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DERIVATIONS ON CERTAIN CSL ALGEBRAS

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All derivations from a CSL algebra $\mathscr{A}$ into itself were shown by E. Christensen to be norm continuous [2]. For nest algebras they are all inner and in fact the nth Hochschild cohomology group $H^n(\mathscr{A}, \mathscr{B})$ is trivial for all n and all dual $\mathscr{A}$-modules $\mathscr{B}$ which are submodules of $\mathscr{L}(H)$ [7]. On the other hand, $H^1(\mathscr{A}, \mathscr{A})$ need not be zero even for certain width two CSL algebras (the intersection of two nest algebras with mutually commuting nests) [6]. However, if $\mathscr{A}$ is the finite intersection of nest algebras with mutually commuting and independent nests then $H^1(\mathscr{A}, \mathscr{A})$ is trivial [6].

In this note we investigate the obstruction to $H^1(\mathscr{A}, \mathscr{A}) = 0$ when $\mathscr{A}$ is an irreducible CSL algebra containing a purely atomic masa. The irreducible tridiagonal algebra $\mathscr{A}_\infty$ introduced in [4] as well as its finite analogies $\mathscr{A}_n$ are shown to be the only obstructions to $H^1(\mathscr{A}, \mathscr{A}) = 0$ for this class of CSL algebras.

A derivation δ of $\mathscr{A}$ into $\mathscr{A}$ is said to be *quasi-inner* if there exists bounded operator $T \in \mathscr{A}$ and a possibly unbounded operator S affiliated with the core $\mathscr{C}$ of $\mathscr{A}$ so that $\delta = \delta_{T+S}$ ($\delta_R(A) = AR - RA$). We show that whenever every derivation of $\mathscr{A}$ into $\mathscr{A}$ is quasi-inner then $H^1(\mathscr{A}, \mathscr{A}) \neq 0$ if and only if $\mathscr{A}$ is a tridiagonal algebra. If we denote $M_n = \text{span}\,(\mathscr{A} + \mathscr{A}^*)^n$, then it turns out that $H^1(\mathscr{A}, \mathscr{A}) \neq 0$ is also equivalent to the condition that for all n, M_n is not weakly dense in $\mathscr{L}(H)$. Moreover if $H^1(\mathscr{A}, \mathscr{A}) \neq 0$ because there exists a non quasi-inner derivation δ then it is shown that there exists a core projection P so that $P\mathscr{A}P$ is a canonical minimal finite dimensional CSL algebra $\mathscr{A}_n$ and $\delta \mid A_n$ is not inner. These canonical algebras $\mathscr{A}_n$ are isomorphic to certain finite dimensional matrix algebras and are closely related to the tridiagonal CSL algebra.

1. PRELIMINARIES

All Hilbert spaces will be separable. In this paper subspace lattices will all be commutative and are assumed to be closed in the strong operator topology. For a subspace lattice $\mathscr{L}$ we let $\mathscr{C}_{\mathscr{L}}$ denote the *core*, the von Neumann algebra generated by $\mathscr{L}$. A *nest* is a totally ordered subspace lattice while $\mathscr{L}$ is said to have *finite width* if $\mathscr{L}$ is the join of a finite number of mutually commuting nests. As usual we let Lat $\mathscr{A}$ denote the invariant projections for an algebra $\mathscr{A} \subseteq \mathscr{L}(H)$ and Alg $\mathscr{L}$ the operators invariant under a lattice $\mathscr{L}$. We say an operator T is *affiliated* with the core $\mathscr{C}_{\mathscr{L}}$ if there exist core projections P_n converging strongly to the identity so that the domain of T contains the range of each P_n, $P_nTP_n = TP_n$ and $P_nTP_n \in \mathscr{C}_{\mathscr{L}}$. We shall use P to denote either a projection P or its range space PH.

Core projections E and F are called *strictly ordered* ($E \ll F$) if $E\mathscr{L}(H)F \subseteq \mathscr{A}$. A CSL algebra with an atomic core is determined by the relation $\ll$ between its minimal core projections. Thus we describe the simple tridiagonal algebra $\mathscr{A}$ and the finite algebras $\mathscr{A}_n$ by describing the order $\ll$ on the minimal core projections.

EXAMPLE 1.1. To determine the algebra $\mathscr{A}_{2n}$ we let $E_1, \ldots, E_{2n}$ be the set of minimal core projections. Thus $E_i \perp E_j$ if $i \neq j$ and $I = E_1 + \ldots + E_{2n}$. Let $E_{2i-1} + E_{2i+1} \ll E_{2i}$ for $i = 1, \ldots, n$ and $E_1 \ll E_{2n}$ determine the CSL algebra $\mathscr{A}_{2n}$. An operator $A \in \mathscr{A}_{2n}$ precisely if $A: E_i \to E_i$ for i odd and $A: E_{2i} \to E_{2i-1} + E_{2i} + E_{2i+1}$ where E_1 is identified with E_{2n+1}. This algebra has non inner derivations for $n \geq 2$ and a direct calculation shows that $H^1(\mathscr{A}_{2n}, \mathscr{A}_{2n}) = \mathbf{C}$. Clearly $H^1(\mathscr{A}_2, \mathscr{A}_2)$ is trivial and $\mathscr{A}_4$ is the smallest algebra with non inner derivations (if $\dim E_i = 1$, then $\mathscr{A}_4$ is a four dimensional algebra on a four dimensional Hilbert space).

EXAMPLE 1.2. To determine the irreducible tridiagonal algebra $\mathscr{A}_\infty$ let $\{E_i\}_{i=1}^{\infty}$ be the minimal core projections. Thus $E_i \perp E_j$ if $i \neq j$ and $\sum_1^\infty E_i = I$. Let $E_i \ll E_i$ and $E_{2i-1} + E_{2i+1} \ll E_{2i}$ for $i = 1,2 \ldots$ determine the algebra $\mathscr{A}_\infty$. It was shown in [6] that $H^1(\mathscr{A}_\infty, \mathscr{A}_\infty) = \ell_\infty/s$ where $s = \left\{ \{a_i\} \in \ell_\infty \;\; \sum_{i=1}^{n} a_i \text{ is bounded} \right\}$. An operator $A \in \mathscr{A}_\infty$ if and only if $A: E_i \to E_i$ for i odd and $A: E_{2i} \to E_{2i-1} + E_{2i} + E_{2i+1}$.

A representation for a commutative subspace lattice was given by W. Arveson [1]. This representation of the subspace lattice is in terms of the increasing sets on a compact partially ordered measure space $(X, \leqslant, \mu)$. We consider the case when each of the minimal core projections is one dimensional. Then for the algebras $\mathscr{A}_{2n}$ and $\mathscr{A}_\infty$ given above the space X is discrete, μ is counting measure and each minimal core projection corresponds to a point x_i in X. Because the representation of $L \in \mathscr{L}$ is in terms of increasing sets (S is an increasing set if $x \in S$ and $y \geqslant x$ then $y \in S$) the order $\leqslant$ on X is the mirror image of the order $\ll$ on the corresponding minimal projections. Thus to determine the lattice

$\mathscr{L}_{2n} = \operatorname{Lat}\mathscr{A}_{2n}$ we have $X = \{x_1, \ldots, x_{2n}\}$ and $x_{2i} \leqslant x_{2i+1}$ and $x_{2n} \leqslant x_1$. For $\mathscr{L}_\infty = \operatorname{Lat}\mathscr{A}_\infty$ we simply have $x_{2i} \leqslant x_{2i+1}$. In each case $x_i \leqslant x_i$ for all i.

In this paper we shall denote the rank one operator $z \to (z, x)y$ by $x \otimes y$. Thus if $E \ll F$ are strictly ordered core projections for $\mathscr{A}$ with $x \in F$, $y \in E$ then $x \otimes y \in \mathscr{A}$.

Matricially in terms of $\{E_i\}$ these algebras are the following: $\mathscr{A} \in \mathscr{A}_\infty$ has the matrix representation

$$\begin{pmatrix} * & * & & & & & \\ 0 & * & 0 & & & & \\ & * & * & * & & & \\ & & 0 & * & 0 & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix}$$

while $\mathscr{A} \in \mathscr{A}_{2n}$ has the $2n \times 2n$ matrix representation

$$\begin{pmatrix} * & * & & & & & & * \\ 0 & * & 0 & & & & & \\ & * & * & * & & & & \\ & & 0 & * & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & * & * \\ & & & & & & 0 & * \end{pmatrix}.$$

In particular $\mathscr{A}_4$ consists of all matrices of the form

$$\begin{pmatrix} * & * & 0 & * \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}.$$

The algebra $\mathscr{A}_4$ is spanned by the core and the nilpotent of order 2 operators $A_1 = e_2 \otimes e_1$, $A_2 = e_2 \otimes e_3$, $A_3 = e_4 \otimes e_3$, and $A_4 = e_4 \otimes e_1$. If $\delta : \mathscr{A}_4 \to \mathscr{A}_4$ is determined by $\delta(A_i) = \alpha_i A_i$ and $\delta \mid \mathscr{C} = 0$, then δ is an inner derivation precisely if $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$. Thus, as we noted above, $H^1(\mathscr{A}_4, \mathscr{A}_4) = \mathbf{C}$.

Let $\mathscr{A}_\infty$ be the tridiagonal algebra where each E_i is one dimensional and is the span of e_i. Let A_n and B_n be the nilpotent of order two operators $A_n = e_{2n} \otimes e_{2n-1}$ and $B_n = e_{2n} \otimes e_{2n+1}$. If we set $\delta(A_n) = -A_n$ and $\delta(B_n) = B_n$ and let $\delta \mid \mathscr{C} = 0$, then δ extends to a derivation on $\mathscr{A}_\infty$. For δ to be inner would require an operator T such that $Te_i = ie_i$. Such an operator is not bounded but is affiliated with the core $\mathscr{C}$. Hence δ is a quasi-inner derivation. It is shown in [6] that all derivations on $\mathscr{A}_\infty$ are quasi-inner.

In [4] the simple tridiagonal algebra was introduced wherein as above each E_i was a minimal core projection. In general a *tridiagonal algebra* $\mathscr{A}$ is defined as having core projections $\{F_i\}$ so that for all $A \in \mathscr{A}$, $AF_i \subseteq F_{i-1} + F_i + F_{i+1}$. The algebra $\mathscr{A}_\infty$ is "maximal" with respect to $\{E_i\}$ in the sense that it is contained in no larger tridiagonal algebra w.r.t. $\{E_i\}$. Generally many different algebras can be tridiagonal with respect to a family of projections $\{E_i\}$.

Recall that for projections E, F in the core of a CSL algebra $\mathscr{A}$, $E \ll F$ means $E\mathscr{A}F = E\mathscr{L}(H)F$. If E and F are minimal core projections then it is easy to see that $E\mathscr{A}F = 0$ or $E \ll F$. This observation will be useful.

LEMMA 1.3. *Let $\mathscr{A}$ be an* CSL *algebra and E, F distinct minimal core projections. Then $E\mathscr{A}F = 0$ or $E\mathscr{A}F = E\mathscr{L}(H)F$ and $E\mathscr{A}F = 0$ or $F\mathscr{A}E = 0$.*

Proof. Let E' be the projection on the space $[E\mathscr{A}FH]$ which is the closed span of the vectors $EAFx$ for $A \in \mathscr{A}$ and $x \in H$. If $E\mathscr{A}F \neq 0$ then $E' \neq 0$. If $B \in \mathscr{D} = \mathscr{C}' = \mathscr{A} \cap \mathscr{A}^*$ then $BEAFx = EBAFx \in [E\mathscr{A}FH]$. Thus $E' \in \mathscr{D}' = \mathscr{C}$ and since E is minimal in $\mathscr{C}$ we have $E = E'$. Since E and F are minimal core projections $E\mathscr{L}(H)E$ and $F\mathscr{L}(H)F$ are contained in $\mathscr{D}$. Thus $E\mathscr{A}F = E\mathscr{L}(H)F$. If $E\mathscr{A}F \neq 0$ and $F\mathscr{A}E \neq 0$, then $\mathscr{A} \supseteq (E + F)\mathscr{L}(H)(E + F)$ and $E \notin \mathscr{C}$. ▨

REMARK 1.4. The lemma is true if $\mathscr{A}$ is replaced by $\mathscr{M}_n = (\mathscr{A} + \mathscr{A}^*)^n = \text{span}\left\{\prod_{i=1}^{n} (A_i + B_i^*) : A_i, B_i \in \mathscr{A}\right\}$. That is, if $E\mathscr{M}_nF \neq 0$ then $E\mathscr{M}_nF$ is dense in $E\mathscr{L}(H)F$ in the weak operator topology. Equality need not be achieved since $\mathscr{M}_n$ is not necessarily a weakly closed set.

While E. Christensen showed that any derivation of a CSL algebra $\mathscr{A}$ into $\mathscr{L}(H)$ is norm continuous it is sometimes useful to know about weaker continuity.

PROPOSITION 1.5. *Let δ be a derivation from a* CSL *algebra with completely atomic core into $\mathscr{L}(H)$. If $\mathscr{A}_1$ is the unit ball of $\mathscr{A}$, then $\delta \mid \mathscr{A}_1$ is continuous in the weak operator topology.*

Proof. Let $\mathscr{C}$ be the core of $\mathscr{A}$ and $\mathscr{D} = \mathscr{C}'$ be the diagonal of $\mathscr{A}$. Then $\delta \mid \mathscr{D}$ is a derivation from $\mathscr{D}$ into $\mathscr{L}(H)$ and as such is inner. Thus there exists a bounded operator T in $\mathscr{L}(H)$ so that $\delta - \delta_T$ is a derivation which is zero on the von Neumann algebra $\mathscr{D}$. Since δ_T is continuous in the weak operator topologies we may assume $\delta \mid \mathscr{D} = 0$.

Since $\delta \mid \mathscr{D} = 0$ we have $\delta(ABC) = A\delta(B)C$ whenever $A, C \in \mathscr{D}$. Let $A_n \to A$ weakly in $\mathscr{A}_1$ and let P_F be the projection on a finite set of vectors $\{x_i\}_{i=1}^n$ where $P_F \in \mathscr{D}$. Since δ is norm continuous we have that $\{\|\delta(A_n)\|\}$ is bounded and $P_F\delta(A_n)P_F = \delta(P_FA_nP_F) \to \delta(P_FAP_F) = P_F\delta(A)P_F$. By our assumption $\bigvee P_FH$, where P_F is as above, is dense in H. Hence $\delta(A_n) \to \delta(A)$ weakly. ▨

2. $(\mathscr{A}+\mathscr{A}^*)^n$

We show in this section that for an irreducible reflexive algebra $\mathscr{A}$ containing a completely atomic masa, that $\mathscr{A}$ is tridiagonal if and only if for all n the linear span of $(\mathscr{A}+\mathscr{A}^*)^n$ is not dense in $\mathscr{L}(H)$ in the weak operator topology. The following lemma is the obvious half of that result.

LEMMA 2.1. *Let $\mathscr{A}_\infty$ be the irreducible tridiagonal algebra with respect to the minimal core projections $\{E_i\}$. Then for all n, $\mathscr{M}_n$ is not weakly dense in $\mathscr{L}(H)$.*

Proof. If T is in $\mathscr{A}\cap\mathscr{A}^*=\mathscr{D}$, then $T\mathscr{M}_n\subseteq\mathscr{M}_n$ as well as $\mathscr{M}_nT\subseteq\mathscr{M}_n$. Since $\mathscr{C}\subseteq\mathscr{A}\cap\mathscr{A}^*$ then $E_j\mathscr{M}_nE_i\subseteq\mathscr{M}_n$ for all i,j. We show that any non zero rank one operator $y\otimes x$ with $x\in E_1$ and $y\in E_{n+2}$ cannot be in the closure of $\mathscr{M}_n$. If it were then there exists a net A_α in $\mathscr{M}_n$ with $A_\alpha\to y\otimes x$ weakly. Since $x\otimes x$ and $y\otimes y\in\mathscr{D}$ then $(x\otimes x)A_\alpha(y\otimes y)\in\mathscr{M}_n$ as well. We then have that $(x\otimes x)A_\alpha(y\otimes y)\to y\otimes x$. Since $y\otimes x$ is nonzero we conclude that for some α $(x\otimes x)A_\alpha(y\otimes y)\neq 0$ and hence $y\otimes x\in\mathscr{M}_n$. That $y\otimes x$ cannot be in $\mathscr{M}_n$ follows from the fact that $E_j\mathscr{A}E_i=0$ if $j\neq i, i\pm 1$. ▨

An algebra $\mathscr{A}$ is tridiagonal if there exists a family of core projections $\{E_i\}_{i=1}^\infty$ so that $\mathscr{A}$ is contained in a maximal tridiagonal algebra $\mathscr{A}_\infty$ determined by $\{E_n\}$ [6]. If $\mathscr{A}$ is also irreducible then the $*$-algebra generated by $\cup\mathscr{M}_n$ where $\mathscr{M}_n=(\mathscr{A}+\mathscr{A}^*)^n$ is dense in $\mathscr{L}(H)$. Moreover each $\mathscr{M}_n\subseteq(\mathscr{A}_\infty+\mathscr{A}_\infty^*)^n$. Thus we have established the following corollary.

COROLLARY 2.2. *If $\mathscr{A}$ is an irreducible tridiagonal algebra then for all n, $\mathscr{M}_n$ is not weakly dense in $\mathscr{L}(H)$.*

THEOREM 2.3. *Let $\mathscr{A}$ be an irreducible reflexive algebra containing a completely atomic masa. If for all n, $\mathscr{M}_n$ is not dense in $\mathscr{L}(H)$, then $\mathscr{A}$ is a tridiagonal algebra with respect to a family of core projections.*

Proof. Let $E_0, E_1, \dots$ be the minimal core projections. Assume that for some n the subspace $[\mathscr{M}_nE_0H]=H$. Recall that $\mathscr{D}=\mathscr{C}'$ satisfies $\mathscr{D}(\mathscr{M}_n)=$ $=\mathscr{M}_n\mathscr{D}=\mathscr{M}_n$. Since $[\mathscr{M}_nE_0H]=H$ we have $E_i\mathscr{M}_nE_0\neq 0$ for all i and by (1.4), $E_i\mathscr{M}_nE_0$ is dense in $E_i\mathscr{L}(H)E_0$. Let $x_i\in E_i$, $y\in E_j$ and $z\neq 0$ in E_0. Then $z\otimes x$ and $z\otimes y\in\mathscr{M}_n$. Since $\mathscr{M}_n=\mathscr{M}_n^*$ we have $y\otimes z\in\mathscr{M}_n$ as well. So $y\otimes x=$ $=(z\otimes x)(y\otimes z)\in\mathscr{M}_{2n}$. Thus $\mathscr{M}_{2n}$ is dense in $\mathscr{L}(H)$ in the weak operator topology. This is a contradiction and hence $[\mathscr{M}_nE_0H]\neq H$ for all n.

Let F_n be the orthogonal core projection $[\mathscr{M}_nE_0H]$. Since $\vee F_nH$ reduces $\mathscr{A}$ we have $\vee F_nH=H$. If $E_iF_n\neq 0$ then as in the paragraph above $E_i\leqslant F_n$. Clearly $(\mathscr{A}+\mathscr{A}^*)F_nH\subseteq F_{n+1}H$ so that $F_{n+1}^\perp F_n=0$. Let $G_n=F_n\ominus F_{n-1}$ with $G_0=E_0=F_0$. We have $\sum G_n=I$ and $(\mathscr{A}+\mathscr{A}^*)G_n\perp G_{n+i}$ for $i\geqslant 2$. Since $\mathscr{A}+\mathscr{A}^*$ is selfadjoint we conclude that $(\mathscr{A}+\mathscr{A}^*)G_n\perp G_k$ for $|k-n|\geqslant 2$. Thus by definition $\mathscr{A}$ is tridiagonal. ▨

REMARK. The irreducibility of $\mathscr{A}$ is an unnecessary hypothesis in (2.3) however the proof depends on the existence of minimal core projections.

It can happen and is more usual that for an irreducible CSL algebra $\mathscr{A}$ the linear space $\mathscr{M}_n$ is dense for some n. If for some n we have $\mathscr{M}_n$ is dense in $\mathscr{M}_{n+1}$ then in fact $\mathscr{M}_n$ is dense in $\mathscr{L}(H)$.

LEMMA 2.4. *If $\mathscr{M}_n$ is weakly dense in $\mathscr{M}_{n+1}$ then $\mathscr{M}_n$ is weaky dense in $\mathscr{L}(H)$.*

Proof. We will show by induction that $\mathscr{M}_n$ is weakly dense in $\mathscr{M}_{n+k}$ for all k and hence $\mathscr{M}_n$ is weakly dense in $\bigcup_{k=1}^{\infty} \mathscr{M}_k$. However $\bigcup_{k=1}^{\infty} \mathscr{M}_k$ is dense in the von Neumann algebra generated by $\mathscr{A}$. Since $\mathscr{A}$ is irreducible $\bigcup_{k=1}^{\infty} \mathscr{M}_k$ is weakly dense in $\mathscr{L}(H)$.

We only need to show that if $\mathscr{M}_n$ is dense in $\mathscr{M}_{n+1}$ then $\mathscr{M}_{n+1}$ is dense in $\mathscr{M}_{n+2}$. Let $T \in \mathscr{M}_{n+2}$ and let $x, y \in H$ be given. There are operators $S_i \in (\mathscr{A} + \mathscr{A}^*)^{n+2}$, $1 \leqslant i \leqslant N$ so that $T = \sum_{i=1}^{N} S_i$. Each S_i may be considered the product of $n+2$ or fewer operators from $\mathscr{A} \cup \mathscr{A}^*$. Consider only any S_{i_0} which is the product of $n+2$ operators in $\mathscr{A} \cup \mathscr{A}^*$. By the assumption that $\mathscr{M}_n$ is dense in $\mathscr{M}_{n+1}$ the first $n+1$ operators in the product S_{i_0} can be approximated by n operators in $\mathscr{A} \cup \mathscr{A}^*$. If T_{i_0} is the new product with the remaining operator in the product S_{i_0} we can take $|\langle (S_{i_0} - T_{i_0})x, y\rangle| < \varepsilon/N$. If we do this for all the S_i, $1 \leqslant i \leqslant N$, where necessary, we have each T_i and hence $T_0 = \sum_{i=1}^{N} T_i \in \mathscr{M}_{n+1}$. As they were chosen we see that $|\langle (T - T_0)x, y\rangle| < \varepsilon$. ▨

An element L_0 in a subspace lattice $\mathscr{L}$ is called *comparable* if $M \in \mathscr{L}$ implies $M \leqslant L_0$ or $L_0 \leqslant M$. In particular every member of a nest is a comparable member of the nest.

LEMMA 2.5. *Let $\mathscr{A}$ be any CSL algebra and $\mathscr{L} = \operatorname{Lat}\mathscr{A}$. Assume $\mathscr{L}$ contains an infinite rank comparable element L whose complement $L^\perp$ is also an infinite rank projection. Then $\mathscr{M}_2 = \mathscr{L}(H)$.*

Proof. Let $T \in \mathscr{L}(H)$ and $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ with respect to the decomposition of H determined by L and $L^\perp$. We identify $L^\perp$ with L and let a be a constant so that

$$T - \begin{pmatrix} 0 & aI \\ aI & 0 \end{pmatrix} = \begin{pmatrix} T_1 & S_2 \\ S_3 & T_4 \end{pmatrix}$$

where S_2 and S_3 are invertible in $\mathscr{L}(L)$. Then

$$T - \begin{pmatrix} 0 & aI \\ aI & 0 \end{pmatrix} = AB$$

where $A = \begin{pmatrix} 0 & S_2 \\ S_3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} I & S_3^{-1}T_4 \\ S_2^{-1}T_1 & I \end{pmatrix}$. Finally

$$T = AB + \begin{pmatrix} 0 & aI \\ aI & 0 \end{pmatrix}$$

where $A, B, \begin{pmatrix} 0 & aI \\ aI & 0 \end{pmatrix} \in \mathscr{A} + \mathscr{A}^*$. ▨

Since every member of a nest is a comparable element, the above shows that for "most" nests $\mathscr{M}_2 = \mathscr{L}(H)$. A result (unpublished) of D. Larson shows that $\mathscr{M}_2 = \mathscr{L}(H)$ for all nests. The only case not included above is when $\mathscr{N}_0 = \{0 = P_0 < P_1 < \ldots < I\}$ and each P_i is finite rank.

We show this for completeness.

LEMMA 2.6. (Larson). *For a nest algebra* $\mathscr{M}_2 = \mathscr{L}(H)$.

Proof. Let $\mathscr{N}_0$ be as above and let $T \in \mathscr{L}(H)$. Let $T + aI = A + iB$ so that A and B are strictly positive. We will show $A \in \mathscr{M}_2$. Let $A_1 = A^{1/2}$. Since A_1 is invertible and P_iH is finite dimensional, if A_1 maps P_iH onto $\hat{\mathscr{M}}_i$ then $\dim(P_i - P_{i-1})H = \dim(\hat{\mathscr{M}}_i \ominus \hat{\mathscr{M}}_{i-1})$ for all i.

Let S be an operator defined by induction so that $S: (\hat{\mathscr{M}}_i \ominus \hat{\mathscr{M}}_{i-1}) \to (P_i - P_{i-1})H$ is an isometry. Clearly $\bigvee \hat{\mathscr{M}}_i = H$ so S is an isometry on H.

Moreover $SA_1 \in \mathscr{A} = \mathrm{Alg}\mathscr{N}_0$ so that $A = A_1A_1 = (SA_1)^*SA_1 \in \mathscr{M}_2$. ▨

REMARK. That $\mathscr{M}_1$ is dense in $\mathscr{L}(H)$ in the weak operator topology for a nest algebra was first shown by R. Loebl and P. Muhly [8], with further proofs and generalizations given in [3] and [5]. For the irreducible tridiagonal algebra $\mathscr{A}_\infty$ (2.1) shows that $\mathscr{M}_n$ is not dense for any n and we note that $\mathscr{A}_\infty$ is the intersection of two nest algebras with commuting nests. Furthermore the denseness of $\mathscr{M}_1$ in $\mathscr{L}(H)$ for a CSL algebra is in fact equivalent to $\mathscr{A}$ being a nest algebra ((2.8) in [5]).

3. DERIVATIONS

For any derivation δ of a CSL algebra $\mathscr{A}$ into $\mathscr{A}$ the restriction to the diagonal algebra $\mathscr{D} = \mathscr{A} \cap \mathscr{A}^*$ is inner since $\mathscr{D}$ is a type I von Neumann algebra. Thus we may substract that inner part and hence have a derivation of $\mathscr{A}$ into $\mathscr{A}$ which annihilates $\mathscr{D}$. These derivations in turn are constant on the set $E\mathscr{L}(H)F$ where E, F are minimal core projections.

Lemma 3.1. *Let E, F be two minimal projections with $E \ll F$. Let δ be a derivation of $\mathscr{A}$ into $\mathscr{A}$ with $\delta \mid \mathscr{D} = 0$. There exists a constant c so that $\delta(A) = cA$ for all $A \in E\mathscr{A}F$.*

Proof. If E and F are projections with $\dim E = \dim F$ let S be a partial isometry of F onto E. Otherwise let S be a partial isometry of F into E or from a subspace of F onto E whichever is possible. Every $A \in E\mathscr{A}F$ can be factored through S in the sense that $A = SA_1$ where $A_1 \in F\mathscr{L}(H)F$ or $A = A_1S$ where $A_1 \in E\mathscr{L}(H)E$. In either case $S \in \mathscr{A}$ and $A_1 \in \mathscr{D}$. Now $\delta(A) = \delta(A_1S) = A_1\delta(S) + \delta(A_1)\, S = A_1\delta(S)$ or in the other case $\delta(A) = \delta(SA_1) = \delta(S)A_1$. In either case we are done if we can show $\delta(S) = cS$ for some constant c.

Since $ESF = S$ we have $\delta(S) = E\delta(S)F$. To show $\delta(S) = cS$ let $e \in F$, $f \in E$ with f in the range of S and e in the domain of S. Since $e \otimes f \in \mathscr{A}$ we have $\delta(e \otimes f) = f \otimes f\, \delta(e \otimes f)\, e \otimes e = c\, e \otimes f$. Hence $\langle \delta(S)e, f\rangle = \langle cSe, f\rangle$. If f_1 and e_1 are any other vectors in the range and domain respectively of S then $\delta(e_1 \otimes f_1) = \delta((f \otimes f_1)(e \otimes f)(e_1 \otimes e)) = ce_1 \otimes f_1$ and $\langle(\delta(S) - cS)e_1, f_1\rangle = 0$. Since the projections on the domain and range of S are in $\mathscr{D}$ we conclude that $\delta(S) = cS$. ▨

Let δ be a derivation on $\mathscr{A}$ for which $\delta \mid \mathscr{D}$ is zero. Then δ is quasi-inner if there exists a (possibly unbounded) operator T affiliated with the core $\mathscr{C}$ for which $\delta = \delta_T$. This means there exist core projections $E_n \uparrow I$ so that $\bigvee E_nH$ is contained in the domain of T and $TE_n = E_nT \in \mathscr{C}$. Moreover $\delta(A) = \delta_T(A)$ whenever $A \in \mathscr{A}$ and $A = E_nAE_n$. Thus δ agrees with δ_T on $E_n\mathscr{A}E_n$ and by (1.5) this determines δ.

Proposition 3.2. *Assume for some n, $\mathscr{M}_n$ is dense in $\mathscr{L}(H)$ in the weak operator topology. Every quasi-inner derivation of $\mathscr{A}$ into $\mathscr{A}$ is inner.*

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be minimal core projections for $\mathscr{C}$ and let δ be given. Let δ be quasi-inner. We may assume that $\delta \mid \mathscr{D} = 0$ and $\delta = \delta_T$ where T is a (possibly unbounded) operator affiliated with the core. Thus the domain of T contains $\bigvee G_iH$ where G_i are core projections. Since $\{E_i\}$ are the minimal core projections the domain of T contains $\bigvee E_iH$.

Let $R_1 = \{i \mid E_i\mathscr{M}_1E_1 \neq 0\}$. For $i \in R_1$ either $E_i\mathscr{A}E_1$ or $E_1\mathscr{A}E_i$ is not zero. Our goal is to define a bounded operator S so that $\delta = \delta_S$. Let $S \mid E_1 = 0$. Let $i \in R_1$ and assume $A \in E_i\mathscr{A}E_1$ is not zero. By (3.1) $\delta(A) = c_iA$. Thus we define $S \mid E_i = -c_iE_i$. Then $AS - SA = E_i(AS - SA)E_1 = c_iA$. If however $E_1\mathscr{A}E_i \neq 0$ we similarly apply (3.1) to define $S \mid E_i$. If $i, j \in R_1$ and $A \in E_i\mathscr{A}E_j$ is not zero then using the derivation property of δ one can check that $\delta(A) = \delta_S(A)$. Hence δ_S extends to a map on $F\mathscr{M}_nF$ for $F = E_1 + \sum_{i=1}^{k} E_{n_i}$ for $\{n_i\}_{i=1}^{k} \subseteq R_1$ which has the derivation property $\delta_S(AB) = A\delta_S(B) + \delta_S(A)B$ whenever

$A, B \in F\mathcal{M}_nF$. Thus $(\delta - \delta_S) \mid F\mathcal{M}_nF = 0$. Since this is true for any set $\{n_i\}_{i=1}^n \subseteq R_1$ then δ and δ_S agree on $G_1\mathcal{A}G_1$ where $G_1 = \bigvee E_i$, $i \in R_1$. Now SG_1 is a core operator so $\|SG_1\| = \sup|c_i|$, $i \in R_1$. However $|c_i| = \dfrac{\|\delta(A)\|}{\|A\|}$ where $0 \neq A \in E_i\mathcal{A}E_1$ or $E_1\mathcal{A}E_i$. Thus $\|SG_i\| \leqslant \|\delta\|$.

Let $R_2 = \{i \mid E_i\mathcal{M}_2E_1 \neq 0\}$. If $i \in R_2 - R_1$ then there exists a $j \in R_1$ and a nonzero $A \in \mathcal{A}$ so that $A \in E_i\mathcal{A}E_j$ or $A \in E_j\mathcal{A}E_i$. In case $0 \neq E_j\mathcal{A}E_i$ by (3.1) there is a constant d_i so that $\delta(A) = d_iA$ for all $A \in E_j\mathcal{A}E_i$. Let $c_i = d_i + c_j$ and define $SE_i = c_iE_i$. We must show that SE_i is well defined. Hence let $0 \neq B \in E_k\mathcal{A}E_i$ for some $k \in R_1$. By (3.1) $\delta(B) = \alpha B$ and we need show that $\alpha + c_k = d_i + c_j$. However $AB^* \in E_j\mathcal{M}_2E_k$ and since δ is quasi-inner, δ extends to a derivation on $F_0\mathcal{L}(H)F_0$ where $F_0 = E_i + E_j + E_k$. Hence $\delta(AB^*) = A\delta(B^*) + \delta(A)B^* = (-\alpha + d_i)AB^*$. On the other hand $\delta_S(AB^*) = AB^*S - SAB^* = (c_k - c_j)AB^*$ and $\delta_S(AB^*) = \delta(AB^*)$ since $j, k \in R_1$. Thus $c_j + d_i = \alpha + c_k$ and c_i is well defined in this case. A similar calculation shows c_i is well defined in case $E_i\mathcal{A}E_k \neq 0$ for $k \in R_1$ and for the two symmetric cases when $E_i\mathcal{A}E_j \neq 0$. By (1.3) these cases are mutually exclusive. Now as before let $F = \sum_{i=1}^{k} E_{n_i}$ where $\{n_i\}_{i=1}^k \subseteq R_2$. Clearly δ_S extends to a map of $F\mathcal{M}_nF$ into $F\mathcal{M}_nF$ so that if A, B and AB are in $F\mathcal{M}_nF$ then $\delta_S(AB) = A\,\delta_S(B) + \delta_S(A)B$. In particular, since δ is a derivation one can check that δ and δ_S agree on $F\mathcal{A}F$. Thus δ_S extends to $G_2 = \bigvee E_i$, $i \in R_2$ as above and $\|SG_2\| \leqslant 2\|\delta\|$ since $|d_i| \leqslant \|\delta\|$.

Continuing we extend S to G_n so that $\|SG_n\| \leqslant n\|\delta\|$ and $\delta(A) = \delta_S(A)$ for $A \in G_n\mathcal{A}G_n$. Moreover $G_n = \bigvee E_i$ where $i \in R_n = \{i \mid E_i\mathcal{M}_nE_1 \neq 0\}$. By our assumption $\mathcal{M}_n$ is weakly dense in $\mathcal{L}(H)$ so $G_n = H$.

We collect the main results in Section 2 and above into one theorem.

THEOREM 3.3. *Let $\mathcal{A}$ be an irreducible* CSL *algebra which contains a completely atomic masa. Assume every derivation of $\mathcal{A}$ into $\mathcal{A}$ is quasi-inner. Then the following are equivalent:*

i) $H^1(\mathcal{A}, \mathcal{A}) \neq 0$;

ii) *for all n, $\mathcal{M}_n$ is not weakly dense in $\mathcal{L}(H)$;*

iii) *$\mathcal{A}$ is a tridiagonal algebra.*

Proof. The implications (i) implies (ii) is (3.2) while (ii) implies (iii) is (2.3). Finally (iii) implies (i) follows from Theorem 5.1 in [6]. ▨

Now we turn to the case when every derivation need not be quasi-inner. In this case a derivation on $\mathcal{A}$ does not extend to the von Neumann algebra generated by $\mathcal{A}$. Examples are found using any of the algebras $\mathcal{A}_{2n}$ with $n \geqslant 2$. Our main result is that these are the only obstructions to every derivation being quasi-inner.

Next we give a less obvious example to illustrate how it may occur that all derivations are not quasi-inner. This example in another form was used by A. Hopenwasser and R. Moore to illustrate an irreducible CSL algebra $\mathscr{A}$ with a rank two operator which is not the linear combination of rank one operators in $\mathscr{A}$.

EXAMPLE 3.4. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis. Determine $\mathscr{A}$ by the strict ordering on the minimal core projections $E_i = \mathrm{span}\{e_i\}$ (cf. Examples (1.1) and (1.2)). Let $E_{2n+1} \gg E_j$ whenever j is even and $j \neq 2n$ and $E_k \gg E_k$ for all k. It is almost immediate that $\mathscr{M}_3$ is dense in $\mathscr{L}(H)$. To see this consider the rank one map $e_{2i} \otimes e_{2k+1}$. If $i \neq k$ then $e_{2i} \otimes e_{2k+1}$ is in $\mathscr{A}^*$. The map $e_{2i} \otimes e_{2n}$ is in $(\mathscr{A} + \mathscr{A}^*)^2$ by using the product of $e_{2i} \otimes e_{2k+1}$ and $(e_{2n} \otimes e_{2k+1})^*$ for $k \neq i$ or n. Similarly one shows $e_{2i+1} \otimes e_{2n+1} \in \mathscr{M}_2$. To obtain $e_{2i} \otimes e_{2i+1}$ one must use the product of three maps from $\mathscr{A} + \mathscr{A}^*$.

Let P be the core projection $P = E_1 + E_3 + E_4 + E_6$. It is easy to see that $P\mathscr{A}P = \mathscr{A}_4$ given in (1.1). Noting this one can easily construct non quasi-inner derivations on $\mathscr{A}$. Thus $H^1(\mathscr{A}, \mathscr{A}) \neq 0$ yet $\mathscr{M}_3$ is dense in $\mathscr{L}(H)$.

This example is precisely the obstruction encountered if every derivation of $\mathscr{A}$ into $\mathscr{A}$ is not quasi-inner. We recall the algebras A_{2n} defined in Example 1.1.

THEOREM 3.5. *Let $\mathscr{A}$ be an irreducible* CSL *algebra containing a purely atomic masa. If there exists a non quasi-inner derivation δ then there exists a core projection P so that $P\mathscr{A}P = \mathscr{A}_{2n}$ for some n and $\delta \mid P\mathscr{A}P$ is not inner.*

Proof. Let $\{E_i\}_{i=1}^{\infty}$ be the minimal core projections for $\mathscr{A}$. Assume δ is not quasi-inner and as usual we may assume $\delta \mid \mathscr{D} = 0$. Let $F_n = E_1 + \ldots + E_n$. If one assumes that for all n, $\delta \mid F_n\mathscr{A}F_n$ is quasi-inner, then δ is in fact quasi-inner on $\mathscr{A}$. Thus there exists an n so that $\delta \mid F_n\mathscr{A}F_n$ is not quasi-inner. For a subset R of $\{1, \ldots, n\}$ one defines F_R as $\sum E_i$ where $i \in R$. Let R_0 be a subset of $\{1, \ldots, n\}$ of minimal cardinality for which $\delta \mid F_R\mathscr{A}F_R$ is not inner. Let $F_0 = F_{R_0}$ and assume $R_0 = \{1, \ldots, k_0\}$. After reordering $E_1, \ldots, E_{k_0}$ if necessary we claim that $F_0\mathscr{A}F_0 = \mathscr{A}_{k_0}$ where A_{k_0} is given by (1.1). Now only consider $\{E_1, \ldots, E_{k_0}\}$ and A_{k_0}.

Let $S_i = \{j \in R_0 : E_j\mathscr{A}E_i \neq 0,\ j \neq i\}$. If $i \in S_j$ and $j \in S_i$ let $R = R_0 \setminus \{i\}$. Then $\delta \mid F_R\mathscr{A}F_R$ is inner and the implementing operator T can be extended to E_i so that $\delta \mid R_0\mathscr{A}R_0 = \delta_T \mid R_0\mathscr{A}R_0$. Also if $j \in S_i$ and $k \in S_j$ let $R = R_0 \setminus \{j\}$. Let T be a core operator on F_R so that $(\delta_T - \delta) \mid F_R\mathscr{A}F_R = 0$. Choose $TE_j = cE_j$ so that if $R_1 = E_i + E_j + E_k$, $\delta_T \mid R_1\mathscr{A}R_1 = \delta \mid R_1\mathscr{A}R_1$. The constant c is uniquely determined by δ through the relationship $E_i \gg E_j \gg E_k$ and since TE_i and TE_k are known. Now if $m \in S_j \cap R_0$ then $m \in S_i$ or $k \in S_m$. This relationship guarantees that $(\delta - \delta_T) \mid R_0\mathscr{A}R_0 = 0$. Thus we conclude that $i \in S_j$ and $j \in S_i$ cannot occur as well as $i \in S_j$ and $j \in S_k$ cannot occur.

Clearly if $i \in R_0$ is in no S_j for $j \neq i$ and if no j for $j \neq i$ is in S_i then R_0 is not a minimal set. Hence for $i \in R_0$ there exists $j \in R_0$ such that $i \in S_j$ or $j \in S_i$ but not both. Let R_1 be the $j \in R_0$ so that j is in no S_i for $j \neq i$. R_1 is non empty for if $i_0 \in S_{j_0}$ then by the above j_0 is in no S_i and hence $j_0 \in R_1$. If $j \notin R_1$, then there exists i so that $j \in S_i$. Let $R_2 = R_0 \setminus R_1$; then $R_1 \cup R_2 = R_0$.

Let $(E_{n_1}, \ldots, E_{n_k})$ be a *loop* if $\{n_1, \ldots, n_k\} \subseteq R_0$ so that $E_{n_{2i}} \gg E_{n_{2i \pm 1}}$ where k is even and $k+1$ is identified with 1. Loops exist since by the preceding paragraph let n_1 be any index in R_1 and $n_2 \in S_{n_1}$ and continue letting $n_3 \neq n_1$ so that $n_2 \in S_{n_3}$. Since R_0 is finite after k_0+1 steps n_{k_0+1} must be some n_i. One loop obtained is thus $(E_{n_i}, E_{n_{i+1}}, \ldots, E_{n_{k_0}})$.

If δ is inner when restricted to $\mathscr{A}$ compressed to any loop in R_0 then as in the proof of (3.2) one can show that δ is inner on $\mathscr{A}$ compressed to F_{R_0}. Thus let $E_{n_1}, \ldots, E_{n_k}$ be the shortest loop with $n_1, \ldots, n_k$ in R_0 on which δ is not inner. Since R_0 was minimal among sets on which δ is not inner $\{n_1, \ldots, n_k\} = R_0$. Thus $(n_1, \ldots, n_k)$ is simply a permutation of $(1, \ldots, k)$ and we may assume $n_i = i$. We now have $E_{2i} \gg E_{2i+1}$ and $E_k \gg E_1$ and we may assume k is even. If $E_{2i} \gg E_j$ for some other odd j, then δ must fail to be inner on one of two loops; the one between j and $2i$ or the one obtained by bypassing the core projections between $2i$ and j. Consequently $P_{R_0}\mathscr{A}P_{R_0}$ is precisely $\mathscr{A}_n$. Finally R_0 was obtained so that $\delta \mid R_0\mathscr{A}R_0$ is not inner. ▨

One might hope in view of (2.5) that in general a lattice with a comparable element has $H^1(\mathscr{A}, \mathscr{A}) = 0$. Unfortunately $\mathscr{A}_4$ has non inner derivations and has a comparable element, specifically $L = E_1 + E_3$. However for $\mathscr{A}_4$ it is easy to see that every quasi-inner derivation is inner (3.2). This is also the case for arbitrary lattices with comparable elements (no atomic assumption on $\mathscr{C}_{\mathscr{L}}$ is needed).

PROPOSITION 3.6. *Let $\mathscr{L}$ be any commutative subspace lattice with a nontrivial comparable element. Every quasi-inner derivation on $\mathscr{A} = \operatorname{Alg} \mathscr{L}$ is inner.*

Proof. Let L be a nontrivial comparable element. Then $L^\perp \gg L$. Let δ be a quasi-inner derivation on $\mathscr{A}$ so that $\delta \mid \mathscr{D} = 0$ and δ is implemented by the possibly unbounded operator T affiliated with the core. Fix unit vectors $x_0 \in L^\perp$ and $y_0 \in L$. For $y \in L$ define $T_0(y) = -\delta(x_0 \otimes y)(x_0)$. $\|T_0(y)\| \leqslant \|\delta\| \, \|y\|$ and if $P \leqslant L$, $P \in \mathscr{C}$ then $T_0P(y) = PT_0(y)$. Thus $T_0 \mid L \in \mathscr{C} \mid L$. Moreover $T_0(y) = T(y) - \langle Tx_0, x_0 \rangle$ and thus $T \mid L$ is a bounded operator.

In order to show $T \mid L^\perp$ is also bounded we let $x \in L^\perp$ and in the domain of T^* and then set $T_1^*(x) = \delta(x \otimes y_0)^*(y_0)$. As above $\|T_1^* \mid L^\perp\| \leqslant \|\delta\|$. Expanding we get $\delta(x \otimes y_0)^*(y_0) = \delta_T(x \otimes y_0)^*(y_0) = (T^* - \alpha I)(x)$ where $\alpha = \langle y_0, Ty_0 \rangle$. Thus $T^* \mid L^\perp$ is also a bounded operator and thus $\delta = \delta_T$ is an inner derivation. ▨

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