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ON MULTIDIMENSIONAL SINGULAR INTEGRAL OPERATORS. II: THE CASE OF COMPACT MANIFOLDS

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INTRODUCTION

Singular integral operators (equations)

$$A\varphi(x) \equiv a(x)\varphi(x) + \int_{u} \frac{\Omega(x, x-y)}{|x-y|^n}\varphi(y)\,\mathrm{d}y = f(x), \quad x \in M \tag{0.1}$$

are investigated, where M is a compact manifold with the boundary $\partial M \neq \emptyset$; $f(x) \in (H^{sp})^N(M)$, $\varphi \in (H_0^{sp})^N(M)$, $1 < p < \infty$, $-\infty < s < \infty$; proofs here are based on [3], where the first part of the present investigations (the half-space case $M = \mathbf{R}^{n+}$) was published.

As it was already mentioned in [3] the operators (0.1) were investigated by Simonenko [6] (the case $p = 2$, $s = 0$) and by Wišik and Eskin [7] (the case $p = 2$, $-\infty < s < \infty$).

The main tool of investigation here is the local principle (cf. § 3.2°), which is a slight modification of the local principle from [5] (cf. § 1.6), extended with some notions from [6].

We set forth here the numeration of sections; thus the references to §§ 1–2 will mean the reference to [3]; all notations from [3] are used without the further explanations.

3. PRELIMINARIES

1°. ON THE ISOMORPHISM OF SOBOLEV-SLOBODECKIĬ SPACES. Consider the operators

$$\Lambda_{\pm}^{s}\varphi = \mathscr{F}^{-1}g_{\pm}^{s}\mathscr{F}\varphi, \quad \Lambda^{s}\varphi = \mathscr{F}^{-1}g_{0}^{s}\mathscr{F}\varphi, \quad \varphi \in C_0^{\infty}(\mathbf{R}^n), -\infty < s < \infty,$$

where

$$g^s_\pm(\xi) = (i\xi_1 \mp |\xi'| \mp 1)^s, \quad \xi = (\xi_1, \xi') \in \mathbf{R}^n,$$
$$g^s_0(\xi) = (1 + |\xi|)^s;$$

these operators have continuous extensions

$$\Lambda^s, \Lambda^s_\pm : H^{rp}(\mathbf{R}^n) \to H^{(r-s)p}(\mathbf{R}^n) \quad (-\infty < r < \infty)$$

and these extensions are isomorphisms (Λ^s arranges even the isometrical isomorphism $\|\varphi\|_{rp} = \|\Lambda^s\varphi\|_{(r-s)p}$); obviously

$$\Lambda^s\Lambda^{-s} = I, \quad \Lambda^s_\pm\Lambda^{-s}_\pm = I.$$

The operators

$$\tilde{\Lambda}^s_\pm = P_{1+}\Lambda^s l_{1+},$$

where P_{1+} and l_{1+} are restricting and extending operators, also arrange isomorphisms (cf. [2, 4])

$$\tilde{\Lambda}^s_+ : H^{rp}_0(\mathbf{R}^{n+}) \to H^{(r-s)p}_0(\mathbf{R}^{n+}), \quad \tilde{\Lambda}^s_\pm\tilde{\Lambda}^{-s}_\pm = I,$$

$$\tilde{\Lambda}^s_- : H^{rp}(\mathbf{R}^{n+}) \to H^{(r-s)p}(\mathbf{R}^{n+}), \quad (-\infty < s, r < \infty).$$

The following two lemmas are easy to prove (cf., for example, [2, 4]).

LEMMA 3.1. *The operator* $W^0_a \in \mathscr{L}((H^{sp})^N(\mathbf{R}^n))$ *is isomorphic*

$$W^0_a = B^{-s}W^0_aB^s \quad (B^{\pm s} = \Lambda^{\pm s}, \Lambda^{\pm s}_-, \Lambda^{\pm s}_+)$$

to itself $W^0_a \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ *operating in the space* $L^N_p(\mathbf{R}^n)$; *therefore* $W^0_a \in \mathscr{L}((H^{sp})^N(\mathbf{R}^n))$ *if and only if* $a(\xi) \in M^{N\times N}_p(\mathbf{R}^n)$ $(1 < p < \infty, -\infty < s < \infty)$.

LEMMA 3.2. *The operator*

$$W^1_a \in \mathscr{L}((H^{sp}_0)^N(\mathbf{R}^{n+}), (H^{sp})^N(\mathbf{R}^{n+})) \tag{3.1}$$

is isomorphic

$$W^1_a = \Lambda^{-s}_- W^1_{a_s}\Lambda^s_+$$

to the operator

$$W^1_{a_s} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+})), \quad a_s(\xi) = \left(\frac{i\xi_1 + |\xi'| + 1}{i\xi_1 - |\xi'| - 1}\right)^s a(\xi); \tag{3.2}$$

hence (3.1) *is valid if and only if* $a(\xi) \in M^{N\times N}_p(\mathbf{R}^n)$.

We remind that $(C^r_0)^{N\times N}(\mathbf{R}^n)$ denotes the set of $N\times N$ matrix-functions with entries $a(\xi) \in C^r_0(\mathbf{R}^n)$ having continuous derivatives $D^k_\xi a(\xi)$ $(|k| \leqslant r)$ with compact supports.

LEMMA 3.3. *Let* $a(\xi) \in (C^r_0)^{N\times N}(\mathbf{R}^n)$ *and* $|s| < r$.

The operators

$$T = \Lambda^{-s} a \Lambda^{s} - aI, \quad T_{\pm} = \Lambda_{\pm}^{-s} a \Lambda_{\pm}^{s} - aI,$$

$$\tilde{T}_{\pm} = \tilde{\Lambda}_{\pm}^{-s} a \tilde{\Lambda}_{\pm}^{s} - aI, \quad T'_{\pm} = \Lambda_{\pm}^{-s} a \Lambda_{\mp}^{s} - aW^{0}_{g_{\pm s}},$$

$$\tilde{T}'_{\pm} = \tilde{\Lambda}_{\pm}^{-s} a \tilde{\Lambda}_{\mp}^{s} - aW^{1}_{g_{\pm s}},$$

where

$$g_{\mu}(\xi) = \left(\frac{i\xi_1 + |\xi'|}{i\xi_1 - |\xi'|} \right)^{\mu}, \tag{3.3}$$

are compact

$$T,\ T_{\pm},\ T'_{\pm} \in \mathfrak{S}(L_p^N(\mathbf{R}^n)), \quad \tilde{T}_{\pm},\ \tilde{T}' \in \mathfrak{S}(L_p^N(\mathbf{R}^{n+})).$$

Proof. Obviously we can suppose $N = 1$ and due to standard approximation consider only $a \in C_0^{\infty}(\mathbf{R}^n)$; applying further Theorem 1.11 (namely the second part of it on the interpolation of compact operators) we can concentrate only on the case $p = 2$ (due to the boundedness of operators under the consideration in all L_p spaces).

The inclusion $T \in \mathfrak{S}(L_p(\mathbf{R}^n))$ was proved in [1], Chapter III, § 9. We consider only T'_{-}, because T'_{+}, $T_{\pm}$ can be considered similarly and $\tilde{T}'_{\pm} = P_{1+} T'_{\pm} l_{1+}$, $\tilde{T}_{\pm} = P_{1+} T_{\pm} l_{1+}$.

Λ_{+}^{s} is a pseudodifferential operator (cf. [1, 4, 7]) and its symbol $(i\xi_1 - |\xi'| - 1)^s$ belongs to the class $S^{s}_{1,0}(\mathbf{R}^n)$ (cf. [7], Chapter II, § 1); the operator $d\Lambda_{+}^{s} - \Lambda_{+}^{s} dI$ is also pseudodifferential for all $d \in C_0^{\infty}(\mathbf{R}^n)$ and its symbol belongs to $S^{s-1}_{1,0}(\mathbf{R}^n)$; hence $d\Lambda_{-}^{s} - \Lambda^{s} dI \in \mathscr{L}(H^{r}(\mathbf{R}^n), H^{r-s+1}(\mathbf{R}^n))$ (here $H^{r}(\mathbf{R}^{r}) \overset{\text{def}}{=} H^{r2}(\mathbf{R}^n)$; cf. [7], § 6 and [3], § 18).

Let now $b(\xi) \in C_0(\mathbf{R}^n)$ be such that $b(\xi) = 1$ for all $\xi \in \operatorname{supp} a$; then $a(\xi)b(\xi) \equiv a(\xi)$, $a\Lambda_{+}^{s} = AK_1 + a\Lambda_{+}^{s} bI$, $K_1 = b\Lambda_{+}^{s} - \Lambda_{+}^{s} bI$; rewrite now T'_{-} in the form

$$\begin{gathered} T'_{-} = \Lambda_{-}^{-s}(aK_1 + K_2 bI) + K_3 aI, \\ K_2 = a\Lambda_{+}^{s} - \Lambda_{+}^{s} aI, \quad K_3 = \Lambda_{-}^{-s}\Lambda_{+}^{s} - W^{0}_{g_{-s}}. \end{gathered} \tag{3.4}$$

As we already noticed K_1 and K_2 are pseudodifferential operators of order $s-1$ and therefore K_1, $K_2 \in \mathscr{L}(H^{s}(\mathbf{R}^n), H^{1}(\mathbf{R}^n))$; but then (cf. [4], Theorem 18.4 and [1], Chapter III, Lemma 8.1)

$$aK_1,\ K_2 bI \in \mathfrak{S}(H^{s}(\mathbf{R}^n), L_2(\mathbf{R}^n)),$$

since $\operatorname{supp} a \subset \operatorname{supp} b$ are compact. On the other hand

$$\Lambda_{-}^{-s} \in \mathscr{L}(L_2(\mathbf{R}^n),\ H^{s}(\mathbf{R}^n))$$

and therefore

(3.5) $$\Lambda_-^{-s} a K_1,\ \Lambda_-^{-s} K_2 b I \in \mathfrak{S}(L_2(\mathbf{R}^n)).$$

Since

$$\Lambda_-^{-s}\Lambda_+^s = W^0_{\tilde{g}_{-s}}, \quad \tilde{g}_{-s}(\xi) = \left(\frac{i\xi_1 + |\xi'| + 1}{i\xi_1 - |\xi'| - 1}\right)^{-s};$$

we obtain

$$K_3 = W^0_{d_{-s}}, \quad d_{-s}(\xi) = \tilde{g}_{-s}(\xi) - g_{-s}(\xi);$$

clearly $d_{-s} \in m_2(\mathbf{R}^n)$ and $d_{-s}(\infty) = 0$; Theorem 1.12 yields then

(3.6) $$aK_3 \in \mathfrak{S}(L_2(\mathbf{R}^n));$$

due to (3.4)—(3.6) the desired inclusion $T'_- \in \mathfrak{S}(L_2(\mathbf{R}^n))$ is evident. ▨

2°. Local principle. II. We present here a local principle which is a slight modification of the local principles from [5,6] (cf. § 1.6°).

Let X_1, X_2 be the Banach spaces and

$$\mathscr{L}' = \mathscr{L}(X_1, X_2), \quad \mathfrak{S}' = \mathfrak{S}(X_1, X_2),$$

$$\mathscr{L}'' = \mathscr{L}(X_2, X_1), \quad \mathfrak{S}'' = \mathfrak{S}(X_2, X_1);$$

$\tilde{\mathscr{L}}' = \mathscr{L}/\mathfrak{S}'$ denotes the factor-algebra with the norm

$$\|\hat{A}\| = |||A||| \overset{\text{def}}{=} \inf_{T\in\mathfrak{S}'} \|A + T\|.$$

A set

$$\Delta \subset \mathscr{L}(X_1) \cap \mathscr{L}(X_2)$$

is called a *localizing class* provided

$$\mathfrak{S}(X_1) \cap \mathfrak{S}(X_2) \cap \Delta = \emptyset$$

and for each $A_1, A_2 \in \Delta$ there exists $A \in \Delta$ such that $A_jA = AA_j = A \quad (j = 1,2)$.

Two elements $A, B \in \mathscr{L}'$ are called *Δ-equivalent*, written $A \overset{\Delta}{\sim} B$, provided

$$\inf_{E\in\Delta} |||(A - B)E||| = \inf_{E\in\Delta} |||E(A - B)||| = 0.$$

An element $A \in \mathscr{L}'$ is called *left* (*right*) *Δ-regularizable* if there exist $R \in \mathscr{L}''$ and $B \in \Delta$ such that $RAB = B + T$, $T \in \mathfrak{S}(X_1)$ $(BAR = B + T,\ T \in \mathfrak{S}(X_2))$.

A is called *Δ-regularizable* provided it is left and right Δ-regularizable.

Similarly *Δ-invertibility*, *left* and *right Δ-invertibility* are defined provided $T = 0$ (cf. § 1.6°).

A system $\{\Delta_\omega\}_{\omega\in\Omega}$ of localizing classes is called a *covering* if from each set $\{A_\omega\}_{\omega\in\Omega}$, $A_\omega \in \Delta_\omega$, finite number of elements can be selected $\{A_{\omega_k}\}_{k=1}^l$ such that $A = \sum_{k=1}^{l} A_{\omega_k}$ has the regularizer

$$RA = I + T_1, \quad AR = I + T_2, \quad T_1, T_2 \in \mathfrak{S}(X_1) \cap \mathfrak{S}(X_2).$$

THEOREM 3.4. *Let $\{\Delta_\omega\}_{\omega\in\Omega}$ be a covering system of localizing classes and $A \overset{\Delta}{\sim} B_\omega$ $(A, B_\omega \in \mathscr{L}', \omega \in \Omega)$, $AB - BA \in \mathfrak{S}'$ for all $B \in \bigcup_{\omega\in\Omega} \Delta_\omega$.*

The operator A is Fredholm (has a left or a right regularizer) if and only if B_ω have Δ_ω-regularizers (B_ω have left, have right Δ_ω-regularizers) for all $\omega \in \Omega$.

Proof is completely similar to the proof of Theorem 1.1 from [5], Chapter XII (cf. Theorem 1.17 above).

Consider now the particular case

$$X_1 = (H_0^{sp})^N(M), \quad X_2 = (H^{sp})^N(M), \quad 1 < p < \infty, \ -\infty < s < \infty,$$

where M is a r-smooth $(r > s)$ n-dimensional compact manifold with the boundary $\partial M \neq \emptyset$; denote

$$(3.7) \qquad \begin{aligned} \mathscr{L}_{sp}^N(M) &= \mathscr{L}((H_0^{sp})^N(M), (H^{sp})^N(M)), \\ \mathfrak{S}_{sp}^N(M) &= \mathfrak{S}((H_0^{sp})^N(M), (H^{sp})^N(M)). \end{aligned}$$

Let $x \in M$ and consider

$$(3.8) \quad \Delta_x = \{v_x I : v_x \in C^r(M), \ v_x(t) = 1 \text{ in some neighborhood of } x \in M\}.$$

Clearly Δ_x is a localizing class in $\mathscr{L}_{sp}^N(M)$ and $\{\Delta_x\}_{x\in M}$ (due to the compactness of M) is a covering.

Clearly

$$\sup_{v_x I \in \Delta_x} \|v_x I\|_{sp} = \infty \quad \text{for } s \neq 0;$$

nevertheless

$$(3.9) \qquad \sup_{v_x I \in \Delta_x} |||v_x I|||_{sp} = \sup_{v_x I \in \Delta_x} \inf_{\gamma} \|v_x I + T\|_{sp} < \infty,$$

where the norm is taken either in $(H_0^{sp})^N(M)$ or in $(H^{sp})^N(M)$.

Let us prove (3.9).

Due to the definition of the norms we can suppose $N = 1$, $M = \mathbf{R}^n$ (or $M = \mathbf{R}^{n+}$) and $\operatorname{supp} v_x \subset U$ for all $v_x I \in \Delta_x$ and a certain compact U.

Let $B^s = \Lambda^s$, $B^s = \tilde{\Lambda}^s_+$ or $B^s = \tilde{\Lambda}^s_-$ for the cases of the spaces $H^{sp}(\mathbf{R}^n)$ $(M = \mathbf{R}^n)$, $H_0^{sp}(\mathbf{R}^{n+})$ or $H^{sp}(\mathbf{R}^{n+})(M = \mathbf{R}^{n+})$, respectively; due to Lemma 3.3 and isomorphical properties of B^s (cf. § 3.1) we easily get

$$\|v_x I\|_{sp} = C\|B^s v_x B^{-s}\| = C\|v_x I\|_p = C\|v_x I\|_p = C \sup_\xi |v_x(\xi)| = C,$$

where C depends only on $B^{\pm s}$.

Let now M' be another r-smooth n-dimensional (not necessarily compact) manifold with or without boundary; suppose there exists some r-diffeomorphism

$$\beta(t) : U_x \to U_y, \quad \beta(x) = y, \quad x \in M,\, y \in M'$$

of neighborhoods of the points x and y.

The operator

$$\beta_* \varphi(t) = \begin{cases} \varphi(\beta(t)), & t \in U_x, \\ 0, & t \notin U_x \end{cases} \tag{3.10}$$

is bounded

$$\beta_* \in \mathscr{L}((H_0^{sp})^N(M'), (H_0^{sp})^N(M)) \cap \mathscr{L}((H^{sp})^N(M'), (H^{sp})^N(M))$$

and its restrictions on $(H_0^{sp})^N(U_y)$ and on $(H^{sp})^N(U_y)$ are isomorphisms (β_* and its inverse β_*^{-1} are obviously bounded in $(H_0^{sp})^N(U_y)$ and $(H^{sp})^N(U_y)$ for $s =$ integer; for $s \neq$ integer their boundedness follows from interpolation Theorem 1.11).

Similarly to Δ_x ($x \in M$; cf. (3.8)) the localizing class Δ_y ($y \in M'$) is defined.

The operators $A \in \mathscr{L}_{sp}^N(M)$ and $B \in \mathscr{L}_{sp}^N(M')$ are called *$(\Delta_x, \beta, \Delta_y)$-equivalen* (or *quasiequivalent*; cf. [6]), written $A \overset{x}{\sim} \beta \overset{y}{\sim} B$, if there exist two neighborhoods $U_x \subset M$, $U_y \subset M'$ and a diffeomorphism $\beta : U_x \to U_y$, such that

$$y = \beta(x), \quad \beta_*^{-1} A \beta_* \overset{y}{\sim} B.$$

All properties of quasiequivalence listed in [6], page 577 (with exception of the property b)) are valid in the considered case as well.

With the help of Theorem 3.4 we easily prove the following.

THEOREM 3.5. *Let $A \in \mathscr{L}_{sp}^N(M)$, $B \in \mathscr{L}_{sp}^N(M')$ and*

$$A \overset{x}{\sim} \beta \overset{y}{\sim} B;$$

the operator A has a left (right) Δ_x-regularizer if and only if B has a left (right) Δ_y-regularizer.

In the case $s = 0$ we can add to Theorem 3.4 the following.

LEMMA 3.6. *If the operator* $A \in \mathscr{L}(L_p^N(M)) \equiv \mathscr{L}_{0p}^N(M)$ *has a left (right)* Δ_x*-regularizer, it is left (right)* Δ_x*-invertible.*

Proof. Let, for definiteness, A has a left Δ_x-regularizer

$$RAv_xI = v_xI + T, \quad T \in \mathfrak{S}(L_p^N(M)).$$

Since

$$\inf_{vI \in \Delta_x} \|v\varphi\|_p = 0$$

for any $\varphi \in L_p^N(M)$, there exists such $v_0I \in \Delta_x$ that

(3.11) $$\|Tv_0I\|_p < 1;$$

let $\tilde{v}_xI \in \Delta_x$ be such that

$$\tilde{v}_x v_0 = v_x\tilde{v}_x = \tilde{v}_x;$$

then

$$RA\tilde{v}_xI = (I + Tv_0)\tilde{v}_xI$$

and $I + Tv_0$ is invertible (cf. (3.11)); therefore

$$(I + Tv_0)^{-1}RA\tilde{v}_xI = \tilde{v}_xI$$

and A is left Δ_x-invertible. ▨

LEMMA 3.7. *Let* $A \in \mathscr{L}(L_p^N(\mathbf{R}^n))$ *(or* $A \in \mathscr{L}(L_p^N(\mathbf{R}^{n+}))$*) and*

$$V_\lambda A = AV_\lambda, \quad V_\lambda\varphi(t) = \varphi(\lambda t) \quad (\lambda > 0).$$

The operator A has a left (has a right) Δ_0*-regularizer if and only if A is normally solvable and* $\dim \operatorname{Ker} A = 0$ *(is normally solvable and* $\dim \operatorname{Coker} A = 0$*, respectively).*

In particular A has a Δ_0*-regularizer if and only if A is the invertible operator.*

Proof. Let A has a left Δ_0-regularizer; then A is left Δ_0-invertible (cf. Lemma 3.6)

(3.12) $$RAv_0I = v_0I;$$

we prove now that

(3.13) $$\inf_{\|\varphi\|_p=1} \|A\varphi\|_p > 0,$$

which is equivalent to the normal solvability and $\dim \operatorname{Ker} A = 0$ (cf. [5]).

Let (3.13) does not hold: there exists a sequence $\{\varphi_j\}_{j=1}^\infty$ such that

(3.14) $$\|\varphi_j\|_p = 1, \quad \lim_{j\to\infty} \|A\varphi_j\|_p = 0;$$

due to (3.12)

$$1 = \lim_{\lambda\to 0} \|v_\lambda \varphi_j\|_p = \lim_{\lambda\to 0} \|V_\lambda R V_{1/\lambda} A v_\lambda \varphi_j\|_p \leqslant$$

(3.15)

$$\leqslant \|R\|_p[\|A\varphi_j\|_p + \|A\|_p \lim_{\lambda\to 0} \|v_\lambda\varphi_j - \varphi_j\|_p] = \|R\|_p \|A\varphi_j\|_p,$$

where

$$v_\lambda(t) = v_0(\lambda t) = V_\lambda v_0(t).$$

Due to (3.14) the inequality (3.15) is a contradiction; thus (3.13) holds.

If A has a right Δ_0-regularizer the conjugate operator A^* will have a left Δ_0-regularizer and the result follows from duality (A is normally solvable and $\dim \operatorname{Coker} A = 0$ if and only if A^* is normally solvable and $\dim \operatorname{Ker} A^* = 0$).

A will be normally solvable and $\dim \operatorname{Ker} A = \dim \operatorname{Coker} A = 0$ provided A has both (left and right) Δ_0-regularizers; therefore A will be invertible. ◻

4. SINGULAR INTEGRAL OPERATORS ON A COMPACT MANIFOLD

1°. Definitions and simplest properties. Let M be a compact n-dimensional r-smooth manifold with the boundary $\partial M \neq \emptyset$; $\{u_j\}_{j=1}^l$ be a covering of M, $\bigcup_{j=1}^l u_j = M \cup \partial M$, $\bigcup_{j=k+1}^l U_j \supset \partial M$; $\beta_j : u_j \to u_j^0$ be homeomorphisms on compact domains $u_1^0, \dots, u_k^0 \subset \mathbf{R}^n$, $u_{k+1}^0, \dots, u_n^0 \subset \mathbf{R}^{n+}$.

The operator $A \in \mathscr{L}_{sp}^N(M)$, $|s| \leqslant r$, $1 < p < \infty$ (cf. (3.7)) is called the *singular integral operator* provided (cf. § 3.2°):

(i) $AgI - gA \in \mathfrak{S}_{sp}^N(M)$ for all $g \in C^r(M)$;

(ii) $A \overset{x}{\sim} \beta_j \overset{y}{\sim} W_{a_x}^0$ for $x \notin \partial M$, $x \in U_j \cap M$ ($y \in \beta_j(x)$, $a_x(\xi) \in (Hm_p)^{N\times N}(\mathbf{R}^n)$);

(iii) $A \overset{x}{\sim} \beta_j \overset{y}{\sim} W_{a_x}^1$ for $x \in \partial M \cap U_j$ ($y = \beta_j(x) = (0, y_2, \dots, y_n)$, $a_x(\xi) \in$ $\in (Hm_p)^{N\times N}(\mathbf{R}^N)$).

$a_x(\xi)$ ($x \in M \cup \partial M \overset{\text{def}}{=} \overline{M}$) is called the *symbol* of the operator A.

Lemma 4.1. *The symbol $a_x(\xi) \in (Hm_p)^{N\times N}(\mathbf{R}^n)$ ($x \in \overline{M}$) of the singular integral operator $A \in \mathscr{L}_{sp}^N(M)$ is uniquelly defined.*

Proof. We have to prove that if for a fixed $x \in \overline{M}$

$$A \overset{x}{\sim} \beta_j \overset{y}{\sim} W_b^k, \quad A \overset{x}{\sim} \beta_j \overset{y}{\sim} W_d^k, \tag{4.1}$$

where $b, d \in Hm_p(\mathbf{R}^n)$ and $k = 0$, $k = 1$ for $x \notin \partial M$, $x \in \partial M$, respectively, then $b(\xi) \equiv d(\xi)$; this clearly suffices.

Let first $x \in \partial M$ (hence $k = 1$ in (4.1)); from (4.1) we get

$$W^1_{b-d} = W^1_b - W^1_d \overset{\Delta_y}{\sim} 0 \quad (y = \beta_j(x));$$

hence (cf. Definition in § 3.2°) for any $\varepsilon > 0$ there exist an operator $T_\varepsilon \in \mathfrak{S}_{sp}(\mathbf{R}^{n+})$ and a function $v_0(t) \in C_0^\infty(\mathbf{R}^n)$, $v_0(t) \equiv 1$ in some neighborhood of $y \in \mathbf{R}^{n+}$ such that

$$\| v_0 W^1_g + T_\varepsilon \|_{sp} < \varepsilon \quad (g = b - d);$$

due to the isomorphism properties of $\Lambda^s_\pm$, and Lemmas 3.2 and 3.3 we obtain

$$\text{(4.2)} \quad \| v_0 W^1_{\hat{g}_s} + T \|_p = \| \Lambda^s_- v_0 \Lambda^{-s}_- \Lambda^s_- W^1_g \Lambda^{-s}_+ + \Lambda^s_- T_\varepsilon \Lambda^{-s}_+ \|_p = \| v_0 W^1_g + T_\varepsilon \|_{sp} < \varepsilon,$$

where $T \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+1}))$ and

$$\hat{g}_s(\xi) = \left(\frac{i\xi_1 + |\xi'| + 1}{i\xi_1 - |\xi'| - 1} \right)^s g(\xi);$$

due to the compactness of $\operatorname{supp} v_0$ we conclude (cf. (3.3) and Theorem 1.12)

$$v_0 W^1_{(\hat{g}_s - g \cdot g_s)} \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+})),$$

since

$$\lim_{|\xi| \to \infty} |\hat{g}_s(\xi) - g(\xi) g_s(\xi)| \equiv 0;$$

(4.1) yields then

$$|||v_0 W^1_{gg_s}|||_p < \varepsilon$$

and due to (1.14)

$$\text{(4.3)} \qquad \sup_\xi |g(\xi) g_s(\xi)| \leqslant \|\chi W^1_{gg_s}\|_p = |||\chi W^1_{gg_s}|||_p \leqslant |||v_0 W^1_{gg_s}|||_p < \varepsilon,$$

where $\chi(\xi)$ is the characteristic function of the set $\{\xi \in \mathbf{R}^n : v_0(\xi) = 1\}$ (therefore $\chi v_0 = \chi$); from (4.3) immediately follows $b(\xi) \equiv d(\xi)$.

The case $x \notin \partial M$ ($k = 0$ in (4.1)) is completely similar. ▨

LEMMA 4.2. *The symbol $a_x(\xi)$ of the singular integral operator $A \in \mathscr{L}^N_{sp}(M)$ is a continuous function of $x \in \partial M$.*

Proof. Let $A \in \mathscr{L}^N_{sp}(M)$; by the definition of equivalence (cf. § 3.2°)

$$A \overset{x}{\sim} \beta_j \overset{y}{\sim} W^1_{a_x} \quad (y = \beta_j(x)),$$

for any $\varepsilon > 0$ there exists $v_\varepsilon I \in \Delta_y$ (cf. (3.8)) such that

$$\| v_\varepsilon[\beta_{j*}^{-1} A \beta_{j*} - W^1_{a_x}] \|_{sp} < \varepsilon;$$

let $x, z \in U_j$, $\beta_j(x) \in U_\varepsilon = \{\xi \in \mathbf{R}^{n+} : v_\varepsilon(\xi) = 1\}$; if $\tilde{v}_\varepsilon I \in \Delta_{\beta_j(z)}$ is such that $\operatorname{supp} \tilde{v}_\varepsilon \subset U_\varepsilon$ and

$$\| \tilde{v}_\varepsilon[\beta_{j*}^{-1} A \beta_{j*} - W^1_{a_z}] \|_{sp} < \varepsilon,$$

then

$$\| \tilde{v}_\varepsilon[W^1_{a_z} - W^1_{a_x}] \|_{sp} \leqslant C_1 \| \tilde{v}_\varepsilon v_\varepsilon[\beta_{j*}^{-1} A B_{j*} - W^1_{a_x}] \|_{sp} +$$

$$+ C_1 \| \tilde{v}_\varepsilon[\beta_{j*}^{-1} A \beta_{j*} - W^1_{a_z}] \|_{sp} < C_1(1 + \| v_\varepsilon \|_{sp})\, \varepsilon,$$

where C_1 depends only on s and p; similarly to (4.2) — (4.3) we obtain

$$\sup_{\xi} |[a_z(\xi) - a_x(\xi)] g_s(\xi)| < C_1(1 + \| v_\varepsilon \|_{sp})\, \varepsilon; \tag{4.4}$$

due to (3.9), from (4.4) it follows

$$\lim_{\substack{z \to x \\ z,\, x \in \partial M}} \sup_{\xi} |a_z(\xi) - a_x(\xi)| = 0.$$

▨

REMARK 4.3. Similarly to Lemma 3.2 one can prove that the symbol $a_x(\xi)$ is a continuous function of $x \in M \setminus \partial M$ as well.

REMARK 4.4. If $a_x(\xi) \in (Hm_p)^{N \times N}(\mathbf{R}^n)$ is the symbol of the singular integral operator $A \in \mathscr{L}^N_{sp}(M)$ and $\inf_{|\theta|=1} |\det a_x(\theta)| > 0$ for all $x \in \partial M$, the numbers

$$\delta_j(a_x) = \frac{\ln \lambda_j(a_x)}{2\pi}, \quad \frac{1}{p} - 1 - s < \operatorname{Re} \delta_j(a_x) \leqslant \frac{1}{p} - s$$

$$(j = 1, 2, \ldots, N,\ 1 < p < \infty)$$

can be defined (cf. § 3.1), where $\lambda_1(a_x), \ldots, \lambda_N(a_x)$ are the eigenvalues of the matrix $a_x^{-1}(-1,0,\ldots,0)a_x(+1,0,\ldots 0)$ (with regard of their multiplicities); *partial* (p,s)-*-indices* $\varkappa_N(\theta', x) \leqslant \ldots \leqslant \varkappa_1(\theta', x)$ $(\theta' \in S^{n-2}, x \in \partial M)$ are also defined as in § 2.1.

In the scalar case $N = 1$ all numbers $\delta_j(a_x)$ and $\varkappa(\theta', x) \equiv \varkappa_1(\theta', x)$ are continuous functions of $x \in \partial M$ (since $a_x(\xi)$ is continuous with respect to $x \in \partial M$) and therefore $\varkappa(\theta', x) \equiv$ const. (for $n > 2$), $\varkappa(\theta', x) \equiv \varkappa(\pm 1)$ (for $n = 2$).

REMARK 4.5. The above given definition of the singular integral operator (with slight variations) is well-known; in [6], for example, the condition (i) for the operator $A \in \mathscr{L}(L^N_p(M))$ is replaced by: (i') $\chi_1 A \chi_2$ is compact for all characteristic functions χ_1 and χ_2 with $\operatorname{supp} \chi_1 \cap \operatorname{supp} \chi_2 = \emptyset$.

Conditions (i) and (i′) are equivalent (cf. Seeley's remark in Mathematical Reviews **31**, # 3876).

2°. EXAMPLE. Let $M \subset \mathbf{R}^n$ be a compact domain with m-smooth ($m \geqslant 2$) boundary ∂M, let r be as in Theorem 1.1 and $|s| < l \leqslant m$.

Suppose

$$(4.5)\qquad D_\xi^k \Omega(\xi, \theta) \in (H^{r_2})^{N\times N}(M, S^{n-1}), \quad |k|_1 \leqslant l, \quad \int_{S^{n-1}} \Omega(\xi, \theta)\, d\theta \equiv 0$$

and for $\varepsilon > 0$ there exists a function

$$(4.6)\qquad \Omega_\varepsilon(\xi, \theta) = \sum_{j=1}^{q} \Omega_{j1}(\xi)\Omega_{j2}(\theta),$$

$$\int_{S^{n-2}} \Omega_{j2}(\theta)\, d\theta = 0, \quad \Omega_{j1} \in (C^l)^{N\times N}(M), \quad \Omega_{j2} \in (H^{r_2})^{N\times N}(S^{n-1}),$$

such that

$$(4.7)\qquad \|\Omega - \Omega_\varepsilon\|_{l,r} = \sum_{|k|^1 \leqslant l} \max_\xi \|D_\xi^k[\Omega(\xi, \cdot) - \Omega_\varepsilon(\xi, \cdot)\|_{H^{r_2}(S^{n-1})} < \varepsilon.$$

Consider the operator

$$A_M \varphi(\xi) = \int_M \frac{\Omega(\xi, \xi - \eta)}{|\xi - \eta|^n} \varphi(\eta)\, d\eta;$$

due to Theorem 1.1 $A_M \in \mathscr{L}_{sp}^N(M)$ (cf. (3.7)).

The operator

$$T = A_M g I - g A_M \quad (g \in C^m(M))$$

has a weak singular kernel and therefore is compact on $L_p^N(M)$; due to Theorem 1.11 (cf. (3.7)) $T \in \mathfrak{S}_{sp}^N(M)$.

Let $\beta_x : u_x \to u_0$ be some homeomorphism of the neighborhoods $u_x \subset M$, $u_0 \subset M_x$, where $\beta_x(x) = 0$ and

$$(4.8)\qquad M_x = \begin{cases} \mathbf{R}^{n+}, & \text{if } x \in \partial M \\ \mathbf{R}^n\ , & \text{if } x \in M \setminus \partial M; \end{cases}$$

let J_x be the Jakobian of $\beta_x(t)$ at the point x (if $x \in M \cup \partial M$ we can suppose $\beta_x(t) \equiv$ $\equiv t - x$ and hence $J_x \equiv 1$).

Let now (cf. (4.8))

$$A_x \psi(t) = \int_{M_x} \frac{\Omega_x(x, t - \tau)}{|t - \tau|^n} \psi(\tau)\, d\tau, \quad \Omega_x(x, \xi) \equiv \Omega(x, J_x^{-1}\xi);$$

in virtue of Theorem 1.1 $A_x \in \mathscr{L}_{sp}^N(M_x)$.

The equivalence

$$A_M \overset{x}{\sim} \beta_x \overset{y}{\sim} A_x \quad (x \in M,\ y = \beta_x(x)) \tag{4.9}$$

is valid.

Proof. We prove first the equivalence (cf. (3.8))

$$A_M \overset{\Delta_x}{\sim} \tilde{A}_x, \tag{4.10}$$

where

$$\tilde{A}_x\varphi(\xi) = \int\limits_M \frac{\Omega(x, \xi - \eta)}{|\xi - \eta|^n}\, \varphi(\eta)\, d\eta.$$

Suppose $\Omega = \Omega_\varepsilon$ (cf. (4.5)), $x \in M \setminus \partial M$ and $\operatorname{supp} v_x \subset M$ $(v_x I \in \Delta_x)$; then in virtue of Theorem 1.1 and Lemma 3.3

$$\|v_x^2[A_M - \tilde{A}_x]\|_{sp} = \|v_x[A_M - \tilde{A}_x]v_x\|_{sp} \leqslant$$

$$\leqslant \sum_{j=1}^{q} \|v_x \tilde{\Omega}_{j1} A_{j2} v_x\|_{sp} = \sum_{j=1}^{q} \|\Lambda^s v_x \Omega_{j1} A_{j2} v_x \Lambda^{-s}\|_p \leqslant$$

$$\leqslant \sum_{j=1}^{q} \|\Lambda^s v_k \tilde{\Omega}_{j1} \Lambda^{-s}\|_p \|\Lambda^s A_{j2} \Lambda^{-s}\|_p \|\Lambda^s v_k \Lambda^{-s}\|_p \leqslant \sum_{j=1}^{q} \|v_k \tilde{\Omega}_{j1} I\|_p \|A_{j2}\|_{sp} \|v_k I\|_p \leqslant$$

$$\leqslant \sum_{j=1}^{q} \sup_{\xi \in M} |v_k(\xi)[\Omega_{j1}(\xi) - \Omega_{j1}(x)]| \|A_{j2}\|_{sp} \leqslant$$

$$\leqslant \sum_{j=1}^{q} \|\Omega_{j2}\|_{l,r} \sup |v_k(\xi)[\Omega_{j1}(\xi) - \Omega_{j1}(x)]|$$

where $\tilde{\Omega}_{j1}(\xi) = \Omega_{j1}(\xi) - \Omega_{j1}(x)$,

$$A_{j2}\varphi(\xi) = \int\limits_M \frac{\Omega_{j2}(\xi - \eta)}{|\xi - \eta|^n}\, \varphi(\eta)\, d\eta$$

and $\|\cdot\|_{l,r}$ is defined by (4.7).

Thus

$$\inf_{v_k I \in \Delta_x} \|v_k[A_M - A_x I]\|_{sp} = 0$$

and (4.10) is proved for $x \in M \setminus \partial M$, $\Omega = \Omega_\varepsilon$.

Simple continuity arguments prove (4.10) for all $\Omega(\xi, \theta)$.

For proving (4.10) we need to consider only the case $x \in \partial M$, $\Omega = \Omega_\varepsilon$.

Let supp $v_x \subset u_x$ ($v_x I \in \Delta_x$; cf. (3.8)), $v_x^0 = \beta_{x*}^{-1} v_x$, $\Omega_{j1}^0 = \beta_{x*}^{-1} \tilde{\Omega}_{j1}$ ($\tilde{\Omega}_{j1}(\xi) = \Omega_{j1}(\xi) - \Omega_{j1}(x)$) and $A_{j2}^0 = \beta_{x*}^{-1} A_{j2} \beta_{x*}$ (cf. (3.10)); applying Theorem 1.1, Lemma 3.3 and inequality (3.9) we get

$$|||v_x^2[A_M - \tilde{A}_x]|||_{sp} \leqslant C_1 \sum_{j=1}^{q} |||v_x^0 \Omega_{j1}^0 A_{j2}^0 v_x^0|||_{sp} \leqslant C_1 \sum_{j=1}^{q} |||\tilde{A}_-^s v_x^0 \Omega_{j1}^0 A_{j2}^0 v_k^0 \tilde{A}_+^{-s}|||_p \leqslant$$

$$\leqslant C_1 \sum_{j=1}^{q} |||\tilde{A}_-^s v_x^0 \Omega_{j1}^0 \tilde{A}_-^{-s}|||_p \, |||\tilde{A}_-^s A_{j2}^0 \tilde{A}_+^{-s}|||_p \, |||A_+^s v_x^0 A_+^{-s}|||_p =$$

$$= C_2 \sum_{j=1}^{q} \sup |v_x(\xi) [\Omega_{j1}(\xi) - \Omega_{j1}(x)]| \, ||\Omega_{j2}||_{lr};$$

hence

$$\inf_{v_x I \in \Delta_x} |||v_x[A_M - \tilde{A}_x]|||_{sp} = 0$$

and (4.10) is proved.

Next the equivalence

$$\tilde{A}_x \overset{x}{\sim} \beta_x \overset{y}{\sim} A_x \quad (y = \beta(x)) \tag{4.11}$$

will be proved, which together with (4.10) yields (4.9).

It suffices to consider $\Omega(x, \xi) \in C^\infty(S^{n-1})$ for all $x \in M$ (otherwise we can approximate Ω and use the continuity property of the equivalence (4.11)); rewrite (4.11) as (cf. § 3.2)

$$\beta_{x*}^{-1} \tilde{A}_x \beta_{x*} \overset{\Delta_y}{\sim} A_x;$$

in [6], Chapter I, § 4 was proved that

$$\beta_{x*}^{-1} \tilde{A}_x \beta_{x*} = A_x + B + T, \tag{4.12}$$

where $T \in \mathfrak{S}(L_p^N(M_x))$,

$$B\psi(t) = \int_{M_x} \frac{\Omega_0(t, t - \tau) \psi(\tau) \, d\tau}{|\tau - t|^n},$$

$$\Omega_0(\beta_x(x), \xi) \equiv 0, \quad D_t^k \Omega_0(t, \xi) \in (C^\infty)^{N \times N}(S^{n-1}), \quad |k|_1 \leqslant m.$$

Clearly $B, T \in \mathscr{L}_{lp}^N(M_x)$ and in virtue of Theorem 1.11 $T \in \mathfrak{S}_{sp}^N(M_x)$. Similarly to (4.10) we get $B \overset{\Delta_y}{\sim} 0$ ($y = \beta_x(x)$) and, therefore, (4.12) yields (4.11). ▨

3°. STATEMENT OF THEOREMS.

THEOREM 4.6. *Let M be a compact n-dimensional r-smooth manifold with the boundary $\partial M \neq \emptyset$; $A \in \mathscr{L}^N_{sp}(M)$ be a singular integral operator with the symbol $a_x(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$ $(x \in M \cup \partial M,\ m > n/2,\ -\infty < s < \infty,\ 1 < p < \infty)$ which is elliptic* $\inf\limits_{\substack{x\in M\\ |\theta|=1}} |a_x(\theta)| > 0$.

The operator $A \in \mathscr{L}^N_{sp}(M)$ is Fredholm if and only if (cf. Remark 4.4*)*

$$(4.13)\qquad \delta_j(a_x) \neq \frac{1}{p} - s \quad \textit{for all } j = 1,2,\ldots,N$$

and all partial (p, s)-indices disappear

$$(4.14)\qquad \varkappa_1(\theta', x) \equiv \ldots \equiv \varkappa_N(\theta', x) \equiv 0 \quad (\theta' \in S^{n-2}, x \in M).$$

THEOREM 4.7. *Let all preliminaries of* Theorem 4.6 *together with* (4.13) *held.*

If $\varkappa_j(\theta', x) \geqslant 0$ $(j = 1,2,\ldots,N;\ \theta' \in S^{n-2};\ x \in M)$ but (4.14) *does not hold the operator $A \in \mathscr{L}^N_{sp}(M)$ has a left regularizer and* $\dim \operatorname{Coker} A = \infty$.

If $\varkappa_j(\theta', x) \leqslant 0$ $(j = 1,2,\ldots,N;\ \theta' \in S^{n-2};\ x \in M)$ but (4.14) *does not hold, the operator $A \in \mathscr{L}^N_{sp}(M)$ has a right regularizer and* $\dim \operatorname{Ker} A = \infty$.

THEOREM 4.8. *Let M be a compact n-dimensional r-smooth manifold with the boundary $\partial M \neq \emptyset$; $A \in \mathscr{L}^N_{sp}(M)$ be a singular integral operator with the symbol $a_x(\xi) \in (HC^{m+2})^{N\times N}(\mathbf{R}^n)$ $(x \in M \cup \partial M,\ m > n/2,\ -\infty < s < \infty,\ 1 < p < \infty)$.*

If $\inf |\det a_x(\theta)| = 0$ $(x \in M \cup \partial M,\ \theta \in S^{n-1})$, *the operator $A \in \mathscr{L}^N_{sp}(M)$ has no left and no right regularizer.*

4°. PROOF OF THEOREM 4.6. Due to the definition of singular integral operators $A \in \mathscr{L}^N_{sp}(M)$ and to Theorem 3.5, A has a regularizer (i.e. A is a Fredholm operator) if and only if $W^0_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^n)$ for all $x \in M\setminus\partial M$ and $W^1_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^{n+})$ for all $x \in \partial M$ have Δ_y-regularizers (cf. (3.8); here $y = \beta_j(x)$ for $x \in u_j$, $\bigcup\limits_{j=1}^{l} u_j = M \cup \partial M$; cf. § 4.1°).

We prove now that $W^0_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^n)$ $(x \in M\setminus\partial M)$ and $W^1_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^{n+})$ $(x \in \partial M)$ have Δ_y-regularizers if and only if $W^0_{a_x} \in \mathscr{L}^N_{0p}(\mathbf{R}^n) = \mathscr{L}(L^N_p(\mathbf{R}^n))$ and $W^1_{a_x g_s} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+}))$ have Δ_0-regularizers respectively, ($g_s(\xi)$ is defined by (3.3)).

Let $W^0_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^n)$ has a left Δ_0-regularizer:

$$(4.15)\qquad RW^0_{a_x} v_y I = v_y I + T, \quad v_y I \in \Delta_y,\ T \in \mathfrak{S}^N_{sp}(\mathbf{R}^n);$$

then (cf. § 3.1°)

$$V_y \Lambda^s R \Lambda^{-s} V_{-y} (V_y \Lambda^s W^0_{a_x} \Lambda^{-s} V_{-y}) V_y \Lambda^s a_0 \Lambda^{-s} V_{-y} = V_y \Lambda^s V_0 \Lambda^{-s} V_{-y} + V_y \Lambda^s T \Lambda^{-s} V_{-y},$$

where

$$V_{\pm y}\varphi(t) = \varphi(t \mp y);$$

obviously

$$V_y W^0_{a_x} V_{-y} = W^0_{a_x}, \quad V_y a_y V_{-y} \in \Delta_0$$

and using Lemmas 3.1, 3.3 we get

$$R_0 W^0_{a_x} v_0 I = v_0 I + T_0; \tag{4.16}$$

$R_0 = V_y \Lambda^s R \Lambda^{-s} V_{-y} \in \mathscr{L}^N_p(\mathbf{R}^n)$, $T_0 \in \mathfrak{S}(L^N_p(\mathbf{R}^n))$; hence $W^0_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ has a left Δ_0-regularizer.

Converting the argumentation from (4.16) we easily obtain (4.15).

Similarly, if $W^0_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^n)$ has a right Δ_0-regularizer, $W^0_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ will have a Δ_0-regularizer and vice versa.

Let now $W^1_{a_x} \in \mathscr{L}^N_{sp}(\mathbf{R}^n)$ has a left Δ-regularizer ($x \in \partial M, y = \beta_j(x)$),

$$R W^1_{a_x} v_y I = v_y T + T, \quad v_y I \in \Delta_y, \quad T \in \mathfrak{S}((H^{sp}_0)^N(\mathbf{R}^{n+})); \tag{4.17}$$

then (cf. § 3.1°)

$$R_0(V_y \tilde{\Lambda}^s_- W^1_{a_x} \tilde{\Lambda}^{-s}_+ V_{-y})(V_y \tilde{\Lambda}^s_+ a_0 \tilde{\Lambda}^{-s}_+ V_{-y}) = V_y \tilde{\Lambda}^s a_0 \tilde{\Lambda}^{-s} V_{-y} + T_0,$$

$$R_0 = V_y \tilde{\Lambda}^s_+ R \tilde{\Lambda}^{-s}_- V_{-y} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+})),$$

$$T_0 = V_y \tilde{\Lambda}^s_+ T \tilde{\Lambda}^{-s}_+ V_{-y} \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+})),$$

since $x \in \partial M$, $y = \beta_j(x) = (0, y_2, \dots, y_n)$ and therefore

$$V_y W^1_{a_x} V_{-y} = W^1_{a_x}, \quad V_y a_y V_{-y} \in \Delta_0$$

(if $y_1 \neq 0$, $V_{\pm y} \notin \mathscr{L}(L^N_p(\mathbf{R}^{n+}))$); using Lemmas 3.2 and 3.3 we get

$$R_0 W^1_{a_x} v_0 I = v_0 I + T_1, \quad T_1 \in \mathfrak{S}(L^N_p(\mathbf{R}^{n+})). \tag{4.18}$$

Converting the argumentation from (4.18) we easily obtain (4.17).

Similarly is considered the case of a right Δ_y-regularizer of $W^1_{a_x}$.

Summarizing we conclude: $A \in \mathscr{L}^N_{sp}(M)$ is a Fredholm operator if and only if $W^0_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ have Δ_0-regularizers for all $x \in M \setminus \partial M$ and $W^1_{a_x g_s} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+}))$ have Δ_0-regularizers for all $x \in \partial M$.

The proof of the theorem is completed now with the help of Lemma 3.7 and Theorem 2.7. ▨

5°. Proofs of theorems 4.7 and 4.8. These theorems, similarly to Theorem 4.6, are simple consequences of Theorems 2.7 — 2.8 and Lemma 3.7 if we notice (cf. (4.15) — (4.18)):

(i) if all $W^0_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ $(x \in M \setminus \partial M)$ and all $W^1_{a_x g_s} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+}))$ have a left (have a right) Δ_0-regularizer, A has a left (has a right) regularizer;

(ii) if $W^0_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^n))$ $(x \in M \setminus \partial M)$ or $W^1_{a_x} \in \mathscr{L}(L^N_p(\mathbf{R}^{n+}))$ $(x \in \partial M)$ is not a Fredholm operator for some $x \in M$, A will be not a Fredholm operator as well. ▨

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