorem 3.4 that $\pi(\alpha_g(\mathfrak{A}^T))'' = \pi(\mathfrak{A})''$. Hence, $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$. Done.

REMARK. As we remarked after Theorems 3.4 and 3.6 if the hypotheses of Corollary 3.7 are satisfied then $\pi(\widetilde{\mathfrak{A}}^G)'' = \pi(\widetilde{\mathfrak{A}})''$.

We conclude this section with some results for arbitrary factor states of $\mathfrak{A}$.

THEOREM 3.8. *Suppose* ω *is a factor state of* $\mathfrak{A}$ *and* $(\pi, \mathscr{H}, f_0)$ *is a cyclic *-representation of* $\mathfrak{A}$ *induced by* ω. *Suppose that* $\omega(\sigma_{kx})^2 + \omega(\sigma_{ky})^2$ *does not tend to zero as* $k \to \infty$. *Then* $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$ *and* ω^T *is a factor state of the same type as* ω.

Proof. Assume the hypothesis of the theorem is satisfied. Then there is a subsequence $i \to k(i)$ so that $\omega(\sigma_{k(i)x}) \to a$ and $\omega(\sigma_{k(i)y}) \to b$ as $i \to \infty$ and $a^2 + b^2 > 0$. Let

$$A_n = 2(a^2 + b^2)^{-1} (a + ib)\, e_{12}^{(1)} e_{21}^{(k(n))}$$

$$A_n = (a^2 + b^2)^{-1} (a + ib)\, e_{12}^{(1)} (\sigma_{k(n)x} - i\sigma_{k(n)y}).$$

We will show that $\pi(A_n)$ tends weakly to $\pi(e_{12}^{(1)})$ as $n \to \infty$. Since the A_n are uniformly bounded it is sufficient to show $(f, \pi(A_n - e_{12}^{(1)})g) \to 0$ for all f and g in a dense set in $\mathscr{H}$. We take for our dense set vectors f and g of the form $f = \pi(x) f_0$ and $g = \pi(y) f_0$ with $x, y \in \mathfrak{A}_m$ for some $m \in \mathbf{N}$. Then we have for $n > m$

$$\begin{aligned}(f, \pi(e_{12}^{(1)} - A_n)\, g) &= \omega(x^*(e_{12}^{(1)} - A_n)\, y) = \\ &= \omega(x^* e_{12}^{(1)} y) - 2(a^2 + b^2)^{-1} (a + ib)\, \omega(x^* e_{12}^{(1)} y e_{21}^{(k(n))}).\end{aligned} \tag{3.4}$$

Since ω is a factor state it has the cluster property (see [9]) so $\omega(A e_{21}^{(p)}) \to \omega(A)\omega(e_{21}^{(p)})$ as $p \to \infty$. Hence,

$$2\omega(x^* e_{12}^{(1)} y e_{21}^{(k(n))}) \to \omega(x^* e_{12}^{(1)} y)\; \omega(\sigma_{k(n)x} - i\sigma_{k(n)y}) \to \omega(x^* e_{12}^{(1)} y)\,(a - ib)$$

as $n \to \infty$. Combining this with equation 3.4 we see $(f, \pi(e_{12}^{(1)} - A_n) g) \to 0$ as $n \to \infty$. Since $A_n \in \mathfrak{A}^T$ and $\pi(\mathfrak{A}^T)''$ is weakly closed we have $\pi(e_{12}^{(1)}) \in \pi(\mathfrak{A}^T)''$. Since $\pi(e_{ii}^{(1)}) \in \in \pi(\mathfrak{A}^T)$ for $i = 1, 2$ and $\pi(\mathfrak{A}^T)''$ is closed under taking adjoints we have $\pi(e_{ij}^{(1)}) \in \in \pi(\mathfrak{A}^T)''$ for $i, j = 1, 2$. Recalling from Lemma 3.1 that the transposition operators $P_{rs} \in \mathfrak{A}^G \subset \mathfrak{A}^T$ it follows that $\pi(e_{ij}^{(k)}) = \pi(P_{1k} e_{ij}^{(1)} P_{1k}) \in \pi(\mathfrak{A}^T)''$ for $i, j = 1, 2$ and $k \in \mathbf{N}$. Since the $e_{ij}^{(k)}$ generate $\mathfrak{A}$ we have $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$.

Since f_0 is cyclic in $\mathscr{H}$ for $\pi(\mathfrak{A})''$ and $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$, f_0 is cyclic in $\mathscr{H}$ for $\pi(\mathfrak{A}^T)$. Hence, if π^T is a cyclic *-representation induced by ω^T then π^T is unitarily equivalent to the restriction of π to $\mathfrak{A}^T$, i.e., $\pi^T \sim \pi|\mathfrak{A}^T$. Hence, $\pi^T(\mathfrak{A}^T)''$ is isomorphic to $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$. Hence, ω^T is a factor state of the same type as ω. Done.

THEOREM 3.9. *Suppose ω is a factor state of $\mathfrak{A}$ and $(\pi, \mathscr{H}, f_0)$ is a cyclic *-representation of $\mathfrak{A}$ induced by ω. Suppose the sequence of vectors $\omega(\sigma_k)$ has at least two linearly independent accumulation points. Then $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$ and ω^G is a factor state of the same type as ω.*

Proof. Assume the hypothesis of the theorem is satisfied. Then there are subsequences $i \to k(i)$ and $i \to p(i)$ so that $\omega(\sigma_{k(i)}) \to a$ and $\omega(\sigma_{p(i)}) \to b$ where a and b are linearly independent vectors. Let

$$A_n = \sigma_1 \cdot \sigma_{k(n)} \quad \text{and} \quad B_n = \sigma_1 \cdot \sigma_{p(n)}.$$

Arguing as in the previous theorem we see $\pi(A_n) \to \pi(a \cdot \sigma_1)$ and $\pi(B_n) \to \pi(b \cdot \sigma_1)$ weakly as $n \to \infty$. Since $A_n, B_n \in \mathfrak{A}^G$ and $\pi(\mathfrak{A}^G)''$ is weakly closed we have $\pi(a \cdot \sigma_1)$ $\pi(b \cdot \sigma_1) \in \pi(\mathfrak{A}^G)''$. Since a and b are linearly independent $a \cdot \sigma_1$ and $b \cdot \sigma_1$ generate $\mathscr{B}_1$. Since $\mathfrak{A}^G$ contains the transpositions P_{rs} it follows that $\pi(\mathscr{B}_k) = \pi(P_{1k} \mathscr{B}_1 P_{1k}) \subset \subset \pi(\mathfrak{A}^G)''$. Since the $\mathscr{B}_k$ generate $\mathfrak{A}$ it follows $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$.

Repeating the argument at the end of the last theorem we find ω^G is a factor state of the same type as ω.

4. FACTOR STATES OF $\mathfrak{A}^G$ AND $\mathfrak{A}^T$ AND QUASI-EQUIVALENCE

In this section we analyze when the restriction of a product state $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ to $\mathfrak{A}^T$ and $\mathfrak{A}^G$ is a factor state (i.e., it induces a factor representation of $\mathfrak{A}^T$ or $\mathfrak{A}^G$). We classify these factor states as to type and obtain necessary and sufficient conditions that two such states be quasi-equivalent. We also obtain results for the restrictions of general factor states to $\mathfrak{A}^T$ and $\mathfrak{A}^G$.

THEOREM 4.1. *Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$. Then ω^T is a factor state if and only if*

$$\sum_{k=1}^{\infty} (1 - a_{kz}^2) = 0 \text{ or } \infty. \tag{4.1}$$

Proof. Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$ and the sum (4.1) is finite and greater than zero. We show $\mathfrak{A}^T$ is not a factor state following an argument of [14] and [12]. To this end let

$$J_{zn} = \sum_{k=1}^{n} (\sigma_{kz} - a_{kz} I).$$

Let $(\pi, \mathscr{H}, f_0)$ be a cyclic *-representation of $\mathfrak{A}^T$ induced by ω^T. Let $U_n(t) = \exp(it\pi(J_{zn}))$. We will show that $U_n(t)$ converges strongly to a unitary group $U(t)$ in the center of $\pi(\mathfrak{A}^T)''$. Suppose $A \in \mathfrak{A}_r^T$ and $f = \pi(A) f_0$. Then for $n > m > r$ we have

$$\|U_n(t)f - U_m(t)f\|^2 = \|(U_m(t)^* U_n(t) - I)f\|^2 \leqslant t^2 \|\pi(K)f\|^2$$

where $K = J_{nz} - J_{mz}$ is the generator of the unitary group $U_m(t)^* U_n(t)$. Then, we have

$$\|U_n(t)f - U_m(t)f\|^2 \leqslant t^2\omega(A^*K^2A) = t^2\omega(A^*A)\,\omega(K^2) =$$

$$= t^2\omega(A^*A) \sum_{k=m+1}^{n} (1 - a_{kz}^2).$$

Since the sum (4.1) converges the above sum converges to zero as $n, m \to \infty$. Since such vectors $f = \pi(A) f_0$ are dense in $\mathscr{H}$ the unitaries $U_n(t)$ converge strongly to a one parameter group $U(t)$ of unitaries.

Note the unitaries $U(t)$ are in the center of $\pi(\mathfrak{A}^T)''$ since $U(t)\,\pi(A)\,U(t)^{-1} = \pi(A)$ for all $A \in \mathfrak{A}_n^T$ and such $\pi(A)$ are strongly dense in $\pi(\mathfrak{A}^T)''$. One sees that $U(t) \neq \lambda I$ since $|(f_0, U(t)f_0)| = |\prod_{k=1}^{\infty} \exp(ia_{kz}t)(\cos(t) + ia_{kz}\sin(t))| < 1$ for $0 < t < \pi$. Hence, ω^T is not a factor state.

Suppose the sum (4.1) is zero (so that $a_k = \pm(0, 0, 1)$ for all $k \in \mathbf{N}$). Then by [14] or [12] ω^T is pure and, hence, a factor state.

Next suppose the sum (4.1) is divergent. We show ω^T is a factor state following the line of argument in [1]. From [5] it suffices to show that for each $A \in \mathfrak{A}_n^T$ that $\Gamma_m(A) \to 0$ as $m \to \infty$ where

$$\Gamma_m(A) = \sup\{|\omega(AB) - \omega(A)\omega(B)|;\ B \in (\mathfrak{A}_m^T)^c \cap \mathfrak{A}^T,\ \|B\| \leqslant 1\},$$

where $(\mathfrak{A}_m^T)^c$ is the set of elements of $\mathfrak{A}$ commuting with $\mathfrak{A}_m^T$. Following [1] we have

$$\Gamma_m(A) = \sum_{k=1}^{r} |\omega(AE_k) - \omega(A)\omega(E_k)|$$

where the E_k are the minimal central projections of $\mathfrak{A}_m^T$ which can be expressed

$$E_k = \frac{1}{2\pi} \int_0^{2\pi} \exp(-\, ikt + iN_m t)\, dt$$

where

$$N_m = \sum_{k=1}^{m} e_{22}^{(k)} = \sum_{k=1}^{m} \frac{1}{2} (I - \sigma_{kz}).$$

Using this formula for the E_k we find

$$\Gamma_m(A) = \sum_{k=1}^{r} |\alpha_k|$$

where α_k is the k^{th} Fourier coeficient of the function $f(t)P_m(t)$,

$$f(t) = \omega(A \exp(itN_n)) - \omega(A)\, \omega(\exp(itN_n))$$

and

$$P_m(t) = \prod_{k=n+1}^{m} \frac{1}{2} ((1 + a_{kz}) + (1 - a_{kz})\, e^{it}).$$

Summarizing we have $\Gamma_m(A) = \|fP_m\|_1$ where $\|h\|_1$ is the sum of the absolute values of the Fourier coefficients of h.

Note f can have at most n non-zero Fourier coefficients. Furthermore, since $f(0) = 0$, we can write $f(t) = (e^{it} - 1)\, g(t)$ where g has at most $n - 1$ non-zero Fourier coefficients. Note $\|g\|_1$ is finite. Since $\|hk\|_1 \leqslant \|h\|_1 \|k\|_1$ we have

$$\|fP_m\|_1 = \|(e^{it} - 1) g\, P_m\|_1 \leqslant \|g\|_1 \|(e^{it} - 1)\, P_m\|_1.$$

To show $\Gamma_m(A) \to 0$ as $m \to \infty$ it suffices to show $\|(e^{it} - 1)\, P_m\|_1 \to 0$ as $m \to \infty$. Let $P_m(t) = \sum_{k=0}^{r} \alpha_k e^{ikt}$. Then

$$\|(e^{it} - 1) P_m\|_1 = \sum_{k=0}^{r} |\alpha_{k+1} - \alpha_k|.$$

From [1] there is an integer q so that $\alpha_{k+1} \geqslant \alpha_k$ for $k < q$ and $\alpha_{k+1} \leqslant \alpha_k$ for $k \geqslant q$. It follows from the telescoping property of the sum that

$$\|(e^{it} - 1) P_m\|_1 \leqslant 2 \max\{\alpha_k;\ k = 0, \ldots, r\}.$$

One easily sees (see [1] for details) that this max tends to zero as $m \to \infty$ provided the sum (4.1) diverges. Hence, ω^T is a factor state. Done.

LEMMA 4.2. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *so that* ω^T *is a factor state (i.e.,* $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = 0$ *or* ∞*). Let* $b_k = (0, 0, a_{kz})$ *for each* $k \in \mathbf{N}$ *and* $\omega_0 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$*. Suppose* $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 < \infty$*. Then* $\omega^T \underset{q}{\sim} \omega_0^T$ *and* $\omega \underset{q}{\sim} \omega_0$.

Proof. Suppose ω and ω_0 satisfy the hypothesis of the lemma. Note ω_0^T is a factor state since $\sum_{k=1}^{\infty} (1 - b_{kz}^2) = 0$ or ∞. We have using inequality (2.3)

$$\|\omega^T - \omega_0^T\| \leqslant \|\omega - \omega_0\| \leqslant 2\sqrt{1 - s^2}$$

where

$$s = \prod_{k=1}^{\infty} \Gamma(a_k, b_k).$$

Note $\Gamma(a_k, b_k) > 0$ for all $k \in \mathbf{N}$ ($\Gamma(a, b) = 0$ only if $a = -b$ and $|a| = 1$). From Lemma 2.1 we have $1 - \Gamma(a_k, b_k) \leqslant |c_k|^2$ where $c_k = a_k - b_k = (a_{kx}, a_{ky}, 0)$. Then we have

$$\sum_{k=1}^{\infty} (1 - \Gamma(a_k, b_k)) \leqslant \sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 < \infty.$$

Hence, $s > 0$ and $\|\omega^T - \omega_0^T\| \leqslant \|\omega - \omega_0\| < 2$. Hence, the factor states ω and ω_0 and ω^T and ω_0^T are not disjoint, so $\omega \underset{q}{\sim} \omega_0$ and $\omega^T \underset{q}{\sim} \omega_0^T$. Done.

THEOREM 4.3. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *and* $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = 0$ *or* ∞*. Then* ω^T *is a factor state and the type of* ω^T *is given as follows.*

CASE 1. $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 = \infty$. *Then*

ω^T *is of type* I *if and only if* $\sum_{k=1}^{\infty} (1 - |a_k|) < \infty$*;*

ω^T *is not of type* II_1*;*

ω^T *is semi-finite if and only if* $\sum_{k=1}^{\infty} (1 - |a_k|)\,|a_k|^2 < \infty$*;*

ω^T *is of type* III *if and only if* $\sum_{k=1}^{\infty} (1 - |a_k|)\,|a_k|^2 = \infty$.

CASE 2. $\sum_{k=1}^{\infty} a_{kx}^2 + a_{ky}^2 < \infty$. *Then,*

ω^T *is of type* I *if and only if* $a_k = \pm(0, 0, 1)$ *for each* $k \in \mathbf{N}$*;*

ω^T *is of type* II_1 *if and only if* $\sum_{k=1}^{\infty} (a_{kz} - t)^2 < \infty$ *for some* t *with* $-1 < t < 1$*;*

ω^T is semi-finite if and only if $\sum_{k=1}^{\infty} (1 - a_{kz}^2)(a_{kz} - t)^2 < \infty$ *for some t with* $-1 < t < 1$;

ω^T is of type III *if and only if* $\sum_{k=1}^{\infty} (1 - a_{kz}^2)(a_{kz} - t)^2 = \infty$ *for all t with* $-1 < t < 1$.

Proof. Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$ and $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = 0$ or ∞. Then by the previous theorem ω^T is a factor state. Let $(\pi, \mathscr{H}, f_0)$ and $(\pi^T, \mathscr{H}^T, f_0^T)$ be cyclic *-representations of $\mathfrak{A}$ and $\mathfrak{A}^T$ induced by ω and ω^T, respectively. Suppose we are in Case 1 of the theorem. Then by Theorem 3.4 we have $\pi(\mathfrak{A}^T)'' = \pi(\mathfrak{A})''$. Since f_0 is cyclic in $\mathscr{H}$ for $\pi(\mathfrak{A})$, f_0 is cyclic in $\mathscr{H}$ for $\pi(\mathfrak{A}^T)$. By the uniqueness of the Gelfand-Segal construction it follows that the restriction of π to $\mathfrak{A}^T$ is unitarily equivalent to π^T. Hence $\pi(\mathfrak{A})'' = \pi(\mathfrak{A}^T)''$ and $\pi^T(\mathfrak{A}^T)''$ are isomorphic von Neumann algebras. Hence, ω and ω^T are factor states of the same type. The type of ω follows from [8] (see also [10]). This completes the analysis of Case 1.

Suppose we are in Case 2 of the theorem. Then by Lemma 4.2 we have $\omega^T \underset{q}{\sim} \omega_0^T$ where $\omega_0 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ with $b_k = (0, 0, a_{kz})$. The classification of invariant states has been carried out in [13, Theorem 3.1] and translating their results into our notation the conclusion for Case 2 follows. Done.

LEMMA 4.4. *Suppose $\omega_1 = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ and $\omega_2 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ are quasi-equivalent product states of $\mathfrak{A}$. Then $\sum_{k=1}^{\infty} (1 - a_{kz}^2) < \infty$ if and only if $\sum_{k=1}^{\infty} (1 - b_{kz}^2) < \infty$.*

Proof. Suppose ω_1 and ω_2 satisfy the hypothesis of the theorem and $\sum_{k=1}^{\infty} (1 - a_{kz}^2) < \infty$. Let $c_k = (0, 0, 1)$ if $a_{kz} \geqslant 0$ and $c_k = (0, 0, -1)$ if $a_{kz} < 0$. Let $\omega_0 = \bigotimes_{k=1}^{\infty} \omega_{c_k}$. We recall from the discussion in Section 2 that $\omega_1 \underset{q}{\sim} \omega_0$ if and only if $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, c_k)) < \infty$. We have from equation (2.4) and the fact that $|c_k| = 1$ that

$$1 - \Gamma(a_k, c_k) = 1 - 2^{-1/2}\sqrt{1 + a_k} + 2^{-1/2}(|a_k| - a_k \cdot c_k)\left(\frac{\sqrt{1 + |a_k|} - \sqrt{1 - |a_k|}}{2|a_k|}\right).$$

Since $\sqrt{1 + |a_k|} \geqslant \sqrt{2} - (1 - |a_k|)/\sqrt{2}$ and $\sqrt{1 + |a_k|} - \sqrt{1 - |a_k|} \leqslant \sqrt{2}\,|a_k|$ we have

$$\begin{aligned} 1 - \Gamma(a_k, c_k) &\leqslant (1 - |a_k|)/\sqrt{2} + \frac{1}{2}(|a_k| - a_k \cdot c_k) \leqslant \\ &\leqslant (1 - |a_k|) + (|a_k| - a_k \cdot c_k) \leqslant \\ &\leqslant 1 - a_k \cdot c_k \leqslant 1 - a_{kz}^2 . \end{aligned}$$

Since $\sum_{k=1}^{\infty} (1 - a_{kz}^2) < \infty$ by assumption we have $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, c_k)) < \infty$ and $\omega_1 \underset{q}{\sim} \omega_0$. Since $\omega_1 \underset{q}{\sim} \omega_2$ it follows that $\omega_2 \underset{q}{\sim} \omega_0$. Hence, $\sum_{k=1}^{\infty} (1 - \Gamma(b_k, c_\kappa)) < < \infty$. Now, we have

$$1 - \Gamma(b_k, c_k) = 1 - 2^{-1/2}\sqrt{1 + |b_k|} + 2^{-1/2}(|b_k| - b_k \cdot c_k) \cdot \\ \cdot \left(\frac{\sqrt{1 + |b_k|} - \sqrt{1 - |b_k|}}{2|b_k|} \right).$$

Since $2^{-1/2}\sqrt{1 + |b_k|} \leqslant 1 - (1 - |b_k|)/4$ and $\sqrt{1 + |b_k|} - \sqrt{1 - |b_k|} \geqslant |b_k|$ we have

$$1 - \Gamma(b_k, c_k) \geqslant \frac{1}{4}(1 - |b_k|) + 2^{-3/2}(|b_k| - b_k \cdot c_k) \geqslant$$

$$\geqslant \frac{1}{4}(1 - b_k) + \frac{1}{4}(b_k - b_k \cdot c_k) = \frac{1}{4}(1 - b_k \cdot c_k) \geqslant$$

$$\geqslant \frac{1}{4}(1 - |b_{kz}|) \geqslant (1/8)(1 - b_{kz}^2).$$

Since $\sum_{k=1}^{\infty}(1 - \Gamma(b_k, c_k)) < \infty$ we have $\sum_{k=1}^{\infty}(1 - b_{kz}^2) < \infty$. Hence, we have shown that $\sum_{k=1}^{\infty} (1 - a_{kz}^2) < \infty$ implies $\sum_{k=1}^{\infty} (1 - b_{kz}^2) < \infty$. The same argument gives the reverse implication, so we are done.

THEOREM 4.5. *Suppose* $\omega_1 = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *and* $\omega_2 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ *are product states whose restrictions* ω_1^T *and* ω_2^T *are factorial. Then the question of whether* ω_1^T *and* ω_2^T *are quasi-equivalent may be determined as follows.*

CASE 1. $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = \infty$, *then,*

$\omega_1^T \underset{q}{\sim} \omega_2^T$ *if and only if there is a* $g \in T$ *so that* $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$.

CASE 2. $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = 0$, *so* $a_k = \pm(0, 0, 1)$ *for all* $k \in \mathbf{N}$, *then,*

$\omega_1^T \underset{q}{\sim} \omega_2^T$ *if and only if* $b_k = \pm(0, 0, 1)$ *for all* $k \in \mathbf{N}$,

$$\sum_{k=1}^{\infty} |a_{kz} - b_{kz}| < \infty \quad \textit{and} \quad \sum_{k=1}^{\infty} (a_{kz} - b_{kz}) = 0.$$

Proof. Suppose ω_1 and ω_2 satisfy the hypothesis of the theorem. We begin with Case 1 ($\Leftarrow$). Assume $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = \infty$ and there is a $g \in T$ so that $\omega_1 \underset{q}{\sim} \omega_2 \circ \circ \alpha_g$. Then $\omega_3 = \omega_2 \circ \alpha_g = \bigotimes_{k=1}^{\infty} \omega_{c_k}$ where $c_k = R_g b_k$ for all $k \in \mathbf{N}$. Note $b_{kz} = c_{kz}$.

Since $\omega_1 \underset{q}{\sim} \omega_3$ and $\sum_{k=1}^{\infty} (1 - a_{kz}^2) = \infty$ it follows from Lemma 4.4 that $\sum_{k=1}^{\infty} (1 - c_{kz}^2) = \infty$. Since $\omega_1 \underset{q}{\sim} \omega_3$ we have $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, c_k)) < \infty$. Hence, $\Gamma(a_k, c_k) = 0$ for only a finite set Q of $k \in \mathbf{N}$. Let $d_k = 0$ for $k \in Q$ and $d_k = c_k$ for $k \notin Q$. Let $\omega_4 = \bigotimes_{k=1}^{\infty} \omega_{d_k}$. Clearly, we have $\sum_{k=1}^{\infty} (1 - d_{kz}^2) = \infty$ and, thus, by Theorem 4.1 ω_4^T is a factor state. By inequality (2.3) we have

$$\|\omega_1^T - \omega_4^T\| \leqslant \|\omega_1 - \omega_4\| \leqslant 2\sqrt{1 - s_1^2}$$

$$\|\omega_3^T - \omega_4^T\| \leqslant \|\omega_3 - \omega_4\| \leqslant 2\sqrt{1 - s_2^2}$$

with

$$s_1 = \prod_{k=1}^{\infty} \Gamma(a_k, d_k) \quad \text{and} \quad s_2 = \prod_{k=1}^{\infty} \Gamma(c_k, d_k).$$

Since $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, d_k)) < \infty$, $\sum_{k=1}^{\infty} (1 - \Gamma(c_k, d_k)) < \infty$ and $\Gamma(a_k, d_k) > 0$, $\Gamma(c_k, d_k) > 0$ for all $k \in \mathbf{N}$ it follows that $s_1 > 0$ and $s_2 > 0$. Hence, $\|\omega_1^T - \omega_4^T\| < 2$ and $\|\omega_3^T - \omega_4^T\| < 2$ and since these states are factor states we have $\omega_1^T \underset{q}{\sim} \omega_4^T$ and $\omega_3^T \underset{q}{\sim} \omega_4^T$. Hence, $\omega_1^T \underset{q}{\sim} \omega_3^T = \omega_2^T$. Hence, we are done with Case 1 ($\Leftarrow$).

Next we deal with Case 2 ($\Leftarrow$). To this end we assume that $a_k = \pm(0, 0, 1)$, $b_k = \pm(0, 0, 1)$ for all $k \in \mathbf{N}$, $\sum_{k=1}^{\infty} |a_{kz} - b_{kz}| < \infty$ and $\sum_{k=1}^{\infty} (a_{kz} - b_{kz}) = 0$. Then it is clear that a finite permutation of the $k \in \mathbf{N}$ will carry the state ω_2 onto the state ω_1 (see [12] for further details). Done Case 2 ($\Leftarrow$).

Now we prove the implication ($\Rightarrow$). To this end assume $\omega_1^T \underset{q}{\sim} \omega_2^T$. Le $(\pi_i, \mathscr{H}_i, f_i)$ and $(\pi_i^T, \mathscr{H}_i^T, f_i^T)$ be cyclic *-representations of $\mathfrak{A}$ and $\mathfrak{A}^T$ induced by ω_i and ω_i^T, respectively, for $i = 1, 2$.

We consider Case 1 ($\Rightarrow$). We will divide this case into four subcases depending on whether the sums $\sum_{k=1}^{\infty} (a_{kx}^2 + a_{ky}^2)$ and $\sum_{k=1}^{\infty} (b_{kx}^2 + b_{ky}^2)$ are A(infinite, infinite), B(infinite, finite), C(finite, infinite) or D(finite, finite).

We begin with subcase A. Assume we are in subcase A. Then by Theorem 3.4 we have $\pi_i(\mathfrak{A}^T)'' = \pi_i(\mathfrak{A})''$ and, therefore, π_i^T and the restriction of π_i to $\mathfrak{A}^T$ are unitarily equivalent via a unitary operator U_i from $\mathscr{H}_i$ to $\mathscr{H}_i^T$ so that $U_i^* \pi_i^T(A) U_i = \pi_i(A)$ for all $A \in \mathfrak{A}^T$ and $U_i f_i = f_i^T$ for $i = 1, 2$. Since by assumption $\omega_1^T \underset{q}{\sim} \omega_2^T$ there is a σ-strongly bicontinuous *-isomorphism α of $\pi_1^T(\mathfrak{A}^T)''$ onto $\pi_2^T(\mathfrak{A}^T)''$ so that $\alpha(\pi_1^T(A)) = \pi_2^T(A)$ for all $A \in \mathfrak{A}^T$. We define a σ-strongly bicontinuous *-isomorphism φ of $\pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A})''$ by the relation

$$\varphi(\pi_1(A)) = U_2^* \alpha(U_1 \pi_1(A) U_1^*) U_2 = \pi_2(A)$$

for all $A \in \mathfrak{A}^T$. Note that φ is σ-strongly bicontinuous so it extends uniquely to a *-isomorphism of $\pi_1(\mathfrak{A}^T)'' = \pi_1(\mathfrak{A})''$ onto $\pi_2(\mathfrak{A}^T)'' = \pi_2(\mathfrak{A})''$.

We will show $\varphi(\pi_1(\mathscr{B}_1)) = \pi_2(\mathscr{B}_1)$ where we recall $\mathscr{B}_1$ is the first (2×2)-matrix algebra in the tensor product for $\mathfrak{A}$. To this end suppose $A \in \pi_1(\mathscr{B}_1)$. Then A commutes with everything in $\pi_1(\widetilde{\mathfrak{A}}^T)$ (recall that $\widetilde{\mathfrak{A}}$ is the C^*-subalgebra of $\mathfrak{A}$ generated by the $\mathscr{B}_k$ with $k \geqslant 2$). Since φ is a *-isomorphism we have $\varphi(A)$ commutes with everything in $\varphi(\pi_1(\widetilde{\mathfrak{A}}^T)) = \pi_2(\widetilde{\mathfrak{A}}^T)$. But, by the remark following Theorem 3.4 we have $\pi_2(\widetilde{\mathfrak{A}}^T)'' = \pi_2(\widetilde{\mathfrak{A}})''$. Hence $B = \varphi(A) \in \pi_2(\mathfrak{A})'' \cap \pi_2(\widetilde{\mathfrak{A}})'$. Now $B \in \pi_2(\mathfrak{A})''$ can be uniquely expressed in the form $B = \sum_{i,j=1}^{2} \pi_2(e_{ij}^{(1)}) B_{ij}$ with $B_{ij} \in \pi_2(\mathfrak{A})'' \cap \cap \pi_2(\mathscr{B}_1)'$ $\left(\text{in fact, } B_{ij} = \sum_{k=1}^{2} \pi_2(e_{ki}^{(1)}) B \pi_2(e_{jk}^{(1)})\right)$. Since $\pi_2(e_{ij}^{(1)}) \in \pi_2(\widetilde{\mathfrak{A}})'$ and $B \in \in \pi_2(\widetilde{\mathfrak{A}})'$ we have $B_{ij} \in \pi_2(\widetilde{\mathfrak{A}})'$ for $i, j = 1, 2$. We also have $B_{ij} \in \pi_2(\mathscr{B}_1)'$. Since $\mathscr{B}_1$ and $\widetilde{\mathfrak{A}}$ generate $\mathfrak{A}$ we have $B_{ij} \in \pi_2(\mathfrak{A})' \cap \pi_2(\mathfrak{A})''$. Since $\pi_2(\mathfrak{A})''$ is a factor the B_{ij} must be multiples α_{ij} of the identity. Hence, $\varphi(A) = \sum_{i,j=1}^{2} \alpha_{ij}\pi_2(e_{ij}^{(1)}) \in \in \pi_2(\mathscr{B}_1)$. Since φ is a *-isomorphism we have $\varphi(\pi_1(A)) = \pi_2(\beta(A))$ for $A \in \mathscr{B}_1$ and β a *-automorphism of $\mathscr{B}_1$.

We further determine β by noting that $\sigma_{1z} \in \mathfrak{A}^T$ and, thus, $\varphi(\pi_1(\sigma_{1z})) = \pi_2(\sigma_{1z})$. Hence, $\beta(\sigma_{1z}) = \sigma_{1z}$. Thus, β corresponds to a rotation around the z-axis. Hence there is a $g \in T$ so that $\alpha_g(A) = \beta(A)$ for $A \in \mathscr{B}_1$. Recalling from Lemma 3.1 that the transposition elements $P_{rs} \in \mathfrak{A}^T$ we have for $A \in \mathscr{B}_1$,

$$\begin{aligned}\varphi(\pi_1(P_{1k}AP_{1k})) &= \varphi(\pi_1(P_{1k}))\,\varphi(\pi_1(A))\,\varphi(\pi_1(P_{1k})) = \\ &= \pi_2(P_{1k})\,\pi_2(\alpha_g(A))\,\pi_2(P_{1k}) = \\ &= \pi_2(P_{1k}\alpha_g(A)\,P_{1k}) = \pi_2(\alpha_g(P_{1k}AP_{1k})).\end{aligned}$$

Since $P_{1k}\mathscr{B}_1P_{1k} = \mathscr{B}_k$ we have $\varphi(\pi_1(A)) = \pi_2(\alpha_g(A))$ for all $A \in \mathscr{B}_k$ and all $k \in \mathrm{N}$ Since the $\mathscr{B}_k$ generate $\mathfrak{A}$ we have $\varphi(\pi_1(A)) = \pi_2(\alpha_g(A))$ for all $A \in \mathfrak{A}$. Since φ is σ-strongly bicontinuous it follows that π_1 and $\pi_2 \circ \alpha_g$ are quasi-equivalent. Hence, $\omega_1 \underset{\mathrm{q}}{\sim} \omega_2 \circ \alpha_g$. Done, Case 1 ($\Rightarrow$), part A.

Next we will show that if $\omega_1^T \underset{\mathrm{q}}{\sim} \omega_2^T$, then subcase B can not occur. We will assume $\omega_1^T \underset{\mathrm{q}}{\sim} \omega_2^T$, $\sum_{k=1}^{\infty} (a_{kx}^2 + a_{ky}^2) = \infty$ and $\sum_{k=1}^{\infty} (b_{kx}^2 + b_{ky}^2) < \infty$ and arrive at a contradiction. Let $c_k = (0, 0, b_{kz})$ for $k \in \mathrm{N}$ and $\omega_3 = \bigotimes_{k=1}^{\infty} \omega_{c_k}$. By Lemma 4.2 it follows that $\omega_2^T \underset{\mathrm{q}}{\sim} \omega_3^T$. Then $\omega_1^T \underset{\mathrm{q}}{\sim} \omega_3^T$. Let $(\pi_3^T, \mathscr{H}_3^T, f_3^T)$ be a cyclic *-representation of $\mathfrak{A}^T$ induced by ω_3^T. Since $\omega_1^T \underset{\mathrm{q}}{\sim} \omega_3^T$ there is a σ-strongly bicontinuous *-isomorphism θ of $\pi_1^T(\mathfrak{A}^T)''$ onto $\pi_3^T(\mathfrak{A}^T)''$ so that $\theta(\pi_1^T(A)) = \pi_3^T(A)$ for all $A \in \mathfrak{A}^T$.

We recall in the argument of Lemma 3.3 we showed that if $\sum_{k=1}^{\infty} (a_{kx}^2 + a_{ky}^2) = \infty$ then $\pi_1(\sigma_{1x}) \in \pi_1(\mathfrak{A}^T)''$ by constructing $U_n(t) \in \pi_1(\mathfrak{A}^T)$ so that $U_n(t) \to \exp(it\pi_1(\sigma_{1x}))$ strongly as $n \to \infty$. Carrying out the same construction in the representation π_1^T we obtain a sequence $U_n'(t) \in \pi_1^T(\mathfrak{A}^T)$ which converges strongly to a non-trivial one parameter group as $n \to \infty$. Since θ is σ-strongly bicontinuous $\theta(U_n'(t))$ must converge to a non trivial one parameter unitary group. But this is a contradiction because if one uses the construction in Lemma 3.3 and calculates for the representation π_3^T induced by ω_3^T one sees that $\theta(U_n'(t)) \to I$ as $n \to \infty$. Hence, if $\omega_1^T \underset{q}{\sim} \omega_2^T$ then subcase B can not occur. The same argument shows that if $\omega_1^T \underset{q}{\sim} \omega_2^T$ then subcase C can not occur. Done, Case 1 ($\Rightarrow$), parts B and C.

Now we consider subcase D. Assume $\omega_1^T \underset{q}{\sim} \omega_2^T$ and $\sum_{k=1}^{\infty} (a_{kx}^2 + a_{ky}^2) < \infty$ and $\sum_{k=1}^{\infty} (b_{kx}^2 + b_{ky}^2) < \infty$. Let $a_k' = (0, 0, a_{kz})$, $b_k' = (0, 0, b_{kz})$, $\omega_1' = \bigotimes_{k=1}^{\infty} \omega_{a_k'}$ and $\omega_2' = \bigotimes_{k=1}^{\infty} \omega_{b_k'}$. It follows from Lemma 4.2 that $\omega_1^T \underset{q}{\sim} \omega_1'^T$, $\omega_1 \underset{q}{\sim} \omega_1'$, $\omega_2^T \underset{q}{\sim} \omega_2'^T$ and $\omega_2 \underset{q}{\sim} \omega_2'$. Hence, $\omega_1'^T \underset{q}{\sim} \omega_2'^T$. Since ω_1' and ω_2' are α_g-invariant for $g \in T$ it follows from [1] or [13] that $\omega_1' \underset{q}{\sim} \omega_2'$. Since $\omega_1 \underset{q}{\sim} \omega_1'$ and $\omega_2 \underset{q}{\sim} \omega_2'$ we have $\omega_1 \underset{q}{\sim} \omega_2$. Done, Case 1 ($\Rightarrow$) part D.

Finally, we come to case 2 ($\Rightarrow$). Assume $\omega_1^T \underset{q}{\sim} \omega_2^T$ and $a_k = \pm(0, 0, 1)$ for all $k \in \mathbf{N}$. Then $\sum_{k=1}^{\infty} (a_{kx}^2 + a_{ky}^2) = 0 < \infty$. Then by the argument of Case 1 ($\Rightarrow$ subcases B and C, we have $\sum_{k=1}^{\infty} (b_{kx}^2 + b_{ky}^2) < \infty$. We have ω_1^T is of type I (in fact, ω_1^T is pure) so ω_2^T must also be type I. Hence, by Theorem 4.2 we must have $b_k = \pm (0, 0, 1)$ for all $k \in \mathbf{N}$. Again we have α_g-invariant states for $g \in T$ so it follows from [1] or [13] or [14] that $\sum_{k=1}^{\infty} (a_{kz} - b_{kz}) < \infty$ and $\sum_{k=1}^{\infty} a_{kz} - b_{kz} = 0$. Done.

Next we consider product states restricted to $\mathfrak{A}^G$ the rotationally invariant algebra.

LEMMA 4.6. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$ *and* $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. *Then* ω^G *is a factor state and* $\omega^G \underset{q}{\sim} \tau^G$ *where* $\tau = \bigotimes_{k=1}^{\infty} \tau_k$ *is the unique trace state of* $\mathfrak{A}$.

Proof. It follows from [11] that the extremal traces of $\mathfrak{A}^G$ are all of the form ρ^G with $\rho = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ with $b_k = b$ for all $k \in \mathbf{N}$. Hence, $\tau = \bigotimes_{k=1}^{\infty} \tau_k = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ with $b_k = 0$ for all $k \in \mathbf{N}$ is a factor state of $\mathfrak{A}^G$.

Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ is a product state of $\mathfrak{A}$ and $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. We will show ω^G is a factor state which is quasi-equivalent to τ^G. Let $\omega_n = \bigotimes_{k=1}^{\infty} \omega_{a_{nk}}$ where $a_{nk} = a_k$

for $k \leqslant n$ and $a_{nk} = 0$ for $k > n$. Let

$$\Omega_n = \prod_{k=1}^{n} (I + a_k \cdot \sigma_k).$$

Note that $\omega_n(A) = \tau(A\Omega_n)$ for all $A \in \mathfrak{A}$. Let φ be the conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}^G$ given by the group averaging map $\varphi(A) = \int_G \alpha_g(A)\, d\mu(g)$ where μ is Haar measure on G. Note that $\varphi(AB) = A\varphi(B)$ for $A \in \mathfrak{A}^G$ and $B \in \mathfrak{A}$. Since τ is α_g-invariant we have $\tau(A) = \tau(\varphi(A))$ for $A \in \mathfrak{A}$. Hence, for $A \in \mathfrak{A}^G$

$$\omega_n(A) = \tau(A\Omega_n) = \tau(\varphi(A\Omega_n)) = \tau(A\varphi(\Omega_n)).$$

Hence, $\omega_n^G(A) = \tau(\varphi(\Omega_n)^{1/2} A \varphi(\Omega_n)^{1/2})$ for all $A \in \mathfrak{A}^G$ so ω_n^G is a vector state of the representation induced by τ^G. Hence, $\tau^G \underset{q}{\sim} \omega_n^G$. We estimate the norm differences $\|\omega_n^G - \omega_m^G\|$ for $n < m$ from inequality (2.3) as follows :

$$\|\omega_n^G - \omega_m^G\| \leqslant \|\omega_n - \omega_m\| \leqslant 2\sqrt{1 - s_{nm}^2}$$

where

$$s_{nm} = \prod_{k=n+1}^{m} \Gamma(a_k, 0)$$

and

$$1 - \Gamma(a_k, 0) = 1 - \frac{1}{2}\sqrt{1 + |a_k|} - \frac{1}{2}\sqrt{1 - |a_k|}.$$

Since $\sqrt{1 + x} \geqslant 1 + \frac{1}{2}x - \frac{1}{2}x^2$ for $-1 \leqslant x \leqslant 1$ it follows that $1 - \Gamma(a_k, 0) \leqslant \leqslant \frac{1}{2}|a_k|^2$. Hence, we have

$$0 \leqslant 1 - s_{nm} \leqslant \frac{1}{2} \sum_{k=n+1}^{m} |a_k|^2.$$

Hence, $s_{nm} \to 1$ as $n, m \to \infty$ and, thus, $\|\omega_n^G - \omega_m^G\| \to 0$ as $n, m \to \infty$. Hence, the states ω_n^G converge in norm to $\omega^G = \left(\bigotimes_{k=1}^{\infty} \omega_{a_k} \right)^G \cdot$ Since ω^G is the norm limit of states ω_n^G which are vector states of the type II_1 factor representation induced by τ^G it follows that $\omega^G \underset{q}{\sim} \tau^G$ (in fact, ω^G is a vector state of the representation induced by τ^G). Done.

THEOREM 4.7. *Suppose* $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *is a product state of* $\mathfrak{A}$. *Then* ω^G *is a factor state if and only if*

$$\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2) = 0 \text{ or } \infty$$

for all unit vectors $s \in \mathbf{R}^3$. *Furthermore, if* ω^G *is a factor state its type is given as follows.*

CASE 1. $\sum_{k=1}^{\infty} (|a_k|^2 - (s \cdot a_k)^2) = \infty$ *for all unit vectors* $s \in \mathbf{R}^3$, *then,*

ω^G *is of type* I *if and only if* $\sum_{k=1}^{\infty} (1 - |a_k|) < \infty$;

ω^G *is not of type* II_1;

ω^G *is semi-finite if and only if* $\sum_{k=1}^{\infty} (1 - |a_k|) |a_k|^2 < \infty$.

CASE 2. $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ *and* $\sum_{k=1}^{\infty} (|a_k|^2 - (s \cdot a_k)^2) < \infty$ *for a unit vector* $s \in \mathbf{R}^3$, *then,*

ω^G *is of type* I *if and only if* $\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2) < \infty$ *and since* ω^G *is factorial* $a_k = \pm s$ *for all* $k \in \mathbf{N}$;

ω^G *is of type* II_1 *if and only if* $\sum_{k=1}^{\infty} (s \cdot a_k - t)^2 < \infty$ *for some* t *with* $-1 < t < 1$;

ω^G *is semi-finite if and only if* $\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2)(s \cdot a_k - t)^2 < \infty$ *for some* t *with* $-1 < t < 1$.

CASE 3. $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, *then,*

ω^G *is of type* II_1.

(Note that ω^G is of type III if it is not semi-finite.)

Proof. Suppose $\omega = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ and $(\pi, \mathscr{H}, f_0)$ and $(\pi^G, \mathscr{H}^G, f_0^G)$ are cyclic *-representations of $\mathfrak{A}$ and $\mathfrak{A}^G$ induced by ω and ω^G, respectively.

First let us assume we have the situation of Case 1 so that $\sum_{k=1}^{\infty} (|a_k|^2 - (s \cdot a_k)^2) = \infty$ for all unit vector $s \in \mathbf{R}^3$. Since $1 \geqslant |a_k|^2$ we have $\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2) = \infty$. From Corollary 3.7 if follows that $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$. Hence, f_0 is cyclic in $\mathscr{H}$ for $\pi(\mathfrak{A}^G)$ and, hence, the restriction of π to $\mathfrak{A}^G$ is unitarily equivalent to π^G (by the uniqueness of the Gelfand-Segal construction). Hence, $\pi^G(\mathfrak{A}^G)''$ is *-isomorphic to $\pi(\mathfrak{A}^G)'' = \pi(\mathfrak{A})''$. Hence, ω^G is a factor state and its type is the same as ω. Done, Case 1.

Next we assume we are in the situation of Case 2, so $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ and $\sum_{k=1}^{\infty} (|a_k|^2 - (s\cdot a_k)^2) < \infty$ for a unit vector $s \in \mathbf{R}^3$. Since $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ it follows from Theorem 3.6 that $\pi(\mathfrak{A}^G)'' = \pi(\alpha_g(\mathfrak{A}^T))''$ where $g \in G$ is such that $R_g(0, 0, 1) = s$. Let π_1 be the restriction of π to $\alpha_g(\mathfrak{A}^T)$ and let π_1 act on $\mathscr{H}_1$ which is the closed span of $\{\pi(\alpha_g(\mathfrak{A}^T))''f_0\} = \{\pi(\mathfrak{A}^G)''f_0\}$. (Note $f_1 = f_0 \in \mathscr{H}_1$ is a cyclic vector for π_1.) Note f_1 is cyclic in $\mathscr{H}_1$ for $\pi_1(\mathfrak{A}^G)$ so the restriction of π_1 to $\mathfrak{A}^G$ is unitarily equivalent to π^G. Hence, $\pi^G(\mathfrak{A}^G)''$ is *-isomorphic to $\pi_1(\mathfrak{A}^G)''$. Since $\mathscr{H}_1$ is an invariant subspace of $\mathscr{H}$ for $\pi(\alpha_g(\mathfrak{A}^T))$ and $\pi(\alpha_g(\mathfrak{A}^T))'' = \pi(\mathfrak{A}^G)''$ we have $\pi_1(\mathfrak{A}^G)'' = \pi_1(\alpha_g(\mathfrak{A}^T))''$. Hence, we have $\pi^G(\mathfrak{A}^G)''$ is *-isomorphic to $\pi_1(\alpha_g(\mathfrak{A}^T))''$. Now $\pi_1(\alpha_g(\mathfrak{A}^T))''$ is a factor if and only if the restriction of ω to $\alpha_g(\mathfrak{A}^T)$ is a factor state. And this restriction is a factor state if and only if $\omega \circ \alpha_g^{-1}$ is a factor state when restricted to $\mathfrak{A}^T$. Applying Theorem 4.1 we find $(\omega \circ \alpha_g^{-1})^T$ is a factor state if and only if $\sum_{k=1}^{\infty} (1 - (s\cdot a_k)^2) = 0$ or ∞. (Note the condition $\sum_{k=1}^{\infty} (|a_k|^2 - (s\cdot a_k)^2) < \infty$ implies that $\sum_{k=1}^{\infty} (1 - (s'\cdot a_k)^2) = \infty$ for all unit vectors $s' \neq \pm s$.) Hence, ω^G is a factor state if and only if $\sum_{k=1}^{\infty} (1 - (s\cdot a_k)^2) = 0$ or ∞ for all unit vectors $s \in \mathbf{R}^3$. To determine the type of ω^G we note that the type of ω^G in this case is the same as the type of the restriction of ω to $\alpha_g(\mathfrak{A}^T)$ which is the same as the type of $(\omega \circ \alpha_g^{-1})^T$. The type of $(\omega \circ \alpha_g^{-1})^T$ can be determined from Case 2 of Theorem 4.3 by replacing a_{kz} by $s\cdot a_k$. Done, Case 2.

Finally, we assume we are in the situation of Case 3, so $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. Then $\sum_{k=1}^{\infty} (1 - (s\cdot a_k)^2) = \infty$ for all unit vectors $s \in \mathbf{R}^3$ and from Lemma 4.6 we have $\omega^G \underset{q}{\sim} \tau^G$. Hence ω^G is a factor state of type II_1. Done.

THEOREM 4.8. *Suppose* $\omega_1 = \bigotimes_{k=1}^{\infty} \omega_{a_k}$ *and* $\omega_2 = \bigotimes_{k=1}^{\infty} \omega_{b_k}$ *are product states of* $\mathfrak{A}$ *so that* ω_1^G *and* ω_2^G *are factor states. Then the quasi-equivalence of* ω_1^G *and* ω_2^G *may be determined as follows.*

CASE 1. $\sum_{k=1}^{\infty} (1 - (s\cdot a_k)^2) = \infty$ *for all unit vectors* $s \in \mathbf{R}^3$, *then,*

$\omega_1^G \underset{q}{\sim} \omega_2^G$ *if and only if there is a* $g \in G$ *so that* $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$.

CASE 2. $\sum_{k=1}^{\infty} (1 - (s\cdot a_k)^2) = 0$ *for a unit vector* $s \in \mathbf{R}^3$, *then,*

$\omega_1^G \underset{q}{\sim} \omega_2^G$ *if and only if there is a* $g \in G$ *so that* $R_g b_k = \pm s$ *for all* $k \in \mathbf{N}$ *and* $\sum_{k=1}^{\infty} |a_k - R_g b_k| < \infty$ *and* $\sum_{k=1}^{\infty} (a_k - R_g b_k) = 0$.

Proof. Assume ω_1 and ω_2 satisfy the hypothesis of the theorem. We begin with Case 1 ($\Leftarrow$), so we assume there is a $g \in G$ with $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$ and $\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2) = \infty$ for all unit vectors $s \in \mathbf{R}^3$. Let $c_k = R_g b_k$ for all $k \in \mathbf{N}$ and $\omega_3 = \omega_2 \circ \alpha_g = \bigotimes_{k=1}^{\infty} \omega_{c_k}$. Since $\omega_1 \underset{q}{\sim} \omega_3$ we have from the discussion at the end of Section 2 that $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, c_k)) < \infty$. Hence $\Gamma(a_k, c_k) = 0$ for only a finite set Q of $k \in \mathbf{N}$. Let $a'_k = a_k$ for $k \notin Q$ and $a'_k = 0$ for $k \in Q$. Let $\omega'_1 = \bigotimes_{k=1}^{\infty} \omega_{a'_k}$. Clearly, $\sum_{k=1}^{\infty} (1 - (s \cdot a'_k)^2) = \infty$ for all unit vectors $s \in \mathbf{R}^3$. Hence, $\omega_1'^G$ is a factor state by Theorem 4.7. Since $\Gamma(a_k, a'_k) > 0$ and $\Gamma(a'_k, c_k) > 0$ for all $k \in \mathbf{N}$ and $\sum_{k=1}^{\infty} (1 - \Gamma(a_k, a'_k)) < \infty$ and $\sum_{k=1}^{\infty} (1 - \Gamma(a'_k, c_k)) < \infty$ we have from inequality 2.3 and routine estimates of infinite products that

$$\|\omega_1^G - \omega_1'^G\| \leqslant \|\omega_1 - \omega'_1\| < 2$$

and

$$\|\omega_1'^G - \omega_3^G\| \leqslant \|\omega'_1 - \omega_3\| < 2.$$

Hence, $\omega_1^G \underset{q}{\sim} \omega_1'^G$ and $\omega_1'^G \underset{q}{\sim} \omega_3^G$. Hence, $\omega_1^G \underset{q}{\sim} \omega_3^G = \omega_2^G$. Done, Case 1 ($\Leftarrow$).

Next we consider Case 2 ($\Leftarrow$). To this end we assume $\sum_{k=1}^{\infty} (1 - (s \cdot a_k)^2) = 0$ for a unit vector $s \in \mathbf{R}^3$ and there is a $g \in G$ so that $R_g b_k = \pm s$ for all $k \in \mathbf{N}$ and $\sum_{k=1}^{\infty} |a_k - R_g b_k| < \infty$ and $\sum_{k=1}^{\infty} (a_k - R_g b_k) = 0$. Let $c_k = R_g b_k$ for $k \in \mathbf{N}$ and $\omega_3 = \bigotimes_{k=1}^{\infty} \omega_{c_k}$. Then $c_k = b_k$ except for a finite set Q where $c_k = -a_k$. Since $\sum_{k=1}^{\infty} (a_k - c_k) = 0$, Q can be divided into two equal sets Q_1 and Q_2 so that for $k \in Q_1$, $a_k = s = -c_k$ and for $k \in Q_2$, $a_k = -s = -c_k$. As we saw in the discussion in Lemma 3.1, $\mathfrak{A}^G$ contains the transposition elements P_{rs} and, therefore, the finite permutation elements. Since Q_1 and Q_2 are disjoint finite sets with the same number of elements there exists a finite permutation which exchanges the sets Q_1 and Q_2. Let U be a unitary element of $\mathfrak{A}^G$ corresponding to such a permutation, e.g. $U = P_{i_1 j_1} P_{i_2 j_2} \ldots P_{i_r j_r}$ where $Q_1 = (i_1, \ldots, i_r)$ and $Q_2 = (j_1, \ldots \ldots, j_r)$. One sees that for such a unitary $\omega_1(UAU^{-1}) = \omega_3(A)$ for all $A \in \mathfrak{A}^G$. Hence ω_1^G and ω_3^G are unitarily equivalent. Hence, $\omega_1^G \underset{q}{\sim} \omega_3^G = \omega_2^G$. Done, Case 2 ($\Leftarrow$).

Now we consider the implication ($\Rightarrow$). We assume $\omega_1^G \underset{q}{\sim} \omega_2^G$. Let $(\pi_i, \mathscr{H}_i, f_i)$ be the cyclic *-representations of $\mathfrak{A}$ induced by ω_i, for $i = 1, 2$. Let $\mathscr{H}_i^G$ be the norm closure of $\{\pi_i(\mathfrak{A}^G) f_i\}$, let P_i be the orthogonal projection of $\mathscr{H}_i$ onto $\mathscr{H}_i^G$ and let

$(\pi_i^G, \mathscr{H}_i^G, f_i^G)$ be the cyclic representation of $\mathfrak{A}^G$ obtained by restricting π_i to $\mathfrak{A}^G$ and $\mathscr{H}_i^G$. Note π_i^G has a cyclic vector $f_i^G = f_i$ and is characterized by the state ω_i^G. Since $\omega_1^G \underset{q}{\sim} \omega_2^G$ there is a σ-strongly bicontinuous *-isomorphism φ of $\pi_1^G(\mathfrak{A}^G)''$ onto $\pi_2^G(\mathfrak{A}^G)''$ so that $\varphi(\pi_1^G(A)) = \pi_2^G(A)$ for all $A \in \mathfrak{A}^G$.

Now let us assume $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. Then by Lemma 3.5 there is a sequence $A_n \in \mathfrak{A}^G$ of the form

$$A_n = Z_n^{-1} \sum_{i=2}^{n} |a_{k(i)}| \sigma_1 \cdot \sigma_{k(i)}$$

with

$$Z_n = \sum_{i=2}^{n} |a_{k(i)}|^2 \to \infty \quad \text{as } n \to \infty,$$

so that $U_n(t) = \pi_1(\exp(itA_n))$ converges strongly to $V(t) = \pi_1(\exp(its_1 \cdot \sigma_1))$ where $s_1 \in \mathbf{R}^3$ and $|a_{k(i)}|^{-1} a_{k(i)} \to s_1$ as $i \to \infty$. Since $\mathscr{H}_1^G \subset \mathscr{H}_1$ is an invariant subspace for $\pi_1(\mathfrak{A}^G)$ we have

$$U_n^G(t) = \pi_1^G(\exp(itA_n)) = U_n(t)P_1 \to V^G(t) = V(t)P_1$$

strongly as $n \to \infty$.

Since $V(\pi) = -I$ we have $U_n^G(\pi) \to -I$ strongly as $n \to \infty$. Since φ is σ-strongly bicontinuous and the $U_n^G(\pi)$ are unitary we have

$$\varphi(U_n^G(\pi)) = \varphi(\pi_1^G(\exp(i\pi A_n))) = \pi_2^G(\exp(i\pi A_n)) \to -I$$

strongly as $n \to \infty$. Now we have

$$\operatorname{Re}(f_2^G, (I - \pi_2^G(\exp(i\pi A_n)))f_2^G) = \operatorname{Re}\omega_2(I - \exp(i\pi A_n)) \leqslant \frac{1}{2}\pi^2\omega_2(A_n^2) =$$

$$= \frac{1}{2}\pi^2\left(|c_n|^2 + Z_n^{-2} \sum_{i=2}^{n} |a_{k(i)}|^2(3 - 2b_1 \cdot b_{k(i)} - |b_{k(i)}|^2)\right) \leqslant$$

$$\leqslant \frac{1}{2}\pi^2(|c_n|^2 + 3/Z_n)$$

where

$$c_n = Z_n^{-1} \sum_{i=2}^{n} |a_{k(i)}|\, b_{k(i)}.$$

Since the above expression approaches 2 as $n \to \infty$ and $Z_n \to \infty$, we have for each $\varepsilon > 0$

$$\frac{1}{2}\pi^2|c_n|^2 \geqslant 2 - \varepsilon$$

for n sufficiently large. Hence, $|c_n| \geqslant \pi^{-1/2}$ for n sufficiently large. On the other hand we have from the Schwarz inequality

$$|c_n| \leqslant Z_n^{-1} \sum_{i=2}^{n} |a_{k(i)}|\, |b_{k(i)}| \leqslant Z_n^{-1} Z_n^{1/2} Y_n^{1/2} = (Y_n/Z_n)^{1/2}$$

where

$$Y_n = \sum_{i=2}^{n} |b_{k(i)}|^2.$$

Since $|c_n|^2 \geqslant \pi^{-1}$ for n sufficiently large we have $Y_n \geqslant \pi^{-1} Z_n$ for n sufficiently large. Hence, $Y_n \to \infty$ as $n \to \infty$.

We have shown that $\omega_1^G \underset{q}{\sim} \omega_2^G$ and $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ imply that $\sum_{k=1}^{\infty} |b_k|^2 = \infty$.

Now consider the sequence of vector pairs $\{(|a_{k(i)}|^2 + |b_{k(i)}|^2)^{-1/2} b_{k(i)}, |b_{k(i)}|^{-1} b_{k(i)}\}$ where we set the second vector equal to zero if $b_{k(i)} = 0$. These vector pairs lie in a compact set, so by the argument of the first part of Lemma 3.5 there is a subsequence $i \to p(i)$ of the original subsequence $i \to k(i)$ so that

$$|b_{p(i)}|^{-1} b_{p(i)} \to s_2 \quad \text{and} \quad (|a_{p(i)}|^2 + |b_{p(i)}|^2)^{-1/2} |b_{p(i)}| \to \lambda$$

as $i \to \infty$ and $\sum_i |b_{p(i)}|^2 = \infty$. The argument which showed $Y_n \geqslant Z_n/\pi$ for n sufficiently large can be applied to this new subsequence $i \to p(i)$ to show that λ can not be zero or one (in fact, we will see that $\lambda = 2^{-1/2}$). In summary we have a subsequence $i \to p(i)$ so that

$$|a_{p(i)}|^{-1} a_{p(i)} \to s_1, \quad |b_{p(i)}|^{-1} b_{p(i)} \to s_2$$

and

$$|a_{p(i)}|/|b_{p(i)}| \to r = \lambda(1 - \lambda^2)^{-1/2}$$

as $i \to \infty$. Now we redefine A_n as

$$A_n = Z_n^{-1} \sum_{i=2}^{n} |a_{p(i)}| \sigma_1 \cdot \sigma_{p(i)}$$

with

$$Z_n = \sum_{i=2}^{n} |a_{p(i)}|^2.$$

Repeating the calculations of Lemma 3.5 we have

$$\pi_1(\exp(\mathrm{i}tA_n)) \to \pi_1(\exp(\mathrm{i}ts_1 \cdot \sigma_1))$$

$$\pi_2(\exp(\mathrm{i}tA_n)) \to \pi_2(\exp(\mathrm{i}ts_2 \cdot \sigma_1))$$

strongly as $n \to \infty$. Since $\mathscr{H}_i^G \subset \mathscr{H}_i$ and the $\mathscr{H}_i^G$ are invariant under the action of $\pi_i(\mathfrak{A}^G)$ we have strong convergence on $\mathscr{H}_i^G$, for $i = 1, 2$. Since φ is σ-strongly bicontinuous and $\varphi(\pi_1^G(A)) = \pi_2^G(A)$ for all $A \in \mathfrak{A}^G$ we have

$$\varphi(\pi_1(\exp(its_1 \cdot \sigma_1))P_1) = \pi_2(\exp(itrs_2 \cdot \sigma_1))P_2.$$

And since φ is an isomorphism we have $r = 1$ and $\varphi(\pi_1(s_1 \cdot \sigma_1)P_1) = \pi_2(s_2 \cdot \sigma_1)P_2$.

Let $g_i \in G$ be chosen so that $R_{g_i}(0, 0, 1) = s_i$ for $i = 1, 2$. We claim

$$\varphi(\pi_1(\alpha_{g_1}(A))P_1) = \pi_2(\alpha_{g_2}(A))P_2 \quad \text{for all } A \in \mathfrak{A}^T.$$

Since $\mathfrak{A}^G$ is α_g-invariant this relation holds for all $A \in \mathfrak{A}^G$. And since

$$\varphi(\pi_1(\alpha_{g_1}(\sigma_{1z}))P_1) = \varphi(\pi_1(s_1 \cdot \sigma_1)P_1) = \pi_2(s_2 \cdot \sigma_1)P_2 = \pi_2(\alpha_{g_2}(\sigma_{1z}))P_2$$

this relation holds for σ_{1z}. Since from Lemma 3.1 we have that σ_{1z} and $\mathfrak{A}^G$ generate $\mathfrak{A}^T$ and since φ is a *-isomorphism it follows that

$$\varphi(\pi_1(\alpha_{g_1}(A))P_1) = \pi_2(\alpha_{g_2}(A))P_2 \quad \text{for all } A \in \mathfrak{A}^T.$$

Now $A \to \pi_i(\alpha_{g_i}(A))P_i$ are cyclic representations of $\mathfrak{A}^T$ induced by $(\omega_i \circ \alpha_{g_i})^T$ for $i = 1,2$. Since φ is σ-strongly bicontinuous it follows that these two representations are quasi-equivalent. Hence, $(\omega_1 \circ \alpha_{g_1})^T \underset{q}{\sim} (\omega_2 \circ \alpha_{g_2})^T$. Applying Theorem 4.5 to the product states $\omega_1 \circ \alpha_{g_1} = \bigotimes_{k=1}^{\infty} \omega_{a_k'}$ $(a_k' = R_{g_1}^{-1} a_k)$ and $\omega_2 \circ \alpha_{g_2} = \bigotimes_{k=1}^{\infty} \omega_{b_k'}$ $(b_k' = R_{g_2}^{-1} b_k)$ the conclusion of the theorem follows.

We have proven the implication $(\Rightarrow)$ assuming $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. Now we assume $\sum_{k=1}^{\infty} |a_k|^2 < \infty$. We showed earlier that $\omega_1^G \underset{q}{\sim} \omega_2^G$ and $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ imply $\sum_{k=1}^{\infty} |b_k|^2 = \infty$. The same argument with the states ω_1 and ω_2 interchanged shows that $\omega_1^G \underset{q}{\sim} \omega_2^G$ and $\sum_{k=1}^{\infty} |b_k|^2 = \infty$ implies $\sum_{k=1}^{\infty} |a_k|^2 = \infty$. Hence, if $\sum_{k=1}^{\infty} |a_k|^2 < \infty$ (and by assumption $\omega_1^G \underset{q}{\sim} \omega_2^G$) we have $\sum_{k=1}^{\infty} |b_k|^2 < \infty$. Then by Lemma 4.6 we have $\omega_1^G \underset{q}{\sim} \tau^G$ and $\omega_2^G \underset{q}{\sim} \tau^G$. Hence, $\omega_1^G \underset{q}{\sim} \omega_2^G$. Done.

We now prove some results concerning general factor states of $\mathfrak{A}$.

THEOREM 4.9. *Suppose ω_1 and ω_2 are factor states of $\mathfrak{A}$ so that ω_1^T and ω_2^T are factor states of $\mathfrak{A}^T$. Suppose $\omega_1(\sigma_{kx})^2 + \omega_1(\sigma_{ky})^2$ does not tend to zero as $k \to \infty$. Then $\omega_1^T \underset{q}{\sim} \omega_2^T$ if and only if there is a $g \in T$ so that $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$.*

Proof. Suppose ω_1 and ω_2 satisfy the hypothesis of the theorem. Suppose there is a $g \in T$ so that $\omega_1 \underset{q}{\sim} \omega_2' = \omega_2 \circ \alpha_g$. Let $\omega_0 = (1/2)(\omega_1 + \omega_2')$. We have that

ω_0 is a factor state and $\|\omega_1 - \omega_0\| = \|\omega_2' - \omega_0\| = \frac{1}{2}\|\omega_1 - \omega_2'\| \leqslant 1$. Since $\omega_1 \underset{q}{\sim} \omega_2'$ it follows (see [9], Theorem 2.7) that ω_1 and ω_2' are asymptotically equal so $\omega_1(\sigma_k) - \omega_2'(\sigma_k) \to 0$ as $k \to \infty$. Hence, $\omega_0(\sigma_k) \to \omega_1(\sigma_k)$ and $\omega_0(\sigma_{kx})^2 + \omega_0(\sigma_{ky})^2$ does not tend to zero as $k \to \infty$. Hence, it follows from Theorem 3.8 that ω_0^T is a factor state. Since $\|\omega_1^T - \omega_0^T\| \leqslant \|\omega_1 - \omega_0\| \leqslant 1 < 2$ and $\|\omega_2'^T - \omega_0^T\| \leqslant \|\omega_2' - \omega_0\| \leqslant 1 < 2$ we have $\omega_1^T \underset{q}{\sim} \omega_0^T$ and $\omega_2'^T \underset{q}{\sim} \omega_0^T$. Since $\omega_2'^T = \omega_2^T$ we have $\omega_1^T \underset{q}{\sim} \omega_2^T$.

Now we assume $\omega_1^T \underset{q}{\sim} \omega_2^T$. Let $(\pi_i^T, \mathscr{H}_i^T, f_i^T)$ be cyclic representations of $\mathfrak{A}^T$ induced by ω_i^T, for $i = 1, 2$. Since $\omega_1^T \underset{q}{\sim} \omega_2^T$ there is a σ-weakly bicontinuous isomorphism φ of $\pi_1^T(\mathfrak{A}^T)''$ onto $\pi_2^T(\mathfrak{A}^T)''$ so that $\varphi(\pi_1^T(A)) = \pi_2^T(A)$ for $A \in \mathfrak{A}^T$. Recalling the argument of Theorem 3.8 there are elements $A_n \in \mathfrak{A}^T$ so that $\pi_1^T(A_n)$ converge weakly to a non-zero limit (corresponding to the appropriate restriction of $\pi_1(e_{12}^{(1)})$). Then $\varphi(\pi_1^T(A_n)) = \pi_2^T(A_n)$ must weakly converge to a non-zero limit as $k \to \infty$. But if $\omega_2(\sigma_{kx})^2 + \omega_2(\sigma_{ky})^2 \to 0$ as $k \to \infty$ one sees from the calculations of Theorem 3.8 that $\pi_2^T(A_n) \to 0$ weakly as $n \to \infty$. Hence, $\omega_2(\sigma_{kx})^2 + \omega_2(\sigma_{ky})^2$ does not tend to zero as $k \to \infty$. Hence, by Theorem 3.8 we have $\pi_i(\mathfrak{A}^T)'' = \pi_i(\mathfrak{A})''$ where π_i is as cyclic representation of $\mathfrak{A}$ induced by ω_i, for $i = 1, 2$. Now repeating the arguments of Theorem 4.5 Case 1 ($\Rightarrow$) part A we find there is a $g \in T$ so that $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$. Done.

THEOREM 4.10. *Suppose ω_1 and ω_2 are factor states of $\mathfrak{A}$ so that ω_1^G and ω_2^G are factor states of $\mathfrak{A}^G$. Suppose the sequence of vectors $\omega_1(\sigma_k)$ has at least two linearly independent accumulation points. Then $\omega_1^G \underset{q}{\sim} \omega_2^G$ if and only if there is a $g \in G$ so that $\omega_1 \underset{q}{\sim} \omega_2 \circ \alpha_g$.*

Proof. The proof of the theorem is the same as the proof of the previous theorem except here one makes use of Theorem 3.9 instead of Theorem 3.8.

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REFERENCES

1. BAKER, B. M., Free states of the gauge invariant canonical anticommutation relations, *Trans. Amer. Math. Soc.*, 237(1978), 35–61.
2. BAKER, B. M., Free states of the gauge invariant canonical anticommutation relations. II, *Trans. Amer. Math. Soc.*, 254(1979), 133–155.
3. BAKER, B. M., A central decomposition, *J. Functional Analysis*, 35(1980), 1–25.

4. BAKER, B. M.; POWERS, R. T., Product states and C^*-dynamical systems of the product type, *J. Functional Analysis*, **50** (1983), 229 – 266.
5. BRATTELI, O., Inductive limits of finite dimensional C^*-algebras, *Trans. Amer. Math. Soc.*, **171**(1972), 195–234.
6. DIXMIER, J., *Les C*-algèbres et leur représentations*, Cahiers Scientifiques, Fascicule XXIX, Gauthier-Villars, Paris, 1969.
7. GLIMM, J. G., On a certain class of operator algebras, *Trans. Amer. Math. Soc.*, **95**(1960), 318–340.
8. MOORE, C. C., Invariant measures on product spaces, *Proc. of the Fifth Berkeley Symposium on Math. Stat. and Probab.* Vol. II, part II, 447–459, 1967.
9. POWERS, R. T., Representations of uniformly hyperfinite algebras and their associated von Neumann rings, *Ann. of Math.*, **86**(1967), 138–171.
10. POWERS, R. T.; STØRMER, E., Free states of the canonical anticommutation relations, *Comm. Math. Phys.*, **16**(1970), 1–33.
11. PRICE, G., Extremal traces on some group-invariant C^*-algebras, *J. Functional Analysis*, to appear.
12. PRICE, G., Pure states on some group-invariant C^*-algebras, preprint.
13. STRĂTILĂ, Ş.; VOICULESCU, D., On a class of KMS states for the unitary group $U(\infty)$, *Math. Ann.*, **235**(1978), 87–110.
14. STAMATOPOULOS, G., *Thesis*, University of Pennsylvania, 1974.

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