J. OPERATOR THEORY
11(1984), 3–40

ON THE EXISTENCE OF HYPERINVARIANT SUBSPACES

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0. INTRODUCTION

Throughout this paper, E will denote an infinite dimensional complex Banach space and $\mathscr{L}(E)$ the algebra of all bounded linear operators on E. For an operator A in $\mathscr{L}(E)$ we shall denote by A^* its adjoint acting on the dual space E^*, and by $(A)'$ its commutant, that is, the set of all operators in $\mathscr{L}(E)$ which commute with A.

We recall that a (closed) subspace $M \subset E$ is called invariant for an operator A in $\mathscr{L}(E)$ if $Ax \in M$ for every $x \in M$. The subspace M is called hyperinvariant for A, if it is invariant for every operator in $(A)'$. We say that M is not trivial, if $M \neq \{0\}$ and $M \neq E$.

In the sequel, we shall denote by $\mathbf{N}$ the set of all positive integers, by $\mathbf{Z}$ the set of all integers, by $\mathbf{C}$ the set of complex numbers, and by $\mathbf{T}$ the unit circle $\{z \in \mathbf{C} : |z| = 1\}$.

The main result of this paper is a Theorem on the existence of nontrivial hyperinvariant subspaces for certain operators (Theorem 1.1), which extends simultaneously the results of Wermer [23], Sz.-Nagy and Foiaş [22, p. 74], Gellar and Herrero [13], and a recent result of Beauzamy [2].

In general terms, our main result asserts that if A is an operator in $\mathscr{L}(E)$, and there exist sequences $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$ with $x_0 \neq 0$ and $y_0 \neq 0$ such that $\forall n \in \mathbf{Z}$

$$(0.1)\qquad Ax_n = x_{n+1} \quad \text{and} \quad A^*y_n = y_{n+1}$$

then under some additional conditions, either A is a multiple of the identity operator or A has a non trivial hyperinvariant subspace.

An example of such additional conditions (which is a particular case of Theorem 1.1 (a)) is, that for some integer $k \geqslant 0$

$$\|x_n\| + \|y_n\| = \mathrm{O}(|n|^k), \quad n \to \pm\infty.$$

This condition clearly holds, if A is invertible, and there exist non zero vectors $x_0 \in E$ and $y_0 \in E^*$ such that

$$\|A^n x_0\| + \|A^{*n} y_0\| = \mathrm{O}(|n|^k), \quad n \to \pm\infty.$$

In addition of providing a common principle to the results of [23], [22, p. 74] and [13], our hypotheses are considerably weaker than theirs, and hold in some cases in which neither of these results is applicable. One such example (see Section 6) is the class of Bishop operators considered by A. M. Davie in [7].

The contents of this paper are as follows:

In Section 1 we state our main results and some of their consequences.

In Section 2 we assemble some preliminary results from harmonic analysis and the theory of analytic vector functions, which are needed in the proof of Theorem 1.1.

In Section 3 we present the proofs of our main results stated in Section 1. After proving Theorem 1.1 we deduce from it, by using a theorem of Helson [14, Theorem 3], the result of Wermer [23]. Then we prove a general Banach space Lemma which enables us to deduce from Theorem 1.1 the extension of the result of Sz.-Nagy and Foiaş [22, p. 74] which is given in [6, p. 134], and also the following result (which is a particular case of Theorem 1.5):

If E is a Hilbert space and A is an operator in $\mathscr{L}(E)$ such that for some vectors x and y in E,

$$\limsup_{n\to\infty}\|A^n x\| > 0, \quad \limsup_{n\to\infty}\|A^{*n} y\| > 0,$$

$$\sup_{m,n\in\mathbf{N}}\|A^{*m}A^n x\| < \infty \quad \text{and} \quad \sup_{m,n\in\mathbf{N}}\|A^m A^{*n} y\| < \infty,$$

then either A is a multiple of the identity operator or A has a non trivial hyperinvariant subspace.

This result clearly extends the result of [22, p. 74] and does not impose any conditions on the norms of the operators A^n, $n \in \mathbf{N}$.

We conclude Section 3 by proving Theorem 3.6 which extends a result of Beauzamy [2].

In Section 4 we apply the methods of Section 3 to prove some additional results. One such result (which is a particular case of Theorem 4.1) is the following:

If A is a contraction in $\mathscr{L}(E)$ such that the intersection of its spectrum with the unit circle $\mathbf{T}$ is countable, and the sequence $(A^n)_{n\in\mathbf{N}}$ does not converge strongly to the zero operator, then A^* has an eigenvalue. Consequently, if A is not a multiple of the identity operator it posseses a non trivial hyperinvariant subspace.

In Section 4 we also extend the results of [1, Theorem 1, and Proposition 6].

In Section 5 we introduce the class of generalized bilateral weighted shifts and extend the results of Gellar and Herrero [13] concerning the existence of non trivial hyperinvariant subspaces for bilateral weighted shifts.

In Section 6 we give some examples, and apply the results of Section 5 to certain operators on homogeneous Banach spaces on $\mathbf{T}$. We also use these operators to disprove a conjecture of Gellar [12, p. 543]. We conclude with some comments and problems concerning the existence of invariant subspaces in a certain class of operators which contains the Bishop operators.

In considering condition (0.1) we were inspired by the recent paper of Beauzamy [2], although his conditions and methods are different from ours.

We wish to express our thanks to Professor Bernard Beauzamy for providing us with preprints of his papers [2] and [3]. We also thank Professor Domingo Herrero for several comments concerning Section 6.

1. STATEMENT OF MAIN RESULTS

Before stating our main results it will be convenient to introduce the following:

DEFINITION. A sequence of real numbers $(\rho_n)_{n\in\mathbf{Z}}$ such that $\rho_0 = 1$ and $\rho_n \geqslant 1$, $\forall n \in \mathbf{Z}$, will be called a *Beurling sequence* if the following conditions hold:

$$\rho_{m+n} \leqslant \rho_m \rho_n, \quad \forall m, n \in \mathbf{Z}, \tag{1.1}$$

$$\sum_{n\in\mathbf{Z}} \frac{\log \rho_n}{1 + n^2} < \infty. \tag{1.2}$$

Similarly we define *one sided Beurling sequences* $(\rho_n)_{n=0}^{\infty}$ by replacing in the above definition, $\mathbf{Z}$ by $\mathbf{N}$.

We shall also adopt the following convention: We shall say that the sequence of real numbers $(a_n)_{n\in\mathbf{Z}}$ is *dominated* by the sequence of real numbers $(b_n)_{n\in\mathbf{Z}}$ if there exists a constant $c > 0$ such that

$$a_n \leqslant c \cdot b_n, \quad \forall n \in \mathbf{Z}.$$

An analogous convention will be used for one sided sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$.

THEOREM 1.1. *Let A be an operator in $\mathscr{L}(E)$ and assume that there exist sequences $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$, with $x_0 \neq 0$ and $y_0 \neq 0$, such that* (0.1) *holds* $\forall n \in \mathbf{Z}$.

Then each of the following conditions implies that either A is a multiple of the identity operator or A has a non trivial hyperinvariant subspace:

(a) *The sequence $(\|y_n\|)_{n\in\mathbf{Z}}$ is dominated by a Beurling sequence and*

$$\|x_n\| = \mathrm{O}(|n|^k), \quad n \to \pm\infty \tag{1.3}$$

for some integer $k \geqslant 0$.

(b) *The sequence $(\|x_n\|)_{n\in\mathbf{Z}}$ is dominated by a Beurling sequence and*

$$\|y_n\| = \mathrm{O}(|n|^k), \quad n \to \pm\infty \tag{1.4}$$

for some integer $k \geqslant 0$.

(c) *The sequences* $(\|x_n\|)_{n\in\mathbf{Z}}$ *and* $(\|y_n\|)_{n\in\mathbf{Z}}$ *are dominated by Beurling sequences and the union of the singularity sets of the two analytic vector valued functions* G_1 *and* G_2 *defined on* $\mathbf{C}\setminus\mathbf{T}$ *by:*

$$G_1(z) = \begin{cases} \sum_{n=1}^{\infty} x_{-n}z^{n-1}, & |z| < 1 \\ -\sum_{n=-\infty}^{0} x_{-n}z^{n-1}, & |z| > 1 \end{cases} \tag{1.5}$$

and

$$G_2(z) = \begin{cases} \sum_{n=1}^{\infty} y_{-n}z^{n-1}, & |z| < 1 \\ -\sum_{n=-\infty}^{0} y_{-n}z^{n-1}, & |z| > 1 \end{cases} \tag{1.6}$$

contains more than one point.

(d) x_0 *is not contained in the closed span in* E *of the set* $\{x_n : n \in \mathbf{Z},\ n\neq 0\}$, y_0 *is not contained in the closed span in* E^* *of the set* $\{y_n : n \in \mathbf{Z},\ n\neq 0\}$, *and*

$$\sum_{n\in\mathbf{Z}} \frac{1}{1+n^2} (\log^+\|x_n\| + \log^+\|y_n\|) < \infty \tag{1.7a}$$

and for some constant $b > 0$

$$\|x_n\| \leqslant b\|x_{n+1}\| \quad \text{and} \quad \|y_n\| \leqslant b\|y_{n+1}\|, \quad \forall n \in \mathbf{Z}. \tag{1.7b}$$

(e) *For some integer* j

$$\inf_{n\in\mathbf{Z}} \|x_{n+j}\|\,\|y_{-n}\| = 0. \tag{1.8}$$

Condition (c) calls for some explanations. As we shall see in Section 2, the assumption that $(\|x_n\|)_{n\in\mathbf{Z}}$ and $(\|y_n\|)_{n\in\mathbf{Z}}$ are dominated by Beurling sequences implies that the power series defining G_1 and G_2 converge absolutely (in the E norm and E^* norm respectively) in their corresponding domains. Therefore G_1 and G_2 are analytic vector functions in $\mathbf{C}\setminus\mathbf{T}$.

If G is a vector valued analytic function in $\mathbf{C}\setminus\mathbf{T}$, then a singular point of G is a point $\lambda \in \mathbf{T}$, which has no neighborhood into which G admits an analytic continuation.

REMARK. It follows from (0.1) that $(A - z)G_1(z) = x_0$ for $|z| \neq 1$, and therefore if A has the single valued extension property (s.v.e.p.), the singularity set of G_1 coincides with $\sigma_A(x_0)$, the local spectrum of x_0 with respect to A. (For the definition of s.v.e.p. and local spectrum see [6, p. 1].) Similarly, if A^* has the s.v.e.p., the singularity set of G_2 is $\sigma_{A^*}(y_0)$. Thus, assuming that the operators A and A^*

have the s.v.e.p. (which is no loss of generality in considering the existence of hyperinvariant subspaces) we see that part (c) of Theorem 1.1 can be formulated as follows:

(c) The sequences $(\|x_n\|)_{n\in\mathbf{Z}}$ and $(\|y_n\|)_{n\in\mathbf{Z}}$ are dominated by Beurling sequences and $\sigma_A(x_0) \cup \sigma_{A^*}(y_0)$ contains more than one point.

We thank the referee for these observations.

An immediate consequence of Theorem 1.1 is:

THEOREM 1.2. *Let A be an invertible operator in $\mathscr{L}(E)$ and let $x_0 \in E$ and $y_0 \in E^*$ be non zero vectors. If the sequences $(A^n x_0)_{n\in\mathbf{Z}}$ and $(A^{*n} y_0)_{n\in\mathbf{Z}}$ satisfy one of the hypotheses* (a) — (e) *of Theorem* 1.1, *then either A is a multiple of the identity operator or A has a non trivial hyperinvariant subspace.*

As we shall see in Section 3, Theorem 1.2 implies the following result of J. Wermer:

THEOREM 1.3 (Wermer [23]). *If A is an invertible operator in $\mathscr{L}(E)$ then each of the following two conditions implies that A satisfies the conclusion of Theorem* 1.2*:*

$$\|A^n\| = O(|n|^k), \quad n \to \pm\infty \tag{1.9}$$

for some integer $k \geq 0$.

$$\sum_{n\in\mathbf{Z}} \frac{\log\|A^n\|}{1+n^2} < \infty \tag{1.10}$$

and the spectrum of A contains more than one point.

REMARKS 1. An important difference between Theorem 1.2 and Theorem 1.3 is the following: The spectral radius formula implies (see Section 2 or [23]) that the spectrum of an operator which satisfies the hypotheses of Theorem 1.3 is contained in the unit circle $\mathbf{T}$. Similarly all the other known extensions of Wermer's Theorem (cf. [19] or [20, Theorem 6.3]) deal with operators which have a portion of their spectrum (that is, the intersection of the spectrum with some open set in the plane) contained in a smooth arc. On the other hand, no such restrictions on the spectrum are imposed by the hypotheses of Theorem 1.2. This permits for example an application of Theorem 1.2. to certain weighted shifts whose spectrum consists of an anulus (such as the one described in Section 6).

2. The second part of Theorem 1.3 was proved by Wermer in [23] under somewhat more restrictive conditions. However as shown in the different proofs of Wermer's Theorem given in [6, p. 154] and [1, Section 6], these restrictions are not needed.

3. Wermer stated in [23] only the existence of non trivial invariant subspace for A which are also invariant under A^{-1}, but his proof actually produces hyper-

invariant subspaces. This fact is explicitely stated and proved in the above mentioned proofs in [6] and [1].

As we shall show in Section 3, Theorem 1.1 also implies the following extension of the result of Sz.-Nagy and Foiaş [22, p. 74] which is given in [6, p. 134].

THEOREM 1.4 (Colojoară and Foiaş). *Let E be a reflexive Banach space, and $(p_n)_{n\in\mathbf{N}}$ an increasing sequence of positive numbers such that*

$$\limsup_{m\to\infty} \frac{p_{m+n}}{p_m} \leqslant cn^k, \qquad \forall\, n \in \mathbf{N} \tag{1.11}$$

for some constant $c > 0$ and integer $k \geqslant 0$.

Let A be an operator in $\mathscr{L}(E)$ such that

$$\|A^n\| = O(p_n), \quad n \to \infty \tag{1.12}$$

and assume that there exist vectors $x \in E$ and $y \in E^*$ such that

$$\limsup_{n\to\infty} \|p_n^{-1} A^n x\| > 0 \tag{1.13}$$

and

$$\limsup_{n\to\infty} \|p_n^{-1} A^{*n} y\| > 0. \tag{1.14}$$

Then either A is a multiple of the identity operator or A has a non trivial hyperinvariant subspace.

REMARKS 1. A simple condition which implies (1.11) with $c = 1$ and $k = 0$ is:

$$\limsup_{n\to\infty} \frac{p_{n+1}}{p_n} \leqslant 1.$$

This holds in the examples: $p_n = n^j$ for some $j \geqslant 0$; $p_n = \exp(n^\alpha)$, for some $0 \leqslant \alpha < 1$; $p_n = \exp\left(\dfrac{n}{(\log(n+1))^\beta}\right)$, for some $0 < \beta < \infty$.

2. In Section 4 (Theorem 4.5) we extend Theorem 1.4 by showing that the right hand side of (1.11) can be replaced by $K\exp(cn^{1/2})$, for some constants $K > 0$ and $c > 0$.

If E is a Hilbert space, one can replace condition (1.12) by a weaker condition which does not impose restrictions on the norms of the operators A^n, $n \in \mathbf{N}$. More precisely we have the following result:

THEOREM 1.5. *Let E be a complex Hilbert space and $(p_n)_{n\in\mathbf{N}}$ a sequence which satisfies the hypotheses of Theorem 1.4. Let A be an operator in $\mathscr{L}(E)$ and assume that there exist vectors $x \in E$ and $y \in E^*$ such that (1.13) and (1.14) are satisfied* *and that*

$$\sup\{\|p_m^{-1}p_n^{-1}A^{*m}A^n x\| : m, n \in \mathbf{N}\} < \infty \tag{1.15}$$

and

$$\sup\{\|p_m^{-1}p_n^{-1}A^{m}A^{*n} y\| : m, n \in \mathbf{N}\} < \infty. \tag{1.16}$$

Then the conclusion of Theorem 1.4 *holds for* A.

Evidently, (1.12) implies (1.15) and (1.16) but not conversely.

Another consequence of Theorem 1.1 is an extension (Theorem 3.6) of the following result:

THEOREM 1.6 (Beauzamy [2]). *Let* A *be an operator in* $\mathscr{L}(E)$ *such that* $\|A\| = 1$, *and assume that for some vector* $x \in E$

$$\limsup_{n\to\infty} \|A^n x\| > 0. \tag{1.17}$$

Suppose that there exists a sequence of vectors $(u_n)_{n=0}^{\infty} \subset E$ *with* $u_0 \neq 0$ *such that* $(\|u_n\|)_{n=0}^{\infty}$ *is dominated by a (one-sided) Beurling sequence and*

$$Au_n = u_{n-1}, \quad \forall\, n \in \mathbf{N}. \tag{1.18}$$

Then either A *is a multiple of the identity operator or* A *has a non trivial hyperinvariant subspace.*

2. PRELIMINARIES

In this section we assemble some background material from harmonic analysis and the theory of analytic vector functions which will be needed in the sequel. Although all of these results are known, some of them do not seem to be readily available in the literature.

In what follows we shall denote by $C(\mathbf{T})$ the set of all complex continuous functions on $\mathbf{T}$. For $f \in C(\mathbf{T})$ and $n \in \mathbf{Z}$ we denote by $\hat{f}(n)$ the n-th Fourier coefficient of f that is

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})e^{-int}\,dt.$$

We shall require the following:

LEMMA 2.1. *Let* $(\sigma_n)_{n\in\mathbf{Z}}$ *be a sequence of real numbers such that* $\sigma_n \geq 1$, $\forall\, n \in \mathbf{Z}$ *and assume that*

$$\sum_{n\in\mathbf{Z}} \frac{\log \sigma_n}{1+n^2} < \infty \tag{2.1}$$

and that for some constant $c > 0$

$$c^{-1}\sigma_n \leqslant \sigma_{n+1} \leqslant c\sigma_n, \quad \forall n \in \mathbf{Z}. \tag{2.2}$$

Then for every $0 \leqslant a < b < 2\pi$, *there exists a function* $f \not\equiv 0$ *in* $C(\mathbf{T})$ *which is supported by the arc*

$$\Gamma = \{z \in \mathbf{T} : a < \arg z < b\}$$

and which satisfies

$$\sum_{n \in \mathbf{Z}} |\hat{f}(n)|\sigma_n < \infty. \tag{2.3}$$

REMARKS 1. Under the additional assumption that $(\sigma_n)_{n\in\mathbf{Z}}$ is a Beurling sequence the above result is well known and follows from the Paley-Wiener Theorem (cf. [6, p. 149], or [8]).

2. Lemma 2.1, even without the assumption (2.2), appears (in equivalent form) in [13, Lemma 3]. However the proof given there is not correct, since in the estimates of the Fourier coefficients in [13, p. 180], the third inequality holds only if the sequence $\left(\frac{\log \sigma_n}{n}\right)_{n\in\mathbf{N}}$ (the sequence φ_n in the notation there) is eventually decreasing. This assumption in conjunction with (2.1) is stronger than (2.2). We do not know whether or not the conclusion of Lemma 2.1 is true without assumption (2.2).

Proof of Lemma 2.1. Let $\beta_n = \log \sigma_n$, $n \in \mathbf{Z}$, and consider the piecewise linear function φ on $(-\infty, \infty)$ which satisfies $\varphi(n) = \beta_n$ for $n \in \mathbf{Z}$.

It is easy to verify that (2.1) implies that

$$\int_{-\infty}^{\infty} \frac{\varphi(x)}{1+x^2}\, dx < \beta \tag{2.4}$$

and (2.2) implies that for every $-\infty < t < \infty$

$$\sup_{-\infty<x<\infty} |\varphi(x+t) - \varphi(x)| < \infty, \tag{2.5}$$

therefore, it follows from [4, Theorem 1] that there exists a continuous nonidentically zero function g on $(-\infty, \infty)$ which is supported by (a, b), whose Fourier transform $\hat{g}$ satisfies

$$\int_{-\infty}^{\infty} |\hat{g}(x)| \exp(\varphi(x))\, dx < \infty. \tag{2.6}$$

Using the fact that

$$\int_0^1 \Big(\sum_{n\in\mathbf{Z}} |\hat{g}(n-t)| \exp(\varphi(n-t))\Big)\,\mathrm{d}t = \int_{-\infty}^{\infty} |\hat{g}(x)| \exp(\varphi(x))\,\mathrm{d}x$$

we obtain from Fubini's Theorem and (2.6) that for some $0 < \alpha < 1$,

$$\sum_{n\in\mathbf{Z}} |\hat{g}(n-\alpha)| \exp(\varphi(n-\alpha)) < \infty. \tag{2.7}$$

It follows from (2.5) and the definition of φ that there exists a constant $d > 0$ such that

$$\beta_n = \varphi(n) \leqslant \varphi(n-\alpha) + d, \quad \forall\, n \in \mathbf{Z}$$

and therefore by (2.7)

$$\sum_{n\in\mathbf{Z}} |\hat{g}(n-\alpha)|\sigma_n < \infty. \tag{2.8}$$

Let now f be the function in $C(\mathbf{T})$ defined by

$$f(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}\alpha t} g(t), \quad 0 \leqslant t < 2\pi.$$

We claim that f has the required properties. Indeed

$$\hat{f}(n) = \frac{1}{2\pi}\,\hat{g}(n-\alpha), \quad \forall\, n \in \mathbf{Z}$$

and therefore (2.3) follows from (2.8). The assumptions on g imply that $f \not\equiv 0$ and that f is supported by Γ. This completes the proof of the lemma.

Throughout the rest of this section $\rho = (\rho_n)_{n\in\mathbf{Z}}$ will be a Beurling sequence and A_ρ will denote the set of all functions f in $\mathbf{C}(\mathbf{T})$ such that $\sum_{n\in\mathbf{Z}} |\hat{f}(n)|\rho_n < \infty$.

Since $\rho_n \geqslant 1$, $\forall\, n \in \mathbf{Z}$, it follows that $\sum_{n\in\mathbf{Z}} |\hat{f}(n)| < \infty$ for $f \in A_\rho$, and therefore the Fourier series of f converges uniformly on $\mathbf{T}$ to f.

It is well known, and easy to verify, that (1.1) implies that, with norm

$$\|f\| = \sum_{n\in\mathbf{Z}} |\hat{f}(n)|\rho_n, \quad f \in A_\rho,$$

A_ρ is a Banach algebra with respect to pointwise addition and multiplication of functions on $\mathbf{T}$. It is also clear that for every f in A_ρ the sequence of trigonometric polynomials

$$s_n(\mathrm{e}^{\mathrm{i}t}) = \sum_{j=-n}^{n} \hat{f}(j)\mathrm{e}^{\mathrm{i}jt}, \quad n \in \mathbf{N}$$

converges to f in the norm of A_ρ.

It is known (cf. [11], p. 128) that (1.1) implies that the limits

$$R_1 = \lim_{n\to-\infty} \rho_n^{1/n} \quad \text{and} \quad R_2 = \lim_{n\to\infty} \rho_n^{1/n}$$

exist. This fact in conjunction with (1.2) implies that $R_1 = R_2 = 1$. Therefore by [11, p. 130], the maximal ideal space of A_ρ can be identified in the natural way with $\mathbf{T}$.

REMARKS 1. The fact that $R_1 = R_2 = 1$ implies that, if $(x_n)_{n\in Z}$ is a sequence of vectors in a Banach space E and $(\|x_n\|)_{n\in Z}$ is dominated by a Beurling sequence then

$$\limsup_{n\to-\infty} \|x_n\|^{1/n} \leqslant 1 \quad \text{and} \quad \limsup_{n\to\infty} \|x_n\|^{1/n} \leqslant 1.$$

Consequently the functions G_1 and G_2 in (1.5) and (1.6) are analytic in $\mathbf{C}\setminus\mathbf{T}$.

2. It also follows from (1.5) and (1.6) and the previous remark that

$$\lim_{|z|\to\infty} \|G_j(z)\| = 0, \quad j = 1, 2$$

and therefore by Liouville's Theorem (cf. [15], p. 100) and the fact that $x_0\neq 0$ and $y_0\neq 0$, we see that each of the functions G_1 and G_2 has at least one singularity on $\mathbf{T}$.

By virtue of Lemma 2.1, condition (1.2) implies that the algebra A_ρ is regular, that is for every closed set $K\subset \mathbf{T}$ and $\lambda\in \mathbf{T}\setminus K$ there exists a function f in A_ρ such that $f(z) = 0$ for $z\in K$ and $f(\lambda) = 1$ (see also [8] where a more general result is proved).

For every S in A_ρ^* (the dual of A_ρ) we set

$$\hat{S}(n) = \langle e^{-int}, S\rangle, \quad n\in \mathbf{Z}.$$

A simple computation shows that

$$\langle f, S\rangle = \sum_{n\in Z} \hat{f}(n)\hat{S}(-n)$$

for every f in A_ρ and S in A_ρ^*.

Let ℓ_ρ^∞ denote the Banach space of all complex sequences $(c_n)_{n\in Z}$ for which the norm

$$\|c\| = \sup_{n\in Z} \frac{|c_n|}{\rho_{-n}}$$

is finite.

It is easily verified that the mapping

$$S \to (\hat{S}(n))_{n\in \mathbf{Z}}, \quad S\in A_\rho^*$$

establishes an isometric isomorphism between A_ρ^* and ℓ_ρ^∞.

For every $f\in A_\rho$ and $S\in A_\rho^*$ we shall denote by $f\cdot S$ the element of A_ρ^* which is defined by

$$\langle g, f\cdot S\rangle = \langle f\cdot g, S\rangle, \quad g\in A_\rho.$$

It is easy to verify that $\forall f\in A_\rho$ and $\forall S\in A_\rho^*$

$$\widehat{f\cdot S}(n) = \sum_{j\in \mathbf{Z}} \hat{f}(n-j)\hat{S}(j), \quad \forall n\in \mathbf{Z}.$$

For an ideal $I\subset A_\rho$ we shall use the notation

$$h(I) = \{z\in \mathbf{T}: f(z) = 0, \quad \forall f\in I\}.$$

$h(I)$ is called the hull (or co-spectrum) of I.

Since the Banach algebra A_ρ is regular, every element S in A_ρ^* has a well defined support (see [17], p. 230), which is the complement (with respect to $\mathbf{T}$) of the largest open (in the topology of $\mathbf{T}$) subset $U\subset \mathbf{T}$, such that $\langle f, S\rangle = 0$ for every function f in A_ρ whose support is contained in U.

We shall denote the support of $S\in A_\rho^*$ by $\Sigma(S)$. It is clear that $\Sigma(S)$ is empty if and only if $S = 0$, and that $\forall f\in A_\rho$ and $\forall S\in A_\rho^*$

$$\Sigma(f\cdot S) \subset \Sigma(S) \cap \operatorname{support}(f).$$

In the sequel we shall require the following:

Lemma 2.2. *Let $S\in A_\rho^*$ and let J be the ideal in A_ρ which consists of all functions $f\in A_\rho$ such that $f\cdot S = 0$. Then $h(J) = \Sigma(S)$.*

Proof. We show first that $h(J)\subset \Sigma(S)$. Suppose that $\lambda\in \mathbf{T}\setminus\Sigma(S)$, and let L be an open arc on $\mathbf{T}$ which contains λ and is disjoint from $\Sigma(S)$. Let f be a function in A, which is supported by L, and $f(\lambda) = 1$. It follows from the definition of $\Sigma(S)$ that $f\cdot S = 0$, and therefore $f\in J$. Since $f(\lambda) = 1$ we deduce that $\lambda\notin h(J)$. This shows that $h(J)\subset \Sigma(S)$.

To prove that $\Sigma(S)\subset h(J)$, consider $\lambda\in \mathbf{T}\setminus h(J)$. There exists a function f in A_ρ such that $f\cdot S = 0$ and $f(\lambda) = 1$. Let Γ be an open arc on $\mathbf{T}$, which contains λ, such that $|f(z)| \geqslant 1/2$, $\forall z\in\Gamma$. Let I be the principal ideal generated (algebraically) in A_ρ by f. It follows from [17, Corollary 5.7, p. 224] that I contains every function g in A_ρ which is supported by Γ. That is for every such function g, there exists a function $\varphi\in A_\rho$ such that $g = \varphi\cdot f$, and therefore, since $f\cdot S = 0$, we have that

$$\langle g, S\rangle = \langle \varphi, f\cdot S\rangle = 0.$$

This shows that Γ is disjoint from $\Sigma(S)$, hence $\lambda \notin \Sigma(S)$. Thus $\Sigma(S) \subset h(J)$, and the lemma is proved.

DEFINITION. The *Carleman transform* of S in A_ρ^* is the function $\widetilde{S}$ defined on $\mathbf{C}\setminus\mathbf{T}$ by

$$\widetilde{S}(z) = \begin{cases} \sum_{n=1}^{\infty} \hat{S}(n) z^{n-1}, & |z| < 1 \\ -\sum_{n=-\infty}^{0} \hat{S}(n) z^{n-1}, & |z| > 1. \end{cases}$$

Since the sequence $(\hat{S}(n))_{n\in\mathbf{Z}}$ is clearly dominated by the Beurling sequence $(\rho_{-n})_{n\in\mathbf{Z}}$, it follows (from Remark 1, following the definition of A_ρ) that $\widetilde{S}$ is well defined and analytic in $\mathbf{C}\setminus\mathbf{T}$.

We shall denote the set of singular points of $\widetilde{S}$ by $\operatorname{sing}(\widetilde{S})$.

In the proofs of Theorem 1.1 (c) and Theorem 4.1, the following result will be of fundamental importance:

LEMMA 2.3. *For every S in A_ρ^*, $\Sigma(S) = \operatorname{sing}(\widetilde{S})$.*

The idea of this result, in the setting of Fourier transforms (at least for Beurling sequences of polynomial growth) goes back to the work of T. Carleman [5, Ch. II], (see also [17], p. 179).

For general Beurling sequences this result is essentially contained in [9]. Since it is not explicitely stated there, we include a proof.

Proof of Lemma 2.3. Let $S \in A_\rho^*$, and consider the ideal J associated with S as in Lemma 2.2. It follows from Lemma 2.2 and [9, Theorem 2.4 and Example 3.1] that $\operatorname{sing}(\widetilde{S}) \subset \Sigma(S)$, and from Lemma 2.2 and [9, Theorem 8.1 and the example which follows] that $\Sigma(S) \subset \operatorname{sing}(\widetilde{S})$.

In the proof of Theorem 1.1 we shall also require the following result:

LEMMA 2.4. *Let E be a complex Banach space, and let F and G be functions with values in E and E^* respectively, defined and analytic in $\mathbf{C}\setminus\mathbf{T}$. Assume that there exists an open disc D, with center on $\mathbf{T}$, such that $\forall x \in E$ and $\forall y \in E^*$ the complex functions*

$$z \to \langle F(z), y\rangle \quad \text{and} \quad z \to \langle x, G(z)\rangle, \quad z \in \mathbf{C}\setminus\mathbf{T}$$

can be continued analytically into D. Then F and G admit analytic continuations into D.

Proof. Let D_1 be an open disc whose closure is contained in D. Remembering that a (complex) analytic function which is analytic in a neighborhood of a closed disc satisfies Lipschitz condition (of order 1) on that disc, we obtain from the hypotheses that $\forall y \in E^*$

$$\sup\{|\langle F(z_2), y\rangle - \langle F(z_1), y\rangle| \cdot |z_2 - z_1|^{-1}: z_1, z_2 \in D_1 \setminus \mathbf{T},\ z_1 \neq z_2\} < \infty.$$

Therefore by the uniform boundedness principle

$$\sup\{\|F(z_2) - F(z_1)\| \cdot |z_2 - z_1|^{-1}: z_1, z_2 \in D_1 \setminus \mathbf{T},\ z_1 \neq z_2\} < \infty.$$

Consequently, F is uniformly continuous on $D_1 \setminus \mathbf{T}$ and therefore admits a continuous extension to $\bar{D}_1$. Since D_1 is an arbitrary open disc whose closure is contained in D, we conclude that F admits a continuous extension to D, which we denote by F_1. From the hypotheses of the lemma it follows that $\forall y \in E^*$, the complex function

$$z \to \langle F_1(z), y\rangle, \quad z \in (\mathbf{C} \setminus \mathbf{T}) \cup D$$

is analytic in D, and therefore (see also [15], p. 53) F_1 is an analytic continuation of F into D.

A similar argument (see also the proof of Theorem 3.9.1 in [15]) shows that G also admits an analytic continuation into D.

3. PROOFS OF MAIN RESULTS

We begin by introducing a notation which will be used throughout this section.

Let A be an operator in $\mathscr{L}(E)$ and assume that $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$ are sequences such that (0.1) holds.

For functions f and g in $C(\mathbf{T})$ such that

$$\text{(3.1)} \qquad \sum_{n\in\mathbf{Z}} |\hat{f}(n)|\,\|x_n\| < \infty \quad \text{and} \quad \sum_{n\in\mathbf{Z}} |\hat{g}(n)|\,\|y_n\| < \infty$$

we shall denote by $u(f)$ and $v(g)$ the vectors in E and E^*, respectively, defined by

$$u(f) = \sum_{n\in\mathbf{Z}} \hat{f}(n) x_n \quad \text{and} \quad v(g) = \sum_{n\in\mathbf{Z}} \hat{g}(n) y_n.$$

Since E and E^* are Banach spaces it follows from (3.1) that the series defining $u(f)$ and $v(g)$ converge in the respective norms.

In the proof of Theorem 1.1 we shall require the following:

LEMMA 3.1. *Let A be an operator in $\mathscr{L}(E)$, and let $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$ be sequences such that* (0.1) *holds. Assume that f and g are functions in $C(\mathbf{T})$ which* *satisfy* (3.1). *Then* $\forall B \in (A)'$

$$\sum_{n\in\mathbf{Z}} |\widehat{fg}(n)\langle Bx_n, y_0\rangle| < \infty \tag{3.2}$$

and

$$\langle Bu(f),\ v(g)\rangle = \sum_{n\in\mathbf{Z}} \widehat{fg}(n)\langle Bx_n, y_0\rangle. \tag{3.3}$$

Thus in particular, if $f\cdot g = 0$, *then*

$$\langle Bu(f),\ v(g)\rangle = 0, \quad \forall B \in (A)'. \tag{3.4}$$

Proof. We show first that

$$\langle Bx_j, y_k\rangle = \langle Bx_{j+k}, y_0\rangle, \quad \forall B \in (A)',\ \forall j, k \in \mathbf{Z}. \tag{3.5}$$

Assume first that k is a nonnegative integer. Then by (0.1)

$$A^k x_j = x_{j+k}, \quad \forall j \in \mathbf{Z} \text{ and } A^{*k} y_0 = y_k.$$

Therefore $\forall B \in (A)'$ and $\forall j \in \mathbf{Z}$

$$\langle Bx_j, y_k\rangle = \langle Bx_j,\ A^{*k}y_0\rangle = \langle A^k Bx_j, y_0\rangle =$$
$$= \langle BA^k x_j, y_0\rangle = \langle Bx_{j+k}, y_0\rangle.$$

Suppose next that k is a negative integer. Then by (0.1),

$$A^{-k}x_{j+k} = x_j, \quad \forall j \in \mathbf{Z} \text{ and } A^{*-k}y_k = y_0.$$

Therefore $\forall B \in (A)'$ and $\forall j \in \mathbf{Z}$

$$\langle Bx_{j+k}, y_0\rangle = \langle Bx_{j+k}, A^{*-k}y_0\rangle =$$
$$= \langle A^{-k}Bx_{j+k}, y_k\rangle = \langle BA^{-k}x_{j+k}, y_k\rangle = \langle Bx_j, y_k\rangle.$$

Thus (3.5) is proved.

From (3.1) and (3.5) we deduce that $\forall B \in (A)'$,

$$\sum_{j,k\in\mathbf{Z}} |\hat{f}(j)\hat{g}(k)\langle Bx_{j+k}, y_0\rangle| = \sum_{j,k\in\mathbf{Z}} |\hat{f}(j)\hat{g}(k)\langle Bx_j, y_k\rangle| \leqslant$$
$$\leqslant \|B\|\Big(\sum_{j\in\mathbf{Z}} |\hat{f}(j)|\,\|x_j\|\Big)\cdot\Big(\sum_{k\in\mathbf{Z}} |\hat{g}(k)|\,\|y_k\|\Big) < \infty.$$

Thus noticing that

$$\widehat{f\cdot g}(n) = \sum_{j+k=n} \hat{f}(j)\hat{g}(k), \quad \forall n \in \mathbf{Z}$$

we obtain in particular that (3.2) holds. By changing the order of summation (which is permitted by virtue of the absolute convergence of the series above) and using (3.5) once again, we obtain that $\forall B \in (A)'$

$$\langle Bu(f), v(g)\rangle = \sum_{j,k\in \mathbf{Z}} \hat{f}(j)\hat{g}(k)\langle Bx_j, y_k\rangle =$$

$$= \sum_{n\in \mathbf{Z}} \sum_{j+k=n} \hat{f}(j)\hat{g}(k)\langle Bx_n, y_0\rangle = \sum_{n\in \mathbf{Z}} \widehat{f\cdot g}(n)\langle Bx_n, y_0\rangle.$$

This completes the proof of the lemma.

In the sequel we shall also need the following:

LEMMA 3.2. *Let A be an operator in $\mathscr{L}(E)$ and assume that there exist non zero vectors $u \in E$ and $v \in E^*$ such that*

$$\langle Bu, v\rangle = 0, \quad \forall B \in (A)'. \tag{3.6}$$

Then A has a non trivial hyperinvariant subspace.

Proof. Let M be the closure in E of the linear manifold $\{Bu: B\in (A)'\}$. It is clear that M is a hyperinvariant subspace for A. $M\neq\{0\}$ since $u \in M$, and since $v\neq 0$, it follows from (3.6) that $M\neq E$. Thus M is not trivial, and the lemma is proved.

REMARK. Using the Hahn-Banach Theorem it is easy to show that the hypotheses of Lemma 3.2 are also necessary for the existence of a non trivial hyperinvariant subspace for A.

Proof of Theorem 1.1. First we notice that if $A\neq 0$ and A is not injective, then $\ker(A)$ is a non trivial hyperinvariant subspace for A, and if A^* is not injective then the closure of the range of A is a non trivial hyperinvariant subspace for A. Thus in what follows we shall assume that A and A^* are injective. This assumption, in conjunction with (0.1) and the hypothesis that $x_0\neq 0$ and $y_0\neq 0$, implies that

$$x_n\neq 0 \quad \text{and} \quad y_n\neq 0, \qquad \forall n\in \mathbf{Z}. \tag{3.7}$$

Proof of (e). From (3.5) we deduce that

$$\langle Bx_j, y_0\rangle = \langle Bx_{n+j}, y_{-n}\rangle, \qquad \forall B\in (A)',\ \forall n\in \mathbf{Z}$$

and therefore $\forall B\in(A)'$,

$$|\langle Bx_j, y_0\rangle| \leqslant \|B\| \inf_{n\in \mathbf{Z}} \|x_{n+j}\|\,\|y_{-n}\|.$$

Consequently, (1.8) implies that

$$\langle Bx_j, y_0\rangle = 0, \qquad \forall B\in (A)'. \tag{3.8}$$

Since $y_0 \neq 0$ and by (3.7) also $x_j \neq 0$, we obtain from (3.8) and Lemma 3.2 that A has a non trivial hyperinvariant subspace.

Proof of (d). Consider the sequence

$$\sigma_n = \max\{\|x_n\|, 1\} \cdot \max\{\|y_n\|, 1\}, \quad n \in \mathbf{Z}.$$

It follows from (1.7a) that the sequence $(\sigma_n)_{n \in \mathbf{Z}}$ satisfies (2.1), and from (0.1) and (1.7b) we obtain that it also satisfies (2.2) with $c = \|A\|^2 + b^2 + 1$.

Let Γ_1 and Γ_2 be disjoint open arcs on $\mathbf{T}$. By Lemma 2.1 there exist functions $f \not\equiv 0$ and $g \not\equiv 0$ in $C(\mathbf{T})$, supported by Γ_1 and Γ_2 respectively, such that

$$\sum_{n \in \mathbf{Z}} |\hat{f}(n)| \sigma_n < \infty \quad \text{and} \quad \sum_{n \in \mathbf{Z}} |\hat{g}(n)| \sigma_n < \infty.$$

Noticing that $\max\{\|x_n\|, \|y_n\|\} \leqslant \sigma_n$, $\forall n \in \mathbf{Z}$, we obtain that (3.1) holds for f and g. Since Γ_1 and Γ_2 are disjoint, $f \cdot g = 0$, and therefore by Lemma 3.1 also (3.4) holds. Thus by virtue of Lemma 3.2 the assertion of the theorem will follow, if we show that $u(f) \neq 0$ and $v(g) \neq 0$.

Since x_0 is not in the closed span of the set $\{x_n : n \in \mathbf{Z},\ n \neq 0\}$, we deduce from (0.1) that for every negative integer p, x_p is not in the closed span of the set $\{x_n : n \in \mathbf{Z},\ n \neq p\}$. Since $f = 0$ on Γ_2, and $f \not\equiv 0$, there exists a negative integer q, such that $\hat{f}(q) \neq 0$ (see [17], p. 90, Corollary 3.14). Combining these facts we obtain that

$$\hat{f}(q) x_q \neq -\sum_{\substack{n \in \mathbf{Z} \\ n \neq q}} \hat{f}(n) x_n$$

and consequently $u(f) \neq 0$. A similar argument shows that $v(g) \neq 0$, and the proof of (d) is complete.

Proof of (c). Since the product of two Beurling sequences is again a Beurling sequence which dominates both, we may assume that $(\|x_n\|)_{n \in \mathbf{Z}}$ and $(\|y_n\|)_{n \in \mathbf{Z}}$ are dominated by the same Beurling sequence $\rho = (\rho_n)_{n \in \mathbf{Z}}$. Clearly (3.1) holds $\forall f, g \in A_\rho$. Therefore by virtue of Lemma 3.1 and Lemma 3.2, it suffices to show that there exist functions f and g in A_ρ, with disjoint supports, such that $u(f) \neq 0$ and $v(g) \neq 0$.

For this consider the two vector functions G_1 and G_2 defined by (1.5) and (1.6). As noted in Section 2, each of these functions has at least one singularity on $\mathbf{T}$. Therefore the hypotheses of (c) imply that there exist $\lambda_1, \lambda_2 \in \mathbf{T}$, $\lambda_1 \neq \lambda_2$, such that λ_1 is a singular point of G_1 and λ_2 is a singular point of G_2. By Lemma 2.4 there exist vectors $x \in E$ and $y \in E^*$ such that λ_1 is a singular point of the function

$$z \to \langle G_1(z), y \rangle, \quad z \in \mathbf{C} \setminus \mathbf{T}$$

and λ_2 is a singular point of the function

$$z \to \langle x, G_2(z) \rangle, \quad z \in \mathbf{C} \setminus \mathbf{T}.$$

Therefore, if S_1 and S_2 are the elements in A_ρ^* defined by

$$\hat{S}_1(n) = \langle x_{-n}, y\rangle \quad \text{and} \quad \hat{S}_2(n) = \langle x, y_{-n}\rangle, \qquad \forall n \in \mathbf{Z},$$

we deduce from Lemma 2.3 that $\lambda_1 \in \Sigma(S_1)$ and $\lambda_2 \in \Sigma(S_2)$.

Let Γ_1 and Γ_2 be two disjoint open arcs on $\mathbf{T}$ such that $\lambda_1 \in \Gamma_1$ and $\lambda_2 \in \Gamma_2$. Since $\lambda_1 \in \Sigma(S_1)$ and $\lambda_2 \in \Sigma(S_2)$, there exist functions f and g in A_ρ, supported by Γ_1 and Γ_2 respectively, such that $\langle f, S_1\rangle \neq 0$ and $\langle g, S_2\rangle \neq 0$. But a simple computation shows that

$$\langle f, S_1\rangle = \langle u(f), y\rangle \quad \text{and} \quad \langle g, S_2\rangle = \langle x, v(g)\rangle,$$

and therefore $u(f) \neq 0$ and $v(g) \neq 0$. This completes the proof of (c).

Proofs of (a) and (b). If the hypotheses of (a) or (b) are satisfied and one of the functions G_1 or G_2, defined by (1.5) and (1.6), has more than one singularity, the conclusion of the theorem follows from part (c) (since $((1+|n|)^k)_{n\in\mathbf{Z}}$ is a Beurling sequence). Remembering that each of these functions has at least one singularity on $\mathbf{T}$, we see that it suffices to consider the case in which each of them has exactly one singularity on $\mathbf{T}$.

We shall show first that if G_1 has a single singularity at $\lambda_0 \in \mathbf{T}$ and (1.3) holds, then

$$(A - \lambda_0 I)^{k+1} x_0 = 0. \tag{3.9}$$

This will prove the assertion, since (3.9) implies that either $A = \lambda_0 I$ or $\ker(A - \lambda_0 I)$ is a non trivial hyperinvariant subspace for A.

It suffices to prove (3.9) in the case that $\lambda_0 = 1$, since the general case can be deduced from this one, by replacing the operator A by $\lambda_0^{-1}A$ and the sequences $(x_n)_{n\in\mathbf{Z}}$ and $(y_n)_{n\in\mathbf{Z}}$ by the sequences $(\lambda_0^n x_n)_{n\in\mathbf{Z}}$ and $(\lambda_0^n y_n)_{n\in\mathbf{Z}}$. (It is easily verified that these replacements preserve all the hypotheses.)

Thus we assume that $z = 1$ is the only singularity of G_1 and that (1.3) holds, and we shall show that

$$(A - I)^{k+1} x_0 = 0. \tag{3.10}$$

For this we introduce the difference operator Δ defined on sequences $(a_n)_{n=0}^{\infty} \subset E$ by

$$\Delta a_n = a_n - a_{n-1}, \quad n \in \mathbf{N}, \text{ and } \Delta a_0 = a_0.$$

If $(a_n)_{n=0}^{\infty} \subset E$ and F is the (formal) power series $F(z) = \sum_{n=0}^{\infty} a_n z^n$, it is easy to prove by induction that

$$(1 - z)^j F(z) = \sum_{n=0}^{\infty} (\Delta^j a_n) z^n, \quad \forall j \in \mathbf{N} \tag{3.11}$$

where Δ^j denotes the j-th iterate of Δ.

Now if G_1 has a single singularity at $z = 1$ and (1.3) holds, then according to [15, Theorem 3.12.7, p. 60], G_1 is a polynomial in $(1 - z)^{-1}$ of degree not exceeding k (with coefficients in E). Therefore the same is true for the function

$$H(z) = z^{-1}G_1(z^{-1}), \quad z \in \mathbf{C}\setminus\{1\}$$

and consequently the function $p(z) = (1 - z)^{k+1}H(z)$ is a polynomial in z (with coefficients in E) of degree at most k. From (0.1) and the definition of G_1 we obtain that

$$H(z) = \sum_{n=0}^{\infty} (A^n x_0)z^n, \quad |z| < 1$$

and therefore by (3.11)

$$p(z) = \sum_{n=0}^{\infty} \Delta^{k+1}(A^n x_0)z^n, \quad |z| < 1.$$

Since p is a polynomial of degree not exceeding k we conclude that

$$\Delta^{k+1}(A^n x_0) = 0, \quad \forall n > k. \tag{3.12}$$

It is easily verified that

$$\Delta^j(A^n x_0) = A^{n-j}(A - I)^j x_0, \quad \forall n \geqslant j$$

and therefore (3.10) follows by setting $n = k + 1$ in (3.12). Thus (a) is proved.

A similar argument shows that if G_2 has a single singularity at $\lambda_1 \in \mathbf{T}$ and (1.4) holds, then $(A^* - \lambda_1 I)^{k+1}y_0 = 0$. Therefore either $A = \lambda_1 I$, or the closure of the range of $A - \lambda_1 I$ is a non trivial hyperinvariant subspace for A. This proves (b), and completes the proof of Theorem 1.1.

Proof of Theorem 1.3. Assume first that A satisfies (1.9). Then for every two non zero vectors $x_0 \in E$ and $y_0 \in E^*$, the sequences $(A^n x_0)_{n\in\mathbf{Z}}$ and $(A^{*n}y_0)_{n\in\mathbf{Z}}$ satisfy the hypotheses of part (a) of Theorem 1.2 and the assertion follows.

To prove the second part of the theorem, assume that (1.10) is satisfied and consider the sequence $\rho_n = \|A^n\|$, $n \in \mathbf{Z}$. It follows from (1.10) and the fact that $\|A^{n+m}\| \leqslant \|A^n\| \|A^m\|$, $\forall m, n \in \mathbf{Z}$, that $\rho = (\rho_n)_{n\in\mathbf{Z}}$ is a Beurling sequence.

Since $\forall x \in E$ and $\forall y \in E^*$ we have that

$$\|A^n x\| \leqslant \rho_n\|x\|, \quad \text{and} \quad \|A^{*n}y\| \leqslant \rho_n\|y\|, \quad \forall n \in \mathbf{Z}$$

the assertion will follow from part (c) of Theorem 1.2, once we show that there exist non zero vectors $x_0 \in E$ and $y_0 \in E^*$ such that the union of the singularity sets of the functions G_1 and G_2 associated with the sequences $(A^n x_0)_{n\in\mathbf{Z}}$ and $(A^{*n}y_0)_{n\in\mathbf{Z}}$ by (1.5) and (1.6) contains more than one point.

To show this, let $\sigma(A)$ denote the spectrum of A and consider the resolvent

$$\mathrm{R}(A, z) = (A - zI)^{-1}, \quad z \in \mathbf{C}\setminus\sigma(A).$$

Remembering that $\sigma(A) \subset \mathbf{T}$ by (1.10) (see the remarks in Section 1) we obtain that

$$\mathrm{R}(A, z) = \sum_{n=1}^{\infty} A^{-n} z^{n-1}, \quad |z| < 1$$

and

$$\mathrm{R}(A, z) = - \sum_{n=-\infty}^{0} A^{-n} z^{n-1}, \quad |z| > 1.$$

Therefore if $x \in E$, and G_1 is the function associated with the sequence $(A^n x)_{n\in\mathbf{Z}}$ by (1.5), we see that

(3.13) $$G_1(z) = \mathrm{R}(A, z)x, \quad \forall z \in \mathbf{C}\setminus\mathbf{T}.$$

Assume now that $\lambda_1 \in \sigma(A)$. By a Theorem of Helson [14, Theorem 3], there exists a vector $x_0 \in E$ such that the vector function

$$z \to \mathrm{R}(A, z)x_0, \quad |z| < 1$$

has no analytic continuation to any neighborhood of λ_1. Clearly $x_0 \neq 0$. Therefore (3.13) implies that λ_1 is a singular point of the function G_1 associated with the sequence $(A^n x_0)_{n\in\mathbf{Z}}$ by (1.5).

By the hypotheses, $\sigma(A)$ contains more than one point, and therefore there exists $\lambda_2 \in \sigma(A)$ such that $\lambda_1 \neq \lambda_2$. Remembering that $\sigma(A) = \sigma(A^*)$, we obtain by replacing in the above argument λ_1 and A by λ_2 and A^*, that there exists a non zero vector $y_0 \in E^*$, such that λ_2 is a singular point of the function G_2 associated with the sequence $(A^{*n} y_0)_{n\in\mathbf{Z}}$ by (1.6). This completes the proof of the theorem.

The link between Theorem 1.1 and Theorems 1.4, 1.5 and 1.6 is established by means of the following results:

LEMMA 3.3. *Let A be an injective operator in $\mathscr{L}(E)$ and assume that there exists a sequence $(w_n)_{n\in\mathbf{N}} \subset E^*$ and a vector $x \in E$ such that*

(3.14) $$\sup\{\|A^{*m} w_n\| : m, n \in \mathbf{N},\ m \leqslant n\} < \infty$$

and

(3.15) $$\limsup_{n\to\infty} |\langle A^n x, w_n\rangle| > 0.$$

Then there exists a norm bounded sequence $(v_n)_{n=0}^{\infty} \subset E^*$ with $v_0 \neq 0$ such that

(3.16) $$A^* v_n = v_{n-1}, \quad \forall n \in \mathbf{N}.$$

If in addition there exists a sequence of positive numbers $(q_n)_{n\in\mathbf{N}}$ *such that*

$$\limsup_{n\to\infty}\|A^{*m+n}w_n\| \leqslant q_m, \quad \forall m\in\mathbf{N} \tag{3.17}$$

then

$$\|A^{*m}v_0\| \leqslant q_m, \quad \forall m\in\mathbf{N}. \tag{3.18}$$

Proof. It follows from (3.15) that there exists a number $\delta > 0$ and an increasing sequence of integers $(n_k)_{k\in\mathbf{N}}$ such that

$$|\langle x, A^{*n_k}w_{n_k}\rangle| > \delta, \quad \forall k\in\mathbf{N}. \tag{3.19}$$

Since a bounded set in E^* is precompact in the w*-topology we obtain from (3.14) that there exists a subnet $(A^{*\gamma}w_\gamma)_{\gamma\in\Lambda}$ of $(A^{*n_k}w_{n_k})_{k\in\mathbf{N}}$ (where Λ is a directed subset of $\mathbf{N}$) which converges in the w*-topology to some vector $v_0\in E^*$. It follows from (3.19) that $|\langle x, v_0\rangle| > \delta$, and therefore $v_0\neq 0$.

For every $n\in\mathbf{N}$ consider the subset of E^*

$$V_n = \{A^{*\gamma-n}w_\gamma : \gamma\in\Lambda,\ \gamma > n\}.$$

By (3.14) V_n is bounded and therefore has a w*-limit point $v_n\in E^*$. Using (3.14) once again, we see that the sequence $(v_n)_{n=0}^{\infty}$ is norm bounded.

From the definition of V_n and the fact the adjoint of an operator in $\mathscr{L}(E)$ is continuous with respect to the w*-topology of E^*, we deduce that $A^{*n}v_n = v_0$, $\forall n\in\mathbf{N}$ and therefore

$$A^{*n-1}(A^*v_n - v_{n-1}) = 0, \quad \forall n\in\mathbf{N}.$$

Since A^* is injective this implies (3.16).

If (3.7) holds, then (3.18) follows directly from the definition of v_0. This completes the proof of the lemma.

COROLLARY 3.4. *Let A be an injective operator in $\mathscr{L}(E)$ and assume that there exists an increasing sequence of positive numbers $(p_n)_{n\in\mathbf{N}}$ and a vector $x\in E$ such that*

$$\|A^n\| = O(p_n), \quad n\to\infty \tag{3.20}$$

and

$$\limsup_{n\to\infty}\|p_n^{-1}A^n x\| > 0. \tag{3.21}$$

Then there exists a norm bounded sequence $(v_n)_{n=0}^{\infty}\subset E^*$ with $v_0\neq 0$ such that (3.16) *holds.*

If in addition there exists a sequence of positive numbers $(q_n)_{n\in\mathbf{N}}$ such that

$$\limsup_{n\to\infty}\frac{p_{m+n}}{p_n} \leqslant q_m, \quad \forall m\in\mathbf{N}, \tag{3.22}$$

then (3.18) *also holds.*

Proof. By the Hahn-Banach Theorem there exists a sequence of unit vectors $(z_n)_{n\in\mathbf{N}} \subset E^*$ such that

$$\langle p_n^{-1}A^n x, z_n\rangle = \|p_n^{-1}A^n x\|, \quad \forall n \in \mathbf{N}.$$

Thus (3.20) and (3.21) imply (3.14) and (3.15) with $w_n = p_n^{-1}z_n$, $n \in \mathbf{N}$.

If (3.22) is satisfied, then it is easily seen that (3.17) also holds for the sequence $(w_n)_{n\in\mathbf{N}}$. Hence the Corollary follows from Lemma 3.3.

COROLLARY 3.5. *Let E be a Hilbert space, A an injective operator in $\mathscr{L}(E)$, and $(p_n)_{n\in\mathbf{N}}$ an increasing sequence of positive numbers which satisfies* (3.22) *for some sequence of positive numbers $(q_n)_{n\in\mathbf{N}}$. Assume that there exists a vector $x \in E$ such that* (3.21) *holds and that*

$$\sup\{\|p_m^{-1}p_n^{-1}A^{*m}A^n x\| : m, n \in \mathbf{N}\} < \infty. \tag{3.23}$$

Then there exists a norm bounded sequence $(v_n)_{n=0}^{\infty} \subset E$ with $v_0 \neq 0$ such that (3.16) *and* (3.18) *hold.*

Proof. The Corollary follows from Lemma 3.3, by observing that the assumptions imply that the sequence $w_n = p_n^{-2}A^n x$, $n \in \mathbf{N}$ satisfies all the hypotheses of the lemma.

Proof of Theorem 1.4. By the remarks in the beginning of the proof of Theorem 1.1, we may assume that A and A^* are injective. Thus applying Corollary 3.4, we deduce from (1.11), (1.12), (1.13), that there exist a norm bounded sequence $(v_n)_{n=0}^{\infty} \subset E^*$ with $v_0 \neq 0$ such that (3.16) holds and

$$\|A^{*n}v_0\| = \mathrm{O}(n^k), \quad n \to \infty. \tag{3.24}$$

Consider the sequence $(y_n)_{n\in\mathbf{Z}} \subset E^*$ defined by

$$y_n = v_{-n}, \quad n < 0; \quad y_n = A^{*n}v_0, \quad n \geqslant 0.$$

It follows from the properties of $(v_n)_{n=0}^{\infty}$ and (3.24) that (0.1) and (1.4) hold for the sequence $(y_n)_{n\in\mathbf{Z}}$.

Since E is reflexive we obtain by a similar argument, using (1.14), that there exists a sequence $(x_n)_{n\in\mathbf{Z}} \subset E$ with $x_0 \neq 0$ such that (0.1) and (1.3) hold.

Thus A satisfies the hypotheses of part (a) (and part (b)) of Theorem 1.1 and the desired conclusion follows.

Proof of Theorem 1.5. By an argument similar to that in the proof of Theorem 1.4 we obtain from the hypotheses and Corollary 3.5 that A satisfies the hypotheses of part (a) of Theorem 1.1, and the assertion follows.

We conclude this section by proving the following extension of Theorem 1.6.

THEOREM 3.6. *Let A be an operator in $\mathscr{L}(E)$ and let $(p_n)_{n=0}^{\infty}$ be an increasing Beurling sequence which satisfies* (1.11) *and* (1.12). *Assume that there exists a vector $x \in E$ such that* (1.13) *holds, and a sequence $(u_n)_{n=0}^{\infty} \subset E$ which satisfies the hypotheses of Theorem* 1.6. *Then the conclusion of Theorem* 1.6 *holds for A.*

Proof. Using Corollary 3.4 we obtain from (1.11), (1.12) and (1.13) that there exists a norm bounded sequence $(v_n)_{n=0}^{\infty} \subset E^*$ which satisfies (3.16) and (3.18) with $q_n = n^k$, $n \in \mathbf{N}$.

Consider the sequences $(x_n)_{n \in \mathbf{Z}} \subset E$ and $(y_n)_{n \in \mathbf{Z}} \subset E^*$ defined by:

$$x_n = u_{-n}, \quad n < 0; \quad x_n = A^n u_0, \quad n \geqslant 0$$

and

$$y_n = v_{-n}, \quad n < 0; \quad y_n = A^{*n} v_0, \quad n \geqslant 0.$$

It follows from the properties of the sequences $(u_n)_{n=0}^{\infty}$ and $(v_n)_{n=0}^{\infty}$, and the fact that $(p_n)_{n=0}^{\infty}$ is a Beurling sequence, that the hypotheses of part (b) of Theorem 1.1 are satisfied for the operator A and the sequences $(x_n)_{n \in \mathbf{Z}}$ and $(y_n)_{n \in \mathbf{Z}}$. This completes the proof.

4. ADDITIONAL RESULTS

This section contains some additional results which can be proved by the methods of the previous section.

We begin with an extension of the result mentioned in the end of the introduction.

THEOREM 4.1. *Let E be a complex Banach space and let A be an operator in $\mathscr{L}(E)$ with spectrum $\sigma(A)$. Assume that there exists an increasing sequence of positive numbers $(p_n)_{n \in \mathbf{N}}$ and a vector $x \in E$, such that conditions* (1.11), (1.12) *and* (1.13) *hold.*

If $\sigma(A) \cap \mathbf{T}$ is countable, then A^* has an eigenvalue, and consequently, either A is a multiple of the identity operator, or A has a non trivial hyperinvariant subspace.

Proof. First we recall that if A^* has an eigenvalue λ and $A \neq \lambda I$ then the closure of the range of $A - \lambda I$ is a non trivial hyperinvariant subspace for A. Thus the second assertion of the theorem follows from the first.

If A^* is not injective then $\lambda = 0$ is an eigenvalue of A^*. Thus in what follows we shall assume that A^* is injective. We shall show that in this case A^* has an eigenvalue in $\mathbf{T}$.

As shown in the proof of Theorem 1.4, conditions (1.11), (1.12) and (1.13) imply by Corollary 3.4 that there exists a sequence $(y_n)_{n \in \mathbf{Z}} \subset E^*$ with $y_0 \neq 0$, such that (1.4) and the second part of (0.1) hold. Let G_2 be the function associated by (1.6) with this sequence.

We claim that the singularity set of G_2 is included in $\sigma(A) \cap \mathbf{T}$. Using (0.1), a simple computation with power series shows that

$$(A^* - zI)G_2(z) = y_0 \quad \forall z \in \mathbf{C} \setminus \mathbf{T}$$

and therefore

$$G_2(z) = \mathrm{R}(A^*, z)y_0, \quad \forall z \in \mathbf{C} \setminus (\mathbf{T} \cup \sigma(A)) \tag{4.1}$$

where $\mathrm{R}(A^*, z)$ denotes the resolvent of A^*. Since

$$z \to \mathrm{R}(A^*, z)y_0, \quad z \in \mathbf{C} \setminus \sigma(A^*)$$

is an analytic (E^* valued) function, we deduce from (4.1) that $\mathbf{T} \setminus \sigma(A^*)$ is disjoint from the singularity set of G_2, and since $\sigma(A) = \sigma(A^*)$, the claim is proved.

Consequently, since $\sigma(A) \cap \mathbf{T}$ is countable, also the singularity set of G_2 is countable, and since it is clearly closed and not empty (see the remarks in Section 2) it has an isolated point λ. We shall show that λ is an eigenvalue of A^*.

By Lemma 2.4 there exists a vector $x' \in E$ such that λ is a singular point of the function

$$z \to \langle x', G_2(z) \rangle, \quad z \in \mathbf{C} \setminus \mathbf{T}.$$

Consider the Beurling sequence $\rho = ((1 + |n|)^k)_{n \in \mathbf{Z}}$ (where k is the integer in (1.11)) and let S be the element of A_ρ^* defined by

$$\hat{S}(n) = \langle x', y_{-n} \rangle, \quad n \in \mathbf{Z}.$$

Noticing that by (1.6)

$$\widetilde{S}(z) = \langle x', G_2(z) \rangle, \quad z \in \mathbf{C} \setminus \mathbf{T}$$

we deduce that $\lambda \in \operatorname{sing}(\widetilde{S})$ and therefore by Lemma 2.3, $\lambda \in \Sigma(S)$.

Let Γ be an open arc on $\mathbf{T}$ which contains λ but no other singular point of G_2. Since $\lambda \in \Sigma(S)$ there exists a function f in A_ρ which is supported by Γ, such that $\langle f, S \rangle \neq 0$.

Consider the sequence $(y'_n)_{n \in \mathbf{Z}} \subset E^*$ defined by

$$y'_n = v(\mathrm{e}^{-int} f), \quad n \in \mathbf{Z};$$

(we use here the notation introduced in Section 3) that is,

$$y'_n = \sum_{j \in \mathbf{Z}} \hat{f}(n + j) y_j, \quad n \in \mathbf{Z}. \tag{4.2}$$

From (4.2) and the definition of S we deduce that $\langle x', y'_0 \rangle = \langle f, S \rangle$ and remembering that $\langle f, S \rangle \neq 0$, we conclude that $y'_0 \neq 0$. Using the fact that $(y_n)_{n \in \mathbf{Z}}$ satisfies the second part of (0.1), we obtain from (4.2) that

$$A^* y'_n = y'_{n+1}, \quad \forall n \in \mathbf{Z}. \tag{4.3}$$

Since $((1+|n|)^k)_{n\in\mathbf{Z}}$ is a Beurling sequence we obtain that $(y'_n)_{n\in\mathbf{Z}}$ satisfies (1.4).

Let G be the function associated with the sequence $(y'_n)_{n\in\mathbf{Z}}$ by (1.6). We claim that λ is the single singular point of G. For this, consider for every $t \in E$ the elements S_t and L_t of A_p^* defined by

$$\hat{S}_t(n) = \langle t, y_{-n}\rangle, \quad n \in \mathbf{Z} \tag{4.4}$$

and

$$\hat{L}_t(n) = \langle t, y'_{-n}\rangle, \quad n \in \mathbf{Z}. \tag{4.5}$$

It follows from (4.2), (4.4) and (4.5) that

$$\hat{L}_t(n) = \widehat{f\cdot S_t}(n), \quad \forall n \in \mathbf{Z},\ \forall t \in E$$

and therefore

$$L_t = f\cdot S_t, \quad \forall t \in E. \tag{4.6}$$

Observing that $\forall t \in E$

$$\tilde{S}_t(z) = \langle t, G_2(z)\rangle, \quad z \in \mathbf{C}\setminus\mathbf{T}$$

we deduce that $\operatorname{sing}(\tilde{S}_t)$ is included in the singularity set of G_2, $\forall t \in E$, and therefore by Lemma 2.3, $\Sigma(S_t)$ is also included in the same set, $\forall t \in E$. Since the only singular point of G_2 in Γ is λ, and f is supported by Γ, we infer from (4.6) that

$$\sum (L_t) \subset \{\lambda\}, \quad \forall t \in E. \tag{4.7}$$

Therefore, noticing that $\forall t \in E$

$$\tilde{L}_t(z) = \langle t, G(z)\rangle, \quad z \in \mathbf{C}\setminus\mathbf{T}$$

we obtain from (4.7), Lemma 2.3 and Lemma 2.4, that G has no singularity on $\mathbf{T}\setminus\{\lambda\}$. Since $y'_0 \neq 0$, it follows from the remarks in Section 2 that the singularity set of G is not empty, and consequently G has a single singularity at λ. Consequently, using (4.3), we conclude as in the proof of part (b) of Theorem 1.1 that $(A^* - \lambda I)^{k+1} y'_0 = 0$. Since $y'_0 \neq 0$, this shows that λ is an eigenvalue of A^*, and the proof is complete.

REMARK. It follows from [22, p. 79, Corollary 7.9] that if E is a Hilbert space and A a contraction in $\mathscr{L}(E)$ such that $\limsup_{n\to\infty} \|A^n x\| > 0$, for some $x \in E$, then the second conclusion of Theorem 4.1 still holds, if the hypothesis that $\sigma(A) \cap \mathbf{T}$ is countable is replaced by the weaker hypothesis that this set is of measure zero with respect to Lebesgue measure on $\mathbf{T}$.

Combining the methods of this paper with the methods of [1] we obtain the following extension of [1, Theorem 1].

THEOREM 4.2. *Let E be a complex Banach space and let A be an operator in $\mathscr{L}(E)$. Assume that there exist sequences $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$ with $x_0 \neq 0$ and $y_0 \neq 0$ such that* (0.1) *holds. Suppose that $(\|y_n\|)_{n\in\mathbf{Z}}$ is dominated by a Beurling sequence and that for some integer $k \geqslant 0$ and constant $c > 0$*

$$(4.8) \qquad \|x_n\| + \|y_n\| = O(n^k), \quad n \to \infty$$

and

$$(4.9) \qquad \|x_{-n}\| = O(\exp(cn^{1/2})), \quad n \to \infty.$$

Then either A is a multiple of the identity operators or A has a non trivial hyperinvariant subspace.

Proof. First notice that (4.8) and (4.9) imply that $(\|x_n\|)_{n\in\mathbf{Z}}$ is dominated by a Beurling sequence, and therefore if the function G_1, associated with the sequence $(x_n)_{n\in\mathbf{Z}}$ by (1.5), has more than one singularity, then the conclusion of the theorem follows from part (c) of Theorem 1.1. Thus in what follows we shall assume that G_1 has a single singularity λ, and as observed in the proof of part (a) of Theorem 1.1, it suffices to consider the case in which $\lambda = 1$.

As also noticed in previous proofs, if either A or A^* has an eigenvalue, then the assertion of the theorem follows. Thus we shall assume in the sequel that neither of these operators has an eigenvalue.

We shall show that these assumptions imply that there exist non zero vectors $u \in E$ and $v \in E^*$ such that

$$(4.10) \qquad \langle Bu, v\rangle = 0, \quad \forall B \in (A)',$$

and by Lemma 3.2, this will imply the conclusion of the theorem.

To prove (4.10), we consider as in [1, Section 2] the Banach algebra B_k which consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the unit disc $\mathbf{U} = \{z \in \mathbf{C}: |z| < 1\}$, such that $\sum_{n=0}^{\infty} |a_n|(1+n)^k < \infty$, the latter quantity serving as the norm of f in B_k (k denotes here the integer in (4.8)). It follows from (4.8) that for every function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in B_k, the series $\sum_{n=0}^{\infty} a_n x_n$ and $\sum_{n=0}^{\infty} a_n y_n$ converge in the norms of E and E^* respectively, to vectors which we shall denote as in Section 2, by $u(f)$ and $v(f)$.

It is clear that the mappings

$$f \to u(f), \quad f \in B_k \quad \text{and} \quad g \to v(g), \quad g \in B_k,$$

are bounded linear transformations from B_k into E and E^* respectively. By the same argument as in the proof of Lemma 3.1 we obtain that $\forall f, g \in B_k$

$$\langle Bu(f), v(g)\rangle = \langle Bx_0, v(fg)\rangle, \quad \forall B \in (A)'. \tag{4.11}$$

For every $a \geqslant 0$ we denote by f_a the analytic function

$$f_a(z) = (z-1)^m \exp a \frac{z+1}{z-1}, \quad z \in \mathbf{U},$$

where $m = 2k+3$. As noted in [1, Section 2], the functions f_a are in B_k, and for every $b \geqslant 0$, $\lim_{a\to b} f_a = f_b$ in the norm of B_k.

We shall show that for some $s > 0$

$$\langle BG_1(z), v(f_s)\rangle = 0, \quad \forall z \in \mathbf{C}\setminus\mathbf{T}, \ \forall B \in (A)'. \tag{4.12}$$

First we show that (4.12) implies (4.10).

Noticing that $G_1(0) = x_{-1}$, and using (0.1) we obtain from (4.12), by replacing B by BA, that

$$\langle Bx_0, v(f_s)\rangle = 0, \quad \forall B \in (A)'. \tag{4.13}$$

Let

$$\alpha = \inf\{a > 0 : \langle Bx_0, v(f_a)\rangle = 0, \quad \forall B \in (A)'\}.$$

Since $\lim_{a\to\alpha} f_a = f_\alpha$ in the norm of B_k, we obtain that

$$\langle Bx_0, v(f_\alpha)\rangle = 0, \quad \forall B \in (A)'. \tag{4.14}$$

Therefore if $v(f_\alpha)\neq 0$, (4.14) implies (4.10) with $u = x_0$ and $v = v(f_\alpha)$.

Suppose now that $v(f_\alpha) = 0$. It follows from (0.1) that $v(f_0) = (A^* - I)^m y_0$, and therefore by the assumption that A^* has no eigenvalues, $v(f_0)\neq 0$. Consequently, the assumption that $v(f_\alpha) = 0$ implies that $\alpha > 0$.

A similar argument shows that $u(f_0)\neq 0$, and since $\lim_{a\to 0} u(f_a) = u(f_0)$, there exists a number $0 < \beta < \alpha$ such that $u(f_\beta)\neq 0$. Noticing that $\forall a \geqslant 0$, $\forall b \geqslant 0$,

$$f_a(z) f_b(z) = (z-1)^m f_{a+b}(z), \quad z \in \mathbf{U}$$

and that by (0.1), for every polynomial p,

$$v(pf) = p(A^*) v(f), \quad \forall f \in B_k,$$

we obtain from (4.11) and the assumption that $v(f_\alpha) = 0$, that

(4.15) $$\langle Bu(f_\beta),\ v(f_{\alpha-\beta})\rangle = \langle B(A-I)^{m}x_0,\ v(f_\alpha)\rangle = 0, \quad \forall B \in (A)'.$$

Since $0 < \alpha - \beta < \alpha$, it follows from the definition of α that $v(f_{\alpha-\beta}) \neq 0$. Therefore (4.15) implies (4.10) with $u = u(f_\beta)$ and $v = v(f_{\alpha-\beta})$.

Thus it remains to prove (4.12). This is accomplished by the methods used in the proof of [1, Theorem 2], as follows.

First observe that (4.8) implies that

$$\|G_1(z)\| = O(|z|-1)^{-k-1}, \quad |z| \to 1+$$

and by estimates which are similar to those used in [1, Lemma 2(a)] we obtain that (4.9) implies that

$$\|G_1(z)\| = O\left(\exp\frac{d}{1-|z|}\right), \quad |z| \to 1-$$

for some constant $d > 0$. Therefore since $z = 1$ is the only singularity of G_1, we deduce from [1, Lemma 3] that

(4.16) $$\|G_1(z)\| = O(|1-z|)^{-2(k+1)}, \quad |z| \to 1+$$

and

(4.17) $$\|G_1(z)\| = O\left(\exp\frac{b}{|1-z|}\right), \quad |z| \to 1-$$

for some constant $b > 0$.

Following [1, Section 1] we denote for every function f which is analytic in $\mathbf{U}$ and every $w \in \mathbf{U}$ by $L_w f$ the analytic function in $\mathbf{U}$ defined, for $z \in \mathbf{U}\setminus\{w\}$, by $L_w f(z) = \dfrac{f(z)-f(w)}{z-w}$. As noted in [1], for $\forall f \in B_k$ and $\forall w \in \mathbf{U}$, also $L_w f \in B_k$.

Using (3.2) and comparing Taylor coefficients we obtain $\forall f \in B_k$, and $\forall B \in (A)'$ the identity

(4.18) $$\langle BG_1(z),\ v(f)\rangle - f(z)\langle BG_1(z),\ y_0\rangle = \langle Bx_0,\ v(L_z f)\rangle, \quad z \in \mathbf{U}.$$

From this point the proof proceeds exactly as the proof of [1, Theorem 2], by replacing the estimates (16) and (17) in [1] by the estimates (4.16) and (4.17), and the identity (5) in [1] by identity (4.18). We omit the details.

REMARK. The conclusion of Theorem 4.2 also holds if the hypotheses (4.8) and (4.9) are replaced by the hypotheses

(4.19) $$\|x_{-n}\| + \|y_{-n}\| = O(n^k), \quad n \to \infty$$

and

(4.20) $$\|x_n\| = O(\exp(cn^{1/2})), \quad n \to \infty.$$

This is proved in the same way as Theorem 4.2 by replacing the mappings $f \to u(f)$, $f \in B_k$ and $f \to v(f)$, $f \in B_k$ by the mappings $f \to u'(f)$, $f \in B_k$ and $f \to v'(f)$, $f \in B_k$ defined for $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in B_k by $u'(f) = \sum_{n=0}^{\infty} a_n x_{-n}$ and $v'(f) = \sum_{n=0}^{\infty} a_n y_{-n}$, and observing that $\forall m \in \mathbf{N}$,

$$A^m u'(f_0) = (I - A)^m x_0 \quad \text{and} \quad A^{*m} v'(f_0) = (I - A^*)^m y_0 .$$

Theorem 4.2 implies the following:

COROLLARY 4.3. *Let A be an invertible operator in $\mathscr{L}(E)$ and assume that there exist non zero vectors $x_0 \in E$ and $y_0 \in E^*$ such that for some integer $k \geqslant 0$ and some constant $c > 0$,*

$$(4.21) \qquad \|A^n x_0\| + \|A^{*n} y_0\| = O(n^k), \quad n \to \infty$$

and

$$(4.22) \qquad \|A^{-n} x_0\| + \|A^{*-n} y_0\| = O(\exp(cn^{1/2})), \quad n \to \infty.$$

Then the conclusion of Theorem 4.1 holds for A.

Proof. It follows from (4.21) and (4.22) that the sequences

$$x_n = A^n x_0, \ n \in \mathbf{Z} \quad \text{and} \quad y_n = A^{*n} y_0, \ n \in \mathbf{Z}$$

satisfy the hypotheses of Theorem 4.1.

A particular case of Corollary 4.3 is clearly:

COROLLARY 4.4 ([1, Theorem 1]). *Let A be an invertible operator in $\mathscr{L}(E)$ and assume that for some integer $k \geqslant 0$ and constant $c > 0$*

$$\|A^n\| = O(n^k), \quad n \to \infty$$

and

$$\|A^n\| = O(\exp(cn^{1/2})), \quad n \to -\infty.$$

Then the conclusion of Theorem 4.2 holds for A.

Finally, we obtain from Theorem 4.2 the following extension of Theorem 1.4 and [1, Proposition 5]:

THEOREM 4.5. *The conclusion of Theorem 1.4 holds if condition (1.11) in its hypotheses is replaced by the weaker condition*

$$(4.23) \qquad \limsup_{m \to \infty} \frac{p_{n+m}}{p_m} \leqslant K \exp(cn^{1/2}), \quad n \in \mathbf{N}$$

for some constants $c > 0$ and $K > 0$.

Proof. Using Corollary 3.4 with $q_n = K\exp(cn^{1/2})$, $n \in \mathbf{N}$, we obtain from (4.23) by an obvious modification of the proof of Theorem 1.4 that (0.1), (4.19) and (4.20) hold for some sequences $(x_n)_{n\in\mathbf{Z}} \subset E$ and $(y_n)_{n\in\mathbf{Z}} \subset E^*$ with $x_0 \neq 0$ and $y_0 \neq 0$. Thus the conclusion of the theorem follows from the remark following the proof of Theorem 4.2.

5. GENERALIZED BILATERAL WEIGHTED SHIFTS

In this section we apply Theorem 1.1 to obtain an extension of the results in [13] concerning the existence of hyperinvariant subspaces for certain bilateral weighted shifts. We shall consider a larger class of operators, which we call generalized bilateral weighted shifts. Before defining this class, we recall some definitions.

Let E be a Banach space. A pair of sequences $(e_n)_{n\in\mathbf{Z}} \subset E$ and $(e_n^*)_{n\in\mathbf{Z}} \subset E^*$ is called a *biorthogonal system*, if

$$\langle e_m, e_n^* \rangle = \delta_{m,n}, \quad \forall m, n \in \mathbf{Z}. \tag{5.1}$$

A sequence $(e_n)_{n\in\mathbf{Z}} \subset E$ is called *minimal* if there exists a sequence $(e_n^*)_{n\in\mathbf{Z}} \subset E^*$ such that $\{(e_n)_{n\in\mathbf{Z}}, (e_n^*)_{n\in\mathbf{Z}}\}$ forms a biorthogonal system.

It follows from the Hahn-Banach Theorem that a sequence $(e_n)_{n\in\mathbf{Z}} \subset E$ is minimal if and only if for every $j \in \mathbf{Z}$, e_j is not in the closed span of the set $\{e_n : n \in \mathbf{Z},\ n \neq j\}$.

A minimal sequence $(e_n)_{n\in\mathbf{Z}} \subset E$ is called *fundamental* if its closed span coincides with E.

If $(e_n)_{n\in\mathbf{Z}} \subset E$ is a fundamental sequence, there exists a unique sequence $(e_n^*)_{n\in\mathbf{Z}} \subset E^*$ such that (5.1) holds. We shall call $(e_n^*)_{n\in\mathbf{Z}}$ the *dual sequence* of $(e_n)_{n\in\mathbf{Z}}$.

A sequence $(e_n)_{n\in\mathbf{Z}} \subset E$ is called *normalized* if $\|e_n\| = 1$, $\forall n \in \mathbf{Z}$.

DEFINITION. Let E be a complex separable Banach space. An operator A in $\mathscr{L}(E)$ is called a *generalized bilateral weighted shift* (*GBWS*) if there exists a fundamental sequence $(e_n)_{n\in\mathbf{Z}} \subset E$, whose dual sequence $(e_n^*)_{n\in\mathbf{Z}}$ is norm bounded, and a sequence of complex numbers $(\lambda_n)_{n\in\mathbf{Z}}$ such that

$$Ae_n = \lambda_n e_{n+1}, \quad \forall n \in \mathbf{Z}. \tag{5.2}$$

The sequence $(\lambda_n)_{n\in\mathbf{Z}}$ is called the *weight sequence* of A. Clearly $\|\lambda_n\| \leqslant \|A\|$, $\forall n \in \mathbf{Z}$, hence $(\lambda_n)_{n\in\mathbf{Z}} \in \ell^\infty(\mathbf{Z})$.

REMARK. According to a result of Obsepian and Pełczyński (see [18], p. 44) every infinite dimensional separable Banach space contains a fundamental normalized sequence whose dual sequence is norm bounded.

We recall that an operator A in $\mathscr{L}(E)$ is called a *bilateral weighted shift* (*BWS*) if there exists a normalized Schauder basis $(e_n)_{n\in\mathbf{Z}} \subset E$, such that (5.2) holds for some sequence of complex numbers $(\lambda_n)_{n\in\mathbf{Z}}$. For information on bilateral weighted shifts we refer to [13] and the references given there.

Since the dual sequence of a normalized Schauder basis is norm bounded (cf. [18], p. 7), every BWS is also a GBWS.

It is not known whether or not every BWS has a non trivial hyperinvariant subspace. Partial results are proved in [13]. The answer is not known even for bilateral weighted shifts in a Hilbert space, which are defined with respect to an orthonormal basis. The main results known for such operators appear in [21].

REMARK. Every GBWS clearly has a non trivial invariant subspace. If A is a GBWS defined with respect to the sequence $(e_n)_{n\in\mathbf{Z}}$, then the closed span of the set $\{e_n : n\in\mathbf{Z},\ n\geqslant 1\}$ is a non trivial invariant subspace for A.

From Theorem 1.1 we obtain the following results.

THEOREM 5.1. *Let A be a* GBWS *in $\mathscr{L}(E)$ with weight sequence $(\lambda_n)_{n\in\mathbf{Z}}$. If $A\neq 0$ then each of the following two conditions implies that A has a non trivial hyperinvariant subspace.*

(I)

$$\inf_{n\in\mathbf{Z}}|\lambda_n| = 0. \tag{5.3}$$

(II) $\lambda_n\neq 0,\quad \forall n\in\mathbf{Z}$, *and*

$$\sum_{n=0}^{\infty}\frac{1}{1+n^2}\left(\left|\sum_{j=0}^{n-1}\log|\lambda_j|\right|+\left|\sum_{j=0}^{n}\log|\lambda_{-j}|\right|\right)<\infty. \tag{5.4}$$

Proof. Assume that A is defined with respect to the fundamental normalized sequence $(e_n)_{n\in\mathbf{Z}}$, with dual sequence $(e_n^*)_{n\in\mathbf{Z}}$. Using (5.1) and (5.2) we obtain that

$$\langle e_k, A^*e_n^* - \lambda_{n-1}e_{n-1}^*\rangle = 0,\quad \forall n\in\mathbf{Z},\ \forall k\in\mathbf{Z}$$

and therefore since $(e_n)_{n\in\mathbf{Z}}$ is fundamental,

$$A^*e_n^* = \lambda_{n-1}e_{n-1}^*,\quad \forall n\in\mathbf{Z}. \tag{5.5}$$

If $\lambda_j = 0$ for some $j\in\mathbf{Z}$, then A is not injective, and therefore if $A\neq 0$, $\ker(A)$ is a non trivial hyperinvariant subspace for A.

Assume that $\lambda_n\neq 0$, $\forall n\in\mathbf{Z}$ and consider the sequence of complex numbers $(\alpha_n)_{n\in\mathbf{Z}}$ defined by:

$$\alpha_0 = 1;\quad \alpha_n = \prod_{j=0}^{n-1}\lambda_j,\ n\in\mathbf{N};\quad \alpha_{-n} = \prod_{j=1}^{n}\lambda_{-j}^{-1},\ n\in\mathbf{N},$$

and the sequences $(x_n)_{n\in\mathbf{Z}}\subset E$ and $(y_n)_{n\in\mathbf{Z}}\subset E^*$ defined by

$$x_n = \alpha_n e_n,\quad n\in\mathbf{Z};\quad y_n = \alpha_{-n}^{-1}e_{-n}^*,\quad n\in\mathbf{Z}. \tag{5.6}$$

Using (5.2) and (5.5) we obtain that (0.1) holds for A and the sequences defined by (5.6).

Part I of the theorem follows from part (e) of Theorem 1.1, by observing that the sequences defined by (5.6) satisfy

$$\|x_{n-1}\| \, \|y_{-n}\| = |\lambda_n| \|e_{n+1}\| \, \|e_n^*\|, \qquad \forall n \in \mathbf{Z}.$$

We turn now to the proof of part II.

Noticing that for every complex number $a \neq 0$

$$\big|\log |a|\big| = \log^+ |a| + \log^+ \left|\frac{1}{a}\right|$$

and remembering that $\|e_n\| = 1$, $\forall n \in \mathbf{Z}$ and that $\|e_n^*\| \leqslant c$, $\forall n \in \mathbf{Z}$ for some constant $c > 0$, we obtain from (5.6) that

$$\sum_{n \in \mathbf{Z}} \frac{1}{1+n^2} (\log^+ \|x_n\| + \log^+ \|y_n\|) \leqslant$$

$$\leqslant \sum_{n \in \mathbf{Z}} \frac{1}{1+n^2} \big|\log |\alpha_n|\big| + \log c \sum_{n \in \mathbf{Z}} \frac{1}{1+n^2}$$

and therefore if (5.3) holds then (1.7a) is also satisfied. By part (I) we may assume that (5.3) is not satisfied, and it is easily verified that this assumption implies that (1.7b) holds for some constant $b > 0$. From (5.1) we see that x_0 is not contained in the closed span of the set $\{x_n : n \in \mathbf{Z},\ n \neq 0\}$ and that y_0 is not contained in the closed span of the set $\{y_n : n \in \mathbf{Z},\ n \neq 0\}$. Thus the desired conclusion follows from part (d) of Theorem 1.1.

COROLLARY 5.2. *Let A be a* GBWS *with weight sequence $(\lambda_n)_{n \in \mathbf{Z}}$. If $A \neq 0$ and*

$$\sum_{n \in \mathbf{Z}} \frac{\big|1 - |\lambda_n|\big|}{1 + |n|} < \infty, \tag{5.7}$$

then A has a non trivial hyperinvariant subspace.

Proof. If (5.3) holds the assertion follows from part I of Theorem 5.1. Thus we may assume that $\inf_{n \in \mathbf{Z}} |\lambda_n| > 0$. Remembering that $(\lambda_n)_{n \in \mathbf{Z}} \in \ell^\infty(\mathbf{Z})$, we obtain that there exists a constant $d > 1$ such that

$$d^{-1} \leqslant |\lambda_n| \leqslant d \qquad \forall n \in \mathbf{Z}.$$

Since

$$\sup \left\{ \left| \frac{\log |x|}{1 - |x|} \right| : d^{-1} \leqslant |x| \leqslant d \right\} < \infty,$$

we deduce from (5.7) that

$$\sum_{n=0}^{\infty} \frac{1}{1+n} (|\log|\lambda_n|| + |\log|\lambda_{-n}||) < \infty$$

and therefore

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} \left(\left| \sum_{j=0}^{n-1} \log|\lambda_j| \right| + \left| \sum_{j=1}^{n} \log|\lambda_{-j}| \right| \right) \leqslant$$

$$\leqslant \sum_{n=0}^{\infty} \left[(|\log|\lambda_n|| + |\log|\lambda_{-n}||) \sum_{j=n}^{\infty} \frac{1}{1+j^2} \right] \leqslant$$

$$\leqslant 5 \sum_{n=0}^{\infty} \frac{1}{1+n} (|\log|\lambda_n|| + |\log|\lambda_{-n}||) < \infty,$$

and the assertion follows from part II of Theorem 5.1.

Theorem 5.1 extends the result of [13, Theorem 4], where the existence of non trivial hyperinvariant subspaces is proved for bilateral weighted shifts which satisfy certain conditions which are more restrictive than (5.3).

For bilateral weighted shifts which are defined with respect to a rotation invariant basis (see definition in [13], p. 776), Corollary 5.2 coincides with [13, Corollary 2].

REMARKS 1. If $|\lambda_n| \geqslant 1$, $\forall n \in \mathbf{Z}$, then the proof of Corollary 5.2 shows that conditions (5.3) and (5.7) are equivalent.

2. Condition (5.7) in Corollary 5.2 can be replaced by the more general condition, that for some constant $a \geqslant 0$

$$(*) \qquad \sum_{n \in \mathbf{Z}} \frac{||\lambda_n| - a|}{1 + |n|} < \infty.$$

Indeed for $a = 0$, condition $(*)$ implies (5.3) and the assertion follows from part I of Theorem 5.1. If $a > 0$, the assertion follows from Corollary 5.2 by observing that $(a^{-1}\lambda_n)_{n \in \mathbf{Z}}$ is the weight sequence of operator $a^{-1}A$.

3. If A is a GBWS which is invertible, we obtain from (5.2) that

$$|\lambda_n| \geqslant \|A^{-1}\|^{-1}, \qquad \forall n \in \mathbf{Z}.$$

Thus a GBWS (in particular a BWS) which satisfies (5.3) is not invertible. If A is a BWS on a Hilbert space, which is defined with respect to an orthonormal basis, the converse is also true. That is (5.3) holds if and only if A is not invertible (see [21], Proposition 10). The existence of non trivial hyperinvariant subspaces for such an operator A was proved (see [21], p. 91 and p. 102) by using the fact that in this case every element in $(A)'$ is the limit, in the strong operator topology, of a sequence of polynomials in A. Thus part II of Theorem 5.1 gives an alternate proof of this fact, which does not use the structure of $(A)'$.

4. If A is a BWS on a Banach space which is defined with respect to an unconditional normalized basis (see definition in [18], p. 15), then it follows from [18, Proposition 1.c.7] that (5.3) holds if and only if A is not invertible. However as we shall show in Proposition 6.2 this is not true in general even for a BWS which is defined with respect to a rotation invariant basis. This will disprove the conjecture in [12, p. 543]. (See also the remark following Proposition 6.2.)

5. It is claimed in [13, p. 772 and p. 777] that if A is a non invertible BWS which is defined with respect to a rotation invariant basis, then every element in $(A)'$ is the limit, in the strong operator topology, of a sequence of polynomials in A. But the proof of this claim is not correct, since it is based on the assertion that it is proved in [12], that every element in $(A)'$ can be identified with a power series (in the sense described in [12]). But this fact is proved in [12, Theorem 4(2)] only for bilateral weighted shifts for which (5.3) holds, and as mentioned in Remark 4, there exist bilateral weighted shifts (even defined with respect to a rotation invariant basis) which are not invertible, but do not satisfy (5.3).

However, as Professor Herrero has shown us, the gap in the proof can be easily corrected, so that the result remains true also if (5.3) is not satisfied. Consequently, the assertion in the proof of [13, Theorem 4], that every non invertible BWS which is defined with respect to a rotation invariant basis has a non trivial hyperinvariant subspace, is correct also if (5.3) does not hold.

6. EXAMPLES, COMMENTS AND PROBLEMS

In this section we give some examples, make some comments and pose some problems.

We begin with an example of an operator which satisfies the hypotheses of part (a) of Theorem 1.1 and those of Theorem 1.2 but does not satisfy the hypotheses of Theorems 1.3 and 1.4.

Let $(n_j)_{j=0}^{\infty}$ be the sequence of integers defined by:

$$n_j = j(j+2),\ j\text{-even}, \quad \text{and} \quad n_j = (j+1)^2,\ j\text{-odd},$$

and consider the sequence $(\lambda_n)_{n\in\mathbf{Z}}$ defined by:

$$\lambda_n = 2^{(-1)^n} \quad \text{for } n_j \leqslant n < n_{j+1},\ j = 0, 1, \ldots,$$

and

$$\lambda_n = \lambda_{-n}^{-1} \quad \text{for } n < 0.$$

Let E be a complex Hilbert space with orthonormal basis $(e_n)_{n\in\mathbf{Z}}$, and let A be the invertible BWS defined on E by

$$Ae_n = \lambda_n e_{n+1}, \quad n \in \mathbf{Z}.$$

Using [21, p. 59 and p. 67] it is easy to see that

$$\|A^n\| = 2^{|n|}, \quad \forall n \in \mathbf{Z} \tag{6.1}$$

and therefore A does not satisfy the hypotheses of Theorems 1.3 and 1.4.

It also follows from [21, Theorem 5] and (6.1) that the spectrum of A consists of the anulus $\{z \in \mathbf{C} : 2^{-1} \leqslant |z| \leqslant 2\}$.

A simple computation shows that

$$\|A^n e_0\| \leqslant 1, \ \forall n \in \mathbf{Z} \quad \text{and} \quad \|A^{*n} e_0\| \leqslant 1, \ \forall n \in \mathbf{Z}$$

and therefore A satisfies the hypotheses of part (a) of Theorem 1.1 and Theorem 1.2.

Next we consider generalized bilateral weighted shifts which are defined on some spaces of functions on $\mathbf{T}$.

In what follows B will denote a homogeneous Banach space on $\mathbf{T}$, in the sense of [17, p. 14], and e_n will denote for every $n \in \mathbf{Z}$, the function on $\mathbf{T}$ defined by $e_n(w) := w^n$, $w \in \mathbf{T}$.

We shall also assume that $e_n \in B$, $\forall n \in \mathbf{Z}$ and that, $\|e_n\|_B = 1$, $\forall n \in \mathbf{Z}$, and also that $\forall f \in B$ and $\forall n \in \mathbf{Z}$, $e_n f \in B$ and $\|e_n f\|_B = \|f\|_B$.

Since $\|f\|_B \geqslant \|f\|_{L^1(\mathbf{T})}$, $f \in B$, the sequence of linear functionals e_n^*, $n \in \mathbf{Z}$ defined on B by

$$\langle f, e_n^* \rangle = \hat{f}(n), \quad f \in B$$

are in B^*, and $\|e_n^*\|_B = 1$, $\forall n \in \mathbf{Z}$.

It is clear that $\{(e_n)_{n \in \mathbf{Z}}, (e_n^*)_{n \in \mathbf{Z}}\}$ is a biorthogonal system, and according to [17, p. 15, Theorem 2.12], $(e_n)_{n \in \mathbf{Z}}$ is a fundamental sequence in B.

Examples of spaces which satisfy all the above conditions are the spaces $C(\mathbf{T})$ and $L^p(\mathbf{T})$ for $1 \leqslant p < \infty$. The sequence $(e_n)_{n \in \mathbf{Z}}$ is a Schauder basis in $L^p(\mathbf{T})$ for $1 < p < \infty$ but not in $L^1(\mathbf{T})$ and $C(\mathbf{T})$ (see [17], Chapter II).

In the rest of this section the term generalized bilateral weighted shift on B will mean a GBWS which is defined with respect to the sequence $(e_n)_{n \in \mathbf{Z}}$.

It follows easily from the hypotheses on B, that the generalized bilateral weighted shifts on B are the operators of the form $V \cdot P$, where V is the isometry defined on B by

$$Vf = e_1 \cdot f, \quad f \in B \tag{6.2}$$

and P is a Fourier multiplier on B, that is an operator in $\mathscr{L}(B)$ such that for some sequence of complex numbers $(p_n)_{n \in \mathbf{Z}}$

$$\widehat{Pf}(n) = p_n \hat{f}(n), \quad \forall n \in \mathbf{Z}, \ f \in B.$$

The weight sequence of $V \cdot P$ is clearly $(p_n)_{n \in \mathbf{Z}}$.

Let $M(\mathbf{T})$ denote the space of all (complex) Borel measures on $\mathbf{T}$. It is known (see [17], p. 39) that every measure $\mu \in M(\mathbf{T})$ defines a Fourier multiplier P on B by

$$Pf = \mu * f, \quad f \in B \tag{6.3}$$

where $\mu * f$ denotes the convolution of μ with f. The sequence $(p_n)_{n\in\mathbf{Z}}$ which corresponds to this multiplier is given by $p_n = \hat{\mu}(n)$, $n \in \mathbf{Z}$, where $\hat{\mu}(n)$ denotes the n-th Fourier-Stieltjes coefficient of μ.

From these remarks we see that every measure $\mu \in M(\mathbf{T})$ defines on B a GBWS A which is given by

$$Af = e_1(\mu * f), \quad f \in B. \tag{6.4}$$

The weight sequence of this shift is clearly $(\hat{\mu}(n))_{n\in\mathbf{Z}}$.

An immediate consequence of Theorem 5.1 and Corollary 5.2 is:

THEOREM 6.1. *Let μ be a measure in $M(\mathbf{T})$, $\mu \neq 0$, and consider the operator A defined on B by* (6.4). *Each of the following two conditions implies that A has a non trivial hyperinvariant subspace:*

$$\sum_{n\in\mathbf{Z}} \frac{\big||\hat{\mu}(n)| - 1\big|}{1 + |n|} < \infty; \tag{6.5}$$

$$\inf_{n\in\mathbf{Z}} |\hat{\mu}(n)| = 0. \tag{6.6}$$

It is known that every Fourier multiplier on the spaces $L^1(\mathbf{T})$ and $C(\mathbf{T})$ is of the form (6.3) for some measure $\mu \in M(\mathbf{T})$ (cf. [10], Section 16.32). Consequently the generalized bilateral weighted shifts on these spaces are exactly the operators of the form (6.4).

It is well known and easily verified, that the Fourier multipliers on $L^2(\mathbf{T})$ can be identified (in the obvious way) with $\ell^\infty(\mathbf{Z})$ (see [10, Section 16.1.2(4)]). This corresponds to the simple known fact that a sequence of complex numbers $(\lambda_n)_{n\in\mathbf{Z}}$ is the weight sequence of a BWS on a Hilbert space, which is defined with respect to an orthonormal basis, if and only if $(\lambda_n)_{n\in\mathbf{Z}} \in \ell^\infty(\mathbf{Z})$. On the other hand no characterization is known for the multipliers on the spaces $L^p(\mathbf{T})$ for $1 < p < \infty$, $p \neq 2$. Partial results can be found in [10, Section 16.4].

R. Gellar conjectured in [12, p. 543] that if A is a BWS on a Banach space, with weight sequence $(\lambda_n)_{n\in\mathbf{Z}}$ and

$$R_1 = \lim_{n\to\infty} \left(\sup_{m\in\mathbf{Z}} \prod_{j=m+1}^{m+n} |\lambda_j| \right)^{1/n}, \quad R_2 = \lim_{n\to\infty} \left(\inf_{m\in\mathbf{Z}} \prod_{j=m+1}^{m+n} |\lambda_j| \right)^{1/n}$$

(as noted in [12] these limits always exist) then the spectrum of A is the set $\{z \in \mathbf{C} : R_2 \leqslant |z| \leqslant R_1\}$.

Noticing that $R_2 = 0$ if and only if $\inf_{z \in \mathbf{Z}} |\lambda_n| = 0$, we see that the truth of the conjecture would imply that if $\inf_{n \in \mathbf{Z}} |\lambda_n| > 0$ then A is invertible. Thus the conjecture is disproved by the following:

PROPOSITION 6.2. *If* $1 < p < 2$, *there exists a* BWS A *on* $L^p(\mathbf{T})$ *whose weight sequence* $(\lambda_n)_{n \in \mathbf{Z}}$ *consists of real numbers such that* $\lambda_n \geqslant 1$, $\forall n \in \mathbf{Z}$, *but* A *is not invertible.*

Proof. Let $1 < p < 2$ be fixed. According to a theorem of Igari [16, Theorem 6], there exists a measure $\mu \in M(\mathbf{T})$ whose Fourier-Stieltjes coefficients are real and satisfy $\hat{\mu}(n) \geqslant 1$, such that the Fourier multiplier defined on $L^p(\mathbf{T})$ by (6.3) is not invertible. Consider the BWS A on $L^p(\mathbf{T})$ defined by (6.9) with this measure. Since the operator V (defined by (6.2)) is invertible, it follows that A is not invertible, and the Proposition is proved.

REMARKS. 1) After a first draft of the paper was circulated, Professor Herrero informed us that Gellar's conjecture was also disproved by R. Gellar and R. Silber in *Proc. Amer. Math. Soc.*, **61**(1976), 225—226. Their example relies on a Banach space introduced by Nakano, and is somewhat less "natural" than the example in Proposition 6.2.

2) The basis $(e_n)_{n \in \mathbf{Z}}$ in $L^p(\mathbf{T})$ for $1 < p < 2$, is clearly rotation invariant according to the definition in [13, p. 776]. This is also the case in the example of Gellar and Silber.

It is natural to consider the more general class of operators on the spaces $L^p(\mathbf{T})$, $1 \leqslant p < \infty$, which is obtained by replacing the function e_1 in (6.4) by an arbitrary $L^\infty(\mathbf{T})$ function. That is, for every function $\varphi \in L^\infty(\mathbf{T})$, and every measure $\mu \in M(\mathbf{T})$, one can consider the operator A on $L^p(\mathbf{T})$, $1 \leqslant p < \infty$, defined by

$$Af = \varphi(\mu * f), \quad f \in L^p(\mathbf{T}). \tag{6.7}$$

For these operators the existence of non trivial invariant subspaces is not known in general even when μ is the unit point mass concentrated at some $w \in \mathbf{T}$. In this case the corresponding operators are given by

$$Af = \varphi \cdot f_w, \quad f \in L^p(\mathbf{T}) \tag{6.8}$$

where $f_w(z) := f(w^{-1}z)$, $z \in \mathbf{T}$.

In the special case that $\varphi(e^{it}) = t$ for $0 \leqslant t < 2\pi$ and $p = 2$, the operators defined by (6.8) are called Bishop operators (see [20]). A. M. Davie [7] proved that for all w belonging to a certain subset of $\mathbf{T}$ of measure 1, the corresponding Bishop operator possesses a non trivial hyperinvariant subspace. Using the techniques of [7] (especially in the proofs of Lemma 2 and Lemma 3 there) one can show that these operators satisfy the hypotheses of part (c) of Theorem 1.1.

It would be interesting to find conditions on φ and μ, in addition to those given by Davie in [7] and by Theorem 6.1 of the present paper, which imply that the operators defined by (6.7) possess non trivial invariant (or hyperinvariant) subspaces. One such condition is that μ is absolutely continuous, or for $p = 2$, more generally, that $\lim_{n \to \pm\infty} \hat{\mu}(n) = 0$. It is easy to show that, in this case, the operator given by (6.7) is compact for every $\varphi \in L^{\infty}(\mathbf{T})$. Therefore by the well know result of Lomonosov (cf. [20], p. 158), these operators have non trivial hyperinvariant subspaces (if $\mu \neq 0$ and $\varphi \not\equiv 0$).

We conclude with the following:

PROBLEM. Let $\varphi = e_1 + e_{-1}$. Does the operator, defined by (6.8) with this function φ, have a non trivial invariant subspace for every $w \in \mathbf{T}$?

This problem is of some interest in view of the fact that one can show that if w is not a rooth of unity, then the two operators given by (6.8) with $\varphi = e_{-1}$ and $\varphi = e_1$ respectively, do not possess a common non trivial invariant subspace.

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Received April 21, 1981; revised December 20, 1982.