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THE ITÔ-CLIFFORD INTEGRAL. IV: A RADON-NIKODYM THEOREM AND BRACKET PROCESSES

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0. INTRODUCTION

The construction and various properties of the Itô-Clifford stochastic integral have been discussed in [1, 2, 3]. In particular, it was shown in [1] that any centred L^2-martingale is given as an Itô-Clifford stochastic integral; $X_t = \int_0^t \tilde{X}(s)\,\mathrm{d}\Psi_s$, where $\Psi_s \equiv \Psi(\chi_{[0,s]})$ is the Fermi-field. It was also shown in [1] that stochastic integrals of the form $\int f\mathrm{d}X$ can be defined as elements of $L^2(\mathscr{C})$, the non-commutative L^2-space associated with the Clifford probability gage space. We consider the relationship between the stochastic integral with respect to Ψ and that with respect to X. Specifically, we prove a Radon-Nikodym theorem in the form: $\int f\mathrm{d}X = \int f\tilde{X}\mathrm{d}\Psi$.

Using the Doob-Meyer decomposition of the submartingale $X_t^*X_t$ given in [1], we define the pointed-bracket L^1-process $\langle X_t, Y_t\rangle$ associated with L^2-martingales (X_t) and (Y_t). The stochastic integral $\int f\,\mathrm{d}X$ is shown to be characterized, as a process, in terms of pointed-bracket processes. These results parallel those of standard (i.e. commutative) probability theory (see, for example [8,9]).

In Sections 1 and 2 we review and generalize some of the results from [1]. The stochastic integral $\int f\mathrm{d}X$ is defined in Section 3 — this being a simplified version of that in [1]. The Radon-Nikodym theorem is presented in Section 4, and in Section 5 an analogous result for stochastic integrals with respect to Wick martingales is proved. The pointed-bracket process is considered in Section 6, together with a characterization of the stochastic integral as a process.

Finally, in Section 7, we give a summary of the analogous results valid for "left" rather than "right" integrals.

A Doob-Meyer decomposition and stochastic integration with respect to martingales over an arbitrary probability gage space is considered in [4].

1. FOCK SPACE AND THE CLIFFORD ALGEBRA

We recall some notation and definitions from [1]. Let $\Lambda(L^2(\mathbf{R}^+))$ denote the antisymmetric Fock space over $L^2(\mathbf{R}^+)$, and, for $u \in L^2(\mathbf{R}^+)$, let $C(u)$ and $A(u) = C(u)^*$ denote the creation and annihilation operators on $\Lambda(L^2(\mathbf{R}^+))$. The fermion field is defined as $\Psi(u) = C(u) + A(\bar{u})$, where $\bar{u}$ is the complex-conjugate of u in $L^2(\mathbf{R}^+)$. The fermion fields satisfy the canonical anticommutation relations

$$\Psi(u)\Psi(v) + \Psi(v)\Psi(u) = 2(\bar{v}, u)\mathbf{1}. \tag{1.1}$$

For each $t \geqslant 0$, $\mathscr{C}_t$ denotes the von Neumann algebra generated by the fields $\Psi(u)$ for $u \in L^2(\mathbf{R}^+)$ with $\operatorname{supp} u \subseteq [0, t]$, and $\mathscr{C}$ is the von Neumann algebra generated by the increasing family $\{\mathscr{C}_t : t \geqslant 0\}$. $\mathscr{C}$ is the weakly closed Clifford algebra over $L^2(\mathbf{R}^+)$ [6, 11].

Let m denote the vector state $m(x) = (\Omega, x\Omega)$, $x \in \mathscr{C}$, where Ω is the Fock vacuum vector. Then m is a faithful, central state on $\mathscr{C}$.

For $1 \leqslant p < \infty$, $L^p(\mathscr{C})$ is the completion of $\mathscr{C}$ with respect to the norm $\|x\|_p = m(|x|^p)^{1/p}$, $x \in \mathscr{C}$, and $L^\infty(\mathscr{C})$ is $\mathscr{C}$ equipped with its C^*-norm.

The elements of $L^p(\mathscr{C})$ can be identified with closed (possibly unbounded) operators on $\Lambda(L^2(\mathbf{R}^+))$ [7,10]. In fact, $L^p(\mathscr{C})$ consists of those closed operators Y on $\Lambda(L^2(\mathbf{R}^+))$ affiliated to $\mathscr{C}$ such that Ω is in the domain of $|Y|^{p/2}$. Similarly, one defines $L^p(\mathscr{C}_t)$ for $t \geqslant 0$ and $1 \leqslant p \leqslant \infty$, so that $L^p(\mathscr{C}_t)$ is a closed subspace of $L^p(\mathscr{C})$. Indeed, if $\mathscr{B}$ is any von Neumann subalgebra of $\mathscr{C}$, $L^p(\mathscr{B})$ is a closed subspace of $L^p(\mathscr{C})$. The conditional expectation given $\mathscr{B}$ is a contraction of $L^p(\mathscr{C})$ onto $L^p(\mathscr{B})$ for all $1 \leqslant p \leqslant \infty$, and is denoted $m(\cdot|\mathscr{B})$. We will write M_t for $m(\cdot|\mathscr{C}_t)$, $t \geqslant 0$.

Definition 1.1. An *$L^p(\mathscr{C})$-martingale* is a family $\{X_t : t \geqslant 0\}$ with $X_t \in L^p(\mathscr{C})$ for all $t \geqslant 0$ and such that $M_t X_s = X_t$ for all $0 \leqslant t \leqslant s$. It follows that $X_t \in L^p(\mathscr{C}_t)$, $t \geqslant 0$.

It is convenient to define here the parity operator β and establish some of its properties. Let Q^0 denote the linear space of even polynomials in the fields $\Psi(u)$, $u \in L^2(\mathbf{R}^+)$, and let Q denote the von Neumann subalgebra of $\mathscr{C}$ generated by Q^0.

Definition 1.2. The *parity operator* β is the map $\beta : L^p(\mathscr{C}) \to L^p(\mathscr{C})$, $1 \leqslant p \leqslant \infty$, given by $\beta(f) = 2m(f \mid Q) - f$, $f \in L^p(\mathscr{C})$.

Evidently, $\beta\colon L^p(\mathscr{C}) \to L^p(\mathscr{C})$ is continuous, and $\beta(f)^* = \beta(f^*)$ for $f \in L^p(\mathscr{C})$. Moreover, using the fact that $m(m(\cdot|Q)|Q) = m(\cdot|Q)$, we see that $\beta^2(f) = f$ for $f \in L^p(\mathscr{C})$. Furthermore, since $m(\cdot|Q)$ defines the orthogonal projection of $L^2(\mathscr{C})$ onto $L^2(Q)$, it follows that $\beta : L^2(\mathscr{C}) \to L^2(\mathscr{C})$ is self-adjoint and unitary.

PROPOSITION 1.3. (i) *$\beta(f) = f$ for all $f \in L^p(Q)$, $1 \leqslant p \leqslant \infty$.*

(ii) *$\beta(x) = -x$ for any odd polynomial x in the fields $\Psi(u)$, $u \in L^2(\mathbf{R}^+)$.*

(iii) *$\beta\colon \mathscr{C} \to \mathscr{C}$ is σ-weakly continuous.*

(iv) *$\beta(hg) = \beta(h)\beta(g)$ for $h \in L^p(\mathscr{C})$, $g \in L^q(\mathscr{C})$, with $1/p + 1/q = 1$.*

(v) *$\beta\colon L^p(\mathscr{C}) \to L^p(\mathscr{C})$ is isometric for $1 \leqslant p \leqslant \infty$.*

(vi) *If $f \in L^1(\mathscr{C})$ and $f \geqslant 0$, then $\beta(f) \geqslant 0$.*

(vii) *$\beta\colon L^p(\mathscr{C}_t) = L^p(\mathscr{C}_t)$ for all $t \geqslant 0$, $1 \leqslant p \leqslant \infty$.*

Proof. (i) Trivial using $m(f \mid Q) = f$ for $f \in L^p(Q)$.

(ii) If x is an odd polynomial in the fields and $y \in Q^0$, one sees that $y\Omega$ and $x\Omega$ are orthogonal in $\Lambda(L^2(\mathbf{R}^+))$. Hence $m(y^*x) = 0$. By continuity it follows that $m(y^*x) = 0$ for all $y \in Q$, and so $m(x \mid Q) = 0$. Thus $\beta(x) = -x$.

(iii) By continuity and self-adjointness of β on $L^2(\mathscr{C})$, we have $m(g\beta(h)) = m(\beta(g)h)$ for $h \in L^p(\mathscr{C})$, $g \in L^q(\mathscr{C})$ with $1/p + 1/q = 1$. In particular, this holds for $q = 1$, $p = \infty$. But $L^1(\mathscr{C})$ is the predual of $\mathscr{C}$ [10] under the pairing $L^1(\mathscr{C}) \times L^\infty(\mathscr{C}) \to \mathbf{C}$, $(g, h) \to m(gh)$, and so we deduce that $\beta : L^\infty(\mathscr{C}) \to L^\infty(\mathscr{C})$ is σ-weakly continuous.

(iv) Any polynomial in the fields can be written as a linear combination of an odd and an even polynomial. Now if x', x'' are odd and y', y'' are even polynomials in the fields, we have that $x'x''$, $y'y''$ are even and $x'y''$ and $x''y'$ are odd. Hence, using (i) and (ii),

$$\beta((x' + y')(x'' + y'')) = \beta(x'x'' + x'y'' + y'x'' + y'y'') =$$

$$= x'x'' - x'y'' - y'x'' + y'y'' =$$

$$= \beta(x' + y')\,\beta(x'' + y'').$$

By continuity, it follows that $\beta(xy) = \beta(x)\beta(y)$ for any $x, y \in \mathscr{C}$ and, again by continuity, the result follows.

(v) From (iv), and the fact that $\beta(x) = 0$ implies that $\beta^2(x) = x = 0$, we see that β is an automorphism of $\mathscr{C}$. Hence $\beta : \mathscr{C} \to \mathscr{C}$ is isometric. By duality it follows that $\beta\colon L^1(\mathscr{C}) \to L^1(\mathscr{C})$ is a contraction, and so by interpolation [7] $\beta\colon L^p(\mathscr{C}) \to L^p(\mathscr{C})$ is a contraction for $1 \leqslant p < \infty$. Now, for $f \in L^p(\mathscr{C})$, $1 \leqslant p < \infty$,

$$\|f\|_p = \|\beta^2(f)\|_p \leqslant \|\beta(f)\|_p \leqslant \|f\|_p.$$

It follows that $\beta\colon L^p(\mathscr{C}) \to L^p(\mathscr{C})$ is an isometry, for $1 \leqslant p \leqslant \infty$.

(vi) If $f \in L^1(\mathscr{C})$ with $f \geqslant 0$, then, there is $g \in L^2(\mathscr{C})$ such that $f = g^* g$. Indeed, we can take $g = f^{1/2}$. Then, by (iv), $\beta(f) = \beta(g^* g) = \beta(g)^* \beta(g) \geqslant 0$.

(vii) This follows from (i), (ii), (iii) and (v). Q.E.D.

REMARK 1.4. Properties (i) and (ii) imply that the definition of β given here agrees with that in [1]. Indeed, the above results are readily obtained using the definition of β given in [1], but the proofs given here seem to be interesting in their own right.

2. THE ITÔ-CLIFFORD INTEGRAL

DEFINITION 2.1. An *$L^p(\mathscr{C})$-valued process on* $[0, t]$ is a map $f : [0, t] \to L^p(\mathscr{C})$ such that $f(s) \in L^p(\mathscr{C}_s)$ for $0 \leqslant s \leqslant t$. Such a process is said to be *elementary* if it is of the form $f = g\chi_{[r, \tau)}$ for some $0 \leqslant r < \tau \leqslant t$ and $g \in L^p(\mathscr{C})$. Note that $f(s) \in L^p(\mathscr{C}_s)$ implies that $g \in L^p(\mathscr{C}_r)$. An L^p-process on $[0, t]$ is said to be *simple* if, on $[0, t)$, it is a finite sum of elementary L^p-processes. Denote by $\mathscr{S}([0, t], L^p(\mathscr{C}))$ the linear space of simple $L^p(\mathscr{C})$-valued processes on $[0, t]$.

For $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$, set $\Psi_t(u) = \Psi(u\chi_{[0, t]})$. Then it was shown in [1] that $\{\Psi_t(u) : t \geqslant 0\}$ is an L^∞-martingale adapted to the filtration $\{\mathscr{C}_t : t \geqslant 0\}$.

DEFINITION 2.2. If $f = g\chi_{[r, \tau)}$, $0 \leqslant r < \tau \leqslant t$, is an elementary L^∞-valued process on $[0, t]$, the *Itô-Clifford stochastic integral of f with respect to $\Psi_s(u)$* is defined to be

$$\int_0^t f(s)\, \mathrm{d}\Psi_s(u) = g(\Psi_\tau(u) - \Psi_r(u)).$$

The integral $\int_0^t f(s)\, \mathrm{d}\Psi_s(u)$ for $f \in \mathscr{S}([0, t], L^\infty)$ is defined by linearity.

Let $\mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$ denote the subspace of processes in $L^2([0, t], |u(s)|^2 \mathrm{d}s; L^2(\mathscr{C}))$. Then as in [1], one sees that $\mathscr{S}([0, t], L^\infty)$ is dense in $\mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$, and that the Itô-Clifford stochastic integral is well-defined, by continuity, as an element of $L^2(\mathscr{C})$ for $f \in \mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$ and satisfies the isometry property:

$$\left\| \int_0^t f(s)\, \mathrm{d}\Psi_s(u) \right\|_2^2 = \int_0^t \|f(s)\|_2^2 |u(s)|^2 \mathrm{d}s. \tag{2.1}$$

Let $\mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, |u(s)|^2 \mathrm{d}s)$ denote those maps (i.e. classes of maps) f such that the restriction of f to each $[0, t]$, $t \geqslant 0$, belongs to $\mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$. Then it was shown in [1] that for each $f \in \mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, |u(s)|^2 \mathrm{d}s)$, $\left\{\int\limits_0^t f(s)\,\mathrm{d}\Psi_s(u) : t \geqslant 0\right\}$ is a centred L^2-martingale.

If $u(s) = 1$ for $s \in \mathbf{R}^+$, write Ψ_t for $\Psi_t(u)$. Using Theorem 4.1 of [1], we have the following representation theorem.

THEOREM 2.3. *Let (X_t) be an L^2-martingale (adapted to the family $\{\mathscr{C}_t : t \geqslant 0\}$). Then there is a unique element $\tilde{X}$ of $\mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, \mathrm{d}s)$ such that*

$$X_t = X_0 + \int\limits_0^t \tilde{X}(s)\,\mathrm{d}\Psi_s \quad \textit{for } t \geqslant 0.$$

Proof. Since $(X_t - X_0)$ is centred, the existence of $\tilde{X}$ follows from Theorem 4.1 of [1]. The uniqueness follows immediately from the isometry property equation (2.1).
Q.E.D.

Our first aim is to generalize this theorem.

LEMMA 2.4. *For $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$ and $h \in \mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$, we have*

$$\int\limits_0^t h(s)\,\mathrm{d}\Psi_s(u) = \int\limits_0^t h(s)u(s)\,\mathrm{d}\Psi_s. \tag{2.2}$$

Proof. We first note that $hu \in \mathfrak{H}([0, t], \mathrm{d}s)$ and so the right-hand side is well-defined. If h is elementary and $u = \chi_{[s_1, s_2]}$ for some $0 \leqslant s_1 \leqslant s_2 \leqslant t$, the result is clear. Now, for any $v \in L^2(\mathbf{R}^+)$, $\|\Psi(v)\|_\infty = \|v\|$, so $v \mapsto \Psi(v)$ is a linear continuous map from $L^2(\mathbf{R}^+)$ into $L^\infty(\mathscr{C})$. Hence, by linearity and continuity, the result holds for h elementary and $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$. But then again by linearity and continuity the result holds for arbitrary $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$ and $h \in \mathfrak{H}([0, t], |u(s)|^2 \mathrm{d}s)$. Q.E.D.

REMARK. This is a Radon-Nikodym theorem.

THEOREM 2.5. *Let $u \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$ and let (X_t) be an L^2-martingale. There is $g \in \mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, |u(s)|^2 \mathrm{d}s)$ such that $X_t = X_0 + \int\limits_0^t g(s)\,\mathrm{d}\Psi_s(u)$, $t \geqslant 0$, if and only if $\{s : \tilde{X}(s) \neq 0\} \subseteq \{s : u(s) \neq 0\}$ up to a set of (Lebesgue) measure zero. If such g exists, it is unique.*

Proof. Let $E = \{s \in \mathbf{R}^+ :\ u(s) \neq 0\}$, and set $v = u + \chi_{E^c}$. Then $v \neq 0$ (Lebesgue) almost everywhere, and so $h(s) = v(s)^{-1}\tilde{X}(s)$ is well-defined (Lebesgue) almost everywhere, and determines an element of $\mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, |v(s)|^2 \mathrm{d}s)$. If

$\{s : \tilde{X}(s)\neq 0\}\subseteq\{s : u(s)\neq 0\}$ up to a set of Lebesgue measure zero, we have $\tilde{X}=\chi_E\tilde{X}=\chi_E vh=\chi_E uh$ (Lebesgue) almost everywhere. Putting $g=\chi_E h$, we see that $g\in\mathfrak{H}_{\text{loc}}(\mathbf{R}^+, |u(s)|^2\,\mathrm{d}s)$, and by Lemma 2.4, for $t\geqslant 0$,

$$X_t - X_0 = \int_0^t \tilde{X}(s)\,\mathrm{d}\Psi_s = \int_0^t \chi_E(s)u(s)h(s)\,\mathrm{d}\Psi_s = \int_0^t g(s)\mathrm{d}\Psi_s(u).$$

The uniqueness of g follows from the isometry property, equation 2.1.

Conversely, suppose that there is $g\in\mathfrak{H}_{\text{loc}}(\mathbf{R}^+, |u(s)|^2\,\mathrm{d}s)$ such that $X_t = X_0 + \int_0^t g(s)\,\mathrm{d}\Psi_s(u)$, $t\geqslant 0$. Then, by Lemma 2.4,

$$X_t - X_0 = \int_0^t g(s)u(s)\,\mathrm{d}\Psi_s, \quad t\geqslant 0.$$

Hence, since $gu\in\mathfrak{H}_{\text{loc}}(\mathbf{R}^+,\mathrm{d}s)$, it follows from Theorem 2.3 that $\tilde{X}=gu$, and the proof is complete. Q.E.D.

REMARK. As is customary, for convenience (and brevity) we have treated elements of $L^2_{\text{loc}}(\mathbf{R}^+)$ and $\mathfrak{H}$ as if they were maps rather than equivalence classes of maps.

3. EXTENSION OF THE ITÔ-CLIFFORD INTEGRAL

We shall construct stochastic integrals with respect to an arbitrary L^2-martingale and, in the next section, relate them to the Itô-Clifford integral by means of a Radon-Nikodym theorem. The construction here is a simplified version of that given in § 7 of [1].

DEFINITION 3.1. For an L^2-martingale (X_t), let μ_X denote the Borel measure on $\mathbf{R}^+$ given by $\mu_X([0,t]) = \int_0^t \|\tilde{X}(s)\|_2^2\,\mathrm{d}s$, where $\tilde{X}$ is given by Theorem 2.3.

We shall sometimes use λ to denote Lebesgue measure on $\mathbf{R}^+$.

DEFINITION 3.2. Let $f=g\chi_{[r,\tau)}$ be an elementary L^∞-valued process on $[0,t]$. The *stochastic integral of f with respect to the L^2-martingale* (X_t) is $\int_0^t f(s)\,\mathrm{d}X_s = g(X_\tau - X_r)\in L^2(\mathscr{C})$. $\int_0^t f(s)\,\mathrm{d}X_s$ for $f\in\mathscr{S}([0,t], L^\infty)$ is defined by linearity.

We shall extend the definition of the stochastic integral to more general integrands using a contraction property. First we recall the following result from [1].

THEOREM 3.3. *If* (X_t) *is an* L^2*-martingale, then* $Z_t = X_t^* X_t - \int_0^t |\beta(\tilde{X}(s))|^2 \, ds$ *is an* L^1*-martingale.*

Proof. Use Theorem 2.3 and [1, Theorem 3.18]. Q.E.D.

THEOREM 3.4 (Contraction property). *For* $f \in \mathscr{S}([0, t], L^\infty)$, *we have*

(3.1) $$\left\| \int_0^t f(s) \, dX_s \right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2 \, d\mu_X(s).$$

Proof. Suppose that $f = \sum_{k=1}^n h_{k-1} \chi_{[t_{k-1}, t_k)}$ on $[0, t)$, with $0 \leqslant t_0 \leqslant \ldots \leqslant t_n = t$, is a simple L^∞-valued process. Then, writing ΔX_k for $X_{t_k} - X_{t_{k-1}}$, we have

$$\left\| \int_0^t f(s) \, dX_s \right\|_2^2 = \sum_{k,j} m(\Delta X_k^* h_{k-1}^* h_{j-1} \Delta X_j) = \sum_k m(\Delta X_k^* |h_{k-1}|^2 \Delta X_k) =$$

(using the martingale property)

$$= \sum_k m(|h_{k-1}|^2 \Delta X_k \Delta X_k^*) \leqslant \sum_k \|h_{k-1}\|_\infty^2 \|\Delta X_k^*\|_2^2 =$$

$$= \sum_k \|h_{k-1}\|_\infty^2 \|\Delta X_k\|_2^2 = \sum_k \|h_{k-1}\|_\infty^2 (\|X_{t_k}\|_2^2 - \|X_{t_{k-1}}\|_2^2) =$$ (using the martingale property)

$$= \sum_k \|h_{k-1}\|_\infty^2 \int_{t_{k-1}}^{t_k} \|\beta \tilde{X}(s)\|_2^2 \, ds =$$ (using Theorem 23.3)

$$= \sum_k \|h_{k-1}\|_\infty^2 \int_{t_{k-1}}^{t_k} \|\tilde{X}(s)\|_2^2 \, ds =$$ (since $\beta : L^2 \to L^2$ is isometric)

$$= \int_0^t \|f(s)\|_\infty^2 \, d\mu_X(s).$$ Q.E.D.

DEFINITION 3.5. Let $\mathscr{K}([0, t], \mu_X)$ denote the closure of $\mathscr{S}([0, t], L^\infty)$ in $L^2([0, t], d\mu_X; L^\infty(\mathscr{C}))$.

COROLLARY 3.6. *For any $f \in \mathscr{K}([0, t], \mu_X)$, let (g_n) be a sequence in $\mathscr{S}([0, t], L^\infty)$ such that $g_n \to f$ in $\mathscr{K}([0, t], \mu_X)$. Then there exists $L^2(\mathscr{C})$-$\lim_n \int_0^t g_n(s)\,\mathrm{d}X_s$. This limit is independent of the particular sequence (g_n) converging to f; and is denoted $\int_0^t f(s)\,\mathrm{d}X_s$. Furthermore*

$$\left\| \int_0^t f(s)\,\mathrm{d}X_s \right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2\,\mathrm{d}\mu_X(s).$$

Proof. This is an immediate consequence of Theorem 3.4. Q.E.D.

REMARK 3.7. It is not clear whether $\mathscr{K}([0, t], \mu_X)$ is the set of all processes in $L^2([0, t], \mathrm{d}\mu_X; L^\infty(\mathscr{C}))$ or not. The obstruction to the analogue of the proof in [1] for $\mathfrak{H}$ concerns the continuity of the map $s \mapsto M_s g$ for fixed $g \in L^\infty(\mathscr{C})$. This is continuous as a map: $\mathbf{R}^+ \to L^2(\mathscr{C})$ but we do not know whether or not it is also continuous as a map: $\mathbf{R}^+ \to L^\infty(\mathscr{C})$. However, using the continuity of $v \mapsto \Psi(v) \in L^\infty(\mathscr{C})$, it is not difficult to see that $s \mapsto M_s g$ is continuous as a map from $\mathbf{R}^+$ into $\mathfrak{A}$ for any $g \in \mathfrak{A}$, the C^*-algebra generated by the fields $\Psi(v)$, $v \in L^2(\mathbf{R}^+)$. Thus, by copying the proof of Theorem 3.9 in [1], we see that $\mathscr{K}([0, t], \mu_X)$ contains the set of processes in $L^2([0, t], \mathrm{d}\mu_X; \mathfrak{A})$.

If $f: \mathbf{R}^+ \to L^\infty(\mathscr{C})$ is such that the restriction of f to $[0, t]$ belongs to $\mathscr{K}([0, t], \mu_X)$ for each $t \geqslant 0$, then $f\tilde{X} \in \mathfrak{H}_{\text{loc}}(\mathbf{R}^+, \mathrm{d}s)$ and it is easy to see [1] that $\int_0^t f\mathrm{d}X$ is a centred $L^2(\mathscr{C})$-martingale.

4. A RADON-NIKODYM THEOREM

In this section, we shall relate the stochastic integral $\int_0^t f\mathrm{d}X$ to the Itô-Clifford stochastic integral. Indeed, Theorem 2.3 suggests that formally $\mathrm{d}X = \tilde{X}\mathrm{d}\Psi_s$ and so we would expect $\int_0^t f\,\mathrm{d}X = \int_0^t f\tilde{X}\mathrm{d}\Psi_s$. We shall see that this is the case.

LEMMA 4.1. *Let (X_t) be an L^2-martingale, and let $f \in \mathscr{S}([0, t], L^\infty(\mathscr{C}))$. Then*

$$\int_0^t f(s)\,\mathrm{d}X_s = \int_0^t f(s)\tilde{X}(s)\,\mathrm{d}\Psi_s$$

where $\tilde{X}$ is given by Theorem 2.3.

Proof. By linearity, we may suppose that f is elementary: say, $f = g\chi_{[r,\tau)}$, $0 \leqslant r < \tau \leqslant t$, $g \in L^\infty(\mathscr{C}_r)$. Then

$$\int_0^t f(s)\,dX_s = g(X_\tau - X_r) = g\int_r^\tau \tilde{X}(s)\,d\Psi_s = \int_r^\tau g\tilde{X}(s)\,d\Psi_s =$$

(since left-multiplication by an element of L^∞ is continuous from $L^2(\mathscr{C}) \to L^2(\mathscr{C})$)

$$= \int_0^t f(s)\tilde{X}(s)\,d\Psi_s. \qquad \text{Q.E.D.}$$

THEOREM 4.2 (Radon-Nikodym theorem for stochastic integrals). *Let (X_t) be an L^2-martingale and let $f \in \mathscr{K}([0, t], \mu_X)$. Then $f\tilde{X} \in \mathfrak{H}([0, t], ds)$ and*

$$\int_0^t f(s)\,dX_s = \int_0^t f(s)\tilde{X}(s)\,d\Psi_s.$$

Proof. Let (g_n) be a sequence in $\mathscr{S}([0, t], L^\infty(\mathscr{C}))$ such that $g_n \to f$ in $\mathscr{K}([0, t], \mu_X)$. By passing to a subsequence if necessary we may suppose that $(g_n(s) - f(s))\|\tilde{X}(s)\|_2^2 \to 0$ in $L^\infty(\mathscr{C})$ λ a.e., and hence $g_n\tilde{X} \to f\tilde{X}$ in $L^2(\mathscr{C})$ λ a.e. on $[0, t]$.

Now

$$\int_0^t f\,dX = L^2\text{-lim}\int_0^t g_n\,dX =$$

$$= L^2\text{-lim}\int_0^t g_n\tilde{X}d\Psi, \qquad \text{by Lemma 4.1.}$$

In particular, $(g_n\tilde{X})$ is a Cauchy sequence in $\mathfrak{H}([0, t], ds)$ and thus there is G such that $g_n\tilde{X} \to G$ in $\mathfrak{H}([0, t], ds)$.

Again, by passing to a subsequence if necessary, we may suppose that $g_n\tilde{X} \to G$ in $L^2(\mathscr{C})$ λ a.e. on $[0, t]$.

Hence $f\tilde{X} = G$ λ a.e. on $[0, t]$; i.e. $f\tilde{X} \in \mathfrak{H}([0, t], ds)$ and $g_n\tilde{X} \to f\tilde{X}$ in $\mathfrak{H}([0, t], ds)$.

By the isometry property, equation (2.1), it follows that $\int_0^t g_n\tilde{X}\,d\Psi \to \int_0^t f\tilde{X}\,d\Psi$ in $L^2(\mathscr{C})$ and the proof is complete. Q.E.D.

COROLLARY 4.3. *Let* (X_t) *be an* L^2*-martingale and let* $g \in \mathscr{K}([0, t], \mu_X)$ *for each* $t \geqslant 0$. *Then if* $Y_t = \int_0^t g(s)\mathrm{d}X_s$, $t \geqslant 0$, *we have*

$$\int_0^t f(s)\,\mathrm{d}Y_s = \int_0^t f(s)g(s)\tilde{X}(s)\,\mathrm{d}\Psi_s$$

for any $f \in \mathscr{K}([0, t], \mu_Y)$.

Proof. By the theorem, $\int f\mathrm{d}Y = \int f\tilde{Y}\mathrm{d}\Psi$. But $Y = \int g\,\mathrm{d}X = \int g\tilde{X}\mathrm{d}\Psi$, again by the theorem. Hence $\tilde{Y} = g\tilde{X}$ and the result follows. Q.E.D.

5. STOCHASTIC INTEGRALS WITH RESPECT TO WICK MONOMIAL MARTINGALES

For given real $u_1, \ldots, u_n \in L^2_{\mathrm{loc}}(\mathbf{R}^+)$, the Wick monomial martingale $W(u_1, \ldots, u_n; s)$, $s \geqslant 0$ is defined as $W(u_1, \ldots, u_n; s) = :\Psi(u_1\chi_{[0,s]}) \ldots \Psi(u_n\chi_{[0,s]}):$, where $:\ldots:$ denotes Wick ordering. It was shown in [1] that $W_s = W(u_1, \ldots, u_n; s)$, $s \geqslant 0$ is an L^∞-martingale, and that stochastic integrals $\int_0^t f(s)\,\mathrm{d}W_s$ can be constructed as elements of $L^2(\mathscr{C})$ for $f \in \mathfrak{H}([0, t], \mathrm{d}\nu)$, the set of processes in $L^2([0, t], \mathrm{d}\nu; L^2(\mathscr{C}))$, where ν is the measure on $\mathbf{R}^+$ given by $\nu([s, t]) = a_t - a_s$ and $a_s \in \mathbf{R}$ is $a_s = W_s^* W_s = W_s W_s^*$.

Writing $W_t = \int_0^t \tilde{W}(s)\,\mathrm{d}\Psi_s$, we have, by the isometry property

$$a_t = m(W_t^* W_t) = \|W_t\|_2^2 = \int_0^t \|\tilde{W}(s)\|_2^2\,\mathrm{d}s.$$

Hence ν is the measure μ_W.

The closure of $\mathscr{S}([0, t], L^\infty(\mathscr{C}))$ in $L^2([0, t], \mathrm{d}\mu_W; L^2(\mathscr{C}))$ is equal to $\mathfrak{H}([0, t], \mathrm{d}\mu_W)$ and so Theorem 4.3 has a sharper analogue here.

THEOREM 5.1. *Let* $f \in \mathfrak{H}([0, t], \mathrm{d}\mu_W)$. *Then* $f\tilde{W} \in \mathfrak{H}([0, t], \mathrm{d}s)$ *and*

$$\int_0^t f(s)\,\mathrm{d}W_s = \int_0^t f(s)\tilde{W}(s)\,\mathrm{d}\Psi_s.$$

Proof. This is analogous to that of Theorem 4.2. Q.E.D.

REMARK 5.2. The isometry property for $f \in \mathfrak{H}([0, t], \mathrm{d}\mu_W)$ can be written as

$$\left\| \int_0^t f \,\mathrm{d}W \right\|_2^2 = \int_0^t \|f(s)\|_2^2 \, \|\tilde{W}(s)\|_2^2 \,\mathrm{d}s.$$

This follows because $\mathrm{d}\nu = \mathrm{d}\mu_W = \|\tilde{W}(s)\|_2^2 \,\mathrm{d}s$. We see from the theorem that

$$\left\| \int_0^t f \,\mathrm{d}W \right\|_2^2 = \int_0^t \|f(s)\tilde{W}(s)\|_2^2 \,\mathrm{d}s$$

and so

$$\int_0^t \|f(s)\tilde{W}(s)\|_2^2 \,\mathrm{d}s = \int_0^t \|f(s)\|_2^2 \|\tilde{W}(s)\|_2^2 \,\mathrm{d}s.$$

We can formulate an analogue of Theorem 3.18 of [1].

THEOREM 5.3. *Let* $f \in \mathfrak{H}([0, t], \mathrm{d}\mu_W)$. *Then* $Z_t = \left| \int_0^t f \,\mathrm{d}W \right|^2 - \int_0^t |\beta(f(s)\tilde{W}(s))|^2 \,\mathrm{d}s$ *is a centred* L^1*-martingale (on* [0, t]*)*.

Proof. Use Theorem 5.1 together with [1, Theorem 3.18]. Q.E.D.

As a corollary, we note that $\left(\int_0^t f \,\mathrm{d}W\right)$ has a Doob-Meyer type decomposition for any $f \in \mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, \mathrm{d}\mu_W)$.

6. THE POINTED-BRACKET PROCESS

We shall define the so-called pointed-bracket process corresponding to a pair of L^2-martingales, and give a characterization of the process given by the Itô--Clifford integral.

If $f, g \in L^2_{\mathrm{loc}}(\mathbf{R}^+, \mathrm{d}\lambda; L^2(\mathscr{C}))$, then $fg \in L^1_{\mathrm{loc}}(\mathbf{R}^+, \mathrm{d}\lambda; L^1(\mathscr{C}))$, and so we may make the following definition.

DEFINITION 6.1. Let $X_t = X_0 + \int_0^t \tilde{X} \,\mathrm{d}\Psi$ and $Y_t = Y_0 + \int_0^t \tilde{Y} \,\mathrm{d}\Psi$, $t \geqslant 0$, be L^2-martingales. The *pointed-bracket* between (X_t) and (Y_t) is the $L^1(\mathscr{C})$-process

$$\langle X_t, Y_t \rangle = \int_0^t \beta(\tilde{Y}(s))^* \beta(\tilde{X}(s)) \,\mathrm{d}s, \quad t \geqslant 0.$$

Notice that $\langle X_t, X_t\rangle$ is the increasing L^1-process given in the Doob-Meyer decomposition of $X_t^* X_t$ by Theorem 3.3.

Clearly, $\langle\cdot,\cdot\rangle$ is a sesquilinear map from $\mathfrak{M}\times\mathfrak{M}$ into the set of processes in $L^1_{\text{loc}}(\mathbf{R}^+, \mathrm{d}\lambda; L^1(\mathscr{C}))$, where $\mathfrak{M}$ denotes the set of L^2-martingales.

For any bounded Borel set E in $\mathbf{R}^+$, set $\langle X, Y\rangle(E) = \int_E \beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s.$
Then $\langle X, Y\rangle$ defines an $L^1(\mathscr{C})$-valued countably additive Borel vector measure on any bounded interval $[0, t]$ in $\mathbf{R}^+$.

LEMMA 6.2. *For $X, Y\in\mathfrak{M}$, $\langle X, Y\rangle$ has bounded variation on any interval* $[0, t]$ *in* $\mathbf{R}^+$, *and is continuous with respect to Lebesgue measure on the Borel sets in* $[0, t]$.

Proof. By polarization, it is enough to show that $\langle X, X\rangle$ has bounded variation on $[0, t]$. Now, the variation of $\langle X, X\rangle$ on $[0, t]$ is defined to be the set function

$$E \mapsto |\langle X, X\rangle|(E) = \sup \sum_i \|\langle X, X\rangle(E_i)\|_1$$

where E is any Borel set in $[0, t]$, and the supremum is taken over all partitions of E into a finite number of disjoint Borel sets.

But for any such partition $\{E_i\}$ of a Borel set E in $[0, t]$, we have

$$\sum_i \|\langle X, X\rangle(E_i)\|_1 = \sum_i \left\|\int_{E_i}\beta(\tilde{X}(s))^*\beta(\tilde{X}(s))\mathrm{d}s\right\|_1 =$$

$$= \sum_i m\left(\int_{E_i}|\beta(\tilde{X}(s))|^2\,\mathrm{d}s\right) =$$

$\left(\text{since} \int_{E_i}\beta(\tilde{X}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s \text{ is a non-negative element of } L^1(\mathscr{C})\right)$,

$$= m\left(\sum_i \int_{E_i}|\beta(\tilde{X}(s))|^2\,\mathrm{d}s\right) = \left\|\int_E|\beta(\tilde{X}(s))|^2\,\mathrm{d}s\right\|_1 = \|\langle X, X\rangle(E)\|_1.$$

Hence, for any $t\geqslant 0$, $|\langle X, X\rangle|([0, t]) = \|\langle X, X\rangle([0, t])\|_1$, and so $\langle X, X\rangle$ has finite variation on $[0, t]$. The same is then true of $\langle X, Y\rangle$.

If E is a Borel set in $[0, t]$ with Lebesgue measure $\lambda(E) = 0$, then

$$\langle X, Y\rangle(E) = \int_E \beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s = 0. \qquad \text{Q.E.D.}$$

For suitable $L^\infty(\mathscr{C})$-valued maps f, one can define the $L^1(\mathscr{C})$-valued Bartle-integral $\int f\,\mathrm{d}\langle X, Y\rangle$.

LEMMA 6.3. *Let $t \geqslant 0$, and suppose that f: $[0, t] \to L^{\infty}(\mathscr{C})$ is the limit λ a.e. on $[0, t]$ of a uniformly bounded sequence (g_n) in $\mathscr{S}([0, t], L^{\infty}(\mathscr{C}))$. Then, for any $X, Y \in \mathfrak{M}$, f is integrable with respect to $\langle X, Y\rangle$ and*

$$L^1(\mathscr{C})\text{-lim}\int_E g_n \,\mathrm{d}\langle X, Y\rangle = \int_E f\mathrm{d}\langle X, Y\rangle$$

exists, where $\int_E f\mathrm{d}\langle X, Y\rangle$ is the Bartle integral.

Proof. This follows immediately from Bartle's bounded convergence theorem [5]. Q.E.D.

DEFINITION 6.4. Let $\mathscr{P}[0, t]$ denote the set of maps f: $[0, t] \to L^{\infty}(\mathscr{C})$ satisfying the hypothesis of Lemma 6.3.

LEMMA 6.5. *For $f \in \mathscr{P}[0, t]$ and $X \in \mathfrak{M}$, we have*

(i) $\beta(f) \in \mathscr{P}[0, t]$;

(ii) $\beta(f)\tilde{X} \in \mathfrak{H}([0, t], \mathrm{d}s; L^2(\mathscr{C}))$;

(iii) $\int_0^t \tilde{X}(s)\,\mathrm{d}\Psi_s = \left(\int_0^t \beta(\tilde{X}(s))^*\mathrm{d}\Psi_s\right)^*.$

Proof. (i) If (g_n) is a uniformly bounded sequence in $\mathscr{S}([0, t], L^{\infty}(\mathscr{C}))$ such that $\|g_n(s) - f(s)\|_{\infty} \to 0$ λ a.e. on $[0, t]$, then, using Proposition 1.3, it follows that $(\beta(g_n))$ is also a uniformly bounded sequence in $\mathscr{S}([0, t], L^{\infty}(\mathscr{C}))$ and $\beta(g_n) \to \beta(f)$ in $L^{\infty}(\mathscr{C})$ λ a.e. on $[0, t]$. In other words, $\beta(f) \in \mathscr{P}[0, t]$.

(ii) There is a sequence (h_n) in $\mathscr{S}([0, t], L^{\infty}(\mathscr{C}))$ such that $h_n \to \tilde{X}$ in $\mathfrak{H}([0, t], \mathrm{d}s; L^2(\mathscr{C}))$. Then with the notation of part (i) above, and passing to a subsequence if necessary, we see that $\beta(g_n)h_n \to \beta(f)\tilde{X}$ in $L^2(\mathscr{C})$ λ a.e. on $[0,t]$. Hence $\beta(f)\tilde{X}$ is a λ measurable L^2-process on $[0, t]$.

It is easy to see that $\beta(g_n)h_n$ is a Cauchy sequence in $\mathfrak{H}([0, t], \mathrm{d}s)$ and so the result follows.

(iii) Using the canonical anticommutation relations, equation (1.1), and Proposition 1.3, one sees [1, Lemma 3.15] that for any $0 \leqslant r \leqslant s$ and $g \in L^{\infty}(\mathscr{C}_r)$,

$$(g\Psi(\chi_{(r,s]}))^* = \Psi(\chi_{(r,s]})g^* = \beta(g)^*\Psi(\chi_{(r,s]}).$$

That is, $(g(\Psi_s - \Psi_r))^* = \beta(g)^*(\Psi_s - \Psi_r)$.

From the definition of the stochastic integral and the continuity of the adjoint $*: L^2(\mathscr{C}) \to L^2(\mathscr{C})$, it follows that $\left(\int_0^t h(s)\,\mathrm{d}\Psi_s\right)^* = \int_0^t \beta(h(s))^*\,\mathrm{d}\Psi_s$ for any $t \geqslant 0$ and $h \in \mathfrak{H}([0, t], \mathrm{d}s)$. Q.E.D.

THEOREM 6.6. *Let (X_t), (Y_t) be $L^2(\mathscr{C})$-martingales. Then for any $t \geqslant 0$ and $f \in \mathscr{K}([0, t], \mu_X) \cap \mathscr{K}([0, t], \mu_Y)$ we have*

$$\left\langle X_t, \int_0^t f(s)\,\mathrm{d}Y_s \right\rangle = \left\langle \int_0^t f(s)^*\,\mathrm{d}X_s, Y_t \right\rangle.$$

Proof. For $f \in \mathscr{K}([0, t], \mu_X) \cap \mathscr{K}([0, t], \mu_Y)$, we have $\int_0^t f\,\mathrm{d}Y = \int_0^t f\tilde{Y}\,\mathrm{d}\Psi$ and $\int_0^t f^*\,\mathrm{d}X = \int_0^t f^*\tilde{X}\,\mathrm{d}\Psi$, by Theorem 4.2. Hence, by definition

$$\left\langle X_t, \int_0^t f\,\mathrm{d}Y \right\rangle = \int_0^t \beta(f(s)\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s =$$

$$= \int_0^t \beta(\tilde{Y}(s))^*\beta(f(s)^*\tilde{X}(s))\,\mathrm{d}s = \left\langle \int_0^t f^*\,\mathrm{d}X, Y_t \right\rangle. \qquad \text{Q.E.D.}$$

THEOREM 6.7. *Let (X_t), (Y_t) be $L^2(\mathscr{C})$-martingales and let $f \in \mathscr{P}[0, t]$. Then*

(i) $\int_0^t f\,\mathrm{d}\langle X, Y\rangle = \left\langle X_t, \left(\int_0^t f(s)\,\mathrm{d}Y_s^*\right)^* \right\rangle,$

and

(ii) $\int_0^t f\,\mathrm{d}\langle X, Y\rangle^* = \left\langle \left(\int_0^t f(s)\,\mathrm{d}X_s^*\right)^*, Y_t \right\rangle^*.$

Proof. Suppose that $f = g\chi_{[r,\tau)}$ is elementary, where $0 \leqslant r \leqslant \tau \leqslant t$ and $g \in L^\infty(\mathscr{C}_r)$. Then

$$\int_0^t f\,\mathrm{d}\langle X, Y\rangle = g\int_r^\tau \beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s =$$

$$= \int_0^t g\chi_{[r,\tau)}\,\beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s = \int_0^t f(s)\beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s.$$

By linearity and continuity, this equality holds for any $f \in \mathscr{P}[0, t]$.

Now, by Lemma 6.5 (iii),

$$Y_t^* = Y_0^* + \left(\int_0^t \tilde{Y}(s)\,\mathrm{d}\Psi_s\right)^* = Y_0^* + \int_0^t \beta(\tilde{Y}(s))^*\,\mathrm{d}\Psi_s.$$

But, for $h \in L^2(\mathscr{C})$, $\|\beta(h)^*\|_2 = \|h\|_2$ and so $\mu_Y = \mu_{Y^*}$. Moreover, $\mathscr{P}[0,t] \subseteq \subseteq \mathscr{K}([0,t], \mu_Y)$ and thus $\int_0^t f\,\mathrm{d}Y^* = \int_0^t f(s)\beta(\tilde{Y}(s))^*\,\mathrm{d}\Psi_s$. Thus, again by Lemma 6.5 (iii), we have

$$\left(\int_0^t f\,\mathrm{d}Y^*\right)^* = \int_0^t \beta\{\beta(\tilde{Y}(s))f(s)^*\}\,\mathrm{d}\Psi_s = \int_0^t \tilde{Y}(s)\beta(f(s))^*\,\mathrm{d}\Psi_s.$$

Hence

$$\left\langle X_t, \left(\int_0^t f\,\mathrm{d}Y^*\right)^*\right\rangle = \int_0^t \beta(\tilde{Y}(s)\beta(f(s))^*)^*\beta(\tilde{X}(s))\,\mathrm{d}s = \int_0^t f(s)\beta(\tilde{Y}(s))^*\beta(\tilde{X}(s))\,\mathrm{d}s.$$

This proves (i).

The proof of (ii) follows from (i) using $\langle X_t, Y_t\rangle^* = \langle Y_t, X_t\rangle$. Q.E.D.

This theorem has as a corollary a characterization of the stochastic integral in terms of pointed-bracket processes.

THEOREM 6.8. *Let (X_t) be an $L^2(\mathscr{C})$-martingale, and let $f\colon \mathbf{R}^+ \to L^\infty(\mathscr{C})$ be such that the restriction of f to $[0, t]$ belongs to $\mathscr{P}[0, t]$ for all $t \geqslant 0$. Then $\left(\int_0^t f\,\mathrm{d}X\right)^*$ is the unique centred $L^2(\mathscr{C})$-martingale, Z_t, say, such that*

$$\int_0^t f\mathrm{d}\langle Y, X^*\rangle = \langle Y_t, Z_t\rangle$$

for any $L^2(\mathscr{C})$-martingale (Y_t).

Proof. By Theorem 6.7, (Z_t) satisfies

$$\int_0^t f\mathrm{d}\langle Y, X^*\rangle = \langle Y_t, Z_t\rangle.$$

If (Z'_t) is a centred L^2-martingale also satisfying this equation, then $\langle Y_t, (Z_t - Z'_t)\rangle = 0$ for all $t \geqslant 0$ and for all $Y \in \mathfrak{M}$. Hence

$$\langle Z_t - Z'_t, Z_t - Z'_t\rangle = 0 \quad \text{for all } t \geqslant 0,$$

which implies that $\int_0^t |\beta(\tilde{Z}(s) - \tilde{Z}'(s))|^2 \mathrm{d}s = 0$ for $t \geqslant 0$. Thus $\tilde{Z} = \tilde{Z}'$ in $\mathfrak{H}_{\mathrm{loc}}(\mathbf{R}^+, \mathrm{d}s)$. Since (Z_t) and (Z'_t) are both centred, it follows that they are equal. Q.E.D.

7. THE LEFT STOCHASTIC INTEGRAL

Results analogous to those of the preceding sections can be obtained for the left stochastic integral $\int \mathrm{d}Xf$ which is defined similarly to the right stochastic integral. Indeed, if (X_t) is an L^2-martingale, then, in terms of the left stochastic integral, Theorem 2.3 becomes

$$X_t = X_0 + \int_0^t \tilde{X}(s)\,\mathrm{d}\Psi_s = X_0 + \int_0^t \mathrm{d}\Psi_s\,\beta(\tilde{X}(s)), \quad t \geqslant 0.$$

The analogues of Theorem 3.4 and Corollary 3.6 yield the contraction property:

$$\left\| \int_0^t \mathrm{d}X_s f(s) \right\|_2^2 \leqslant \int_0^t \|f(s)\|_\infty^2 \,\mathrm{d}\mu_X(s)$$

for $f \in \mathscr{K}([0, t], \mu_X)$.

The analogue of Theorem 4.2 is:

THEOREM 7.1. *Let (X_t) be an L^2-martingale and let $f \in \mathscr{K}([0, t], \mu_X)$. Then $\beta(\tilde{X})f \in \mathfrak{H}([0, t], \mathrm{d}s)$ and*

$$\int_0^t \mathrm{d}X_s f(s) = \int_0^t \mathrm{d}\Psi_s\,\beta(\tilde{X}(s))f(s).$$

For left integrals, Theorem 6.6 becomes as follows.

THEOREM 7.2. *Let (X_t), (Y_t) be $L^2(\mathscr{C})$-martingales. Then for $t \geqslant 0$ and $f \in \mathscr{K}([0, t], \mu_X) \cap \mathscr{K}([0, t], \mu_Y)$ we have*

$$\left\langle \left(\int_0^t \mathrm{d}Y_s f(s) \right)^*, X_t \right\rangle = \left\langle Y_t^*, \left(\int_0^t \mathrm{d}X_s^* f(s)^* \right)^* \right\rangle.$$

THEOREM 7.3. *Let (X_t), (Y_t) be $L^2(\mathscr{C})$-martingales and let $f \in \mathscr{P}[0, t]$. Then*

$$\int_0^t \mathrm{d}\langle X, Y\rangle f = \left\langle \int_0^t \mathrm{d}X_s f(s), Y_t \right\rangle.$$

Proof. As for that of Theorem 6.7.

THEOREM 7.4. *Let (X_t) be an $L^2(\mathscr{C})$-martingale, and suppose that $f \in \mathscr{P}[0, t]$ for all $t \geqslant 0$. Then $\left(\int_0^t \mathrm{d}Xf\right)$ is the unique centred $L^2(\mathscr{C})$-martingale, N_t, say, such that*

$$\int_0^t \mathrm{d}\langle X, Y\rangle f = \langle N_t, Y_t\rangle.$$

Proof. Just as for that of Theorem 6.8.

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